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On the Pro-Lie Group Theorem and the Closed Subgroup Theorem

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Abstract. Let H and M be closed normal subgroups of a pro-Lie group G and assume that H is connected and that G/M is a Lie group. Then there is a closed normal subgroup N of G such that $N \subseteq M$, that G/N is a Lie group, and that HN is closed in G. As a consequence, $H/(H \cap N) \to HN/N$ is an isomorphism of Lie groups.
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1. Theorem and Proof

In a pro-Lie group G the set $\mathcal{N}(G)$ of normal subgroups N such that G/N is a Lie group is a filterbasis, and G is naturally isomorphic to the projective limit $\lim_{N\in\mathcal{N}(G)} G/N$ (see [4], p. 149ff.). The *Pro-Lie Group Theorem* states that *every limit of a projective system of Lie groups is a pro-Lie group* ([4], Theorem 3.34, p. 157, [5], [1]). The *Closed Subgroup Theorem for Pro-Lie Groups* states that *a closed subgroup of a pro-Lie group is a pro-Lie group* ([4], Theorem 3.5, p. 158); its proof is based on the Pro-Lie Group Theorem and the following facts ([4], Theorem 1.34(i,ii), p. 96f.):

Theorem 1.1. *Assume that $\mathcal{N}(G)$ is a filterbasis of closed normal subgroups of the complete topological group G and assume that $\lim \mathcal{N} = 1$. Let H be a closed subgroup of G. Then the following conclusions hold*

(i) *The isomorphism $\gamma_G\colon G \to \lim_{N\in\mathcal{N}} G/N$ maps H isomorphically onto $\lim_{N\in\mathcal{N}} \overline{HN}/N$.*

(ii) *Under the present hypotheses,*

$$H \cong \lim_{N\in\mathcal{N}} H/(H\cap N) \cong \lim_{N\in\mathcal{N}} HN/N \cong \lim_{N\in\mathcal{N}} \overline{HN}/N.$$

The additional assumption, made in [4], that all factor groups G/N are complete is superfluous, as the referee points out. Recall that a topological group

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G is called *almost connected* if its component factor group G/G_0 is compact. We shall prove the following

Theorem 1.2. (a) *Let H be an almost connected closed subgroup of a pro-Lie group G and let $M \in \mathcal{N}(G)$. Then there is a closed normal subgroup N of $\overline{HM}$ such that $N \subseteq M$ and the standard bijection*

$$f_N\colon H/(H\cap N) \to HN/N, \quad f_N(h(H\cap N)) = hN,$$

is an isomorphism of Lie groups.

(b) *If H is normal in G, then N is constructed to be normal in G, that is, $N \in \mathcal{N}(G)$.*

We shall prove this theorem in several steps through a sequence of lemmas. We are given the pro-Lie group G with its filter base $\mathcal{N}(G)$ of closed normal subgroups N such that G/N is a Lie group. Then G may be identified with the projective limit of the system

$$\{p_{MN}\colon G/N \to G/M : N \subseteq M,\ M, N \in \mathcal{N}(G)\}.$$

By Theorem 1.1, a closed subgroup H of G gives rise to three projective systems of topological groups:

$$\begin{aligned}
&\{q_{MN}\colon \overline{HN}/N \to \overline{HM}/M : N \subseteq M,\ M, N \in \mathcal{N}(G)\},\\
&\{r_{MN}\colon HN/N \to HM/M : N \subseteq M,\ M, N \in \mathcal{N}(G)\},\\
&\{s_{MN}\colon H/(H\cap N) \to H/(H\cap M) : N \subseteq M,\ M, N \in \mathcal{N}(G)\},
\end{aligned}$$

and all of them have H as limit by Theorem 1.1., as is illustrated in the following diagram:

