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The Bohr Topology of Discrete Nonabelian Groups

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Abstract. We look at finitely generated *Bohr groups* $G^\sharp$, i. e., groups G equipped with the topology inherited from their Bohr compactification bG. Among other things, the following results are proved: every finitely generated group without free nonabelian subgroups either contains nontrivial convergent sequences in $G^\sharp$ or is a finite extension of an abelian group; every group containing the free nonabelian group with two generators does not have the extension property for finite dimensional representations, therefore, it does not belong to the class $\mathcal{D}$ introduced by D. Poguntke in: Zwei Klassen lokalkompakter maximal fastperiodischer Gruppen, Monatsh. Math. 81 (1976), 15–40; if G is a countable FC group, then the topology that the commutator subgroup $[G, G]$ inherits from $G^\sharp$ is residually finite and metrizable.
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1. Introduction

We look at finitely generated *Bohr groups* $G^\sharp$. That is, groups G equipped with the topology inherited from their Bohr compactification bG. This topology is called *Bohr topology* and coincides with the weak or initial topology generated by all finite dimensional unitary representations of G. It is a well known fact (due to Glicksberg [13] and Leptin [23]) that, when G is abelian, then $G^\sharp$ contains no nontrivial Bohr convergent sequences. Improving this result considerably, van Douwen proved [10] that, for every infinite subset A of an abelian group G there is $B \subset A$ with $|B| = |A|$ such that B is an interpolation set for bG. This means that every bounded function on B is the restriction to B of a continuous function on bG (in other words, B is C^*-embedded in bG). As a matter of fact, these interpolation sets were called I_0-*sets* by Hartman and Ryll-Nardzewski who were the first ones to investigate them. Subsequently I_0-sets have also been called *Hartman and Ryll-Nardzewski* sets by some authors. It turns out that

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van Douwen and Glicksberg-Leptin results do not extend trivially to nonabelian groups. Indeed, let us say that a sequence $\{x_n\}_{n<\omega}$ is *nontrivial* when it takes infinitely many distinct values. The following alternative *a la Rosenthal* holds for nonabelian groups [11]:

Remark 1.1. Let $\{x_n\}_{n<\omega}$ be a nontrivial sequence in a group G then either $\{x_n\}_{n<\omega}$ has a Bohr Cauchy subsequence, or $\{x_n\}_{n<\omega}$ has an infinite interpolation subset.

As consequence, we say that a group G is *van Douwen* (*vD* group, for short) when every nontrivial sequence $\{x_n\}_{n<\omega}$ in G contains an infinite interpolation subset. The main question we are concerned here is the identification of vD groups. Van Douwen himself proved that abelian groups are vD and, from [25] or [29] and using statement 1.1, it follows that every finite extension of an abelian group is vD. Moreover, it is readily seen that a group G is vD if, and only if, it contains no nontrivial convergent sequence in $G^\sharp$ (that is, it contains no Bohr convergent sequence). Hence, the question of identifying vD groups is equivalent to characterize which groups contain nontrivial Bohr convergent sequences (see Question 971 in [12]). On the other hand, following the terminology in [3], it is easily verified that the class vD is defined by a Markov property. Therefore, according to a well known result of Adyan, there is no algorithm which decides (by means of an effective process) whether or not any finitely presented group is vD see [1, 2, 28]). In other words, even for finitely presented groups, one should not expect to find a sequence $\{w_n\}_{n<\omega}$ of abstract group words that verifies the existence of Bohr convergent sequences in every group which is not vD. Therefore, one must find convergent sequences *ad hoc* for different classes of groups. Our main result here is as follows: every finitely generated group containing no free nonabelian subgroup is vD if, and only if, the group is a finite extension of an abelian group. In a different direction Poguntke defined in [27] the class $\mathcal{D}$ of groups G such that every finite dimensional representation defined on a subgroup H of G may be extended to a finite dimensional representation (of different dimension in general) defined on the whole group G. (this class $\mathcal{D}$ is also characterized by a Markov property in the sense of [3]). Poguntke himself has shown that every solvable locally compact group in $\mathcal{D}$ is Moore (irreducible unitary representations are finite dimensional). Along this line, we prove that the free group of two generators is not in $\mathcal{D}$. This yields the following characterization of (finitely generated) groups which are finite extensions of abelian groups: a finitely generated group G is a finite extension of an abelian group if and only if it belongs to $vD \cap \mathcal{D}$. Finally, we consider groups with finite conjugacy classes (FC groups). We show that, if G is a countable FC group, then the topology that $[G, G]$ inherits from $G^\sharp$ is residually finite and metrizable. Other properties of Bohr groups are also discussed.

