Some basic results concerning $G$-invariant Riemannian metrics

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Abstract. In this paper we study complete $G$-invariant Riemannian metrics. Let $G$ be a Lie group and let $M$ be a proper smooth $G$-manifold. Let $\alpha$ be a smooth $G$-invariant Riemannian metric of $M$, and let $\tilde{K}$ be any $G$-compact subset of $M$. We show that $M$ admits a complete smooth $G$-invariant Riemannian metric $\beta$ such that $\beta|\tilde{K} = \alpha|\tilde{K}$. We also prove the existence of complete real analytic $G$-invariant Riemannian metrics for proper real analytic $G$-manifolds. Moreover, we show that for any given smooth (real analytic) $G$-invariant Riemannian metric there exists a complete smooth (real analytic) $G$-invariant Riemannian metric conformal to the original Riemannian metric. To prove the real analytic results we need the assumption that $G$ can be embedded as a closed subgroup of a Lie group which has only finitely many connected components.

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1. Introduction

Let $G$ be a Lie group acting properly and smoothly on a smooth manifold $M$. According to a well-known result (Theorem 4.3.1 in [8]) of R.S. Palais, $M$ has a smooth $G$-invariant Riemannian metric. An alternative proof of this result is given in [1], Theorem 5.5.2. Moreover, it can be assumed that the Riemannian metric is complete, see Theorem 0.2 in [4].

In this paper we continue the study of $G$-invariant Riemannian metrics on proper smooth and real analytic $G$-manifolds. By a result of K. Nomizu and H. Ozeki ([7], Theorem 1), for any given Riemannian metric $\alpha$ of a smooth manifold there exists a complete Riemannian metric conformal to $\alpha$. We begin by proving an equivariant version of this result (Theorem 3.1).

Let $M$ be a connected smooth manifold with Riemannian metric $\alpha$. According to a result of J.A. Morrow (Theorem in [6]), $M$ has a complete Riemannian metric which agrees with $\alpha$ on a given compact subset of $M$. In Section 4, we prove an equivariant version of Morrow’s result (Theorem 4.1).

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In Section 5, we study real analytic \(G\)-invariant Riemannian metrics. Constructing real analytic Riemannian metrics is more complicated than constructing smooth Riemannian metrics, in particular, when we want the Riemannian metrics to be invariant under a group action. Assume that the Lie group \(G\) can be embedded as a closed subgroup of a Lie group which has only finitely many connected components. It is known by Proposition 1.2 in [3], that under this assumption every proper real analytic \(G\)-manifold has a real analytic \(G\)-invariant Riemannian metric. In this paper we prove a real analytic version of Theorem 3.1 by using \(G\)-equivariant real analytic approximations.

In Section 6, we study \(G\)-invariant Riemannian metrics on a proper \(G\)-manifold \(M\), where \(G\) is a countable discrete group. If \(G\) acts freely on \(M\), or if, more generally, \(M\) has only one orbit type, then the orbit space \(M/G\) is a manifold of the same dimension as \(M\). Every Riemannian metric \(\alpha\) on \(M/G\) induces a \(G\)-invariant Riemannian metric \(\pi^*\alpha\) on \(M\). We show that \(\pi^*\alpha\) is complete if and only if \(\alpha\) is complete. Finally, we prove yet another real analytic version of Theorem 3.1, in the case where \(M\) has only one orbit type.

2. Preliminaries

Let \(G\) be a Lie group and let \(M\) be a smooth, i.e., \(C^\infty\) (real analytic, i.e., \(C^\omega\)) manifold. We assume all the manifolds to be finite dimensional and without boundary and to have at most countably many connected components. Thus they are paracompact. Let \(G\) act on \(M\). If the action \(G \times M \to M\) is smooth (real analytic), we say that \(M\) is a smooth (real analytic) \(G\)-manifold. If the action is also proper, i.e., if the map \(G \times M \to M \times M\), \((g, x) \mapsto (gx, x)\), is proper, we call \(M\) a proper smooth (real analytic) \(G\)-manifold.

Let \(M/G\) denote the orbit space and let \(\pi: M \to M/G\) be the natural projection. We call a subset \(\tilde{K}\) of \(M\) \(G\)-compact, if \(\pi(\tilde{K})\) is compact.

