Journal of Lie Theory
Volume **18** (2008) 141–160

Complex Manifolds Admitting Proper Actions of High-Dimensional Groups

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Communicated by S. Gindikin

Abstract. We explicitly classify all pairs (M, G), where M is a connected complex manifold of dimension $n \geq 2$ and G is a connected Lie group acting properly and effectively on M by holomorphic transformations and having dimension d_G satisfying $n^2 + 2 \leq d_G < n^2 + 2n$. We also consider the case $d_G = n^2 + 1$. In this case all actions split into three types according to the form of the linear isotropy subgroup. We give a complete explicit description of all pairs (M, G) for two of these types, as well as a large number of examples of actions of the third type. These results complement a theorem due to W. Kaup for the maximal group dimension $n^2 + 2n$ and generalize some of the author's earlier work on Kobayashi-hyperbolic manifolds with high-dimensional holomorphic automorphism group.
Mathematics Subject Classification 2000: 53C30, 32M10, 32Q57.
Key Words and Phrases: Proper group actions, complex manifolds.

1. Introduction

Let G be a topological group acting on a C^∞-smooth manifold M by diffeomorphisms. The action is called proper, if the map

$$\Phi : G \times M \to M \times M, \quad (g, p) \mapsto (gp, p),$$

is proper, that is, for every compact subset $C \subset M \times M$ its inverse image $\Phi^{-1}(C) \subset G \times M$ is compact as well. In this paper we only consider effective actions. The properness of the action implies that: (i) G is locally compact, hence by the results due to Bochner and Montgomery (see [28]) it carries the structure of a Lie transformation group; (ii) G is isomorphic to a closed subgroup of the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M endowed with the compact-open topology (see [2] for a brief survey on proper actions). Thus, one can assume that G is a Lie group acting smoothly and properly on the manifold M and that it is realized as a closed subgroup of $\mathrm{Diff}(M)$. Due to the results of [29], [33] (see also [1]) all such groups can be characterized precisely as closed subgroups of the isometry groups for all possible smooth Riemannian metrics on M.

ISSN 0949–5932 / \$2.50

If G acts properly on M, then for every $p \in M$ its isotropy subgroup G_p is compact in G. Then by [3] the isotropy representation

$$\alpha_p : G_p \to GL(\mathbb{R}, T_p(M)), \quad g \mapsto dg(p)$$

is continuous and faithful, where $T_p(M)$ denotes the tangent space to M at p and $dg(p)$ is the differential of g at p. In particular, the linear isotropy subgroup

$$LG_p := \alpha_p(G_p)$$

is a compact subgroup of $GL(\mathbb{R}, T_p(M))$ isomorphic to G_p. In some coordinates in $T_p(M)$ the group LG_p becomes a subgroup of the orthogonal group $O_m(\mathbb{R})$, where $m := \dim M$. Hence $\dim G_p \leq \dim O_m(\mathbb{R}) = m(m-1)/2$. Furthermore, for every $p \in M$ its orbit Gp is a closed submanifold of M of dimension not exceeding m. Thus, setting $d_G := \dim G$ we see that $d_G \leq m(m+1)/2$. It is a classical result (see [7], [4], [6]) that if G acts properly on a smooth manifold M of dimension $m \geq 2$ and $d_G = m(m+1)/2$, then M is isometric (with respect to some G-invariant metric) either to one of the standard complete simply-connected spaces of constant sectional curvature $\mathbb{R}^m$, S^m, $\mathbb{H}^m$ (where $\mathbb{H}^m$ is the hyperbolic space), or to $\mathbb{RP}^m$.

Groups of lower dimensions were extensively studied in the 1950's-70's. It was shown in [36] (see also [5], [39]) that a group G with $m(m-1)/2+1 < d_G < m(m+1)/2$ cannot act properly on a smooth manifold M of dimension $m \neq 4$. The exceptional 4-dimensional case was considered in [18]; it turned out that a group of dimension 9 cannot act properly on a 4-dimensional manifold, and that if a 4-dimensional manifold admits a proper action of an 8-dimensional group G, then it has a G-invariant complex structure. There exists also an explicit classification of pairs (M, G), where $m \geq 4$, G is connected, and $d_G = m(m-1)/2+1$ (see [39], [24], [31], [18]). Further, in [22] a reasonably explicit classification of pairs (M, G), where $m \geq 6$, G is connected, and $(m-1)(m-2)/2+2 \leq d_G \leq m(m-1)/2$, was given. We also mention a classification of G-homogeneous manifolds for $m = 4$, $d_G = 6$ (see [18]) and a classifications of G-homogeneous simply-connected manifolds in the cases $m = 3$, $d_G = 3, 4$ and $m = 4$, $d_G = 5$ (see [4], [34]) obtained by E. Cartan's method of adapted frames. There are many other results, especially for compact groups, but – to the best of our knowledge – no complete classifications exist beyond dimension $(m-1)(m-2)/2+2$ (see [21], [40] and references therein for further details).

We study proper group actions in the complex setting. From now on, M will denote a complex manifold of complex dimension n and G will be a subgroup of $\text{Aut}(M)$, the group of all holomorphic automorphisms of M. If for complex manifolds M_j and subgroups $G_j \subset \text{Aut}(M_j)$, $j = 1, 2$, there exists a biholomorphic map $F : M_1 \to M_2$ such that $F \circ G_1 \circ F^{-1} = G_2$, we say that the pairs (M_1, G_1) and (M_2, G_2) are equivalent. We will be classifying pairs (M, G) up to this equivalence relation, but we will not be concerned with determining G-invariant Riemannian metrics on M.

Proper actions by holomorphic transformations are found in abundance. Due to a result by Kaup (see [19]), Lie groups acting properly and effectively on M by holomorphic transformations are precisely those closed subgroups of $\text{Aut}(M)$

that preserve continuous distances on M. In particular, if M is a Kobayashi-hyperbolic manifold, then $\mathrm{Aut}(M)$ is a Lie group acting properly on M (see also [20]).

In the complex setting, in some coordinates in $T_p(M)$ the group LG_p becomes a subgroup of the unitary group U_n. Hence $\dim G_p \le \dim U_n = n^2$, and therefore $d_G \le n^2 + 2n$. We note that $n^2 + 2n < (m-1)(m-2)/2 + 2$ for $m = 2n$ and $n \ge 5$. Thus, the group dimension range that arises in the complex case, for $n \ge 5$ lies strictly below the dimension range investigated in the classical real case and therefore is not covered by the existing classification results. Furthermore, overlaps with these results for $n = 3, 4$ and $n = 2$, $d_G = 6$ occur only in relatively easy situations and do not lead to any significant simplifications in the complex case. The only interesting overlap with the real case occurs for $n = 2$, $d_G = 5$ (see [34]). Note that in the situations when overlaps do occur, the existing classifications in the real case do not necessarily immediately lead to classifications in the complex case, since the determination of all G-invariant complex structures on the corresponding real manifolds may be a non-trivial task.

The case $d_G = n^2 + 2n$ was considered by Kaup in [19]. In this situation (M, G) is equivalent to to one of the pairs $(\mathbb{B}^n, \mathrm{Aut}(\mathbb{B}^n))$, $(\mathbb{C}^n, G(\mathbb{C}^n))$, $(\mathbb{CP}^n, G(\mathbb{CP}^n))$. Here $\mathbb{B}^n := \{z \in \mathbb{C}^n : ||z|| < 1\}$, $\mathrm{Aut}(\mathbb{B}^n) \simeq PSU_{n,1} := SU_{n,1}/(\mathrm{center})$ is the group of all transformations

$$z \mapsto \frac{Az + b}{cz + d},$$

where

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU_{n,1};$$

$G(\mathbb{C}^n) \simeq U_n \ltimes \mathbb{C}^n$ is the group of all holomorphic automorphisms of $\mathbb{C}^n$ of the form

$$z \mapsto Uz + a, \tag{1.1}$$

where $U \in U_n$, $a \in \mathbb{C}^n$ (we usually write $G(\mathbb{C})$ instead of $G(\mathbb{C}^1)$); and $G(\mathbb{CP}^n) \simeq PSU_{n+1} := SU_{n+1}/(\mathrm{center})$ is the group of all holomorphic automorphisms of $\mathbb{CP}^n$ of the form

$$\zeta \mapsto U\zeta, \tag{1.2}$$

where ζ is a point in $\mathbb{CP}^n$ written in homogeneous coordinates, and $U \in SU_{n+1}$ (this group is a maximal compact subgroup of the complex Lie group $\mathrm{Aut}(\mathbb{CP}^n) \simeq PSL_{n+1}(\mathbb{C}) := SL_{n+1}(\mathbb{C})/(\mathrm{center})$). We remark that the groups $\mathrm{Aut}(\mathbb{B}^n)$, $G(\mathbb{C}^n)$, $G(\mathbb{CP}^n)$ are the full groups of holomorphic isometries of the Bergman metric on $\mathbb{B}^n$, the flat metric on $\mathbb{C}^n$, and the Fubini-Study metric on $\mathbb{CP}^n$, respectively, and that the above result due to Kaup can be obtained directly from E. Cartan's classification of Hermitian symmetric spaces.