2. Basic Definitions and Facts

The Bohr compactification and topology of abelian groups has been extensively studied since the publication of a seminal paper due to van Douwen [10]. As a consequence we now know of many of its properties and how they relate to other areas, especially Harmonic Analysis (see [5, 12]). Nevertheless, there has been no comparable accomplishments in the research of the Bohr compactification for nonabelian groups and the main obstruction here is the existence of several basic questions that are still waiting for a solution (cf. [12]). Here, we look at the Bohr compactification of discrete nonabelian groups and we focus on the question of determining to what extent the results concerning the Bohr topology of abelian groups can be extended to the noncommutative context. Previous contributions to these program have been given by Chu [4], Heyer [19, 20], Landstad [22], Moran [25], Moskowitz [26], and Poguntke [27]. Recent contributions to the subject can be found in [6, 8, 11, 14, 15, 16, 17, 29, 31, 35].

In principle, all groups are assumed to be maximally almost periodic (MAP groups, for short). That is, groups that can be injected into compact groups. For any group G and x, y in G, recall that the *commutator* of x and y is $[x,y] = xyx^{-1}y^{-1}$. In general, if A and B are subsets of G, the symbol $[A,B]$ denotes the subgroup generated by the elements of the form $[a,b]$ with $a \in A$ and $b \in B$. The symbol G' means the *commutator or derived subgroup* of G; that is $[G,G] = [G,G]$. The group G is *residually finite* if for every non-identity element g of G there exists a normal subgroup N of finite index in G such that $g \notin N$. Incidently, every finitely generated MAP group is residually finite (see [9, p. 169]). A group G is called an FC-group if every conjugacy class of G is finite.

The Bohr compactification of an arbitrary topological group can be defined as a pair (bG, b^G) where bG is a compact Hausdorff group and b^G is a continuous homomorphism from G onto a dense subgroup of bG with the following universal property: for every continuous homomorphism h from G into a compact group K there is a continuous homomorphism h^b from bG into K extending h in the sense that $h = h^b \circ b^G$, that is, making the following diagram commutative:

$$\begin{tikzcd} G \arrow[rr, "b^G"] \arrow[dr, "h"'] & & bG \arrow[dl, "h^b"] \\ & K & \end{tikzcd}$$

The group bG is essentially unique but it can be realized in different ways. Perhaps the most illuminating is the one which depends on finite dimensional unitary representations. Next, we sketch the basic facts of this construction.

Let G be a topological group, denote by G^x_n the set of all continuous n-dimensional unitary representations of G, i. e., the set of all continuous homomorphisms of G into the unitary group $U(n)$. The discrete space $G^x = \sqcup_{n<\omega} G^x_n$ will be called the *Bohr dual* of G (see [4] or [19] for details).

If we define $\mathcal{U} = \sqcup_{n<\omega} U(n)$ (topological sum), a *Bohr-representation* of G^x is a mapping $p : G^x \longrightarrow \mathcal{U}$ conserving the main operations between unitary representations: direct sums, tensor products, unitary equivalence and sending the elements of G^x_n into $U(n)$ for all $n \in \mathbb{N}$. The set of all Bohr-representations of G equipped with the point-open topology is a compact group with pointwise multiplication as the composition law. This compact group is a realization of the Bohr compactification bG introduced above. Thus, a neighbourhood base of the identity in bG consists of sets of the form $[F_n, V] = \{p \in bG : p(F_n) \subset V\}$, where V is any neighbourhood of the identity in $U(n)$ and F_n is any finite subset of G^x_n, $n \in \mathbb{N}$. Heyer [19, V, §14] contains a careful examination of bG and its properties. The group G inherits the topology induced by the above homomorphism b, the so-called *Bohr topology*, which is Hausdorff precisely when G is maximally almost periodic (*MAP* group), equivalently, when b is one-to-one. Here, we will be mainly concerned with this class of groups; that is to say, groups whose finite dimensional representations separate points. The term *Bohr group*, denoted $G^\sharp$, stands for a group G equipped with the Bohr topology. It is readily seen that each member of G^x defines a continuous mapping on $G^\sharp$ that extends to bG. Thus, the representation spaces G^x and $(bG)^x$ have exactly the same underlying set.

Let G be a topological group, and let H be a subgroup of G. If D and E are representations of G and H in the Hilbert spaces $H(D)$ and $H(E)$ respectively. It is said that D *extends* E if $H(E)$ is contained in $H(D)$ and if $D(x)[\xi] = E(x)[\xi]$ for all $x \in H$ and all $\xi \in H(E)$. According to Poguntke [27], a locally compact group G is in the *class* $\mathcal{D}$ when, for each closed subgroup H of G, every continuous finite dimensional representation of H can be extended to representation of G. More generally, let G be a topological group and let H be a (topological) subgroup of it. We say that H is *dually embedded* in G when every continuous finite dimensional representation of H can be extended to a continuous finite dimensional representation of G. Thus, Poguntke's class $\mathcal{D}$ can also be defined as the class of locally compact groups such that every closed subgroup is dually embedded. We have the following characterization of dually embedded subgroups (see [24, 27]).