Let \(X\) be a topological space. By the support \(\text{supp}(f)\) of a map \(f: X \to \mathbb{R}\) we mean the closure of the set \(\{x \in X \mid f(x) > 0\}\).

Let \(F\) be a subset of \(M\). If every point \(x \in M\) has a neighbourhood \(U\) such that \(G(U, F) = \{g \in G \mid gU \cap F \neq \emptyset\}\) is relatively compact, we call \(F\) small.

Definition 2.1. Let \(G\) be a Lie group and let \(M\) be a proper smooth \(G\)-manifold. If \(F\) is small and \(GF = M\), we say that \(F\) is a fundamental set for \(G\) in \(M\). If, in addition, \(F\) is closed in \(M\), we say that it is a closed fundamental set. We call a closed fundamental set \(F\) in \(M\) fat, if \(G\tilde{F} = M\), where \(\tilde{F}\) denotes the interior of \(F\).

By Lemma 3.6 in [2], every proper smooth \(G\)-manifold has a fat closed fundamental set.

A euclidean space on which \(G\) acts linearly is called a linear \(G\)-space.

Lemma 2.2. Let \(G\) be a Lie group and let \(M\) be a proper smooth \(G\)-manifold. Let \(f: M \to V\) be a smooth map into a linear \(G\)-space \(V\). Assume the support of \(f\) is small. Then

\[\text{Av}(f): M \to V, \quad x \mapsto \int_G gf(g^{-1}x)dg,\]

where the integral is the left Haar integral over \(G\), is a smooth \(G\)-equivariant map.
Proof. Lemma 2.4 in [5]. □

We denote by $\mathbb{R}^+$ the set $\{x \in \mathbb{R} \mid x > 0\}$.

Lemma 2.3. Let $G$ be a Lie group and let $M$ be a proper smooth $G$-manifold. Let $d: M \to \mathbb{R}^+$ be a continuous $G$-invariant map. Then there is a smooth $G$-invariant map $\tilde{f}: M \to \mathbb{R}^+$ such that $\tilde{f}(p) > \frac{1}{d(p)}$ for all $p \in M$.

Proof. Let $\tilde{f}: M \to \mathbb{R}^+$ be a smooth map such that $\tilde{f}(p) > \frac{1}{d(p)}$ for all $p \in M$. Let $\gamma: M \to \mathbb{R}^+ \cup \{0\}$ be a smooth map whose support is small. We may assume that the intersection of the set $\{x \in M \mid \gamma(x) > 0\}$ with each orbit of $M$ contains an open subset of the orbit. By Lemma 2.2, the map

$$\tilde{f}: M \to \mathbb{R}^+, \quad p \mapsto \frac{\int_G \gamma(g^{-1}p)\tilde{f}(g^{-1}p)dg}{\int_G \gamma(g^{-1}p)dg}$$

is smooth and $G$-invariant. Clearly, $\tilde{f}(p) > \frac{1}{d(p)}$, for all $p \in M$. □

Recall that a metric space $X$ is complete if every Cauchy sequence in $X$ converges. A metric $d$ on a $G$-space $X$ is called $G$-invariant if $d(gx, gy) = d(x, y)$ for all $x, y \in X$ and for all $g \in G$. A $G$-invariant metric $d$ on $X$ induces a metric $\hat{d}$ on $X/G$, where $\hat{d}(\pi(x), \pi(y)) = \inf \{d(x, gy) \mid g \in G\}$.

Lemma 2.4. Let $G$ be a Lie group and let $X$ be a Hausdorff space on which $G$ acts properly. Let $d$ be a $G$-invariant metric on $X$ and let $\hat{d}$ be the metric $d$ induces on $X/G$. Then $\hat{d}$ is complete if and only if $d$ is complete.

Proof. Assume first that $\hat{d}$ is a complete metric. We first assume that $X$ has only one orbit, $X = Gy$. Then $X/G$ is a point. Let $(g_ny)$ be a Cauchy sequence in $X$. Since $G$ acts properly on $X$, $y$ has a neighbourhood $U$ such that the closure of the set

$$G(U \mid U) = \{g \in G \mid gU \cap U \neq \emptyset\}$$

is compact. For sufficiently small $\varepsilon > 0$, the ball $B_y(\varepsilon)$ with center $y$ and radius $\varepsilon$ is in $U$. Since $(g_ny)$ is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that

$$d(y, g_n^{-1}g_ny) = d(g_ny, g_ny) < \varepsilon$$

for all $n \geq n_0$. Thus $g_n^{-1}g_ny \in U$ for all $n \geq n_0$ and, consequently, $g_n^{-1}g_n \in G(U \mid U)$. It follows that $(g_n^{-1}g_n)$ has a subsequence converging to some $g \in G(U \mid U)$. Thus $(g_n^{-1}g_n)$ has a subsequence converging to $gy$ and $(g_ny)$ has a subsequence converging to $g_ny$. Since $(g_ny)$ is a Cauchy sequence, it now follows that it converges.

