Journal of Lie Theory
Volume **18** (2008) 961–978

Proper actions on corank-one reductive homogeneous spaces

Fanny Kassel

Communicated by E. B. Vinberg

Abstract. Let $\mathbf{k}$ be a local field, G the set of $\mathbf{k}$-points of a connected semisimple algebraic $\mathbf{k}$-group $\mathbf{G}$, and H the set of $\mathbf{k}$-points of a connected reductive algebraic $\mathbf{k}$-subgroup $\mathbf{H}$ of $\mathbf{G}$ such that $\mathrm{rank}_{\mathbf{k}}(\mathbf{H}) = \mathrm{rank}_{\mathbf{k}}(\mathbf{G}) - 1$. We consider discrete subgroups Γ of G acting properly discontinuously on G/H and we examine their images under a Cartan projection $\mu : G \to V^+$, where V^+ is a closed convex cone in a real finite-dimensional vector space. We show that if Γ is neither a torsion group nor a virtually cyclic group, then $\mu(\Gamma)$ is almost entirely contained in one connected component of $V^+ \setminus C_H$, where C_H denotes the convex hull of $\mu(H)$ in V^+. As an application, we describe all torsion-free discrete subgroups of $G \times G$ acting properly discontinuously on G by left and right translation when $\mathrm{rank}_{\mathbf{k}}(\mathbf{G}) = 1$.
Mathematics Subject Classification 2000: 20G25, 22E40, 57S30.
Key Words and Phrases: discrete subgroups of Lie groups, discrete subgroups of p-adic groups, reductive groups over local fields, properly discontinuous action, Cartan decomposition.

1. Introduction

Let $\mathbf{k}$ be a local field, G the set of $\mathbf{k}$-points of a connected semisimple algebraic $\mathbf{k}$-group of rank one, and Δ_G the diagonal of $G \times G$. In this paper we describe all torsion-free discrete subgroups of $G \times G$ acting properly discontinuously on $(G \times G)/\Delta_G$ (Theorem 1.3). To this end, we prove a general result on the Cartan projection of discrete groups acting properly discontinuously on corank-one reductive homogeneous spaces (Theorem 1.2). This result holds for algebraic groups over any local field, but we first state it in the setting of real Lie groups (Theorem 1.1).

1.1. The main result in the real case. Let G be a real connected semisimple linear Lie group and H a closed connected reductive subgroup of G. It is known that G contains an infinite discrete subgroup Γ acting properly discontinuously on G/H if and only if $\mathrm{rank}_{\mathbb{R}}(H) < \mathrm{rank}_{\mathbb{R}}(G)$; this is the Calabi-Markus phenomenon ([12], Cor. 4.4). In this paper we consider the case when $\mathrm{rank}_{\mathbb{R}}(H) = \mathrm{rank}_{\mathbb{R}}(G) - 1$.

ISSN 0949–5932 / $2.50

Let us introduce some notation. Fix a Cartan subgroup A of G with Lie algebra $\mathfrak{a}$. Denote by $\Phi = \Phi(A, G)$ the system of restricted roots of A in G, by Φ^+ a system of positive roots, by $A^+ = \{a \in A,\ \chi(a) \geq 1\ \forall \chi \in \Phi^+\}$ the corresponding closed Weyl chamber, and set $V^+ = \log A^+ \subset \mathfrak{a}$. There is a maximal compact subgroup K of G such that the Cartan decomposition $G = KA^+K$ holds: every element $g \in G$ may be written as $g = k_1 a k_2$ for some $k_1, k_2 \in K$ and a unique $a \in A^+$ ([9], Chap. 9, Th. 1.1). Setting $\mu(g) = \log a$ defines a map $\mu : G \to V^+$, which is continuous, proper, and surjective. It is called the *Cartan projection* relative to the Cartan decomposition $G = KA^+K$.

Since $\mathrm{rank}_{\mathbb{R}}(H) = \mathrm{rank}_{\mathbb{R}}(G) - 1$, the set $\mu(H)$ separates V^+ into finitely many connected components, which are permuted by the opposition involution ι. (Recall that for every $a \in A^+$ we have $\iota(\log a) = \log a'$, where a' is the unique element of A^+ conjugate to a^{-1}.)