Theorem 2.1. *Let G be a topological group, and let H be a closed subgroup of G. The following conditions are equivalent.*

(i) H is dually embedded in G.

(ii) Every almost periodic positive-definite continuous function on H has an almost periodic positive-definite continuous extension to G.

(iii) Every finite-dimensional irreducible continuous representation of H extends to a finite-dimensional irreducible continuous representation of G.

(iv) The map $j^b : bH \longrightarrow bG$, where j is the natural injection of H into G, is 1-1, and hence an isomorphism of bH into bG.

3. Bohr groups

In this section we recall some basic properties and examples of Bohr groups $G^\sharp$ in order to show the differences between abelian and nonabelian groups. For the proofs one may consult [5, 12] and the references therein.

Theorem 3.1. *Let G and H be discrete groups. The following assertions are satisfied:*

1. *$G^\sharp$ has the countable chain condition*
2. *Every homomorphism $\phi : G \longrightarrow H$ is continuous as a map from $G^\sharp$ into $H^\sharp$.*
3. *If G is abelian, then every subgroup is closed in $G^\sharp$.*

4. *If $G = \sum_{i=1}^{\infty} F_i$, being every F_i a finite, simple, nonabelian group, then $bG = \prod_{i=1}^{\infty} F_i$.*

5. *If G is abelian, then for every subgroup H of G we have that $H^\sharp$ is topologically embedded in $G^\sharp$.*

6. *$(G \times H)^\sharp \cong G^\sharp \times H^\sharp$.*

7. *If G is abelian, the $G^\sharp$ is neither compact or Baire.*

8. *Let $\{p_i\}$ be an infinite sequence of distinct prime numbers ($p_i > 2$). If we consider the discrete group $G = \prod_{i=1}^{\infty} F_i$, being every F_i the projective special linear group of dimension two over the Galois field $GF(p_i)$ of order p_i, then $G^\sharp \cong bG \cong \prod_{i=1}^{\infty} F_i$, the latter group equipped with the product topology.*

9. *If G is abelian, then $G^\sharp$ is 0-dimensional.*

10. *If G is a compact connected semi simple Lie group and G_d denotes the same group equipped with the discrete topology, then $(G_d)^\sharp \cong bG_d \cong G$.*

11. *Every abelian group is vD.*

12. *If G is an LCA group, then every $A \subset G$ that is compact in bG, it is compact in G. As a consequence, when G is also discrete, we have that every convergent sequence in $G^\sharp$ is trivial.*

13. *Every LCA group satisfies Pontryagin-van Kampen duality.*

We have seen that the commutativity of the groups involved has been required in some of the assertions of Theorem 3.1. This constraint may not be relaxed in general. Indeed, if we take the subgroup $\sum_{i=1}^{\infty} F_i$ in (8) above, then it is dense in $G^\sharp$. This proves that assertion (3) in Theorem 3.1 does not extend to noncommutative groups in general. On the other hand, (8) and (10) show that assertions (7) and (9) do not hold for nonabelian groups. In order to see that (5), (11), and (12) are not always satisfied by nonabelian groups, we need to work a bit further (we show it below). That assertion (13) does not hold for nonabelian groups is a well-known fact that concerns noncommutative duality that we will not touch on here, see [17]. The following result is consequence of those in [7].

Proposition 3.2. *Let G be an LCA group. Let $\hat{G}$ be the dual group of G and let X be a subgroup of $\hat{G}$. If we denote by X_d the group X equipped with the discrete topology, then its dual group $\hat{X}$ is compact. We have that $\hat{X}$ is a compactification of G if and only if X separates points of G.*

Proof. Define $\sigma : G \longrightarrow \hat{X}_d$ by $\langle \sigma(g), \chi \rangle = \chi(g) = \langle \chi, g \rangle$, $\chi \in X$. Clearly σ is 1-to-1 if and only if X separates points. ∎

Let A be a normal abelian subgroup of a group G. Given any $g \in G$, we denote by $\theta_g : A \longrightarrow A$ the automorphism $\theta_g(a) = gag^{-1}$. If χ is character on A and $g \in G$, we define the character χ^g on A by $\chi^g(a) = \chi(g^{-1}ag)$ for all $a \in A$. Thus, the group G acts on $\widehat{A}$, the dual group of A, as $\theta_g(\chi) = \chi^g$ for all $g \in G$

and $\chi \in \widehat{A}$. We denote by $\theta_G(\chi)$ (resp. $Iso(\chi)$) the orbit (resp. isotropy or stabilizer group) of χ under this action. Next follows a lemma that is consequence of [19, 8.2.1] or [27].

