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Some basic results concerning G-invariant Riemannian metrics

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Abstract. In this paper we study complete G-invariant Riemannian metrics. Let G be a Lie group and let M be a proper smooth G-manifold. Let α be a smooth G-invariant Riemannian metric of M, and let $\tilde{K}$ be any G-compact subset of M. We show that M admits a complete smooth G-invariant Riemannian metric β such that $\beta|\tilde{K} = \alpha|\tilde{K}$. We also prove the existence of complete real analytic G-invariant Riemannian metrics for proper real analytic G-manifolds. Moreover, we show that for any given smooth (real analytic) G-invariant Riemannian metric there exists a complete smooth (real analytic) G-invariant Riemannian metric conformal to the original Riemannian metric. To prove the real analytic results we need the assumption that G can be embeddded as a closed subgroup of a Lie group which has only finitely many connected components.
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1. Introduction

Let G be a Lie group acting properly and smoothly on a smooth manifold M. According to a well-known result (Theorem 4.3.1 in [8]) of R.S. Palais, M has a smooth G-invariant Riemannian metric. An alternative proof of this result is given in [1], Theorem 5.5.2. Moreover, it can be assumed that the Riemannian metric is complete, see Theorem 0.2 in [4].

In this paper we continue the study of G-invariant Riemannian metrics on proper smooth and real analytic G-manifolds. By a result of K. Nomizu and H. Ozeki ([7], Theorem 1), for any given Riemannian metric α of a smooth manifold there exists a complete Riemannian metric conformal to α. We begin by proving an equivariant version of this result (Theorem 3.1).

Let M be a connected smooth manifold with Riemannian metric α. According to a result of J.A. Morrow (Theorem in [6]), M has a complete Riemannian metric which agrees with α on a given compact subset of M. In Section 4, we prove an equivariant version of Morrow's result (Theorem 4.1).

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In Section 5, we study real analytic G-invariant Riemannian metrics. Constructing real analytic Riemannian metrics is more complicated than constructing smooth Riemannian metrics, in particular, when we want the Riemannian metrics to be invariant under a group action. Assume that the Lie group G can be embedded as a closed subgroup of a Lie group which has only finitely many connected components. It is known by Proposition 1.2 in [3], that under this assumption every proper real analytic G-manifold has a real analytic G-invariant Riemannian metric. In this paper we prove a real analytic version of Theorem 3.1 by using G-equivariant real analytic approximations.

In Section 6, we study G-invariant Riemannian metrics on a proper G-manifold M, where G is a countable discrete group. If G acts freely on M, or if, more generally, M has only one orbit type, then the orbit space M/G is a manifold of the same dimension as M. Every Riemannian metric α on M/G induces a G-invariant Riemannian metric $\pi^*\alpha$ on M. We show that $\pi^*\alpha$ is complete if and only if α is complete. Finally, we prove yet another real analytic version of Theorem 3.1, in the case where M has only one orbit type.

2. Preliminaries

Let G be a Lie group and let M be a smooth, i.e., C^∞ (real analytic, i.e., C^ω) manifold. We assume all the manifolds to be finite dimensional and without boundary and to have at most countably many connected components. Thus they are paracompact. Let G act on M. If the action $G \times M \to M$ is smooth (real analytic), we say that M is a smooth (real analytic) G-manifold. If the action is also proper, i.e., if the map $G \times M \to M \times M$, $(g,x) \mapsto (gx,x)$, is proper, we call M a *proper smooth (real analytic) G-manifold.*

Let M/G denote the orbit space and let $\pi\colon M \to M/G$ be the natural projection. We call a subset $\tilde{K}$ of M *G-compact*, if $\pi(\tilde{K})$ is compact.

Let X be a topological space. By the support $\operatorname{supp}(f)$ of a map $f\colon X \to \mathbb{R}$ we mean the closure of the set $\{x \in X \mid f(x) > 0\}$.