$$\begin{tikzcd}
\frac{H}{H\cap M} \arrow[d, "f_M"'] & \frac{H}{H\cap N} \arrow[l, "s_{MN}"'] \arrow[d, "f_N"'] & H \cong \lim_{P\in\mathcal{N}(G)} \frac{H}{H\cap P} \arrow[l, "s_N"'] \arrow[d, "\lim_{P\in\mathcal{N}(N)} f_P"] \\
\frac{HM}{M} \arrow[d, "\mathrm{inc}"'] & \frac{HN}{N} \arrow[l, "r_{MN}"'] \arrow[d, "\mathrm{inc}"'] & H = \lim_{P\in\mathcal{N}(G)} \frac{HP}{P} \arrow[l, "r_N"'] \arrow[d, "\mathrm{id}_H"] \\
\frac{\overline{HM}}{M} \arrow[d, "\mathrm{inc}"'] & \frac{\overline{HN}}{N} \arrow[l, "q_{MN}"] \arrow[d, "\mathrm{inc}"'] & H = \lim_{P\in\mathcal{N}(G)} \frac{\overline{HP}}{P} \arrow[l, "q_N"] \arrow[d, "\mathrm{inc}"] \\
\frac{G}{M} & \frac{G}{N} \arrow[l, "p_{MN}"] & G = \lim_{P\in\mathcal{N}(G)} \frac{G}{P}. \arrow[l, "p_N"]
\end{tikzcd}$$

We note, in particular, that $q_N(H) = \frac{\overline{HN}}{N}$. For the Lie algebra $\mathfrak{L}(G)$ we write $\mathfrak{g}$, and so on.

Lemma 1.3. *For each $N \in \mathcal{N}(G)$, the quotient $H/(H\cap N)$ is a Lie group with Lie algebra $\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{n})$.*

Proof. For $N \in \mathcal{N}(G)$ the morphism $f\colon H \to G/N$, $f(h) = hN$ has kernel $H\cap N$. The Lie group G/N has an identity neighborhood V in which $\{1\}$ is the only subgroup. Then $U \stackrel{\text{def}}{=} f^{-1}(V)$ is an identity neighborhood of H in which every subgroup is contained in $H \cap N$. By the Closed Subgroup Theorem of Pro-Lie Groups [4], 3.35, p. 158, H is a pro-Lie group. [If H is assumed to be connected, then this follows already from the arguments yielding [4], 3.29(i) on p. 152.] Since $\lim \mathcal{N}(H) = 1$, there is a $P \in \mathcal{N}(H)$ such that $P \subseteq U$, and thus $P \subseteq H \cap N$. Then $H/(H \cap N) \cong (H/P)/((H \cap N)/P)$ is a Lie group as a quotient of a Lie group.

We have $\mathfrak{L}(H \cap N) = \mathfrak{h} \cap \mathfrak{n}$ since $\mathfrak{L}$ preserves limits, hence intersections (cf. [4], Theorem 2.25(ii)). By [4], Corollary 4.21(i) $\mathfrak{L}(H/(H \cap N)) = \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{n})$. [As Glöckner has pointed out in [1], Corollary 4.21(i) of [4] is already available after [4], Lemma 3.24 on p.147f.] ∎

Remark. We note that for *connected* closed subgroups H of a pro-Lie group G, Lemma 1.3 is available in [4] after Lemma 3.29 (i) and (ii) on p. 152. Thus we see that 1.3 is valid for connected H without invoking the independent proofs of the Pro-Lie Group Theorem in [5] and [1].

For the next step we accept the following facts: An arcwise connected subgroup of a Lie group is an analytic subgroup by the Theorem of Yamabe-Gotô ([2], [6]). If A is an analytic subgroup of a Lie group G, then there is a unique connected Lie group topology on A, possibly finer than the topology induced from G, making it into a Lie group A_{Lie} such that the inclusion map $A \to G$ induces an injective morphism of topological groups $f: A_{\text{Lie}} \to G$, where $\mathfrak{L}(f): \mathfrak{L}(A_{\text{Lie}}) \to \mathfrak{L}(G)$ is an injection of Lie algebras and $\operatorname{im} \mathfrak{L}(f) = \mathfrak{L}(A)$. See for instance [3], Theorem 5.52. The topology of A_{Lie} is the *arc component topology* of A (see [3], A4.1ff.).