Lemma 3.3. *Let A be a normal abelian subgroup of an arbitrary group G. If χ is a character on A that can be extended to a finite dimensional representation defined on G, then the orbit $\theta_G(\chi)$ is finite.*

Proposition 3.4. *Let $G = A \rtimes B$ be a semi-direct product of groups A and B. Assume G is MAP and discrete and let $b : G \longrightarrow bG$ be the injection of G into its Bohr compactification. We have:*

1. *$bG \cong \overline{A}^{bG} \rtimes bB$*
2. *If A is abelian, a character $\chi \in \hat{A}$ extends to a finite dimensional representation on G if and only if $\theta_B(\chi) = \theta_G(\chi)$ is finite.*

Proof. (1) If we identify A and B to subgroups of G, we can assume WLOG that $G = A \cdot B$. Thus, we have $bG = \overline{A \cdot B}^{bG} = \overline{A}^{bG} \cdot \overline{B}^{bG}$. Consider the diagram

$$\begin{tikzcd} B \arrow[r, "j"] & G \arrow[r, "\pi"] \arrow[d, "b^G"'] & B \arrow[d, "b^B"] \\ & bG \arrow[r, "\pi^b"'] & bB \end{tikzcd}$$

Clearly the map π^b is onto and $\ker \pi^b \supseteq \overline{A}^{bG}$. This means that $\pi^b_{|\overline{B}^{bG}}$ is onto, $\overline{A}^{bG} \cap \overline{B}^{bG} = \{e\}$ and $\overline{B}^{bG} \cong bB$. Otherwise, bB would be a proper quotient of $\overline{B}^{bG}$, which is impossible. Therefore, we have $bG \cong \overline{A}^{bG} \rtimes bB$.

(2) By Lemma 3.3, we know that, if χ extends to finite dimensional representation $D \in G_n^x$, then $\theta_G(\chi)$ is finite.

Conversely, let us suppose that $\chi \in \hat{A}$ and $\theta_B(\chi)$ is finite. Then $F = Iso(\chi)$ is a cofinite subgroup of B. Moreover, since F is the isotropy group of χ, it is easily verified that we can extend it to a character $\overline{\chi} \in \widehat{A \rtimes F}$ by setting $\overline{\chi}(ab) := \chi(a)$ for all $a \in A$ and $b \in F$. Now, $A \rtimes F$ is a cofinite subgroup of G and, as a consequence, $\overline{\chi}$ can be extended to finite dimensional representation $D \in G_n^x$. This completes the proof. ∎

Incidently, from the result above it is easy to find a counter example to the assertion (5), (11), and (12) in Theorem 3.1 for nonabelian groups.

Example 3.5. Let G be the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$, that is to say, $G = A \rtimes B$, where $A = \sum_{n \in \mathbb{Z}} F_n$, $B = \mathbb{Z}$, F_n is the cyclic finite group $\mathbb{Z}_2$ for all $n \in \mathbb{Z}$ and the action is the shift automorphism. Then G provides a counter example to (5), (11) and (12) in Theorem 3.1.

Proof. The topology that A inherits from $G^\sharp$ stems from the characters on A that can be extended to G and, according to Proposition 3.2, a character $\chi \in \hat{A}$ can be extended to G when $\theta_B(\chi)$ is finite. Now, by the duality theory of abelian groups we have that $\chi = (x_n)_{n\in\mathbb{Z}} \in \prod_{n\in\mathbb{Z}} \widehat{F_n} \cong \prod_{n\in\mathbb{Z}} F_n$. Thus, identifying the coordinates x_n to elements of $\mathbb{Z}_2$ and taking into account that the action of $\mathbb{Z}$ in $\sum_{n\in\mathbb{Z}} F_n$ is the shift automorphism, it follows that $\theta_B(\chi)$ is finite if, and only if, there is a positive $p \in \mathbb{Z}$ such that $x_{n+pj} = x_n$ for all $0 \leq n \leq p-1$ and $j \in \mathbb{Z}$. As a consequence, the topology that A inherits from $G^\sharp$ is defined by a subgroup of $\widehat{A}$ consisting of *countably* many characters. Therefore, the subgroup A inherits a metrizable topology from $G^\sharp$. Thus, we are done with the proof. Indeed, no infinite abelian Bohr group can be metrizable, which means that (5) is not true for the subgroup A of G. Moreover, being A metrizable with respect to the Bohr topology of G, it follows that there must be nontrivial convergent sequences, what contradicts (11) and (12) respectively. ∎

4. Finitely generated groups without free nonabelian subgroups

The main result of this section follows next. We prove that, if G is a finitely generated vD group with no free nonabelian subgroups then G is a finite extension of an abelian group.