In this setting our main result is the following.

Theorem 1.1. *Let G be a real connected semisimple linear Lie group and H a closed connected reductive subgroup of G such that $\mathrm{rank}_{\mathbb{R}}(H) = \mathrm{rank}_{\mathbb{R}}(G) - 1$. For every discrete subgroup Γ of G acting properly discontinuously on G/H, there exists a connected component C of $V^+ \backslash \mu(H)$ such that $\mu(\gamma) \in C \cup \iota(C)$ for almost all $\gamma \in \Gamma$. If Γ is not virtually cyclic, then $\iota(C) = C$.*

Recall that a group Γ is said to satisfy some property *virtually* if it contains a subgroup of finite index satisfying this property. A property is said to be true for *almost all* $\gamma \in \Gamma$ if it is true for all $\gamma \in \Gamma$ with at most finitely many exceptions.

By results of Chevalley ([7], Chap. 2, Th. 14 & 15), if G is a real connected semisimple linear Lie group and H a closed connected reductive subgroup of G, then G (resp. H) is the identity component (for the real topology) of the set of $\mathbb{R}$-points of a connected semisimple linear algebraic $\mathbb{R}$-group $\mathbf{G}$ (resp. of a connected reductive algebraic $\mathbb{R}$-subgroup $\mathbf{H}$ of $\mathbf{G}$). Theorem 1.1 is equivalent to the analogous result where G (resp. H) is replaced by $\mathbf{G}(\mathbb{R})$ (resp. by $\mathbf{H}(\mathbb{R})$). We prove this result not only for $\mathbb{R}$-groups, but more generally for algebraic groups over any local field $\mathbf{k}$.

1.2. The main result in the general case. Let $\mathbf{k}$ be a local field, *i.e.*, $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, or the field $\mathbb{F}_q((t))$ of formal Laurent series over a finite field $\mathbb{F}_q$. Let G be the set of $\mathbf{k}$-points of a connected semisimple algebraic $\mathbf{k}$-group $\mathbf{G}$ and H the set of $\mathbf{k}$-points of a connected reductive algebraic $\mathbf{k}$-subgroup $\mathbf{H}$ of $\mathbf{G}$ such that $\mathrm{rank}_{\mathbf{k}}(\mathbf{H}) = \mathrm{rank}_{\mathbf{k}}(\mathbf{G}) - 1$. There is a Cartan projection μ of G to a closed convex cone V^+ in some real finite-dimensional vector space (see Section 2). The convex hull C_H of $\mu(H)$ in V^+ separates V^+ into finitely many connected components. The opposition involution $\mu(G) \to \mu(G)$, which maps $\mu(g)$ to $\mu(g^{-1})$ for all $g \in G$, extends to an involution ι of V^+ preserving C_H and permuting the connected components of $V^+ \setminus C_H$ (see Section 3.1). Our main result in this general setting is the following.

Theorem 1.2. *Let $\mathbf{k}$ be a local field, G the set of $\mathbf{k}$-points of a connected semisimple algebraic $\mathbf{k}$-group $\mathbf{G}$, and H the set of $\mathbf{k}$-points of a connected reduc-*

tive algebraic $\mathbf{k}$*-subgroup* $\mathbf{H}$ *of* $\mathbf{G}$ *such that* $\operatorname{rank}_{\mathbf{k}}(\mathbf{H}) = \operatorname{rank}_{\mathbf{k}}(\mathbf{G}) - 1$*. For every discrete subgroup* Γ *of* G *that acts properly discontinuously on* G/H *and that is not a torsion group, there exists a connected component* C *of* $V^+ \setminus C_H$ *such that* $\mu(\gamma) \in C \cup \iota(C)$ *for almost all* $\gamma \in \Gamma$*. If* Γ *is not virtually cyclic, then* $\iota(C) = C$*.*

When $\mathbf{k}$ has characteristic zero, Theorem 1.2 holds without assuming that Γ is not a torsion group: indeed, in this case every discrete torsion subgroup of G is finite (Lemma 3.1). This is not true when $\mathbf{k} = \mathbb{F}_q((t))$ for some finite field $\mathbb{F}_q$: in positive characteristic there are infinite discrete torsion subgroups of G that do not satisfy the conclusions of Theorem 1.2. We will give an example of such a group in Section 5.2.