Let F be a subset of M. If every point $x \in M$ has a neighbourhood U such that $G(U,F) = \{g \in G \mid gU \cap F \neq \emptyset\}$ is relatively compact, we call F *small.*

Definition 2.1. Let G be a Lie group and let M be a proper smooth G-manifold. If F is small and $GF = M$, we say that F is a *fundamental set* for G in M. If, in addition, F is closed in M, we say that it is a *closed fundamental set.* We call a closed fundamental set F in M *fat*, if $G\dot{F} = M$, where $\dot{F}$ denotes the interior of F.

By Lemma 3.6 in [2], every proper smooth G-manifold has a fat closed fundamental set.

A euclidean space on which G acts linearly is called a *linear G-space.*

Lemma 2.2. *Let G be a Lie group and let M be a proper smooth G-manifold. Let $f\colon M \to \mathbb{V}$ be a smooth map into a linear G-space $\mathbb{V}$. Assume the support of f is small. Then*

$$\operatorname{Av}(f)\colon M \to \mathbb{V}, \;\; x \mapsto \int_G g f(g^{-1}x)\,dg,$$

where the integral is the left Haar integral over G, is a smooth G-equivariant map.

Proof. Lemma 2.4 in [5]. ■

We denote by $\mathbb{R}^+$ the set $\{x \in \mathbb{R} \mid x > 0\}$.

Lemma 2.3. *Let G be a Lie group and let M be a proper smooth G-manifold. Let $d\colon M \to \mathbb{R}^+$ be a continuous G-invariant map. Then there is a smooth G-invariant map $\tilde{f}\colon M \to \mathbb{R}^+$ such that $\tilde{f}(p) > \frac{1}{d(p)}$ for all $p \in M$.*

Proof. Let $\hat{f}\colon M \to \mathbb{R}^+$ be a smooth map such that $\hat{f}(p) > \frac{1}{d(p)}$ for all $p \in M$. Let $\gamma\colon M \to \mathbb{R}^+ \cup \{0\}$ be a smooth map whose support is small. We may assume that the intersection of the set $\{x \in M \mid \gamma(x) > 0\}$ with each orbit of M contains an open subset of the orbit. By Lemma 2.2, the map

$$\tilde{f}\colon M \to \mathbb{R}^+, \;\; p \mapsto \frac{\int_G \gamma(g^{-1}p)\hat{f}(g^{-1}p)dg}{\int_G \gamma(g^{-1}p)dg}$$

is smooth and G-invariant. Clearly, $\tilde{f}(p) > \frac{1}{d(p)}$, for all $p \in M$. ■

Recall that a metric space X is complete if every Cauchy sequence in X converges. A metric d on a G-space X is called G-invariant if $d(gx, gy) = d(x, y)$ for all $x, y \in X$ and for all $g \in G$. A G-invariant metric d on X induces a metric $\tilde{d}$ on X/G, where $\tilde{d}(\pi(x), \pi(y)) = \inf\{d(x, gy) \mid g \in G\}$.

Lemma 2.4. *Let G be a Lie group and let X be a Hausdorff space on which G acts properly. Let d be a G-invariant metric on X and let $\tilde{d}$ be the metric d induces on X/G. Then $\tilde{d}$ is complete if and only if d is complete.*

Proof. Assume first that $\tilde{d}$ is a complete metric. We first assume that X has only one orbit, $X = Gy$. Then X/G is a point. Let $(g_n y)$ be a Cauchy sequence in X. Since G acts properly on X, y has a neighbourhood U such that the closure of the set

$$G(U \mid U) = \{g \in G \mid gU \cap U \neq \emptyset\}$$

is compact. For sufficiently small $\varepsilon > 0$, the ball $\mathrm{B}_y(\varepsilon)$ with center y and radius ε is in U. Since $(g_n y)$ is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that

$$d(y, g_{n_0}^{-1} g_n y) = d(g_{n_0} y, g_n y) < \varepsilon$$

for all $n \geq n_0$. Thus $g_{n_0}^{-1} g_n y \in U$ for all $n \geq n_0$ and, consequently, $g_{n_0}^{-1} g_n \in G(U \mid U)$. It follows that $(g_{n_0}^{-1} g_n)$ has a subsequence converging to some $g \in \overline{G(U \mid U)}$. Thus $(g_{n_0}^{-1} g_n y)$ has a subsequence converging to gy and $(g_n y)$ has a subsequence converging to $g_{n_0} g y$. Since $(g_n y)$ is a Cauchy sequence, it now follows that it converges.