Assume then that $X$ may have more than one orbit. Let $(x_n)$ be a Cauchy sequence in $X$. Then $(\pi(x_n))$ is a Cauchy sequence in $X/G$. Thus $(\pi(x_n))$ converges to some point $\pi(y) \in X/G$, where $y \in X$. Let $(\varepsilon_n)$ be a sequence converging to zero, $\varepsilon_n > 0$ for all $n$, with the property that $d(\pi(x_n), \pi(y)) < \varepsilon_n$ for all $n$. Then, for every $n$, there exists $y_n \in Gy$ such that $d(x_n, y_n) < \varepsilon_n$. It is easy to see that $(y_n)$ is a Cauchy sequence. By the first part of the proof, $(y_n)$
converges to some $h y$, where $h \in G$. Since $d(x_n, h y) \leq d(x_n, y_n) + d(y_n, h y)$ for all $n$, it follows that $(x_n)$ converges to $h y$. Thus $d$ is complete.

Assume next that $d$ is a complete metric. Let $(\pi(x_n))$ be a Cauchy sequence in $X/G$. Let $(\varepsilon_q)$ be a decreasing sequence of positive numbers whose sum is finite. For every $q \in \mathbb{N}$, there exists $n(q)$ such that $d(\pi(x_m), \pi(x_p)) < \varepsilon_q$ for all $m, p \geq n(q)$. We may assume that $(n(q))$ is increasing. We choose a sequence $(g_q)$ of elements in $G$ as follows: Let $g_1 = e$. Inductively, we choose $g_q$ to be such that $d(g_{q-1}x_{n(q-1)}, g_qx_{n(q)}) < \varepsilon_{q-1}$. Then $(g_qx_{n(q)})$ is a Cauchy sequence. Since $d$ is complete, it follows that $(g_qx_{n(q)})$ converges. Since the orbit map $\pi: X \to X/G$ is continuous, it follows that $(\pi(x_{n(q)}))$ converges. Then $(\pi(x_n))$ converges, since it is a Cauchy sequence having a convergent subsequence. Thus $\tilde{d}$ is complete. ■

3. Constructing complete $G$-invariant Riemannian metrics

Recall that a Riemannian metric on a manifold $M$ is called complete, if it induces a complete metric on each connected component of $M$.

Let $G$ be a Lie group and let $M$ be a proper smooth $G$-manifold. Let $TM$ denote the tangent bundle of $M$, and let $TM \oplus TM$ denote the Whitney sum. Then $G$ acts by differentials on $TM$ and $TM \oplus TM$. Let

$$\alpha: TM \oplus TM \to \mathbb{R}$$

be a Riemannian metric of $M$. We call $\alpha$ $G$-invariant if $\alpha_{gx}(gv, gw) = \alpha_x(v, w)$, for all $v, w \in T_xM$, for all $x \in M$ and for all $g \in G$.

K. Nomizu and H. Ozeki have shown (Theorem 1 in [7]) that for any given Riemannian metric $\alpha$ on a smooth manifold there exists a complete Riemannian metric conformal to $\alpha$. The ideas in their proof also work in the equivariant case. We briefly explain an equivariant version of their result.