1.3. An application to $(G \times G)/\Delta_G$**.** Our first application of Theorem 1.2, which is actually the main motivation of this paper, concerns homogeneous spaces of the form $(G\times G)/\Delta_G$, where G is the set of $\mathbf{k}$-points of a connected semisimple algebraic $\mathbf{k}$-group $\mathbf{G}$ with $\operatorname{rank}_{\mathbf{k}}(\mathbf{G}) = 1$, and where Δ_G is the diagonal of $G\times G$. In this situation, if μ is a Cartan projection of G, then $\mu\times\mu$ is a Cartan projection of $G \times G$; we identify V^+ with $\mathbb{R}^+ \times \mathbb{R}^+$ and C_H with the diagonal of $\mathbb{R}^+ \times \mathbb{R}^+$.

Theorem 1.3. *Let* $\mathbf{k}$ *be a local field,* G *the set of* $\mathbf{k}$*-points of a connected semisimple algebraic* $\mathbf{k}$*-group* $\mathbf{G}$ *with* $\operatorname{rank}_{\mathbf{k}}(\mathbf{G}) = 1$*, and* Δ_G *the diagonal of* $G\times G$*. Let* Γ *be a discrete subgroup of* $G \times G$*.*

1. *Assume that* Γ *is torsion-free. Then it acts properly discontinuously on* $(G \times G)/\Delta_G$ *if and only if, up to switching the factors of* $G \times G$*, it is a graph of the form*
$$\{(\gamma, \varphi(\gamma)),\ \gamma \in \Gamma_0\},$$
where Γ_0 *is a discrete subgroup of* G *and* $\varphi : \Gamma_0 \to G$ *is a group homomorphism such that for all* $R > 0$*, almost all* $\gamma \in \Gamma_0$ *satisfy* $\mu(\varphi(\gamma)) < \mu(\gamma) - R$*.*

2. *Assume that* Γ *is residually finite and is not a torsion group. Then it acts properly discontinuously on* $(G \times G)/\Delta_G$ *if and only if, up to switching the factors of* $G \times G$*, it has a finite-index subgroup* Γ' *that is a graph as in 1.*

Note that $(g, h)\Delta_G \mapsto gh^{-1}$ defines a $(G \times G)$-equivariant isomorphism from $(G \times G)/\Delta_G$ to G, where $G \times G$ acts on G by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. Thus Theorem 1.3 describes all torsion-free discrete subgroups of $G\times G$ acting properly discontinuously on G by left and right translation.

Recall that a group is said to be *residually finite* if the intersection of its normal finite-index subgroups is trivial. It is known that if $\Gamma \subset G \times G$ is finitely generated, then it is residually finite ([1], Cor. 1); if moreover $\mathbf{k}$ has characteristic zero, then Γ has a finite-index subgroup that is torsion-free by Selberg's lemma ([21], Lem. 8).

In the case of $G = \mathrm{PSL}_2(\mathbb{R})$, Theorem 1.3 has been proved for torsion-free groups by Kulkarni and Raymond [15]. In [13], Kobayashi considered the more general case when G is a real connected semisimple linear Lie group with $\operatorname{rank}_{\mathbb{R}}(G) = 1$: he showed that every torsion-free discrete subgroup of $G\times G$ acting properly discontinuously on $(G\times G)/\Delta_G$ is a graph, and asked whether one of the two projections of this graph is always discrete in G. Theorem 1.3 above answers this question positively and generalizes Kobayashi's result to all local fields. It gives a complete description of all torsion-free discrete subgroups of $G\times G$ acting properly discontinuously on $(G\times G)/\Delta_G$ in terms of a Cartan projection of G.