Lemma 1.4. *Assume that H is a* connected *closed subgroup of the pro-Lie group G. Then for each $N \in \mathcal{N}(G)$, the group $\frac{HN}{N}$ is an analytic subgroup of the Lie group G/N, and its Lie algebra is $\frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}}$. The Lie group $\left(\frac{HN}{N}\right)_{\text{Lie}}$ is isomorphic to $\frac{H}{H \cap N}$; indeed it is the image of $\frac{H}{H \cap N}$ under the standard bijective morphism $f_N: \frac{H}{H \cap N} \to \frac{HN}{N}$, $f_N(h(H \cap N)) = hN$, given the topology making f_N a homeomorphism.*

Proof. Let us write $L = H/(H \cap N)$, whence we may write $\mathfrak{l} = \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{n})$ by Lemma 1.3. Set $\mathfrak{a} \stackrel{\text{def}}{=} \mathfrak{L}(f_N)(\mathfrak{l})$ and $A \stackrel{\text{def}}{=} HN/N$. By Lemma 1.3, L is a Lie group, and since H is connected, it is connected. Then $L = \langle \exp_L(\mathfrak{l}) \rangle$ and we have

$$A = f_N(\langle \exp_L(\mathfrak{l}) \rangle) = \langle f_N(\exp_L(\mathfrak{l})) \rangle = \langle \exp_G \mathfrak{L}(f_N)(\mathfrak{l}) \rangle = \langle \exp_G \mathfrak{a} \rangle.$$

This means that A is the unique analytic subgroup of G/N generated by the (closed) subalgebra $\mathfrak{a}$ of $\mathfrak{L}(G/N)$. By [3], Theorem 5.52(iii), $\mathfrak{a} = \mathfrak{L}(HN/N)$. ∎

We recall that $\mathfrak{f}_N: \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{n}) \to (\mathfrak{h} + \mathfrak{n})/\mathfrak{n}$ is an isomorphism of pro-Lie algebras (more generally, cf. [4], Theorem A2.12(c), p. 640f.).

We abbreviate $\mathfrak{L}(\overline{HN})$ by $\mathfrak{h}_N$ and note $\mathfrak{L}(\overline{HN/N}) = \mathfrak{h}_N/\mathfrak{n}$ (see [4], Corollary 4.21(i), p. 190). Our passing to the Lie Algebras in our big commutative diagram yields

$$\begin{tikzcd}
\frac{\mathfrak{h}}{\mathfrak{h}\cap\mathfrak{m}} \arrow[d, "\mathfrak{f}_M"'] & \frac{\mathfrak{h}}{\mathfrak{h}\cap\mathfrak{n}} \arrow[l, "\mathfrak{s}_{MN}"'] \arrow[d, "\mathfrak{f}_N"'] & \mathfrak{h} \cong \lim_{P\in\mathcal{N}(G)} \frac{\mathfrak{h}}{\mathfrak{h}\cap\mathfrak{p}} \arrow[l, "\mathfrak{s}_N"'] \arrow[d, "\lim_{P\in\mathcal{N}(G)} \mathfrak{f}_P"] \\
\frac{\mathfrak{h}+\mathfrak{m}}{\mathfrak{m}} \arrow[d] & \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}} \arrow[l, "\mathfrak{r}_{MN}"] \arrow[d] & \mathfrak{h} = \lim_{P\in\mathcal{N}(G)} \frac{\mathfrak{h}+\mathfrak{p}}{\mathfrak{p}} \arrow[l, "\mathfrak{r}_N"'] \arrow[d, "\mathrm{id}_{\mathfrak{h}}"] \\
\frac{\mathfrak{h}_M}{\mathfrak{m}} \arrow[d, "\mathrm{inc}"'] & \frac{\mathfrak{h}_N}{\mathfrak{n}} \arrow[l, "\mathfrak{q}_{MN}"] \arrow[d, "\mathrm{inc}"'] & \mathfrak{h} = \lim_{P\in\mathcal{N}(G)} \frac{\mathfrak{h}_P}{\mathfrak{p}} \arrow[l, "\mathfrak{q}_N"] \arrow[d, "\mathrm{inc}"] \\
\frac{\mathfrak{g}}{\mathfrak{m}} & \frac{\mathfrak{g}}{\mathfrak{n}} \arrow[l, "\mathfrak{p}_{MN}"] & \mathfrak{g} = \lim_{P\in\mathcal{N}(G)} \frac{\mathfrak{g}}{\mathfrak{p}}. \arrow[l, "\mathfrak{p}_N"]
\end{tikzcd}$$

Again we observe that $\mathfrak{q}_N(\mathfrak{h}) = \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}}$.