We are interested in characterizing pairs (M, G) for $d_G < n^2 + 2n$. In [16], [11], [12], [13] we considered the special case where M is a Kobayashi-hyperbolic manifold and $G = \mathrm{Aut}(M)$, and explicitly determined all manifolds with $n^2 - 1 \le d_{\mathrm{Aut}(M)} < n^2 + 2n$, $n \ge 2$ (see [14] for a comprehensive exposition of these results). The case $d_{\mathrm{Aut}(M)} = n^2 - 2$ represents the first obstruction to the existence of an explicit classification for all n, namely, there is no good description of hyperbolic manifolds with $n = 2$, $d_{\mathrm{Aut}(M)} = 2$ (see [13], [14]). Our immediate goal is to generalize these results to arbitrary proper actions on not necessarily Kobayashi-hyperbolic manifolds by classifying all pairs (M, G) with $n^2 - 1 \leq d_G < n^2 + 2n$, $n \geq 2$, where G is connected.

In this paper we assume that $n^2 + 1 \leq d_G < n^2 + 2n$ (hence by [19] the action of G on M is transitive). For $n^2 + 2 \leq d_G < n^2 + 2n$ we completely describe all pairs (M, G) in Theorem 2.1 in Section 2 This theorem follows almost immediately from the general theory of Hermitian symmetric spaces (see [9]) and the classification of isotropy irreducible homogeneous manifolds (see [25], [26], [27], [38], [23], [37]). Note that one can also give an elementary (but longer) proof of Theorem 2.1 that does not refer to this general theory and is based almost solely on the analysis of the fundamental vector fields of the G-action (see [15]).

Further, we consider the case $d_G = n^2 + 1$. Firstly, we determine the connected identity components of all possible linear isotropy subgroups in Proposition 3.1 – see Section 3 According to this description, every action with $d_G = n^2 + 1$ falls into one of three types. We deal with actions of types I and II in Section 4 Complete lists of the corresponding pairs (M, G) are obtained in Theorems 4.1 and 4.2, respectively. Actions of type III are the hardest to deal with. We give a large number of examples of such actions in Section 5 It is our conjecture that these examples in fact cover all possible actions of type III (see Conjecture 5.1).

For comparison, we note that the determination of homogeneous Kobayashi-hyperbolic manifolds with $n^2 - 1 \leq d_{\mathrm{Aut}(M)} < n^2 + 2n$, $n \geq 2$, in [16], [11], [13], [14] was an easier task. Indeed, due to [30] every homogeneous Kobayashi-hyperbolic manifold is holomorphically equivalent to a Siegel domain of the second kind, and therefore such manifolds can be studied by using techniques available for Siegel domains (see e.g. [35]). Clearly, this approach cannot be applied to general transitive proper actions, and one motivation for the present work is to re-obtain the classification of homogeneous Kobayashi-hyperbolic manifolds without using the non-trivial result of [30].

Acknowledgements. Part of this work was done while the author was visiting the Ruhr-Universität Bochum in January-February 2007 and the Max-Plank Institut für Mathematik in Bonn in April-May 2007. We would like to thank G. Fels for making a large number of useful comments and interest in our work.

2. The case $n^2 + 2 \leq d_G < n^2 + 2n$

In this section we prove the following theorem.

Theorem 2.1. *Let M be a connected complex manifold of dimension $n \geq 2$ and $G \subset Aut(M)$ a connected Lie group that acts properly on M and has dimension d_G satisfying $n^2+2 \leq d_G < n^2+2n$. Then the pair (M, G) is equivalent to one of the following:*

(i) $(\mathbb{C}^n, G_1(\mathbb{C}^n))$, where the group $G_1(\mathbb{C}^n)$ consists of all maps of the form (1.1) with $U \in SU_n$ (here $d_G = n^2 + 2n - 1$);[1]

(ii) $(\mathbb{C}^4, G_2(\mathbb{C}^4))$, where the group $G_2(\mathbb{C}^4)$ consists of all maps of the form (1.1)

[1]We usually write $G_1(\mathbb{C})$ instead of $G_1(\mathbb{C}^1)$.

for $n=4$ with $U \in e^{i\mathbb{R}}Sp_2$ (here $d_G = n^2+3 = 19$);[2]

(iii) $(M' \times M'', G' \times G'')$, where M' is one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$, M'' is one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, G' is one of $Aut(\mathbb{B}^{n-1})$, $G(\mathbb{C}^{n-1})$, $G(\mathbb{CP}^{n-1})$, and G'' is one of $Aut(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively (here $d_G = n^2+2$);

(iv) $(\mathbb{C}^4, G_3(\mathbb{C}^4))$, where the group $G_3(\mathbb{C}^4)$ consists of all maps of the form (1.1) for $n=4$ with $U \in Sp_2$ (here $d_G = n^2+2 = 18$).

Proof: Fix $p \in M$. Since the action of G on M is transitive (see [19]), we have $n^2 - 2n + 2 \le \dim LG_p < n^2$. Choose coordinates in $T_p(M)$ so that $LG_p \subset U_n$. Then Lemma 2.1 in [16] (see also Lemma 1.4 in [14]) implies that the connected identity component LG_p^0 of LG_p either is SU_n, or for $n=4$ is conjugate in U_4 to $e^{i\mathbb{R}}Sp_2$, or is conjugate in U_n to $U_{n-1} \times U_1$, or for $n=4$ is conjugate in U_4 to Sp_2.

Suppose first that $LG_p^0 = SU_n$. Since LG_p^0 acts $\mathbb{R}$-irreducibly on $T_p(M)$, Theorem 13.1 of [38] gives that M equipped with a G-invariant Hermitian metric is a Hermitian symmetric space. Clearly, either it is an irreducible Hermitian symmetric space of compact or non-compact type, or it is holomorphically isometric to $\mathbb{C}^n$ equipped with the flat metric. The explicit classification of irreducible Hermitian symmetric spaces (see [9]) rules out the first possibility, and therefore M is holomorphically equivalent to $\mathbb{C}^n$ by means of a map that transforms G into a codimension 1 subgroup of $G(\mathbb{C}^n)$. Clearly, this subgroup must coincide with $G_1(\mathbb{C}^n)$, and we have obtained (i) of the theorem.

Assume now that $n=4$ and LG_p^0 is conjugate in U_4 to $e^{i\mathbb{R}}Sp_2$. Then for every $q \in M$ the subgroup LG_q^0 contains the element $-\mathrm{id}$. Therefore, M equipped with a G-invariant Hermitian metric is a Hermitian symmetric space. Further, it follows, as above, that M is holomorphically equivalent to $\mathbb{C}^4$ by means of a map F that transforms G into a subgroup of $G(\mathbb{C}^4)$. Let $p_0 \in M$ be such that $F(p_0) = 0$. Then F transforms $G_{p_0}^0$ into a closed subgroup of $U_4 \subset G(\mathbb{C}^4)$ isomorphic to $e^{i\mathbb{R}}Sp_2$. This subgroup must be conjugate in U_4 to the standard embedding of $e^{i\mathbb{R}}Sp_2$ in U_4 (see Lemma 2.1 in [16]), and hence there exists an equivalence map $\hat{F}$ between M and $\mathbb{C}^4$ that transforms $G_{p_0}^0$ into $e^{i\mathbb{R}}Sp_2$.