Assume then that X may have more than one orbit. Let (x_n) be a Cauchy sequence in X. Then $(\pi(x_n))$ is a Cauchy sequence in X/G. Thus $(\pi(x_n))$ converges to some point $\pi(y) \in X/G$, where $y \in X$. Let (ε_n) be a sequence converging to zero, $\varepsilon_n > 0$ for all n, with the property that $\tilde{d}(\pi(x_n), \pi(y)) < \varepsilon_n$ for all n. Then, for every n, there exists $y_n \in Gy$ such that $d(x_n, y_n) < \varepsilon_n$. It is easy to see that (y_n) is a Cauchy sequence. By the first part of the proof, (y_n)

converges to some hy, where $h \in G$. Since $d(x_n, hy) \leq d(x_n, y_n) + d(y_n, hy)$ for all n, it follows that (x_n) converges to hy. Thus d is complete.

Assume next that d is a complete metric. Let $(\pi(x_n))$ be a Cauchy sequence in X/G. Let (ε_q) be a decreasing sequence of positive numbers whose sum is finite. For every $q \in \mathbb{N}$, there exists $n(q)$ such that $\tilde{d}(\pi(x_m), \pi(x_p)) < \varepsilon_q$ for all $m, p \geq n(q)$. We may assume that $(n(q))$ is increasing. We choose a sequence (g_q) of elements in G as follows: Let $g_1 = e$. Inductively, we choose g_q to be such that $d(g_{q-1}x_{n(q-1)}, g_q x_{n(q)}) < \varepsilon_{q-1}$. Then $(g_q x_{n(q)})$ is a Cauchy sequence. Since d is complete, it follows that $(g_q x_{n(q)})$ converges. Since the orbit map $\pi\colon X \to X/G$ is continuous, it follows that $(\pi(x_{n(q)}))$ converges. Then $(\pi(x_n))$ converges, since it is a Cauchy sequence having a convergent subsequence. Thus $\tilde{d}$ is complete. ∎

3. Constructing complete G-invariant Riemannian metrics

Recall that a Riemannian metric on a manifold M is called complete, if it induces a complete metric on each connected component of M.

Let G be a Lie group and let M be a proper smooth G-manifold. Let $\mathrm{T}M$ denote the tangent bundle of M, and let $\mathrm{T}M \oplus \mathrm{T}M$ denote the Whitney sum. Then G acts by differentials on $\mathrm{T}M$ and $\mathrm{T}M \oplus \mathrm{T}M$. Let

$$\alpha\colon \mathrm{T}M \oplus \mathrm{T}M \to \mathbb{R}$$

be a Riemannian metric of M. We call α *G-invariant* if $\alpha_{gx}(gv, gw) = \alpha_x(v, w)$, for all $v, w \in \mathrm{T}_x M$, for all $x \in M$ and for all $g \in G$.

K. Nomizu and H. Ozeki have shown (Theorem 1 in [7]) that for any given Riemannian metric α on a smooth manifold there exists a complete Riemannian metric conformal to α. The ideas in their proof also work in the equivariant case. We briefly explain an equivariant version of their result.