Lemma 1.5. *For each $M \in \mathcal{N}(G)$ there is an $N = N_M \in \mathcal{N}(G)$ contained in M such that $\frac{\mathfrak{h}_N}{\mathfrak{n}} = \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}} + \frac{\mathfrak{h}_N\cap\mathfrak{m}}{\mathfrak{n}}$.*

Proof. Applying [4], Lemma 3.18 = Theorem A2.12 we find that for each $M \in \mathcal{N}(G)$ there is an $N = N_M \in \mathcal{N}(G)$ such that $\mathfrak{q}_{MN}(\frac{\mathfrak{h}_N}{\mathfrak{n}}) \subseteq \mathfrak{q}_M(\mathfrak{h}) = \frac{\mathfrak{h}+\mathfrak{m}}{\mathfrak{m}}$. Let $X + \mathfrak{n} \in \frac{\mathfrak{h}_N}{\mathfrak{n}}$. Then we find an element $Y \in \mathfrak{h}$ such that $X \in X + \mathfrak{m} = \mathfrak{p}_{MN}(X+\mathfrak{n}) = \mathfrak{q}_{MN}(X+\mathfrak{n}) = \mathfrak{q}_M(Y) = Y + \mathfrak{m}$. Thus there is a $U \in \mathfrak{m}$ such that $X - Y = U$. Now $U + \mathfrak{n} \subseteq \mathfrak{h}_N \cap \mathfrak{m}$ since $\mathfrak{h} + \mathfrak{n} \subseteq \mathfrak{h}_N$. So $X + \mathfrak{n} = Y + U + \mathfrak{n} \in \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}} + \frac{\mathfrak{h}_N\cap\mathfrak{m}}{\mathfrak{n}}$. This implies the claim. ■

Now we let M^* be that subgroup of $\overline{HN} \cap M \subseteq M$ containing N for which $\frac{M^*}{N} = \left(\frac{\overline{HN}\cap M}{N}\right)_0$ in the Lie group G/N. Then M^* is normal in $\overline{HN}$ and characteristic in $\overline{HN}\cap M)$ modulo N.

As a consequence of Lemma 1.5, we get

Lemma 1.6. *For each $M \in \mathcal{N}(G)$ there is an $N = N_M \in \mathcal{N}(G)$ contained in M such that*

$$\frac{\overline{HN}}{N} = \frac{HN}{N}\cdot\left(\frac{\overline{HN}\cap M}{N}\right)_0 = \frac{HN}{N}\cdot\frac{M^*}{N} = \frac{HM^*}{N}.$$

Proof. We have $\frac{\overline{HN}}{N} = \langle \exp_{G/N} \mathfrak{h}_N/\mathfrak{n}\rangle$, further $\frac{HN}{N} = \langle \exp_{G/N} \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}}\rangle$ by Lemma 1.6. Finally

$$\frac{\mathfrak{h}_N}{\mathfrak{n}} \cap \frac{\mathfrak{m}}{\mathfrak{n}} = \ker \mathfrak{q}_{MN} \text{ and } \frac{\overline{HN}}{N} \cap \frac{M}{N} = \ker q_{MN},$$