Theorem 4.1. *Let G be a finitely generated discrete group without free non-abelian subgroups. Then G is a vD group if and only if G is a finite extension of an abelian group.*

The proof of this result depends on several previous facts that we recall here for the reader's sake.

Theorem 4.2. *(Tit's alternative, [33]) Let G be a subgroup of $GL(n, \mathbb{K})$, for some integer $n \geq 1$ and some field $\mathbb{K}$ of characteristic zero. Then either G has a non-abelian free subgroup or G possesses a soluble normal group of finite index.*

Theorem 4.3. *(Lie, Kolchin, Mal'cev, [32, p. 451]) Let V be a vector space of dimension n over an algebraically closed field F. Suppose that G is a solvable subgroup of $GL(V)$. If G is irreducible, there is a normal diagonalizable subgroup D with finite index not exceeding $f(n)$ for some function f.*

Now we present several auxiliary lemmas.

Lemma 4.4. *Let G be a group without free nonabelian subgroups. If D is an arbitrary representation in G_n^x, with $n \in \mathbb{N}$, then there exists an abelian normal subgroup A with finite index in $D(G)$.*

Proof. We have that $D(G)$ is a subgroup of $U(n)$ without free non abelian subgroups. By Theorem 4.2, there exists a normal solvable subgroup L with finite index in $D(G)$; that is to say $D(G) = F \cdot L$, being F a (finite) transversal subset for L in $D(G)$. Moreover, we assume WLOG that L is closed in $D(G)$ and $1 \in F$. Let K be the closure of $D(G)$ in $U(n)$. Then K is a compact Lie group such that $K = F \cdot cl_{U(n)}L$. Because L is solvable, it follows that $cl_{U(n)}L$ is also solvable an normal in K. Observe that, for any $1 \neq t \in F$, we have $t \notin cl_{U(n)}L$ since $t \in F \subset D(G)$ and L is closed in $D(G)$. Hence K_0, the connected component of K, must coincide with $cl_{U(n)}L_0$, which is the connected component of $cl_{U(n)}L$. This implies that K_0 is a compact connected solvable group and, by [18, 29.44], it follows that K_0 is an abelian normal subgroup with finite index in K. It suffices now to take $A = K_0 \cap D(G)$ and the proof is done. ∎

From here on, we denote by G^n the subgroup of G generated by the elements of the form g^n, with $g \in G$. We recall that G^n is a normal subgroup of G.

Lemma 4.5. *Let G be a group without free non-abelian subgroups. If G^n is nonabelian for all $n \in \mathbb{N}$ then $G^\sharp$ contains nontrivial convergent sequences.*

Proof. Consider two sequences $\{x_n\}$ and $\{y_n\}$ in G such that $[x_n^{n!}, y_n^{n!}] \neq 1$ for all $n \in \mathbb{N}$. By Lemma 4.4, for each $D \in G_m^x$, $m \in \mathbb{N}$, there is a normal abelian subgroup A with finite index in $D(G)$. Hence, there is $n_0 \in \mathbb{N}$ such that $D(g^{n_0}) \in A$ for all $g \in G$. This means that $D([x_n^{n!}, y_n^{n!}]) = I_m$ for all $n \geq n_0$, which implies that the nontrivial sequence $\{[x_n^{n!}, y_n^{n!}]\}$ must converge to the neutral element in $G^\sharp$. This completes the proof. ∎

Lemma 4.6. *If G is an infinite finitely generated torsion group then $G^\sharp$ is nondiscrete and metrizable. Therefore, it contains nontrivial Bohr convergent sequences.*

Proof. For all $D \in G_m^x$ we have that $D(G)$ is soluble by finite and, since G is a finitely generated torsion group, this means that $D(G)$ is finite. Now, the group G is finitely generated and, therefore, it contains finitely many co-finite normal subgroups of index less or equal than n for every $n \in \mathbb{N}$ (cf. [30, Prop. 2.5.1(a)]). Let $\mathcal{L}_n$ be the family of all subgroups H of G satisfying that $[G : H] \leq n$. Since G is MAP and $D(G)$ is finite for all $D \in G^x$, it follows that $\mathcal{L} = \cup_{n<\omega}\mathcal{L}_n$ is a countable basis for the neighborhood base of the neutral element in $G^\sharp$. It follows that $G^\sharp$ is metrizable. Moreover, the group $G^\sharp$ is always precompact and, being infinite, may not be discrete. This completes the proof.

∎

Proof. [Proof of Theorem 4.1] *Sufficiency:* It is known (see [25, 29]) that if G is a finite extension of an abelian group then $G^\sharp$ contains no nontrivial convergent sequence. Applying 1.1 (see [11]), we deduce that G is vD.