Let α be a smooth G-invariant Riemannian metric of a proper smooth G-manifold M. Assume α is not complete. Without loss of generality we may assume that α is not complete on any connected component of M. Let d_α denote the metric α induces on the connected components of M. Then d_α is G-invariant. Let M_p denote the connected component containing p and let

$$\mathrm{B}_p(r) = \{q \in M_p \mid d_\alpha(p, q) < r\}$$

denote the metric ball with p as a center and with radius r. Moreover, let

$$d(p) = \sup\{r \mid \mathrm{B}_p(r) \text{ is relatively compact}\}.$$

Then $d\colon M \to \mathbb{R}$ is a continuous G-invariant real-valued map, and $d(p) > 0$ for all $p \in M$. By Lemma 2.3, there exists a smooth G-invariant map $\tilde{f}\colon M \to \mathbb{R}^+$, such that $\tilde{f}(p) > \frac{1}{d(p)}$ for all $p \in M$. Let $f = (\tilde{f})^2$. Then $f\alpha$ is a complete smooth G-invariant Riemannian metric of M. The proof of the completeness can be found in [7].

We obtain:

Theorem 3.1. *Let G be a Lie group and let M be a proper smooth G-manifold. For any smooth G-invariant Riemannian metric α of M there exists a complete smooth G-invariant Riemannian metric of M which is conformal to α.*

Notice that the existence of a complete smooth G-invariant Riemannian metric of a proper smooth G-manifold is known, Theorem 0.2 in [4], where it was obtained as a corollary of an embedding result. That result tells nothing about conformality and the approach would not work in the real analytic case, which we study in Section 5.

Let α be a G-invariant Riemannian metric of M. Let M_i, $i \in I \subset \mathbb{N}$, be the connected components of M and let d_i be the metric α induces on M_i. Then d_i induces a metric $\tilde{d}_i$ on $\pi(M_i) = \pi(GM_i)$, where $\tilde{d}_i(\pi(x), \pi(y)) = \inf\{d_i(x, gy) \mid g \in G, gM_i = M_i\}$ for all $x, y \in M_i$. Notice that if $M_j = gM_i$, for some $g \in G$, then $\tilde{d}_j = \tilde{d}_i$.

Lemma 3.2. *Let G be a Lie group and let M be a proper smooth G-manifold. Let $\tilde{K}$ be a G-compact subset of M and let M_0 be a connected component of M. Let α be a smooth G-invariant Riemannian metric on M. Then α induces a complete metric on $\tilde{K} \cap M_0$.*

Proof. Similar to the proof of Lemma 2.4. ∎

4. Extending G-invariant Riemannian metrics

In this section we prove the following theorem:

Theorem 4.1. *Let G be a Lie group and let M be a proper smooth G-manifold. Let α be a smooth G-invariant Riemannian metric of M. Then given a G-compact subset $\tilde{K} \subset M$, there is a complete smooth G-invariant Riemannian metric γ on M such that $\gamma|\tilde{K} = \alpha|\tilde{K}$.*

Theorem 4.1 is an equivariant version of a result by J.A. Morrow (Theorem in [6]). The proof is based on Morrow's proof. We begin by proving two lemmas.

Lemma 4.2. *Let G be a Lie group and let M be a proper smooth G-manifold. Then M has an exhaustion by G-compact sets.*

Proof. Let F be a fat closed fundamental set of M. Then F has an exhaustion by compact sets K_i, i.e., we can write $F = \bigcup K_i$, where $K_i \subset \operatorname{int}(K_{i+1})$ for all i. But then $M = \bigcup GK_i$, each set GK_i is G-compact and $\overline{GK_i} = GK_i \subset G(\operatorname{int}(K_{i+1})) \subset \operatorname{int}(GK_{i+1})$. ∎

Lemma 4.3. *Let G be a Lie group and let M be a proper smooth G-manifold. Let M_0 be a connected component of M. Let $\tilde{K}$ and $\hat{K}$ be G-compact subsets of M such that $\tilde{K} \subset \operatorname{int}(\hat{K})$. Fix a G-invariant Riemannian metric on M and assume (x_n) is a Cauchy sequence in $(M \setminus \tilde{K}) \cap M_0$ with the property that only finitely many of the x_n are in $\hat{K}$. Then $M \setminus \tilde{K}$ has a connected component containing infinitely many of the x_n.*