whence $\left(\frac{\overline{HN}\cap M}{N}\right)_0 = \langle \exp_{G/N}(\frac{\mathfrak{h}_N\cap\mathfrak{m}}{\mathfrak{n}})\rangle$. Therefore, $\frac{\overline{HN}}{N} = \langle \exp_{G/N} \mathfrak{h}_N/\mathfrak{n}\rangle = \left\langle \exp_{G/N}\left(\frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}} + \frac{\mathfrak{h}_N\cap\mathfrak{m}}{\mathfrak{n}}\right)\right\rangle = \langle \exp_{G/N} \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}}\rangle\langle \exp_{G/N} \frac{\mathfrak{h}_N\cap\mathfrak{m}}{\mathfrak{n}}\rangle = \frac{HN}{N}\cdot\left(\frac{\overline{HN}\cap M}{N}\right)_0 = \frac{HN}{N}\cdot\frac{M^*}{N} = \frac{HM^*}{N}$. This completes the proof. ■

Main Lemma 1.7. *Let H be a closed connected subgroup of the pro-Lie group G. Then for each $M \in \mathcal{N}(G)$ there is an $N_M \in \mathcal{N}(G)$ such that $N_M \subseteq M$*

and that, for the subgroup $M^* \leq \overline{HN_M} \cap M \subseteq \overline{HN_M}$ *containing* N_M *such that* $\frac{M^*}{N_M} = \left(\frac{\overline{HN_M} \cap M}{N_M}\right)_0$ *the following statements are true:*

(i) $N_M \subseteq M^* \subseteq M$ *and* $\frac{\overline{HN_M}}{M^*} = \frac{\overline{HM^*}}{M^*}$ *is a Lie group.*

(ii) *The analytic subgroup* $\frac{HM^*}{M^*}$ *of the Lie group* $\frac{\overline{HM^*}}{M^*}$ *is closed.*

(iii) *The natural morphism* $\frac{H}{H \cap M^*} \to \frac{HM^*}{M^*}$ *is an isomorphism of Lie groups.*

Proof. We choose $N_M \in \mathcal{N}(G)$ as in Lemma 1.6.

Proof of (i). We have $N_M \subseteq M^* \subseteq \overline{HN} \cap M$ by the definition of M^* and note $\overline{HM^*} \subseteq \overline{HN_M} \subseteq \overline{HM^*}$. Since $N_M \in \mathcal{N}(G)$, the quotient G/N_M is a Lie group and thus the quotient $\frac{\overline{HN_M}}{M^*} \cong \frac{\overline{HN_M}/N_M}{M^*/N_M}$ is a Lie group as a quotient of a Lie group.

Proof of (ii). The quotient group $\frac{HN_M}{N_M}$ is an analytic subgroup of the Lie group G/N_M by Lemma 1.4. The group HM^*/M^* is a continuous image of this arcwise connected group under the morphism $hN_M \mapsto hM^*$ and therefore, as an arcwise connected group, an analytic subgroup of $\overline{HM^*}/M^*$. By Lemma 1.6 we have $\frac{\overline{HN_M}}{N_M} = \frac{HM^*}{N_M}$. Passing to the quotient modulo M^*/N_M on both sides yields $\frac{\overline{HN_M}/N_M}{M^*/N_M} = \frac{HM^*/N_M}{M^*/N_M}$, which is equivalent to $\frac{\overline{HN_M}}{M^*} = \frac{HM^*}{M^*}$; this implies that $\frac{HM^*}{M^*}$ is closed and is, therefore, a Lie group.

Proof of (iii). The natural bijective morphism of topological groups

$$h(H \cap M^*) \mapsto hM^* : \frac{H}{H \cap M^*} \to \frac{HM^*}{M^*}$$

between two connected Lie groups is an isomorphism by the Open Mapping Theorem for Locally Compact Groups. ∎