Let $\alpha$ be a smooth $G$-invariant Riemannian metric of a proper smooth $G$-manifold $M$. Assume $\alpha$ is not complete. Without loss of generality we may assume that $\alpha$ is not complete on any connected component of $M$. Let $d_\alpha$ denote the metric $\alpha$ induces on the connected components of $M$. Then $d_\alpha$ is $G$-invariant. Let $M_p$ denote the connected component containing $p$ and let

$$B_p(r) = \{q \in M_p \mid d_\alpha(p, q) < r\}$$

denote the metric ball with $p$ as a center and with radius $r$. Moreover, let

$$d(p) = \sup\{r \mid B_p(r) \text{ is relatively compact}\}.$$

Then $d: M \to \mathbb{R}$ is a continuous $G$-invariant real-valued map, and $d(p) > 0$ for all $p \in M$. By Lemma 2.3, there exists a smooth $G$-invariant map $\tilde{f}: M \to \mathbb{R}^+$, such that $\tilde{f}(p) > \frac{1}{d(p)}$ for all $p \in M$. Let $f = (\tilde{f})^2$. Then $f \alpha$ is a smooth $G$-invariant Riemannian metric of $M$. The proof of the completeness can be found in [7].

We obtain:

**Theorem 3.1.** Let $G$ be a Lie group and let $M$ be a proper smooth $G$-manifold. For any smooth $G$-invariant Riemannian metric $\alpha$ of $M$ there exists a complete smooth $G$-invariant Riemannian metric of $M$ which is conformal to $\alpha$. 
Notice that the existence of a complete smooth $G$-invariant Riemannian metric of a proper smooth $G$-manifold is known, Theorem 0.2 in [4], where it was obtained as a corollary of an embedding result. That result tells nothing about conformality and the approach would not work in the real analytic case, which we study in Section 5.

Let $\alpha$ be a $G$-invariant Riemannian metric of $M$. Let $M_i$, $i \in I \subset \mathbb{N}$, be the connected components of $M$ and let $d_i$ be the metric $\alpha$ induces on $M_i$. Then $d_i$ induces a metric $\tilde{d}_i$ on $\pi(M_i) = \pi(GM_i)$, where $\tilde{d}_i(\pi(x), \pi(y)) = \inf \{d_i(x, gy) : g \in G, gM_i = M_i\}$ for all $x, y \in M_i$. Notice that if $M_j = gM_i$, for some $g \in G$, then $\tilde{d}_j = \tilde{d}_i$.

**Lemma 3.2.** Let $G$ be a Lie group and let $M$ be a proper smooth $G$-manifold. Let $\tilde{K}$ be a $G$-compact subset of $M$ and let $M_0$ be a connected component of $M$. Let $\alpha$ be a smooth $G$-invariant Riemannian metric on $M$. Then $\alpha$ induces a complete metric on $\tilde{K} \cap M_0$.

**Proof.** Similar to the proof of Lemma 2.4. $lacksquare$

### 4. Extending $G$-invariant Riemannian metrics

In this section we prove the following theorem:

**Theorem 4.1.** Let $G$ be a Lie group and let $M$ be a proper smooth $G$-manifold. Let $\alpha$ be a smooth $G$-invariant Riemannian metric of $M$. Then given a $G$-compact subset $\tilde{K} \subset M$, there is a complete smooth $G$-invariant Riemannian metric $\gamma$ on $M$ such that $\gamma|\tilde{K} = \alpha|\tilde{K}$.

Theorem 4.1 is an equivariant version of a result by J.A. Morrow (Theorem in [6]). The proof is based on Morrow’s proof. We begin by proving two lemmas.

**Lemma 4.2.** Let $G$ be a Lie group and let $M$ be a proper smooth $G$-manifold. Then $M$ has an exhaustion by $G$-compact sets.

**Proof.** Let $F$ be a fat closed fundamental set of $M$. Then $F$ has an exhaustion by compact sets $K_i$, i.e., we can write $F = \bigcup K_i$, where $K_i \subset \text{int}(K_{i+1})$ for all $i$. But then $M = \bigcup GK_i$, each set $GK_i$ is $G$-compact and $G(K_i) \subset G(\text{int}(K_{i+1})) \subset \text{int}(GK_{i+1})$. $lacksquare$