Proof. Let 3δ be the distance of $\tilde{K} \cap M_0$ from $(M \setminus \operatorname{int} \hat{K}) \cap M_0$. Then $\delta > 0$. Let $n_0 \in \mathbb{N}$ be such that $x_n \in M \setminus \hat{K}$ and $d(x_n, x_{n+k}) \leq \delta$ for all $n \geq n_0$ and all $k \in \mathbb{N} \cup \{0\}$. Therefore, for given x_n, x_{n+k}, where $n \geq n_0$ and $k \geq 0$, there is a curve in M_0 from x_n to x_{n+k} whose length is less than 2δ. This curve can not touch $\tilde{K}$, since $x_n, x_{n+k} \in M \setminus \hat{K}$. It follows that x_n and x_{n+k} are in the same connected component of $M \setminus \tilde{K}$. ∎

Proof of Theorem 4.1. Let $M = \bigcup_i GK_i$, where the sets K_i are as in Lemma 4.2. Then $\tilde{K} \subset \operatorname{int}(GK_i)$, for some i. The pair $\{\operatorname{int}(GK_{i+1}), M \setminus GK_i\}$ is a covering of M by open G-invariant sets. By Theorem 4.2.4. (4) in [9], there exist smooth G-invariant maps $\varrho_1, \varrho_2 \colon M \to \mathbb{R}$, with the following properties:

1. $\varrho_1, \varrho_2 \geq 0$,
2. $\varrho_1 + \varrho_2 = 1$,
3. $\operatorname{supp}(\varrho_1) \subset \operatorname{int}(GK_{i+1})$, $\operatorname{supp}(\varrho_2) \subset M \setminus GK_i$.

Thus $\varrho_1(x) = 1$ for all $x \in GK_i$ and $\varrho_2(x) = 1$ for all $x \in M \setminus GK_{i+1}$.

By the construction in Section 3, there exists a smooth G-invariant map $f \colon M \setminus GK_i \to \mathbb{R}^+$ such that the G-invariant Riemannian metric $\gamma_1 = f(\alpha | M \setminus GK_i)$ defines a complete metric d_{γ_1} on each connected component of $M \setminus GK_i$. Then $\gamma = (\varrho_1 + \varrho_2 f)\alpha$ is a smooth G-invariant Riemannian metric on M, and $\gamma | \tilde{K} = \alpha | \tilde{K}$.

It remains to show that the metric d_γ induced by γ is complete. Let (x_n) be a d_γ-Cauchy sequence in a connected component M_0 of M. Assume first that infinitely many x_n are in GK_{i+1}. By Lemma 3.2, the restriction $d_\gamma | (GK_{i+1} \cap M_0)$ is complete. Thus (x_n) has a convergent subsequence. Since (x_n) is a d_γ-Cauchy sequence, it must converge.

Assume next that only finitely many x_n are in GK_{i+1}. In this case we may assume that $x_n \in M \setminus GK_{i+1}$ for all n. If infinitely many of the x_n are in GK_{i+2}, then (x_n) has a convergent subsequence, by Lemma 3.2, and we are done. Assume then that only finitely many of the x_n are in GK_{i+2}. By passing to a subsequence, if necessary, we may assume that all the x_n are in the same connected component of $M \setminus GK_{i+1}$ (see Lemma 4.3). Let

$$U_r(GK_{i+1} \cap M_0) = \{x \in M_0 \mid d_\gamma(x, GK_{i+1} \cap M_0) \leq r\}.$$

By choosing r small enough, we may assume that $U_r(GK_{i+1} \cap M_0)$ is G-compact. If infinitely many x_n are in $U_r(GK_{i+1} \cap M_0)$, we are done by Lemma 3.2. Assume $d_\gamma(x_n, GK_{i+1} \cap M_0) > r$ for all n. Let $\varepsilon < r$. Then any curve which begins at some x_n and whose γ-length is less than ε remains in $M_0 \setminus GK_{i+1}$. On $M_0 \setminus GK_{i+1}$, γ equals γ_1. Consequently, the γ_1-length of any curve of γ-length less than ε beginning at x_n equals the γ-length. Thus (x_n) is a d_{γ_1}-Cauchy sequence in $M_0 \setminus GK_{i+1} \subset M_0 \setminus GK_i$. Since d_{γ_1} is complete, the sequence (x_n) converges. ∎