The situation is illustrated by the following diagram:

$$\begin{tikzcd}
\frac{HM}{M} \arrow[d, "\mathrm{inc}"'] & \frac{HM^*}{M^*} \arrow[l] \arrow[d, "="'] & \frac{HN_M}{N_M} \arrow[l] \arrow[d, "\mathrm{inc}"] & H = \lim_{P\in\mathcal{N}(G)} \frac{HP}{P} \arrow[l] \arrow[d, "\mathrm{id}_H"] \\
\frac{\overline{HM}}{M} \arrow[d, "\mathrm{inc}"'] & \frac{\overline{HM^*}}{M^*} \arrow[l] & \frac{\overline{HN_M}}{N_M} \arrow[l] \arrow[d, "\mathrm{inc}"] & H = \lim_{P\in\mathcal{N}(G)} \frac{\overline{HP}}{P} \arrow[l, "q_N"'] \arrow[d, "\mathrm{inc}"] \\
\frac{G}{M} & \frac{G}{N_M} \arrow[l, "p_{MN}"] & \frac{G}{N_M} \arrow[l, "="] & G = \lim_{P\in\mathcal{N}(G)} \frac{G}{P}. \arrow[l, "p_N"]
\end{tikzcd}$$

We observe that M^* need not be a member of $\mathcal{N}(G)$. If H is a normal subgroup, then $\overline{HN_M} \cap M$ is normal in G, and since $\frac{M^*}{N_M} = \left(\frac{\overline{HN_M} \cap M}{N_M}\right)_0$ is characteristic in $\frac{G}{N_M}$, the group M^* is invariant under all automorphisms of G leaving N_M invariant, and so is certainly invariant under all inner automorphisms of G; and thus M^* is a member of $\mathcal{N}(G)$. Therefore we have the

Lemma 1.8. *Let H be a closed connected normal subgroup of the pro-Lie group G. Then for each $M \in \mathcal{N}(G)$ there is an $M^* \in \mathcal{N}(G)$ such that $M^* \subseteq M$ and that the following statements are true:*

(i) *The analytic subgroup $\frac{HM^*}{M^*}$ of the Lie group $\frac{G}{M^*}$ is closed.*

(ii) $h(H \cap M^*) \mapsto hM^* : \frac{H}{H \cap M^*} \to \frac{HM^*}{M^*}$ *is an isomorphism of Lie groups.* ∎

At this time we have proved Theorem 1.2 for *connected* closed subgroups H. We now complete the proof of 1.2 by showing that the results of Lemma 1.7 remain intact for an *almost* connected closed subgroup H of a pro-Lie group G.

Thus let H be a closed almost connected subgroup of the pro-Lie group G and let $M \in \mathcal{N}(G)$. We apply Lemma 1.6 to H_0 in place of H and find that there is a closed normal subgroup M^* of $\overline{H_0M}$ contained in M such that H_0M^*/M^* is a connected Lie group. In particular, H_0M^* is a closed subgroup of G. Since M^* is characteristic in $\overline{H_0N_M} \cap M$ modulo N_M, and $\overline{H_0N_M} \cap M$ is normal in $\overline{HN_M}$, the group M^* is normal in $\overline{HN_M}$. So M^* is normalized by H and thus by HM^*. We must show that HM^*/M^* is a Lie group; the remainder of Theorem 1.2 then follows. Now H/H_0 is assumed to be compact. The continuous morphism

$$hH_0 \mapsto hH_0M^* : H/H_0 \to HM^*/H_0M^*$$

is surjective. Hence HM^*/H_0M^* is a compact group. Thus HM^*/M^* is a locally compact group as an extension of the Lie group H_0M^*/M^* by the compact group HM^*/H_0M^*. The morphism

$$h(H \cap N_M) \mapsto hM^*: H/(H \cap N_M) \to HM^*/M^*$$

is a surjective morphism from a σ-compact group onto a locally compact group and is therefore open. The group $H/(H \cap N_M)$ is a Lie group by Lemma 1.3. Thus HM^*/M^* is a Lie group. This completes the proof of Theorem 1.2.

2. Applications

In the formulation of the Closed Subgroup Theorem for Projective Limits 1.34 in [4], a complete group G was considered for which there is a filter basis $\mathcal{N}$ of closed normal subgroups N such that $\lim \mathcal{N} = 1$ and that G/N is complete for each $N \in \mathcal{N}$. For a closed subgroup H of G it was asserted in 1.34(iv) that

($*$) *for each $N \in \mathcal{N}$ the standard morphism $f_N: H/(H \cap N) \to HN/N$ is an isomorphism of topological groups.*

Glöckner noticed in [1] that ($*$) is false; the error in the proof was the claim that for $N \subseteq M$ in $\mathcal{N}$ the natural morphism $HN/N \to HM/M$ had to be open.