Let $\mathfrak{g}$ be the Lie algebra of fundamental vector fields of the action of the group $\hat{G} := \hat{F} \circ G \circ \hat{F}^{-1}$ on $\mathbb{C}^4$. Since $\hat{G} \subset G(\mathbb{C}^4)$, the algebra $\mathfrak{g}$ is generated by $\langle Z_0 \rangle \oplus \mathfrak{sp}_2$ and some affine holomorphic vector fields V_j, $j = 1, \dots, 8$, that do not vanish at the origin, where

$$Z_0 := i \sum_{k=1}^{4} z_k \, \partial / \partial z_k,$$

and $\mathfrak{sp}_2$ is realized as the algebra of fundamental vector fields of the standard action of Sp_2 on $\mathbb{C}^4$. Considering $[Z_0, [V_j, Z_0]]$ instead of V_j, we can assume that V_j are constant vector fields for all j (cf. the proof of Satz 4.9 in [19]). It is then clear that $\hat{G} = G_2(\mathbb{C}^4)$, and we have obtained (ii) of the theorem.

Assume next that LG_p^0 is conjugate in U_n to $U_{n-1} \times U_1$. As in the preceding case, for every $q \in M$ the subgroup LG_q^0 contains the element $-\mathrm{id}$, and therefore M equipped with a G-invariant Hermitian metric is a Hermitian

[2]Here Sp_2 denotes the standard compact real form of $Sp_4(\mathbb{C})$.

symmetric space. The classification of Hermitian symmetric spaces now yields that (M, G) is equivalent to one of the pairs listed in (iii) of the theorem.

Suppose finally that $n = 4$ and LG_p^0 is conjugate in U_4 to Sp_2. Again, for every $q \in M$ the subgroup LG_q^0 contains the element $-\mathrm{id}$. As in case (ii) above, we obtain that there exists an equivalence map $\hat{F}$ between M and $\mathbb{C}^4$ that transforms G into a subgroup of $G(\mathbb{C}^4)$ and $G_{p_0}^0$ into Sp_2, with $\hat{F}(p_0) = 0$. Let $\mathfrak{g}$ be the Lie algebra of fundamental vector fields of the action of the group $\hat{G} := \hat{F} \circ G \circ \hat{F}^{-1}$ on $\mathbb{C}^4$. The algebra $\mathfrak{g}$ is generated by by $\mathfrak{sp}_2$ and some holomorphic vector fields

$$X_j = \sum_{k=1}^{4} f_j^k \, \partial/\partial z_k,$$

$$Y_j = \sum_{k=1}^{4} g_j^k \, \partial/\partial z_k,$$

for $j = 1, 2, 3, 4$. Here f_j^k, g_j^k are affine functions such that

$$f_j^k(0) = \delta_j^k, \quad g_j^k(0) = i\delta_j^k,$$

where δ_j^k is the Kronecker symbol.

We consider the following vector fields from $\mathfrak{sp}_2$:

$$\begin{aligned} Z_1 &:= iz_2 \, \partial/\partial z_2 - iz_4 \, \partial/\partial z_4, \\ Z_2 &:= iz_1 \, \partial/\partial z_1 - iz_3 \, \partial/\partial z_3. \end{aligned}$$

It is straightforward to see that $[X_1, Z_1](0) = 0$ and $[Y_1, Z_1](0) = 0$, and therefore we have

$$\begin{aligned} [X_1, Z_1] &= 0 \pmod{\mathfrak{sp}_2}, \\ [Y_1, Z_1] &= 0 \pmod{\mathfrak{sp}_2}. \end{aligned} \tag{2.1}$$

Next, we observe

$$\begin{aligned} [X_1, Z_2](0) &= (i, 0, 0, 0), \\ [Y_1, Z_2](0) &= (-1, 0, 0, 0). \end{aligned}$$

It then follows that

$$\begin{aligned} X_1 &= -[Y_1, Z_2] \pmod{\mathfrak{sp}_2}, \\ Y_1 &= [X_1, Z_2] \pmod{\mathfrak{sp}_2}, \end{aligned}$$

which yields

$$\begin{aligned} X_1 &= -[[X_1, Z_2], Z_2] \pmod{\mathfrak{sp}_2}, \\ Y_1 &= -[[Y_1, Z_2], Z_2] \pmod{\mathfrak{sp}_2}. \end{aligned} \tag{2.2}$$

Formulas (2.1) and (2.2) imply that the linear parts of X_1 and Y_1 are elements of $\mathfrak{sp}_2$.

Applying the above arguments to X_3, Y_3 in place of X_1, Y_1 we obtain that the linear parts of X_3, Y_3 are elements of $\mathfrak{sp}_2$ as well. Furthermore, if in these arguments we interchange Z_1, Z_2 and use X_2 in place of X_1, Y_2 in place of Y_1, X_4 in place of X_3, and Y_4 in place of Y_3, we obtain that the linear parts of X_2, Y_2, X_4, Y_4 also lie in $\mathfrak{sp}_2$. It then follows that $\hat{G} = G_3(\mathbb{C}^4)$.

The proof is complete. ∎

Remark 2.2. Some parts of Theorem 2.1 can also be derived from the results of [10].

3. Classification of Linear Isotropy Subgroups for $d_G = n^2 + 1$

In this section we prove the following proposition that extends Lemma 2.1 of [17].

Proposition 3.1. *Let H be a connected closed subgroup of U_n of dimension $(n-1)^2$, $n \geq 2$. Then H is conjugate in U_n to one of the following subgroups:*

I. $e^{i\mathbb{R}}SO_3(\mathbb{R})$ (here $n = 3$);

II. $SU_{n-1} \times U_1$ realized as the subgroup of all matrices

$$\begin{pmatrix} A & 0 \\ 0 & e^{i\theta} \end{pmatrix},$$

where $A \in SU_{n-1}$ and $\theta \in \mathbb{R}$, for $n \geq 3$;

III. the subgroup $H^n_{k_1,k_2}$ of all matrices

$$\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}, \tag{3.1}$$

where k_1, k_2 are fixed integers such that $(k_1, k_2) = 1$, $k_1 > 0$, and $A \in U_{n-1}$, $a \in (\det A)^{\frac{k_2}{k_1}} := \exp(k_2/k_1 \, Ln\,(\det A))$.

Remark 3.2. The groups $H^n_{k_1,k_2}$ are pairwise not conjugate to each other for $n \geq 3$, whereas $H^2_{k_1,k_2}$ and $H^2_{k_2,k_1}$ are conjugate provided $k_2 > 0$. Observe also that the group $H^n_{k_1,k_2}$ is a k_1-sheeted cover of U_{n-1} for every k_2 (note that for $k_2 = 0$ we have $k_1 = 1$).

Proof of Proposition 3.1: Since H is compact, it is completely reducible, i.e. $\mathbb{C}^n$ splits into the sum of H-invariant pairwise orthogonal complex subspaces, $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_m$, such that the restriction H_j of H to each V_j is irreducible. Let $n_j := \dim_{\mathbb{C}} V_j$ (hence $n_1 + \cdots + n_m = n$) and let U_{n_j} be the group of unitary transformations of V_j. Clearly, $H_j \subset U_{n_j}$, and therefore $\dim H \leq n_1^2 + \cdots + n_m^2$. On the other hand $\dim H = (n-1)^2$, which shows that $m \leq 2$.

Let $m = 2$. Then there exists a unitary change of coordinates in $\mathbb{C}^n$ such all elements of H take the form (3.1), where $A \in U_{n-1}$ and $a \in U_1$. If $\dim H_2 = 0$, then $H_2 = \{1\}$, and therefore $H_1 = U_{n-1}$. In this case we obtain the group $H^n_{1,0}$. Suppose next that $\dim H_2 = 1$, i.e. $H_2 = U_1$. Then $(n-1)^2 - 1 \leq \dim H_1 \leq (n-1)^2$. If $\dim H_1 = (n-1)^2 - 1$, then $H_1 = SU_{n-1}$, and hence H is conjugate to $SU_{n-1} \times U_1$ for $n \geq 3$ and to $H^2_{1,0}$ for $n = 2$. Now let $\dim H_1 = (n-1)^2$, i.e. $H_1 = U_{n-1}$. Consider the Lie algebra $\mathfrak{h}$ of H. Up to conjugation, it consists of matrices of the form

$$\begin{pmatrix} \mathfrak{A} & 0 \\ 0 & l(\mathfrak{A}) \end{pmatrix}, \tag{3.2}$$

where $\mathfrak{A} \in \mathfrak{u}_{n-1}$ and $l(\mathfrak{A}) \not\equiv 0$ is a linear function of the matrix elements of $\mathfrak{A}$ ranging in $i\mathbb{R}$. Clearly, $l(\mathfrak{A})$ must vanish on the derived algebra of $\mathfrak{u}_{n-1}$, that is, on $\mathfrak{su}_{n-1}$. Hence matrices (3.2) form a Lie algebra if and only if $l(\mathfrak{A}) = c \cdot \operatorname{trace} \mathfrak{A}$, where $c \in \mathbb{R} \setminus \{0\}$. Such an algebra can be the Lie algebra of a closed subgroup of

$U_{n-1}\times U_1$ only if $c\in\mathbb{Q}\setminus\{0\}$. Hence H is conjugate to $H^n_{k_1,k_2}$ for some $k_1,k_2\in\mathbb{Z}$, where one can always assume that $k_1>0$ and $(k_1,k_2)=1$.