**Lemma 4.3.** Let $G$ be a Lie group and let $M$ be a proper smooth $G$-manifold. Let $M_0$ be a connected component of $M$. Let $\tilde{K}$ and $\hat{K}$ be $G$-compact subsets of $M$ such that $\hat{K} \subset \text{int}(\tilde{K})$. Fix a $G$-invariant Riemannian metric on $M$ and assume $(x_n)$ is a Cauchy sequence in $(M \setminus \hat{K}) \cap M_0$ with the property that only finitely many of the $x_n$ are in $\hat{K}$. Then $M \setminus \hat{K}$ has a connected component containing infinitely many of the $x_n$. 
Proof. Let $3\delta$ be the distance of $\tilde{K} \cap M_0$ from $(M \setminus \text{int} \tilde{K}) \cap M_0$. Then $\delta > 0$. Let $n_0 \in \mathbb{N}$ be such that $x_n \in M \setminus \tilde{K}$ and $d(x_n, x_{n+k}) \leq \delta$ for all $n \geq n_0$ and all $k \in \mathbb{N} \cup \{0\}$. Therefore, for given $x_n, x_{n+k}$, where $n \geq n_0$ and $k \geq 0$, there is a curve in $M_0$ from $x_n$ to $x_{n+k}$ whose length is less than $2\delta$. This curve cannot touch $\tilde{K}$, since $x_n, x_{n+k} \in M \setminus \tilde{K}$. It follows that $x_n$ and $x_{n+k}$ are in the same connected component of $M \setminus \tilde{K}$.

Proof of Theorem 4.1. Let $M = \bigcup_i GK_i$, where the sets $K_i$ are as in Lemma 4.2. Then $\tilde{K} \subset \text{int}(GK_i)$, for some $i$. The pair $\{\text{int}(GK_i+1), M \setminus GK_i\}$ is a covering of $M$ by open $G$-invariant sets. By Theorem 4.2.4. (4) in [9], there exist smooth $G$-invariant maps $\varrho_1, \varrho_2 : M \to \mathbb{R}$, with the following properties:

1. $\varrho_1, \varrho_2 \geq 0$,
2. $\varrho_1 + \varrho_2 = 1$,
3. $\text{supp}(\varrho_1) \subset \text{int}(GK_{i+1})$, $\text{supp}(\varrho_2) \subset M \setminus GK_i$.

Thus $\varrho_1(x) = 1$ for all $x \in GK_i$ and $\varrho_2(x) = 1$ for all $x \in M \setminus GK_{i+1}$.

By the construction in Section 3, there exists a smooth $G$-invariant map $f : M \setminus GK_i \to \mathbb{R}^+$ such that the $G$-invariant Riemannian metric $\gamma_1 = f(\alpha|M \setminus GK_i)$ defines a complete metric $d_{\gamma_1}$ on each connected component of $M \setminus GK_i$. Then $\gamma = (\varrho_1 + \varrho_2 f)\alpha$ is a smooth $G$-invariant Riemannian metric on $M$, and $\gamma|\tilde{K} = \alpha|\tilde{K}$.

It remains to show that the metric $d_{\gamma}$ induced by $\gamma$ is complete. Let $(x_n)$ be a $d_{\gamma}$-Cauchy sequence in a connected component $M_0$ of $M$. Assume first that infinitely many $x_n$ are in $GK_{i+1}$. By Lemma 3.2, the restriction $d_{\gamma}(GK_{i+1} \cap M_0)$ is complete. Thus $(x_n)$ has a convergent subsequence. Since $(x_n)$ is a $d_{\gamma}$-Cauchy sequence, it must converge.

Assume next that only finitely many $x_n$ are in $GK_{i+1}$. In this case we may assume that $x_n \in M \setminus GK_{i+1}$ for all $n$. If infinitely many of the $x_n$ are in $GK_{i+2}$, then $(x_n)$ has a convergent subsequence, by Lemma 3.2, and we are done. Assume then that only finitely many of the $x_n$ are in $GK_{i+2}$. By passing to a subsequence, if necessary, we may assume that all the $x_n$ are in the same connected component of $M \setminus GK_{i+1}$ (see Lemma 4.3). Let

$$U_r(GK_{i+1} \cap M_0) = \{x \in M_0 \mid d_{\gamma}(x, GK_{i+1} \cap M_0) \leq r\}.$$ 

By choosing $r$ small enough, we may assume that $U_r(GK_{i+1} \cap M_0)$ is $G$-compact. If infinitely many $x_n$ are in $U_r(GK_{i+1} \cap M_0)$, we are done by Lemma 3.2. Assume $d_{\gamma}(x_n, GK_{i+1} \cap M_0) > r$ for all $n$. Let $\varepsilon < r$. Then any curve which begins at some $x_n$ and whose $\gamma$-length is less than $\varepsilon$ remains in $M_0 \setminus GK_{i+1}$. On $M_0 \setminus GK_{i+1}$, $\gamma$ equals $\gamma_1$. Consequently, the $\gamma_1$-length of any curve of $\gamma$-length less than $\varepsilon$ beginning at $x_n$ equals the $\gamma$-length. Thus $(x_n)$ is a $d_{\gamma_1}$-Cauchy sequence in $M_0 \setminus GK_{i+1} \subset M_0 \setminus GK_i$. Since $d_{\gamma_1}$ is complete, the sequence $(x_n)$ converges.