In circumstances where (a) G is a pro-Lie group, (b) $\mathcal{N} = \mathcal{N}(G)$, and (c) H is almost connected and normal, then Theorem 1.2 shows that ($*$), while possibly not true for *all* $N \in \mathcal{N}$, is true for $N \in \mathcal{N}$ time and again for smaller and smaller N.

In [4], $(*)$ is used in three proofs under the hypotheses (a), (b), and (c) and indeed in these applications the cofinal validity of the conclusion is perfectly sufficient. We explain how in these instances the correction is implemented.

Correcting the proof of [4], 5.17. We refer to [4], Proposition 5.17 and its proof on pp. 225, 226. We replace lines 3, 4 and 5 on page 226 by the following:

(1) $\ldots G_1 \cap U$. Now let $M \in \mathcal{N}(G)$ be contained in U. Then $M \cap G_1 \subseteq U \cap G_1 = \{0\}$. Let $N \stackrel{\text{def}}{=} M^* \subseteq M$ be the subgroup attached to M by Theorem 1.2 of this paper. Since G is abelian, N is normal. Then G/N is a Lie group and $(G_1+N)/N$ is isomorphic to $G_1/(G_1 \cap N) \cong G_1$ by Theorem 1.2 of this paper. So $(G_1+N)/N \ldots$

Correcting the proof of [4], 11.15. We replace lines 2 and 3 of page 475 of [4] by the following:

(2) ... so, since N is finite dimensional, there is a $P \in \mathcal{N}(G)$, such that $N \cap P = \{1\}$. Now let $M = P^* \subseteq P$ be the member of $\mathcal{N}(G)$ attached to P according to Theorem 1.2 of this paper. Then by Theorem 1.2 of this paper, $n \mapsto nM : N \to NM/M$ is an isomorphism, and...

Correcting the proof of [4], 3.29(iii). We replace section (iii) of [4], 3.29 by the following

(3) (iii) *The set* $I = \{j \in J : G_0/(G_0 \cap K_j) \to (G_0K_j)/K_j$ is an isomorphism of topological groups$\}$ *is cofinal in* J. *For* $j \in I$, *the group* $(G_0K_j)/K_j$ *is a Lie group and a closed subgroup of* G/K_j, *and* $G_0 = \lim_{j \in J}(G_0K_j)/K_j$.

We replace the last sentence of the proof of 3.29(iii) in [4] by the following:

Theorem 1.2 of this paper shows that the set I is cofinal in J, and it establishes the other statements of (iii) as well.

This correction reinstates our original proof of the Pro-Lie Group Theorem [4], 3.34, which was independently proved in [5] and [1]. The revised version of 3.29 corroborates the first sentence of the proof of 3.34.

However, this comment requires additional inspection in order that the argument is not circular. We did take care of this inspection in the Remark following Lemma 1.3.

Correcting the proof of [4], 9.54. This occurrence of 1.34(iv) has nothing to do with an application of Theorem 1.2(b) of this paper and is easily remedied directly. Indeed in lines 8 and 7 from below on page 406 of [4], replace "By Theorem 1.34(iv) ... as topological groups" by

(4) "We know that $C/(C \cap N) \cong CN/N$ as groups".

The point is that the algebraic isomorphism suffices in this case.

3. Concluding Remarks

Theorem 1.2 secures another instance of the validity of the so-called Second Isomorphism Theorem $H/(H \cap N) \cong HN/N$ in the category of topological groups. Usually, if it is true at all in the sense that the isomorphism holds algebraically *and topologically*, some application of the Open Mapping Theorem is involved (see e.g. [4], Corollary 9.62, p. 413) which is not the case in this instance. Theorem 1.2 pertains to the theory of projective limits. Normally, hypotheses on the projective system lead to conclusions on the limit. It is rare that, as in Theorem 1.2, assumptions on the limit entail conclusions on the projective system.

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