Now let $m=1$. We shall proceed as in the proof of Lemma 2.1 in [16]. Let $\mathfrak{h}^{\mathbb{C}}:=\mathfrak{h}+i\mathfrak{h}\subset\mathfrak{gl}_n$ be the complexification of $\mathfrak{h}$, where $\mathfrak{gl}_n:=\mathfrak{gl}_n(\mathbb{C})$. The algebra $\mathfrak{h}^{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^n$ and by a theorem of E. Cartan (see, e.g., [8]), $\mathfrak{h}^{\mathbb{C}}$ is either semisimple or the direct sum of the center $\mathfrak{c}$ of $\mathfrak{gl}_n$ and a semisimple ideal $\mathfrak{t}$. Clearly, the action of the ideal $\mathfrak{t}$ on $\mathbb{C}^n$ is irreducible.

Assume first that $\mathfrak{h}^{\mathbb{C}}$ is semisimple, and let $\mathfrak{h}^{\mathbb{C}}=\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_k$ be its decomposition into the direct sum of simple ideals. Then the natural irreducible n-dimensional representation of $\mathfrak{h}^{\mathbb{C}}$ (given by the embedding of $\mathfrak{h}^{\mathbb{C}}$ in $\mathfrak{gl}_n$) is the tensor product of some irreducible faithful representations of the $\mathfrak{h}_j$ (see, e.g., [8]). Let n_j be the dimension of the corresponding representation of $\mathfrak{h}_j$, $j=1,\dots,k$. Then $n_j\ge 2$, $\dim_{\mathbb{C}}\mathfrak{h}_j\le n_j^2-1$, and $n=n_1\cdot\ldots\cdot n_k$. The following observation is simple.

Claim: *If $n=n_1\cdot\ldots\cdot n_k$, $k\ge 2$, $n_j\ge 2$ for $j=1,\dots,k$, then $\sum_{j=1}^k n_j^2\le n^2-2n$.*

Since $\dim_{\mathbb{C}}\mathfrak{h}^{\mathbb{C}}=(n-1)^2$, it follows from the above claim that $k=1$, i.e. $\mathfrak{h}^{\mathbb{C}}$ is simple. The minimal dimensions of irreducible faithful representations of complex simple Lie algebras $\mathfrak{s}$ are well-known (see, e.g., [32]). In the table below V denotes representations of minimal dimension.

$\mathfrak{s}$	$\dim V$	$\dim\mathfrak{s}$
$\mathfrak{sl}_k$ $k\ge 2$	k	k^2-1
$\mathfrak{o}_k$ $k\ge 7$	k	$k(k-1)/2$
$\mathfrak{sp}_{2k}$ $k\ge 2$	$2k$	$2k^2+k$
$\mathfrak{e}_6$	27	78
$\mathfrak{e}_7$	56	133
$\mathfrak{e}_8$	248	248
$\mathfrak{f}_4$	26	52
$\mathfrak{g}_2$	7	14

Since $\dim_{\mathbb{C}}\mathfrak{h}^{\mathbb{C}}=(n-1)^2$, it follows that none of the above possibilities realize. Therefore, $\mathfrak{h}^{\mathbb{C}}=\mathfrak{c}\oplus\mathfrak{t}$, where $\dim\mathfrak{t}=n^2-2n$. Then, if $n=2$, we obtain that H coincides with the center of U_2 which is impossible since its action on $\mathbb{C}^2$ is then not irreducible. Assuming that $n\ge 3$ and repeating the above argument for $\mathfrak{t}$ in place of $\mathfrak{h}^{\mathbb{C}}$, we see that $\mathfrak{t}$ can only be isomorphic to $\mathfrak{sl}_{n-1}$. But $\mathfrak{sl}_{n-1}$ does not have an irreducible n-dimensional representation unless $n=3$.

Thus, $n=3$ and $\mathfrak{h}^{\mathbb{C}}\simeq\mathbb{C}\oplus\mathfrak{sl}_2\simeq\mathbb{C}\oplus\mathfrak{so}_3$. Further, we observe that every irreducible 3-dimensional representation of $\mathfrak{so}_3$ is equivalent to its defining representation. This implies that H is conjugate in $GL_3(\mathbb{C})$ to $e^{i\mathbb{R}}SO_3(\mathbb{R})$. Since $H\subset U_3$ it is straightforward to show that the conjugating element can be chosen to belong to U_3.

The proof of the proposition is complete. ∎

Let M be a connected complex manifold of dimension $n\ge 2$, and suppose that a connected Lie group $G\subset\mathrm{Aut}(M)$ with $d_G=n^2+1$ acts properly on M.

Fix $p \in M$, consider the linear isotropy subgroup LG_p, and choose coordinates in $T_p(M)$ so that $LG_p \subset U_n$. We say that the pair (M, G) (or the action of G on M) is of type I, II or III, if the connected identity component LG_p^0 of the group LG_p is conjugate in U_n to a subgroup listed in I, II or III of Proposition 3.1, respectively. Since M is G-homogeneous, this definition is independent of the choice of p.

We will now separately consider actions of each type.

4. Actions of Types I and II

Actions of type I are described in the following theorem.

Theorem 4.1. *Let M be a connected complex manifold of dimension 3 and $G \subset Aut(M)$ a connected Lie group with $d_G = 10$ that acts properly on M. If the pair (M, G) is of type I, then it is equivalent to one of the following:*

(i) $(\mathscr{S}, \mathrm{Aut}(\mathscr{S}))$, where $\mathscr{S}$ is the Siegel space

$$\mathscr{S} := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : Z\overline{Z} \ll id\right\},$$

with

$$Z := \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$$

(here $Aut(\mathscr{S})$ is isomorphic to $Sp_4(\mathbb{R})/\mathbb{Z}_2$);

(ii) $(\mathcal{Q}_3, SO_5(\mathbb{R}))$, where $\mathcal{Q}_3$ is the complex quadric in $\mathbb{CP}^4$, and $SO_5(\mathbb{R})$ is realized as a maximal compact subgroup of $Aut(\mathcal{Q}_3)^0 \simeq PSO_5(\mathbb{C})$;

(iii) $(\mathbb{C}^3, G_2(\mathbb{C}^3))$, where $G_2(\mathbb{C}^3)$ is the group that consists of all maps of the form (1.1) with $U \in e^{i\mathbb{R}}SO_3(\mathbb{R})$.

Theorem 4.1 follows from the theory of Hermitian symmetric spaces in the same way as Parts (ii), (iii) and (iv) of Theorem 2.1 do, and we omit details.

We now turn to actions of type II and classify them in the following theorem.

Theorem 4.2. *Let M be a connected complex manifold of dimension $n \geq 3$ and $G \subset Aut(M)$ a connected Lie group with $d_G = n^2+1$ that acts properly on M. If the pair (M, G) is of type II, then it is equivalent to $(\mathbb{C}^{n-1} \times M', G_1(\mathbb{C}^{n-1}) \times G')$, where M' is one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, and G' is one of the groups $Aut(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively.*

We start with the following lemma that clarifies the structure of full linear isotropy subgroups for actions of type II.

Lemma 4.3. *Let $H \subset U_n$, with $n \geq 3$, be a closed subgroup, and let $H^0 = SU_{n-1} \times U_1$. Then for some $m \in \mathbb{N}$ the group H consists of all matrices of the form*

$$\begin{pmatrix} \alpha A & 0 \\ 0 & a \end{pmatrix},$$

where $A \in SU_{n-1}$, $a \in U_1$, $\alpha^m = 1$.