Theorem 1.3 applies to three-dimensional compact *anti-de Sitter* manifolds, *i.e.*, to three-dimensional compact Lorentz manifolds with constant negative sectional curvature. Indeed, such manifolds are modeled on

$$\mathrm{AdS}^3 = \big\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4,\quad x_1^2+x_2^2-x_3^2-x_4^2=1\big\}$$

endowed with the Lorentz metric induced by $x_1^2+x_2^2-x_3^2-x_4^2$, which identifies with $(\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R}))/\Delta_{\mathrm{SL}_2(\mathbb{R})}$ (see Section 5.3). Since three-dimensional compact anti-de Sitter manifolds are complete [11], they are quotients of the universal covering of AdS^3. By [15], up to a finite covering, they may in fact be written as

$$\Gamma\backslash(\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R}))/\Delta_{\mathrm{PSL}_2(\mathbb{R})},$$

where Γ is a torsion-free discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$ acting properly discontinuously on $(\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R}))/\Delta_{\mathrm{PSL}_2(\mathbb{R})}$. We refer the reader to the introduction of [20] for more details.

More generally, for any local field $\mathbf{k}$ and any quadratic form Q of Witt index two on $\mathbf{k}^4$, the quadric

$$S(Q)=\{x\in\mathbf{k}^4,\ Q(x)=1\}$$

identifies with $(\mathrm{SL}_2(\mathbf{k})\times\mathrm{SL}_2(\mathbf{k}))/\Delta_{\mathrm{SL}_2(\mathbf{k})}$ (see Section 5.3). Theorem 1.3 therefore applies to the discrete subgroups of $\mathrm{SL}_2(\mathbf{k})\times\mathrm{SL}_2(\mathbf{k})$ acting properly discontinuously on $S(Q)$.

Note that Theorem 1.3 cannot be generalized to groups G of higher rank. Indeed, take for instance $G=\mathrm{SO}(2,2n)$, and let Γ_1 (resp. Γ_2) be a torsion-free discrete subgroup of $\mathrm{SO}(1,2n)$ (resp. of $\mathrm{U}(1,n)$), where $\mathrm{SO}(1,2n)$ (resp. $\mathrm{U}(1,n)$) is seen as a subgroup of G. By [12], Prop. 4.9, $\Gamma_1\times\Gamma_2$ acts properly discontinuously on $(G\times G)/\Delta_G$. Other examples are obtained by replacing the triple $(\mathrm{SO}(2,2n),\mathrm{SO}(1,2n),\mathrm{U}(1,n))$ by $(\mathrm{SO}(4,4n),\mathrm{SO}(3,4n),\mathrm{Sp}(1,n))$ or by $(\mathrm{U}(2,2n),\mathrm{U}(1)\times\mathrm{U}(1,n),\mathrm{Sp}(1,n))$ (see [12]).

1.4. An application to $\mathrm{SL}_n(\mathbf{k})/\mathrm{SL}_{n-1}(\mathbf{k})$. As another application of Theorem 1.2, we give a simpler proof of the following result, due to Benoist [2].

Corollary 1.4. *Let $\mathbf{k}$ be a local field of characteristic zero. If $n\geq 3$ is odd, then every discrete subgroup of $\mathrm{SL}_n(\mathbf{k})$ acting properly discontinuously on $\mathrm{SL}_n(\mathbf{k})/\mathrm{SL}_{n-1}(\mathbf{k})$ is virtually abelian.*

Theorem 1.2 actually implies a slightly stronger version of Corollary 1.4: we may replace "virtually abelian" by "virtually cyclic".

One consequence of Corollary 1.4 is that in characteristic zero if $n\geq 3$ is odd, then the homogeneous space $\mathrm{SL}_n(\mathbf{k})/\mathrm{SL}_{n-1}(\mathbf{k})$ has no compact quotient, *i.e.*, there is no discrete subgroup Γ of $\mathrm{SL}_n(\mathbf{k})$ acting properly discontinuously on $\mathrm{SL}_n(\mathbf{k})/\mathrm{SL}_{n-1}(\mathbf{k})$ with $\Gamma\backslash\mathrm{SL}_n(\mathbf{k})/\mathrm{SL}_{n-1}(\mathbf{k})$ compact (see [2]).