5. The real analytic case

We call a Lie group G *good*, if it can be embedded as a closed subgroup of a Lie group $\hat{G}$, where $\hat{G}$ has only finitely many connected components. Thus, for example, all the closed linear Lie groups are good.

In this section we construct complete real analytic G-invariant Riemannian metrics. We have to make the assumption that G is a good Lie group, since so far the existence of *any* real analytic G-invariant Riemannian metric has been proven only for proper real analytic G-manifolds, where G is a good Lie group (Proposition 1.2 in [3]). Moreover, the approximation result to which we refer in the proof of Lemma 5.1, has so far been proven only in the case where G is a good Lie group.

Lemma 5.1. *Let G be a good Lie group and let M be a proper real analytic G-manifold. Let $d\colon M \to \mathbb{R}^+$ be a continuous G-invariant map. Then there is a real analytic G-invariant map $f^\omega\colon M \to \mathbb{R}^+$ such that $f^\omega(p) > \frac{1}{d(p)}$ for all $p \in M$.*

Proof. Let G act trivially on $\mathbb{R}$ and diagonally on $M \times \mathbb{R}$. Since M is a proper real analytic G-manifold, also $M \times \mathbb{R}$ is a proper real analytic G-manifold. Let id denote the identity map of M and let $\tilde{f}\colon M \to \mathbb{R}^+$ be as in Lemma 2.3. Then

$$(\mathrm{id}, \tilde{f})\colon M \to M \times \mathbb{R}, \;\; x \mapsto (x, \tilde{f}(x)),$$

is a smooth G-equivariant map.

By Theorem II in [3], real analytic G-equivariant maps are dense among the smooth G-equivariant maps $M \to M \times \mathbb{R}$ in the strong-weak topology. Thus there exists a real analytic G-equivariant map $f^*\colon M \to M \times \mathbb{R}$ approximating $(\mathrm{id}, \tilde{f})$ as well as we like. Let $\mathrm{pr}\colon M \times \mathbb{R} \to \mathbb{R}$ denote the projection. Clearly, we may assume that $\mathrm{pr} \circ f^*(x) > \tilde{f}(x)$ for all $x \in M$. Thus we may choose $f^\omega = \mathrm{pr} \circ f^*$. ∎

Theorem 5.2. *Let G be a good Lie group and let M be a proper real analytic G-manifold. Then M admits a complete real analytic G-invariant Riemannian metric. For any real analytic G-invariant Riemannian metric α of M there exists a complete real analytic G-invariant Riemannian metric conformal to α.*

Proof. Let f^ω be as in Lemma 5.1. Then $(f^\omega)^2\alpha$ is a complete real analytic G-invariant Riemannian metric conformal to α. The first claim follows from the second claim and from Proposition 1.2 in [3] according to which M has a real analytic G-invariant Riemannian metric. ∎

Theorem 5.2 is known in the case where G is compact, see Theorem 1.4.5 in [10]. Clearly, no result like Theorem 4.1 can be true in the real analytic case.

6. Discrete G - the case of one orbit type

We begin by recalling a standard way to induce Riemannian metrics from one manifold to another one: Let M and N be smooth (real analytic) manifolds and let $f\colon M \to N$ be a smooth (real analytic) immersion. Assume β is a smooth (real analytic) Riemannian metric on N. Then β induces a smooth (real analytic) Riemannian metric $f^*\beta$ on M, where $(f^*\beta)_x(v,w) = \beta_{f(x)}(df_x(v), df_x(w))$, for every $x \in M$ and for every $v, w \in \mathrm{T}_xM$.