Proof: The proof is similar to that of Lemma 4.4 in [17]. Let $C_1, \dots, C_K$ be the connected components of H with $C_1 = H^0 = SU_{n-1} \times U_1$. Clearly, there exist $g_1 = \mathrm{id}, g_2, \dots, g_K$ in U_n such that $C_j = g_j H^0$, $j = 1, \dots, K$. Moreover, for each pair of indices i, j there exists an index l such that $g_i H^0 \cdot g_j H^0 = g_l H^0$, and therefore

$$g_l^{-1} g_i H^0 g_j = H^0. \tag{4.1}$$

Applying each side of (4.1) to the vector $v := (0, \dots, 0, 1)$, which is an eigenvector of every element of H^0, we obtain that for every $h \in H^0$ there exists $\beta(h) \in \mathbb{C}$ such that

$$g_l^{-1} g_i h g_j v = \beta(h) v,$$

or, equivalently,

$$h g_j v = \beta(h) g_i^{-1} g_l v,$$

which implies that $g_j v = (0, \dots, 0, a_j)$, where $|a_j| = 1$, $j = 1, \dots, K$. Hence g_j has the form

$$g_j = \begin{pmatrix} A_j & 0 \\ 0 & a_j \end{pmatrix},$$

where $A_j \in U_{n-1}$. Multiplying g_j by an appropriate element of H^0, we can assume that $a_j = 1$ and $A_j = \alpha_j \cdot \mathrm{id}$, with $|\alpha_j| = 1$, $j = 1, \dots, K$.

Clearly, all elements in H of the form

$$\begin{pmatrix} t \cdot \mathrm{id} & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.2}$$

where $|t| = 1$ form a finite subgroup and therefore the corresponding numbers t form a group of roots of unity of some order m.

The proof of the lemma is complete. ∎

Proof of Theorem 4.2: Fix $p \in M$, set $H := G_p$, and identify M as a smooth manifold with G/H (in particular, we identify $T_p(M)$ with $T_H(G/H)$). By means of this identification we introduce a G-invariant complex structure on G/H. Let $\Pi_{G,H} : G \to G/H$ be the factorization map. For every element $g \in G$ we denote by $\mathscr{L}_g$ the action of g on G/H. Let $\mathfrak{g}$ be the Lie algebra of G. Since the subgroup $\mathrm{Ad}(H) \subset GL(\mathbb{R}, \mathfrak{g})$ is compact, there exists an $\mathrm{Ad}(H)$-invariant scalar product on $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of H and $\mathfrak{h}^\perp$ the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. Since $\mathfrak{h}$ is $\mathrm{Ad}(H)$-invariant, so is $\mathfrak{h}^\perp$. The map $\Phi := d\Pi_{G,H}(\mathrm{id})|_{\mathfrak{h}^\perp}$ is a linear isomorphism between $\mathfrak{h}^\perp$ and $T_H(G/H)$, and for every $h \in H$ transforms the operator $\mathrm{Ad}(h)$ on $\mathfrak{h}^\perp$ into the operator $d\mathscr{L}_h(H)$ on $T_H(G/H)$. Analogously, for $c \in \mathfrak{h}$, the map $\mathrm{ad}(c)$ on $\mathfrak{h}^\perp$ is transformed into $\mathscr{L}(c)$ on $T_H(G/H)$, where $\mathscr{L}$ is the differential of the map $h \mapsto d\mathscr{L}_h(H)$ at $\mathrm{id} \in H$.

By Lemma 4.3, at every $q \in M$ there are exactly two non-trivial proper LG_q-invariant complex subspaces $\mathcal{L}_1(q)$ and $\mathcal{L}_2(q)$ in $T_q(M)$. Here $\mathcal{L}_1(q)$ denotes the invariant subspace of dimension $n-1$, and $\mathcal{L}_2(q)$ the invariant complex line. Choosing such subspaces at every point $q \in M$ we obtain two real-analytic G-invariant distributions $\mathcal{L}_1$ and $\mathcal{L}_2$ of $(n-1)$- and 1-dimensional complex subspaces on M, respectively. Lifting $\mathcal{L}_1$ and $\mathcal{L}_2$ to G by means of $\Pi_{G,H}$, we obtain distributions $\mathcal{S}_1$ and $\mathcal{S}_2$ of real (n^2-1)- and (n^2-2n+3)-dimensional subspaces on G, respectively. Since these distributions are invariant under left translations on G, we will think of them as linear subspaces of $\mathfrak{g}$. We have $\mathfrak{g} = \mathcal{S}_1 + \mathcal{S}_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathfrak{h}$. Let $\mathfrak{h}_j^{\perp} := \mathfrak{h}^{\perp} \cap \mathcal{S}_j$, $j = 1, 2$. Clearly, $\mathfrak{h}^{\perp} = \mathfrak{h}_1^{\perp} + \mathfrak{h}_2^{\perp}$, $\dim \mathfrak{h}_1^{\perp} = 2(n-1)$, $\dim \mathfrak{h}_2^{\perp} = 2$, and $\mathfrak{h}_j^{\perp}$ is $\mathrm{Ad}(H)$-invariant for each j.

Fix complex coordinates $(\xi_1, \dots, \xi_n)$ in $T_H(G/H)$ in which LH^0 is given by $SU_{n-1} \times U_1$. Accordingly, we have $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_1 := \mathfrak{su}_{n-1}$, $\mathfrak{h}_2 := \mathfrak{u}_1$. Clearly, Φ maps $\mathfrak{h}_1^{\perp}$ and $\mathfrak{h}_2^{\perp}$ onto $\{\xi_n = 0\}$ and $\{\xi_1 = \cdots = \xi_{n-1} = 0\}$, respectively, and the following holds

$$\begin{aligned} &[\mathfrak{h}_j^{\perp}, \mathfrak{h}_j] \subset \mathfrak{h}_j^{\perp}, \quad j = 1, 2, \\ &[\mathfrak{h}_j^{\perp}, \mathfrak{h}_k] = 0, \qquad j \neq k. \end{aligned} \tag{4.3}$$

Set $\mathcal{S}_j' := \mathfrak{h}_j^{\perp} + \mathfrak{h}_j$ for $j = 1, 2$. We will now show that $\mathcal{S}_j'$ is an ideal in $\mathfrak{g}$ for each j. For any two elements $v_1, v_2 \in \mathfrak{g}$ we write

$$[v_1, v_2] = u_1 + u_2 + w_1 + w_2, \tag{4.4}$$

where $u_k \in \mathfrak{h}_k^{\perp}$ and $w_k \in \mathfrak{h}_k$, $k = 1, 2$. Suppose first that $v_1 \in \mathfrak{h}_1^{\perp}$, $v_2 \in \mathfrak{h}_2^{\perp}$. Applying to (4.4) the element φ_0 of $\mathrm{Ad}(H^0)$ that acts trivially on $\mathfrak{h}_1^{\perp}$ and coincides with $-\mathrm{id}$ on $\mathfrak{h}_2^{\perp}$, we obtain that $u_1 = 0$, $w_1 = 0$, $w_2 = 0$. Next, applying to (4.4) an element of $\mathrm{Ad}(H^0)$ that acts trivially on $\mathfrak{h}_2^{\perp}$ and transforms v_1 into $-v_1$ we obtain that $u_2 = 0$. Thus

$$[\mathfrak{h}_1^{\perp}, \mathfrak{h}_2^{\perp}] = 0. \tag{4.5}$$

Let $v_1, v_2 \in \mathfrak{h}_1^{\perp}$. In this case the application of the transformation φ_0 to (4.4) yields $u_2 = 0$. We now apply the Jacobi identity to v_1, v_2, v, where v is an arbitrary element of $\mathfrak{h}_2^{\perp}$. Then (4.3), (4.5) imply that $[w_2, v] = 0$, and hence $w_2 = 0$. Thus

$$[\mathfrak{h}_1^{\perp}, \mathfrak{h}_1^{\perp}] \subset \mathcal{S}_1'. \tag{4.6}$$

Let finally $v_1, v_2 \in \mathfrak{h}_2^{\perp}$. Applying to (4.4) elements of $\mathrm{Ad}(H^0)$ that act trivially on $\mathfrak{h}_2^{\perp}$ we see that u_1 and w_1 are invariant under all such transformations. Therefore $u_1 = 0$, $w_1 = 0$, and we have obtained

$$[\mathfrak{h}_2^{\perp}, \mathfrak{h}_2^{\perp}] \subset \mathcal{S}_2'. \tag{4.7}$$

Identities (4.3), (4.5), (4.6), (4.7) yield that $\mathcal{S}_j'$ is an ideal in $\mathfrak{g}$ for each j. Thus, for each j the distribution $\mathcal{L}_j$ is integrable, and its integral manifolds form a foliation $\mathfrak{F}_j$ of M by connected complex submanifolds of dimension $n-1$ for $j = 1$ and by connected complex curves for $j = 2$. For $q \in M$ we denote by $\mathfrak{F}_j(q)$ the leaf of the jth foliation passing through q.