1.5. Organization of the paper. In Section 2 we recall basic facts about Bruhat-Tits buildings, Cartan decompositions, and Cartan projections. Section 3 is devoted to the proof of Theorem 1.2 ; we also discuss the assumption that Γ is not a torsion group. In Section 4 we show how Theorem 1.2 implies Corollary 1.4 in the case of $G = \mathrm{SL}_n(\mathbf{k})$ and $H = \mathrm{SL}_{n-1}(\mathbf{k})$. In Section 5 we prove Theorem 1.3 ; we also show that the hypothesis that Γ is not a torsion group is necessary in positive characteristic, and we describe our application to three-dimensional quadrics.

Acknowledgements. I warmly thank Yves Benoist for his advice and encouragement.

2. Cartan projections

Throughout this article, we denote by $\mathbf{k}$ a local field, *i.e.*, $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, or the field $\mathbb{F}_q((t))$ of formal Laurent series over a finite field $\mathbb{F}_q$. If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, we denote by $|\cdot|$ the usual absolute value on $\mathbf{k}$; we set $\mathbf{k}^+ = [1, +\infty[$. If $\mathbf{k}$ is nonarchimedean, we denote by $\mathcal{O}$ the ring of integers of $\mathbf{k}$, by q the cardinal of the residue field of $\mathbf{k}$, by ω the (additive) valuation on $\mathbf{k}$ sending any uniformizer to 1, and by $|\cdot| = q^{-\omega(\cdot)}$ the corresponding (multiplicative) absolute value; we set $\mathbf{k}^+ = \{x \in \mathbf{k},\ |x| \geq 1\}$. If $\mathbf{G}$ is an algebraic group, we denote by G the set of its $\mathbf{k}$-points and by $\mathfrak{g}$ the Lie algebra of G.

In this section, we recall a few well-known facts on connected semisimple algebraic $\mathbf{k}$-groups and their Cartan projections.

2.1. Weyl chambers. Fix a connected semisimple algebraic $\mathbf{k}$-group $\mathbf{G}$. Recall that the $\mathbf{k}$-split $\mathbf{k}$-tori of $\mathbf{G}$ are all conjugate over $\mathbf{k}$ ([4], Th. 4.21). Fix such a torus $\mathbf{A}$ and let $\mathbf{N}$ (resp. $\mathbf{Z}$) denote its normalizer (resp. centralizer) in $\mathbf{G}$. The group $X(\mathbf{A})$ of $\mathbf{k}$-characters of $\mathbf{A}$ and the group $Y(\mathbf{A})$ of $\mathbf{k}$-cocharacters are both free $\mathbb{Z}$-modules of rank $r = \mathrm{rank}_{\mathbf{k}}(\mathbf{G})$, and there is a perfect pairing

$$\langle\cdot,\cdot\rangle : X(\mathbf{A}) \times Y(\mathbf{A}) \longrightarrow \mathbb{Z}.$$

If $\mathbf{k}$ is nonarchimedean, we set $A^\circ = A$; if $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, we set

$$A^\circ = \big\{a \in A\,, \quad \chi(a) \in]0, +\infty[\quad \forall \chi \in X(\mathbf{A})\big\}.$$

The set $\Phi = \Phi(\mathbf{A}, \mathbf{G})$ of restricted roots of $\mathbf{A}$ in $\mathbf{G}$, *i.e.*, the set of nontrivial weights of $\mathbf{A}$ in the adjoint representation of $\mathbf{G}$, is a root system of the real vector space $V = Y(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ ([4], Cor. 5.8). The group $W = N/Z$ is finite and identifies with the Weyl group of Φ ([4], §5.1 & Th. 5.3). Choose a basis $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of Φ and let

$$\begin{array}{rrcl} & A^+ & = & \big\{a \in A^\circ, \quad \alpha_i(a) \in \mathbf{k}^+ \quad \forall\, 1 \leq i \leq r\big\} \\ \big(\text{resp.} & V^+ & = & \big\{x \in V, \quad \langle \alpha_i, x\rangle \geq 0 \quad \forall\, 1 \leq i \leq r\big\}\big) \end{array}$$

denote the closed positive Weyl chamber in A° (resp. in V) corresponding to Δ; the set V^+ is a closed convex cone in V. If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, then V identifies with $\mathfrak{a}$

and V^+ with $\log A^+ \subset \mathfrak{a}$; we endow V with the Euclidean norm $\|\cdot\|$ induced by the Killing form of $\mathfrak{g}$. If $\mathbf{k}$ is nonarchimedean, we endow V with any W-invariant Euclidean norm $\|\cdot\|$.