5. The real analytic case

We call a Lie group $G$ good, if it can be embedded as a closed subgroup of a Lie group $\hat{G}$, where $\hat{G}$ has only finitely many connected components. Thus, for example, all the closed linear Lie groups are good.
In this section we construct complete real analytic $G$-invariant Riemannian metrics. We have to make the assumption that $G$ is a good Lie group, since so far the existence of any real analytic $G$-invariant Riemannian metric has been proven only for proper real analytic $G$-manifolds, where $G$ is a good Lie group (Proposition 1.2 in [3]). Moreover, the approximation result to which we refer in the proof of Lemma 5.1, has so far been proven only in the case where $G$ is a good Lie group.

**Lemma 5.1.** Let $G$ be a good Lie group and let $M$ be a proper real analytic $G$-manifold. Let $d : M \to \mathbb{R}^+$ be a continuous $G$-invariant map. Then there is a real analytic $G$-invariant map $f^\omega : M \to \mathbb{R}^+$ such that $f^\omega(p) > \frac{1}{d(p)}$ for all $p \in M$.

**Proof.** Let $G$ act trivially on $\mathbb{R}$ and diagonally on $M \times \mathbb{R}$. Since $M$ is a proper real analytic $G$-manifold, also $M \times \mathbb{R}$ is a proper real analytic $G$-manifold. Let $id$ denote the identity map of $M$ and let $\tilde{f} : M \to \mathbb{R}$ be as in Lemma 2.3. Then $(id, \tilde{f}) : M \to M \times \mathbb{R}, \ x \mapsto (x, \tilde{f}(x)),$ is a smooth $G$-equivariant map.

By Theorem II in [3], real analytic $G$-equivariant maps are dense among the smooth $G$-equivariant maps $M \to M \times \mathbb{R}$ in the strong-weak topology. Thus there exists a real analytic $G$-equivariant map $f^* : M \to M \times \mathbb{R}$ approximating $(id, \tilde{f})$ as well as we like. Let $pr : M \times \mathbb{R} \to \mathbb{R}$ denote the projection. Clearly, we may assume that $pr \circ f^*(x) > \tilde{f}(x)$ for all $x \in M$. Thus we may choose $f^\omega = pr \circ f^*$.

**Theorem 5.2.** Let $G$ be a good Lie group and let $M$ be a proper real analytic $G$-manifold. Then $M$ admits a complete real analytic $G$-invariant Riemannian metric. For any real analytic $G$-invariant Riemannian metric $\alpha$ of $M$ there exists a complete real analytic $G$-invariant Riemannian metric conformal to $\alpha$.

**Proof.** Let $f^\omega$ be as in Lemma 5.1. Then $(f^\omega)^2 \alpha$ is a complete real analytic $G$-invariant Riemannian metric conformal to $\alpha$. The first claim follows from the second claim and from Proposition 1.2 in [3] according to which $M$ has a real analytic $G$-invariant Riemannian metric.

Theorem 5.2 is known in the case where $G$ is compact, see Theorem 1.4.5 in [10]. Clearly, no result like Theorem 4.1 can be true in the real analytic case.

6. Discrete $G$ - the case of one orbit type

We begin by recalling a standard way to induce Riemannian metrics from one manifold to another one: Let $M$ and $N$ be smooth (real analytic) manifolds and let $f : M \to N$ be a smooth (real analytic) immersion. Assume $\beta$ is a smooth (real analytic) Riemannian metric on $N$. Then $\beta$ induces a smooth (real analytic) Riemannian metric $f^*\beta$ on $M$, where $(f^*\beta)_x(v, w) = \beta_{f(x)}(df_x(v), df_x(w))$, for every $x \in M$ and for every $v, w \in T_xM$.