Let G_j be the (possibly non-closed) normal connected subgroup of G with Lie algebra $\mathcal{S}_j'$ for $j = 1, 2$. For every $q \in M$ the leaf $\mathfrak{F}_j(q)$ coincides with the orbit $G_j q$. Since the tangent space to $G_j q$ at $q' \in G_j q$ is spanned by the values of the holomorphic fundamental vector fields of the G_j-action at q', it follows that $\mathfrak{F}_j$ is a holomorphic foliation. Clearly, every two orbits of G_j are holomorphically equivalent. The ineffectivity kernel K_j of the action of G_j on $G_j p$ is discrete for each j. Since G_1/K_1 acts properly on $G_1 p$, Theorem 2.1 gives that the orbit $G_1 p$ is holomorphically equivalent to $\mathbb{C}^{n-1}$ by means of a map that transforms G_1/K_1 into the group $G_1(\mathbb{C}^{n-1})$. Furthermore, $G_2 p$ is holomorphically equivalent to one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$ by means of a map that transforms G_2/K_2 into one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively.

We will now show that each G_j is closed in G. We assume that $j=1$; for $j=2$ the proof is similar. Let $\mathfrak{U}$ be a connected neighborhood of 0 in $\mathfrak{g}$ where the exponential map into G is a diffeomorphism, and let $\mathfrak{V} := \exp(\mathfrak{U})$. To prove that G_1 is closed in G it is sufficient to show that for some neighborhood $\mathfrak{W}$ of $e \in G$, $\mathfrak{W} \subset \mathfrak{V}$, we have $G_1 \cap \mathfrak{W} = \exp(\mathcal{S}_1' \cap \mathfrak{U}) \cap \mathfrak{W}$. Assuming the opposite we obtain a sequence $\{g_j\}$ of elements of G_1 converging to e in G such that for every j we have $g_j = \exp(a_j)$ with $a_j \in \mathfrak{U} \setminus \mathcal{S}_1'$. Let $p_j := g_j p$. For a neighborhood $\mathcal{V}$ of p we denote by N_p and N_{p_j} the connected components of $G_1 p \cap \mathcal{V}$ containing p and p_j, respectively. We will now show that there exists a neighborhood $\mathcal{V}$ of p such that $N_{p_j} \neq N_p$ for large j.

Let $\mathfrak{U}'' \subset \mathfrak{U}' \subset \mathfrak{U}$ be connected neighborhoods of 0 in $\mathfrak{g}$ such that: (a) $\exp(\mathfrak{U}'') \cdot \exp(\mathfrak{U}'') \subset \exp(\mathfrak{U}')$; (b) $\exp(\mathfrak{U}'') \cdot \exp(\mathfrak{U}') \subset \exp(\mathfrak{U})$; (c) $\mathfrak{U}' = -\mathfrak{U}'$; (d) $H \cap \exp(\mathfrak{U}') \subset \exp(\mathcal{S}_1' \cap \mathfrak{U}')$. We now choose $\mathcal{V}$ so that $N_p \subset \exp(\mathcal{S}_1' \cap \mathfrak{U}'')p$. Suppose that $p_j \in N_p$. Then we have $p_j = sp$ for some $s \in \exp(\mathcal{S}_1' \cap \mathfrak{U}'')$ and hence $t := g_j^{-1} s$ is an element of H. For large j we have $g_j^{-1} \in \exp(\mathfrak{U}'')$. Condition (a) now implies that $t \in \exp(\mathfrak{U}')$ and hence by (c), (d) we have $t^{-1} \in \exp(\mathcal{S}_1' \cap \mathfrak{U}')$. Therefore, by (b) we obtain $g_j \in \exp(\mathcal{S}_1' \cap \mathfrak{U})$ which contradicts our choice of g_j. Thus, for large j we have $N_{p_j} \neq N_p$, and thus the orbit $G_1 p$ accumulates to itself. Below we will show that this is in fact impossible thus obtaining a contradiction.

Consider the set $S := G_1 p \cap G_2 p$. The set S contains a non-constant sequence converging to p. Clearly, H^0 preserves S. Since the H^0-orbit of a point in S cannot have positive dimension, the subgroup H^0 fixes every point in S. At the same time, any compact subgroup of dimension $n^2 - 2n$ in $G_1(\mathbb{C}^{n-1})$ fixes exactly one point in $\mathbb{C}^{n-1}$. This contradiction shows that G_j is closed in G for each j. Therefore, the action of G_j on M is proper and hence every leaf of $\mathfrak{F}_j$ is closed in M, for $j = 1, 2$.

We will now show that the subgroup K_j is in fact trivial for each $j = 1, 2$. Let first $j = 1$. Since G_1/K_1 is isomorphic to the simply-connected group $G_1(\mathbb{C}^{n-1}) \simeq SU_{n-1} \ltimes \mathbb{C}^{n-1}$ and since G_1 covers G_1/K_1 with fiber K_1, it follows that K_1 is trivial. Let $j = 2$. If G_2/K_2 is isomorphic to $G(\mathbb{C})$, the triviality of K_2 follows as above. Further, the action of G_{2p}^0 on $G_2 p$ is effective, and thus we have $K_2 \setminus \{e\} \subset G_{2p} \setminus G_{2p}^0$. Suppose that G_2/K_2 is isomorphic to $\mathrm{Aut}(\mathbb{B}^1)$. Every maximal compact subgroup of $\mathrm{Aut}(\mathbb{B}^1)$ is 1-dimensional, hence so is every maximal compact subgroup of G_2. Since G_{2p}^0 is 1-dimensional, it is maximal compact in G_2. Therefore G_{2p} is connected, which implies that K_2 is trivial. Suppose next that G_2/K_2 is isomorphic to $G(\mathbb{CP}^1) \simeq PSU_2$. If K_2 is non-trivial, then $G_2 \simeq SU_2$ and $K_2 \simeq \mathbb{Z}_2$. Then G_{2p}^0 is conjugate in G_2 (upon the identification of G_2 with SU_2) to the subgroup of matrices of the form

$$\begin{pmatrix} 1/b & 0 \\ 0 & b \end{pmatrix},$$

where $|b| = 1$ (see e.g. Lemma 2.1 of [17]). Since this subgroup contains the center of SU_2, the subgroup G_{2p}^0 contains the center of G_2. In particular, $K_2 \subset G_{2p}^0$ which contradicts the non-triviality of K_2. Thus, G_1 is isomorphic to $G_1(\mathbb{C}^{n-1})$ and G_2 is isomorphic to one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$.

Next, since $\mathfrak{g} = \mathcal{S}_1' \oplus \mathcal{S}_2'$ and G_1, G_2 are closed, the group G is a locally direct product of G_1 and G_2. We claim that $\mathscr{T} := G_1 \cap G_2$ is trivial. Indeed, $\mathscr{T}$ is a discrete normal subgroup of each of G_1, G_2. However, every discrete normal subgroup of each of $G_1(\mathbb{C}^{n-1})$, $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$ is trivial, since the center of each of these groups is trivial. Hence $G = G_1 \times G_2$.

We will now observe that for every $q_1, q_2 \in M$ the orbits G_1q_1 and G_2q_2 intersect at exactly one point. Let $g \in G$ be an element such that $gq_2 = q_1$. It can be uniquely represented in the form $g = g_1g_2$ with $g_j \in G_j$ for $j = 1, 2$, and therefore we have $g_2q_2 = g_1^{-1}q_1$. Hence the intersection $G_1q_1 \cap G_2q_2$ is non-empty. Next, the fact that for every $q \in M$ the intersection $G_1q \cap G_2q$ consists of q alone follows by the argument used at the end of the proof of the closedness of G_1, G_2.