Necessity: Reasoning by contradiction, suppose that G is not a finite extension of an abelian group and let us verify that $G^\sharp$ contains nontrivial convergent sequences. Indeed, by Lemma 4.5, we may assume WLOG that there is $m_0 \in \mathbb{N}$ such that G^{m_0} is abelian. Therefore $G^{m!} \subseteq G^{m_0}$ is also abelian for all $m \geq m_0$. As a consequence, for every $m \geq m_0$, the subset $\{x_1^{m!}x_2^{m!}...x_p^{m!} : x_j \in G, p \in \mathbb{N}\}$ coincides with $G^{m!}$ and, furthermore, it is a normal abelian subgroup of G. Assume that $m \geq m_0$ from here on and let $G_m = cl_{bG}G^{m!} \cap G$, which is also abelian.

Since we are supposing that G is not a finite extension of an abelian group, it follows that G/G_m is infinite. Let $\mathcal{L}_n^m$ be the family of all subgroups H of G satisfying that $G_m \subset H$ and $[G:H] \leq n$. Since G/G_m has bounded exponent and is finitely generated, it follows that $\mathcal{L}_n^m$ is finite and, as a consequence, we have $\{1\} \neq L_n^m = \cap \mathcal{L}_n^m$. Observing that $L_n^m \subset L_n^l$ if $m \geq l$ and $L_{n'}^m \subset L_n^l$ if $n' \geq n$, we deduce that the subgroups L_m^m $(m \in \mathbb{N})$ are eventually contained in each subgroup L_n^l previously fixed. Now, no subgroup L_n^m may be abelian because all of them are cofinite in G. Accordingly, every subgroup L_m^m contains two elements x_m and y_m with $[x_m, y_m] \neq \{1\}$ for all $m \geq m_0$. In order to finish the proof, it will suffice to show that the sequence $\{[x_i, y_i]\}_{i \geq m_0}$ converges to the neutral element in $G^\sharp$. Indeed, let D be an arbitrary finite dimensional unitary representation of G, say $D \in G_m^x$. By Lemma 4.4, the group $D(G)$ must contain a normal abelian subgroup A with finite index in $D(G)$. If L denotes the closure of A in $D(G)$, we obtain that L is abelian, has finite index in $D(G)$ and $D^{-1}(L) \in \mathcal{L}_n^m$ for some $n \in \mathbb{N}$. Hence, for some $n_0 \in \mathbb{N}$ we have $\{x_i, y_i\} \subset L_i^i \subset D^{-1}(L)$ for all $i \geq n_0$. This yields $D([x_i, y_i]) = I_m$ for all $i \geq n_0$, which completes the proof. ∎

We finish this section with the following basic open question (see Question 971 in [12]).

Problem 4.7. Is every vD group a finite extension of an abelian group?

5. Dually embedded subgroups

As stated in the introduction, a locally compact group G is in the class $\mathcal{D}$ when every subgroup H of G is dually embedded. Poguntke himself has proved in [27] that every solvable locally compact group in $\mathcal{D}$ is a Moore group (i.e., every irreducible representation is finite dimensional). In this section we prove that every group containing free nonabelian subgroups is not in $\mathcal{D}$. Firstly, we are concerned with the existence of subgroups which are not dually embedded for the free nonabelian groups of two generators.

Theorem 5.1. *The free group with two generators $F(a,b)$ does not belong to the class $\mathcal{D}$.*

Proof. Let $G = F(a,b)$ and let $[G,G]$ be the commutator subgroup of G. The group $[G,G]$ is free on $\{a^m b^n a b^{-n} a^{-(m+1)} : m, n \in \mathbb{Z}, n \neq 0\}$. Let $\rho : [G,G] \longrightarrow \mathbb{T}$ be a representation that maps the generators of $[G,G]$ to rationally independent elements of $\mathbb{T}$. Reasoning by contradiction, let us suppose there is a finite dimensional irreducible representation $D : G \longrightarrow U(n)$ that extends ρ. Then there is a 1-dimensional subspace V of $\mathbb{C}^n$ such that $D([G,G])_{|V} = \rho([G,G])$. Moreover, since $D(G)$ is irreducible, it follows that the subspace generated by $D(G)[V]$ is $\mathbb{C}^n$. Therefore, if we take $0 \neq v \in V$, then the subset $\{D(g)[v] : g \in G\}$ generates $\mathbb{C}^n$. On the other hand, since $[G,G]$ is a normal subgroup of G, for every $h \in [G,G]$ we have $D(h)[D(g)[v]] = D(hg)[v] = D(gg^{-1}hg)[v] = D(g)[D(g^{-1}hg)[v]] = D(g)[\lambda v] = \lambda D(g)[v]$. Thus, $D([G,G])$ yields a 1-dimensional representation on the subspace generated by $D(g)[v]$ for all $g \in G$. Hence, we can put $\mathbb{C}^n$ as the direct sum of n 1-dimensional subspaces on which $D([G,G])$ acts as a character conjugate to the action of $\rho([G,G])$ on V. Therefore, the group $D([G,G])$ is abelian. Since $G/[G,G]$ also is an abelian group, it follows that $D(G)$ is an irreducible solvable subgroup of $U(n)$. So we have that $\overline{D(G)}$, its closure in $U(n)$, is a solvable compact Lie group, which implies that its connected component $\overline{D(G)}_0$ is abelian and compact, therefore, of finite index in $\overline{D(G)}$. Hence, there is some $p \in \mathbb{N}$ such that $\{D(a^p), D(b^p)\} \subset \overline{D(G)}_0$. As a consequence, we have $D(a^{pm}b^{pn}ab^{-pn}a^{-(pm+1)}) = D(b^{pn}ab^{-pn}a^{-1})$ for all $m,n \in \mathbb{Z}, n \neq 0$. This is a contradiction, since D extends ρ and the latter sends the generators of $[G,G]$ into rationally independent elements of $\mathbb{T}$. This completes the proof. ■