Let $G$ be a countable discrete group and let $M$ be a proper smooth (real analytic) $G$-manifold. Assume $M$ has only one orbit type. Let $H$ be a finite
subgroup of $G$ corresponding to that orbit type, and let $N(H)$ denote the normalizer of $H$ in $G$. Let $\Gamma_H = N(H)/H$. We denote the fixed point set of $H$ in $M$ by $M^H$. Then $\Gamma_H$ acts properly and freely on $M^H$. It also acts on $G/H$ by $nH \cdot gH \mapsto gn^{-1}H$. It is well-known that there exists a $G$-equivariant smooth (real analytic) diffeomorphism

$$G/H_{\Gamma_H} \times M^H \to M, \quad [gH, x] \mapsto gx,$$

see e.g. Theorem 4.3.10 in [9]. This induces a diffeomorphism between the orbit spaces $M^H/\Gamma_H$ and $M/G$.

Let $\pi|: M^H \to M^H/\Gamma_H$ denote the restriction of the orbit map $\pi: M \to M/G$. Then $\pi|$ is a local diffeomorphism and $T(M^H/\Gamma_H) \approx TM^H/\Gamma_H$. Let $\alpha$ be a $G$-invariant Riemannian metric on $M$ and let $\alpha|$ denote its restriction to $M^H$. Then $\alpha|$ is $\Gamma_H$-invariant. Thus $\alpha|$ induces a Riemannian metric on $M^H/\Gamma_H$. It follows that every smooth (real analytic) $G$-invariant Riemannian metric $\alpha$ on $M$ induces a smooth (real analytic) Riemannian metric $\tilde{\alpha}$ on $M/G$. We obtain:

**Proposition 6.1.** Let $G$ be a countable discrete group and let $M$ be a proper smooth (real analytic) $G$-manifold having only one orbit type. Then the map

$$\text{Riem}_G(M) \to \text{Riem}(M/G), \quad \alpha \mapsto \tilde{\alpha},$$

is a bijection. The inverse is given by $\beta \mapsto \pi^*\beta$.

Moreover we have:

**Lemma 6.2.** Let $G$ and $M$ be as in Proposition 6.1. Let $\beta$ be a smooth (real analytic) Riemannian metric on $M/G$. Then the metric $d_\beta$ equals the metric $d_{\pi^*\beta}$ that $d_{\pi^*\beta}$ induces on $M/G$.

**Proof.** The proof follows straightforwardly from the definition of distance on a Riemannian manifold and from the fact that the orbit map $\pi: M \to M/G$ satisfies the path lifting property.

**Corollary 6.3.** Let $G$, $M$ and $\beta$ be as in Lemma 6.2. Then $\beta$ is complete if and only if $\pi^*\beta$ is complete.

If $G$ is a discrete group, we can prove the following:

**Theorem 6.4.** Let $G$ be a countable discrete group and let $M$ be a proper real analytic $G$-manifold. Assume $M$ has only one orbit type. Then $M$ has a real analytic $G$-invariant Riemannian metric. Moreover, for any real analytic $G$-invariant Riemannian metric $\alpha$ of $M$ there exists a complete real analytic $G$-invariant Riemannian metric conformal to $\alpha$. 
Proof. The orbit space $M/G$ can be embedded as a closed real analytic submanifold of a euclidean space, by Grauert’s theorem. Any such embedding $f$ induces a complete real analytic Riemannian metric $\alpha_f$ on $M/G$. But then, $\pi$ induces a real analytic $G$-invariant Riemannian metric $\pi^*(\alpha_f)$ on $M$. By Corollary 6.3, $\pi^*(\alpha_f)$ is complete.

Let then $\alpha$ be any real analytic $G$-invariant Riemannian metric on $M$. Then $\alpha$ induces a real analytic Riemannian metric $\tilde{\alpha}$ on $M/G$. By the result of Nomizu and Ozeki, $M/G$ has a complete real analytic Riemannian metric of form $h^2\tilde{\alpha}$, i.e., conformal to $\tilde{\alpha}$, where $h$ is a real analytic map $M/G \to \mathbb{R}^+$. But now $\pi^*(h^2\tilde{\alpha}) = (h \circ \pi)^2\alpha$ is a complete real analytic $G$-invariant Riemannian metric of $M$ conformal to $\alpha$.

Notice that Theorem 6.4 is not a special case of Theorem 5.2 since there are countable discrete groups which are not good.

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