Let F_1 be a biholomorphic map from G_1p onto $\mathbb{C}^{n-1}$ that transforms G_1 into $G_1(\mathbb{C}^{n-1})$, and F_2 a biholomorphic map from G_2p onto M', where M' is one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, that transforms G_2 into G', where G' is one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively. We will now construct a biholomorphic map $\mathcal{F}$ from M onto $\mathbb{C}^{n-1} \times M'$. For $q \in M$ consider G_2q and let r be the unique point of intersection of G_1p and G_2q. Let $g \in G_1$ be an element such that $r = gp$. Then we set $\mathcal{F}(q) := (F_1(r), F_2(g^{-1}q))$. Clearly, $\mathcal{F}$ is a well-defined diffeomorphism from M onto $\mathbb{C}^{n-1} \times M'$. Since the foliation $\mathfrak{F}_j$ is holomorphic for each j, the map $\mathcal{F}$ is in fact holomorphic. By construction, $\mathcal{F}$ transforms G into $G_1(\mathbb{C}^{n-1}) \times G'$.

The proof is complete. ∎

5. Examples of Actions of Type III

In this section we give a large number of examples of actions of type III. Some of the examples can be naturally combined into classes and some of the actions form parametric families. In what follows $n \geq 2$.

(i). Here both the manifolds and the groups are represented as direct products.

(ia). $M = M' \times \mathbb{C}$, where M' is one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$, and $G = G' \times G_1(\mathbb{C})$, where G' is one of the groups $\mathrm{Aut}(\mathbb{B}^{n-1})$, $G(\mathbb{C}^{n-1})$, $G(\mathbb{CP}^{n-1})$, respectively.

(ib). $M = M' \times \mathbb{C}^*$, where M' is as in (ia), and $G = G' \times \mathbb{C}^*$, where G' is as in (ia).

(ic). $M = M' \times \mathbb{T}$, where M' is as in (ia) and $\mathbb{T}$ is an elliptic curve; $G = G' \times \mathrm{Aut}(\mathbb{T})^0$, where G' is as in (ia).

(id). $M = M' \times \mathcal{P}_>$, where M' is as in (ia) and $\mathcal{P}_> := \{\xi \in \mathbb{C} : \mathrm{Re}\,\xi > 0\}$; $G = G' \times G(\mathcal{P}_>)$, where G' as in (ia) and $G(\mathcal{P}_>) \simeq \mathbb{R} \rtimes \mathbb{R}$ is the group of all maps of the form

$$\xi \mapsto \lambda\xi + ia,$$

with $a \in \mathbb{R}$, $\lambda > 0$.

(ii). Parts (iib) and (iic) of this example are obtained by passing to quotients in Part (iia).

(iia). $M=\mathbb{B}^{n-1}\times\mathbb{C}$, and G consists of all maps of the form

$$\begin{aligned} z' &\mapsto \frac{Az'+b}{cz'+d}, \\ z_n &\mapsto z_n+\ln(cz'+d)+a, \end{aligned}$$

where

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU_{n-1,1},$$

$z':=(z_1,\dots,z_{n-1})$ and $a\in\mathbb{C}$. This group is isomorphic to the universal cover of $SU_{n,1}$. In fact, for $T\in\mathbb{C}$ one can consider the following family of groups acting on $\mathbb{B}^{n-1}\times\mathbb{C}$

$$\begin{aligned} z' &\mapsto \frac{Az'+b}{cz'+d}, \\ z_n &\mapsto z_n+T\ln(cz'+d)+a, \end{aligned} \tag{5.1}$$

where A,a,b,c,d are as above. Example (ia) for $M'=\mathbb{B}^{n-1}$ is included in this family for $T=0$. If $T\neq 0$, then conjugating group (5.1) in $\mathrm{Aut}(\mathbb{B}^{n-1}\times\mathbb{C})$ by the automorphism

$$\begin{aligned} z' &\mapsto z' \\ z_n &\mapsto z_n/T, \end{aligned} \tag{5.2}$$

we can assume that $T=1$.

(iib). $M=\mathbb{B}^{n-1}\times\mathbb{C}^*$, and for a fixed $T\in\mathbb{C}^*$ the group G consists of all maps of the form

$$\begin{aligned} z' &\mapsto \frac{Az'+b}{cz'+d}, \\ z_n &\mapsto \chi(cz'+d)^T z_n, \end{aligned} \tag{5.3}$$

where A,b,c,d are as in (iia) and $\chi\in\mathbb{C}^*$. Example (ib) for $M'=\mathbb{B}^{n-1}$ can be included in this family for $T=0$. This family is obtained from (5.1) by passing to a quotient in the last variable.

(iic). $M=\mathbb{B}^{n-1}\times\mathbb{T}$, where $\mathbb{T}$ is an elliptic curve, and for a fixed $T\in\mathbb{C}^*$ the group G consists of all maps of the form

$$\begin{aligned} z' &\mapsto \frac{Az'+b}{cz'+d}, \\ [z_n] &\mapsto \left[\chi(cz'+d)^T z_n\right], \end{aligned} \tag{5.4}$$

where A,b,c,d,χ are as in (iib), $\mathbb{T}$ is obtained from $\mathbb{C}^*$ by taking the quotient with respect to the equivalence relation $z_n\sim dz_n$, for some $d\in\mathbb{C}^*$, $|d|\neq 1$, and $[z_n]\in\mathbb{T}$ is the equivalence class of a point $z_n\in\mathbb{C}^*$. Example (ic) for $M'=\mathbb{B}^{n-1}$ can be included in this family for $T=0$. Clearly, after passing to the quotient, (5.3) turns into (5.4).

(iii). Part (iiib) of this example is obtained by passing to a quotient in Part (iiia).

(iiia). $M = \mathbb{C}^n$, and G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & e^{\mathrm{Re}\, b} U z' + a, \\ z_n & \mapsto & z_n + b, \end{array}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{C}$. This group is isomorphic to $G(\mathbb{C}^{n-1}) \rtimes G_1(\mathbb{C})$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^n$

$$\begin{array}{rcl} z' & \mapsto & e^{\mathrm{Re}\,(Tb)} U z' + a, \\ z_n & \mapsto & z_n + b, \end{array} \tag{5.5}$$

where U, a, b are as above. Example (ia) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. If $T \neq 0$, then conjugating group (5.5) in $\mathrm{Aut}(\mathbb{C}^n)$ by the automorphism

$$\begin{array}{rcl} z' & \mapsto & z' \\ z_n & \mapsto & T z_n, \end{array}$$

we can assume that $T = 1$.

(iiib). $M = \mathbb{C}^{n-1} \times \mathbb{C}^*$, and for a fixed $T \in \mathbb{R}^*$ the group G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & e^{T\,\mathrm{Re}\, b} U z' + a, \\ z_n & \mapsto & e^{b} z_n, \end{array}$$

where U, a, b are as in (iiia). This group is isomorphic to $G(\mathbb{C}^{n-1}) \rtimes \mathbb{C}^*$. Example (ib) for $M' = \mathbb{C}^{n-1}$ can be included in this family for $T = 0$. This family is obtained from (5.5) for $T \in \mathbb{R}^*$ by passing to a quotient in the last variable.

(iv). Parts (ivb) and (ivc) of this example are obtained by passing to quotients in Part (iva).

(iva). $M = \mathbb{C}^n$, and G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & U z' + a, \\ z_n & \mapsto & z_n + \langle U z', a \rangle + b, \end{array}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{C}$, and $\langle \cdot\,, \cdot \rangle$ is the inner product in $\mathbb{C}^{n-1}$. This group is isomorphic to $(U_{n-1} \ltimes \mathcal{H}) \times \mathbb{R}$, where $\mathcal{H}$ is the following Heisenberg group

$$\begin{array}{rcl} z' & \mapsto & z' + a, \\ z_n & \mapsto & z_n + \langle z', a \rangle + ||a||^2/2 + ic, \end{array}$$

with $a \in \mathbb{C}^{n-1}$, $c \in \mathbb{R}$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^n$

$$\begin{array}{rcl} z' & \mapsto & U z' + a, \\ z_n & \mapsto & z_n + T\langle U z', a \rangle + b, \end{array} \tag{5.6}$$

where U, a, b are as above. Example (ia) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. If $T \neq 0$, then conjugating group (5.6) in $\mathrm{Aut}(\mathbb{C}^n)$ by automorphism (5.2), we can assume that $T = 1$.