Corollary 5.2. *No group containing a free nonabelian group belongs to the class $\mathcal{D}$.*

As a consequence, we deduce the following *topological* characterization of (finitely generated) groups which are finite extensions of abelian groups.

Theorem 5.3. *A finitely generated discrete group G is in the class $vD \cap \mathcal{D}$ if, and only if, the group G is a finite extension of an abelian group.*

Proof. The proof follows directly from Theorem 4.1 and Theorem 5.1. ■

6. Other classes of Bohr groups

This section is dedicated to present some results for FC-groups, which has been obtained in collaboration with Professor Ta-Sun Wu. Recall that G is said to be an *FC group* when it has finite conjugacy classes; that is, the orbit $\theta_G(x)$ is finite for all $x \in G$. Equivalently F_x, the isotropy group of x, is co-finite in G. Let us say that a topology is *residually finite* when there is a neighbourhood base of the identity consisiting of cofinite normal subgroups (cf. [30, p. 79]).

Lemma 6.1. *Let G be an FC group. Given an arbitrary element D of G_n^x, $n \in \mathbb{N}$, there is a co-finite subgroup F of G such that $D(F') = \{I_n\}$.*

Proof. Since G is an FC group, we have that $L = Z(D(G))$ is co-finite in $D(G)$ (see [35]). Thus, $D(G)/L = \{D(x_1)L, ..., D(x_m)L\}$. For each x_i, $1 \leq i \leq m$, let F_i be the isotropy group of x_i. The subgroup $F = \cap_{i=1}^m F_i$ is co-finite and $ax_ia^{-1} = x_i$ for $a \in F$, $1 \leq i \leq m$. Hence $D(F) \subset Z(D(G))$. This yields $D(F') = \{I_n\}$. ■

Theorem 6.2. *Let G be a countable FC group. Then, the topology that $[G,G]$ inherits from $G^\sharp$ is residually finite and metrizable.*

Proof. Let $G = \{x_n\}_{n=1}^\infty$ and let F_{x_i} be the isotropy group of x_i, $i \in \mathbb{N}$. For any $n \in \mathbb{N}$, set $F_n = \cap_{i=1}^n F_{x_i}$, which is a co-finite subgroup of G. Let m and D be arbitrary elements of $\mathbb{N}$ and G_m^x, respectively. By Lemma 6.1, there is $n \in \mathbb{N}$ such that $L = D(F_n) \subset Z(D(G))$, $D(G)/L = \langle D(x_1)L, ...D(x_l)L \rangle$ and $F_n' \subset \ker D$. Let us verify now that $[G,G]/\ker D$ is a finite group.

Indeed, since F_n is co-finite in G, we have $\frac{G}{F_n} = \{x_1F_n, ..., x_lF_n\}$, with $x_i \in G$, $1 \leq i \leq l$. On the other hand, $[G,G]$ is generated by elements of the form $[x_ia, x_jb]$, with $a, b \in F_n$. Thus, because $F_n' \subset \ker D$, we have

$$\begin{aligned}
&[x_ia, x_jb] \ker D = \\
&(x_iax_jba^{-1}x_1^{-1}b^{-1}x_j^{-1}) \ker D = \\
&(x_i(ax_ja^{-1})[a,b](bx_i^{-1}b^{-1})x_j^{-1}) \ker D = \\
&(x_i(ax_ja^{-1})(bx_i^{-1}b^{-1})x_j^{-1}) \ker D
\end{aligned}$$

Since G is an FC group, we obtain that there are only finitely many elements of the form $[x_ia, x_jb] \ker D$, with $a, b \in F_n$ and $x_i, x_j \in \{x_1, ..., x_l\}$, which proves that $\frac{[G,G]}{\ker D}$ is finitely generated. Now, by [32, 14.5.9], $[G,G]$ is torsion and, as a consequence, $\frac{[G,G]}{\ker D}$ is torsion too. Hence, the quotient group $\frac{[G,G]}{\ker D}$ is finitely generated, torsion and FC. Applying [32, 14.5.7], we obtain that $\frac{[G,G]}{\ker D}$ is finite.