Let G be a countable discrete group and let M be a proper smooth (real analytic) G-manifold. Assume M has only one orbit type. Let H be a finite subgroup of G corresponding to that orbit type, and let $N(H)$ denote the normalizer of H in G. Let $\Gamma_H = N(H)/H$. We denote the fixed point set of H in M by M^H. Then Γ_H acts properly and freely on M^H. It also acts on G/H by $nH \cdot gH \mapsto gn^{-1}H$. It is well-known that there exists a G-equivariant smooth (real analytic) diffeomorphism

$$G/H_{\Gamma_H} \times M^H \to M, \ \ [gH, x] \mapsto gx,$$

see e.g. Theorem 4.3.10 in [9]. This induces a diffeomorphism between the orbit spaces M^H/Γ_H and M/G.

Let $\pi|\colon M^H \to M^H/\Gamma_H$ denote the restriction of the orbit map $\pi\colon M \to M/G$. Then $\pi|$ is a local diffeomorphism and $\mathrm{T}(M^H/\Gamma_H) \approx \mathrm{T}M^H/\Gamma_H$. Let α be a G-invariant Riemannian metric on M and let $\alpha|$ denote its restriction to M^H. Then $\alpha|$ is Γ_H-invariant. Thus $\alpha|$ induces a Riemannian metric on M^H/Γ_H. It follows that every smooth (real analytic) G-invariant Riemannian metric α on M induces a smooth (real analytic) Riemannian metric $\tilde{\alpha}$ on M/G. We obtain:

Proposition 6.1. *Let G be a countable discrete group and let M be a proper smooth (real analytic) G-manifold having only one orbit type. Then the map*

$$\mathrm{Riem}_G(M) \to \ \mathrm{Riem}(M/G), \ \alpha \mapsto \tilde{\alpha},$$

is a bijection. The inverse is given by $\beta \mapsto \pi^*\beta$.

Moreover we have:

Lemma 6.2. *Let G and M be as in Proposition 6.1. Let β be a smooth (real analytic) Riemannian metric on M/G. Then the metric d_β equals the metric $\tilde{d}_{\pi^*\beta}$ that $d_{\pi^*\beta}$ induces on M/G.*

Proof. The proof follows straightforwardly from the definition of distance on a Riemannian manifold and from the fact that the orbit map $\pi\colon M \to M/G$ satisfies the path lifting property. ■

Lemmas 2.4 and 6.2 imply:

Corollary 6.3. *Let G, M and β be as in Lemma 6.2. Then β is complete if and only if $\pi^*\beta$ is complete.*

If G is a discrete group, we can prove the following:

Theorem 6.4. *Let G be a countable discrete group and let M be a proper real analytic G-manifold. Assume M has only one orbit type. Then M has a real analytic G-invariant Riemannian metric. Moreover, for any real analytic G-invariant Riemannian metric α of M there exists a complete real analytic G-invariant Riemannian metric conformal to α.*

Proof. The orbit space M/G can be embedded as a closed real analytic submanifold of a euclidean space, by Grauert's theorem. Any such embedding f induces a complete real analytic Riemannian metric α_f on M/G. But then, π induces a real analytic G-invariant Riemannian metric $\pi^*(\alpha_f)$ on M. By Corollary 6.3, $\pi^*(\alpha_f)$ is complete.

Let then α be any real analytic G-invariant Riemannian metric on M. Then α induces a real analytic Riemannian metric $\tilde{\alpha}$ on M/G. By the result of Nomizu and Ozeki, M/G has a complete real analytic Riemannian metric of form $h^2\tilde{\alpha}$, i.e., conformal to $\tilde{\alpha}$, where h is a real analytic map $M/G \to \mathbb{R}^+$. But now $\pi^*(h^2\tilde{\alpha}) = (h \circ \pi)^2\alpha$ is a complete real analytic G-invariant Riemannian metric of M conformal to α. ∎

Notice that Theorem 6.4 is not a special case of Theorem 5.2 since there are countable discrete groups which are not good.

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