(ivb). $M = \mathbb{C}^{n-1} \times \mathbb{C}^*$, and for a fixed $0 \leq \tau < 2\pi$ the group G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & U z' + a, \\ z_n & \mapsto & \chi \exp\Big(e^{i\tau} \langle U z', a \rangle\Big) z_n, \end{array} \tag{5.7}$$

where U, a are as in (iva) and $\chi \in \mathbb{C}^*$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^{n-1} \times \mathbb{C}^*$

$$\begin{aligned} z' &\mapsto Uz' + a, \\ z_n &\mapsto \chi \exp\Big(T\langle Uz', a\rangle\Big) z_n, \end{aligned} \tag{5.8}$$

where U, a, χ are as above. Example (ib) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. For $T \neq 0$ this family is obtained from (5.6) by passing to a quotient in the last variable. Furthermore, conjugating group (5.8) for $T \neq 0$ in $\mathrm{Aut}(\mathbb{C}^{n-1} \times \mathbb{C}^*)$ by the automorphism

$$\begin{aligned} z' &\mapsto \sqrt{|T|}z' \\ z_n &\mapsto z_n, \end{aligned}$$

we obtain the group defined in (5.7) for $\tau = \arg T$.

(ivc). $M = \mathbb{C}^{n-1} \times \mathbb{T}$, where $\mathbb{T}$ is an elliptic curve, and for a fixed $0 \leq \tau < 2\pi$ the group G consists of all maps of the form

$$\begin{aligned} z' &\mapsto Uz' + a, \\ [z_n] &\mapsto \Big[\chi \exp\Big(e^{i\tau}\langle Uz', a\rangle\Big) z_n\Big], \end{aligned} \tag{5.9}$$

where U, a, χ are as in (ivb), $\mathbb{T}$ is obtained from $\mathbb{C}^*$ by taking the quotient with respect to the equivalence relation $z_n \sim dz_n$, for some $d \in \mathbb{C}^*$, $|d| \neq 1$, and $[z_n] \in \mathbb{T}$ is the equivalence class of a point $z_n \in \mathbb{C}^*$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^{n-1} \times \mathbb{T}$

$$\begin{aligned} z' &\mapsto Uz' + a, \\ [z_n] &\mapsto \Big[\chi \exp\Big(T\langle Uz', a\rangle\Big) z_n\Big], \end{aligned} \tag{5.10}$$

where U, a, χ are as above. Example (ic) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. For $T \neq 0$ this family is obtained from (5.8) by passing to the quotient described above. Furthermore, conjugating group (5.10) for $T \neq 0$ in $\mathrm{Aut}(\mathbb{C}^{n-1} \times \mathbb{T})$ by the automorphism

$$\begin{aligned} z' &\mapsto \sqrt{|T|}z' \\ \xi &\mapsto \xi, \end{aligned}$$

where $\xi \in \mathbb{T}$, we obtain the group defined in (5.9) for $\tau = \arg T$.

(v). $M = \mathbb{C}^{n-1} \times \mathcal{P}_>$, and for a fixed $T \in \mathbb{R}^*$ the group G consists of all maps of the form

$$\begin{aligned} z' &\mapsto \lambda^T Uz' + a, \\ z_n &\mapsto \lambda z_n + ib, \end{aligned}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{R}$, $\lambda > 0$. This group is isomorphic to $G(\mathbb{C}^{n-1}) \rtimes (\mathbb{R} \rtimes \mathbb{R})$. Example (id) for $M' = \mathbb{C}^{n-1}$ can be included in this family for $T = 0$.

(vi). $M = \mathbb{C}^n$, and for fixed $k_1, k_2 \in \mathbb{Z}$, $(k_1, k_2) = 1$, $k_1 > 0$, $k_2 \neq 0$, the group $G \simeq H^n_{k_1,k_2} \ltimes \mathbb{C}^n$ consists of all maps of the form (1.1) with $U \in H^n_{k_1,k_2}$ (see (3.1)). Example (ia) for $M' = \mathbb{C}^{n-1}$ can be included in this family for $k_2 = 0$.

(vii). Part (viib) of this example is obtained by passing to a quotient in Part (viia).

(viia). $M = \mathbb{C}^{n*}/\mathbb{Z}_l$, where $\mathbb{C}^{n*} := \mathbb{C}^n \setminus \{0\}$, $l \in \mathbb{N}$, and the group G consists of all maps of the form

$$\{z\} \mapsto \{\lambda U z\},$$

where $U \in U_n$, $\lambda > 0$, and $\{z\} \in \mathbb{C}^{n*}/\mathbb{Z}_l$ is the equivalence class of a point $z \in \mathbb{C}^{n*}$. This group is isomorphic to $\mathbb{R} \times U_n/\mathbb{Z}_l$.

(viib). $M = M_d/\mathbb{Z}_l$, where M_d is the Hopf manifold $\mathbb{C}^{n*}/\{z \sim dz\}$, for $d \in \mathbb{C}^*$, $|d| \neq 1$, and $l \in \mathbb{N}$; the group G consists of all maps of the form

$$\{[z]\} \mapsto \{[\lambda U z]\},$$

where U, λ are as in (viia), $[z] \in M_d$ denotes the equivalence class of a point $z \in \mathbb{C}^{n*}$, and $\{[z]\} \in M_d/\mathbb{Z}_l$ denotes the equivalence class of $[z] \in M_d$.

(viii). In this example the manifolds are the open orbits of the action of a group of affine transformations on $\mathbb{C}^n$. Let $G_{\mathcal{P}}$ be the group of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & \lambda U z' + a, \\ z_n & \mapsto & \lambda^2 z_n + 2\lambda \langle U z', a\rangle + |a|^2 + ib, \end{array}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{R}$, $\lambda > 0$. This group is isomorphic to $CU_{n-1} \ltimes \mathcal{H}$, where CU_{n-1} is the conformal unitary group and $\mathcal{H}$ is the Heisenberg group (see (iva)).

(viiia). $M = \mathcal{P}^n_>$, $G = G_{\mathcal{P}}$, where

$$\mathcal{P}^n_> := \left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \operatorname{Re} z_n > |z'|^2\right\}.$$

Observe that $\mathcal{P}^n_>$ is holomorphically equivalent to $\mathbb{B}^n$.

(viiib). $M = \mathcal{P}^n_<$, $G = G_{\mathcal{P}}$, where

$$\mathcal{P}^n_< := \left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \operatorname{Re} z_n < |z'|^2\right\}.$$

Observe that $\mathcal{P}^n_<$ is holomorphically equivalent to $\mathbb{CP}^n \setminus (\overline{\mathbb{B}^n} \cup L)$, where L is a complex hyperplane tangent to $\partial \mathbb{B}^n$ at some point.

(ix). Here $n = 2$, $M = \mathbb{B}^1 \times \mathbb{C}$, and G consists of all maps of the form

$$\begin{array}{rcl} z_1 & \mapsto & \dfrac{a z_1 + b}{\overline{b} z_1 + \overline{a}}, \\[2ex] z_2 & \mapsto & \dfrac{z_2 + c z_1 + \overline{c}}{\overline{b} z_1 + \overline{a}}, \end{array}$$

where $a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$, $c \in \mathbb{C}$. This group is isomorphic to $SU_{1,1} \times \mathbb{C}$.

(x). Here $n = 3$, $M = \mathbb{CP}^3$, and G consists of all maps of the form (1.2) for $n = 3$ with $U \in Sp_2$. This group is isomorphic to $Sp_2/\mathbb{Z}_2$.

(xi). Let $n = 3$ and $(z : w)$ be homogeneous coordinates in $\mathbb{CP}^3$, where $z = (z_1 : z_2)$, $w = (w_1 : w_2)$. Set $M = \mathbb{CP}^3 \setminus \{w = 0\}$ and let G be the group of all maps of the form

$$\begin{array}{rcl} z & \mapsto & Uz + Aw, \\ w & \mapsto & Vw, \end{array}$$

where $U, V \in SU_2$, and

$$A = \begin{pmatrix} a & i\bar{b} \\ b & -i\bar{a} \end{pmatrix},$$

for some $a, b \in \mathbb{C}$.

(xii). Here $n = 3$, $M = \mathbb{C}^3$, and G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & Uz' + a, \\ z_3 & \mapsto & \det U\, z_3 + \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Uz'\right] \cdot a + b, \end{array}$$

where $z' := (z_1, z_2)$, $U \in U_2$, $a \in \mathbb{C}^2$, $b \in \mathbb{C}$, and $\cdot$ is the dot product in $\mathbb{C}^2$.

We conclude the paper with the following conjecture.

Conjecture 5.1. *Let M be a connected complex manifold of dimension $n \geq 2$ and $G \subset Aut(M)$ a connected Lie group with $d_G = n^2 + 1$ that acts properly on M. If the pair (M, G) is of type III, then it is equivalent to one of the pairs listed in (i)–(xii) above.*

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Received September 12, 2007
and in final form December 10, 2007