Now, we can define a representation $\overline{D}$, which makes the following diagram commutative.

$$\begin{tikzcd}
{[G,G]} \arrow[rr, "\pi_m"] \arrow[dr, "D"'] & & {[G,G]/\ker D} \arrow[dl, "\overline{D}"] \\
& U(m) &
\end{tikzcd}$$

Thus, the set $Hom(G_{|[G,G]}, U(m))$ can be injected into $\cup_{n=1}^{\infty} Hom(\frac{[G,G]}{\ker D}, U(m))$. Since, the latter set is the union of countably many finite sets, it follows that the topology that $[G,G]$ inherits from $G^\sharp$ is metrizable residually finite. ∎

Next result is a direct consequence of Theorem 6.2.

Corollary 6.3. (Wu and Riggins) *Let G be an FC group. If G is not Abelian by finite then G contains some infinite sequence that converges in the Bohr topology.*

Finally, we explore the relationship between the Bohr and the profinite completion for some groups. It is well known that any residually finite group G has a largest profinite group completion, which is called *the profinite completion* of G and is denoted by $\overline{G}$ (see [21, 30, 34]). From [21, Ch. 10] we know that for any group G we have $bG \cong (bG)_0 \times \frac{bG}{(bG)_0}$, where $(bG)_0$ denotes the connected component of bG. As a consequence, we derive a simple characterization of the profinite completion of G.

Lemma 6.4. *For any residually finite group G, the profinite completion $\overline{G}$ is canonically isomorphic to $\frac{bG}{(bG)_0}$.*

Proof. Since G is dense in $\frac{bG}{(bG)_0}$, which is a residually finite group, it will suffice to prove that every homomorphism D, of G into a finite group, factors through $\frac{bG}{(bG)_0}$. Now, let $D : G \longrightarrow F$ be a homomorphism of G into the finite group F. We set

$$\begin{tikzcd}
G \arrow[r, "b"] \arrow[d, "D"'] & bG \arrow[d, "\pi"] \arrow[dl, "D^b"] \\
F & \frac{bG}{(bG)_0} \arrow[l, "\tilde{D}"]
\end{tikzcd}$$

This diagram commutes, since $(bG)_0$ is contained in $\ker D^b$. This completes the proof. ∎

The following lemma is well known (see [21]).

Lemma 6.5. *Any subgroup of bounded exponent of the unitary group $U(n)$ is finite.*

Theorem 6.6. *Let G be a discrete group without free non-abelian subgroups. If $[G,G]$ is of bounded exponent then this subgroup inherits a residually finite topology from bG. As a consequence, if G is further assumed to be finitely generated then the topology that $[G,G]$ inherits from bG and $\bar{G}$ coincide.*

Proof. Let D be an arbitrary, unitary, finite dimensional representation of G. Clearly, the group $D([G,G])$ satisfies the hypothesis of Lemma 6.5, which implies that $D([G,G])$ is finite. This means that the topology that $[G,G]$ inherits from bG is residually finite. Suppose now that G is finitely generated. If D is a finite dimensional unitary representation of G, we have that $D(G)$ is a finitely generated MAP linear group. This implies, according to [9, p. 169], that $D(G)$ is residually finite. Hence, since $D([G,G])$ is finite, there is a cofinite normal subgroup L of $D(G)$ such that $D([G,G]) \cap L = \{1\}$. Let E be the finite dimensional unitary representation that makes the following diagram commutative:

$$\begin{tikzcd} D([G,G]) \arrow[rr, "id"] \arrow[dr, "\pi_L"] & & U(n) \\ & \frac{D([G,G])}{L} \arrow[ur, "E"'] & \end{tikzcd}$$

Now, let $\overline{E} : \frac{D(G)}{L} \longrightarrow U(m)$ an extension of E and define $\overline{D} = \overline{E} \circ \pi_L$. We have that $\overline{D}$ an extension of D and $\overline{D}(G)$ is finite. Thus $\overline{D}$ factors through $\frac{bG}{(bG)_0}$, which implies that the topology that $[G,G]$ inherits from bG coincides with the residually finite topology inherited from $\frac{bG}{(bG)_0}$. This completes the proof. ∎

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