2.2. The Bruhat-Tits building. In this subsection we assume $\mathbf{k}$ to be nonarchimedean. We briefly recall the construction of the Bruhat-Tits building of G, which is a metric space on which G acts properly discontinuously by isometries with a compact fundamental domain. We refer to the original articles [5] and [6], but the reader may also find [19] useful.

Let Res denote the restriction homomorphism from $X(\mathbf{Z})$ to $X(\mathbf{A})$, where $X(\mathbf{Z})$ denotes the group of $\mathbf{k}$-characters of $\mathbf{Z}$. There is a unique group homomorphism $\nu : Z \to V$ such that

$$\langle \mathrm{Res}(\chi), \nu(z)\rangle = -\,\omega(\chi(z))$$

for all $\chi \in X(\mathbf{Z})$ and $z \in Z$. The set $\nu(Z)$ is a lattice in V, and $\nu(A)$ is a sublattice of $\nu(Z)$ of finite index. The action of Z on V by translation along $\nu(Z)$ extends to an action of N on V by affine isometries; such an extension is unique up to translation.

For every $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ denote the connected unipotent $\mathbf{k}$-subgroup of $\mathbf{G}$ corresponding to the root α, as defined in [6]; the Lie algebra of U_α is $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{g}_{i\alpha}$ is the subspace of elements $X \in \mathfrak{g}$ such that $\mathrm{Ad}(a)(X) = \alpha(a)^i X$ for all $a \in A$. For every $u \in U_\alpha$, $u \neq 1$, the set $N \cap U_{-\alpha}\, u\, U_{-\alpha}$ has a unique element, which acts on V by the orthogonal reflection in some affine hyperplane $\mathcal{H}_u$, defined by an equation of the form $\langle \alpha, x\rangle + \psi_\alpha(u) = 0$, where $\psi_\alpha(u) \in \mathbb{R}$. For every $x \in V$, set

$$U_{\alpha,x} = \big\{u \in U_\alpha, \quad u = 1 \ \text{ or } \ \langle \alpha, x\rangle + \psi_\alpha(u) \geq 0\big\};$$

by [6] it is a subgroup of U_α. Set $N_x = \{n \in N,\ n \cdot x = x\}$ and let K_x denote the subgroup of G generated by N_x and the subgroups $U_{\alpha,x}$, where $\alpha \in \Phi$. The group K_x is a maximal compact open subgroup of G.

With this notation, the *Bruhat-Tits building* X of G is the set of equivalence classes of $G \times V$ for the relation

$$(g,x) \sim (g',x') \quad \Longleftrightarrow \quad \exists\, n \in N \text{ such that } x' = n \cdot x \text{ et } g^{-1}g'n \in K_x.$$

We endow X with the quotient topology induced by the discrete topology of G and the Euclidean structure of V. By construction, V embeds into X; we identify it with its image in X. The group G acts on X by

$$g' \cdot \overline{(g,x)} \; = \; \overline{(g'g\,,x)},$$

where $\overline{(g,x)}$ denotes the image of $(g,x) \in G \times V$ in X. This action is properly discontinuous, with a compact fundamental domain. By construction, the stabilizer of any point $x \in V$ is K_x. The *apartments* of X are the sets $g \cdot V$, where $g \in G$; the *walls* of X are the sets $g \cdot \mathcal{H}_u$, where $g \in G$ and $u \in U_\alpha$ for some $\alpha \in \Phi$. A *chamber* of X (or *alcove*) is a connected component of X deprived of its walls. The space X has the following property: for any pair (x,x') of points in X, there is an apartment containing both x and x'. We can therefore endow X with a distance d defined as follows: $d(x,x')$ is the Euclidean distance between x and x' in any apartment containing x and x' (it does not depend on the apartment). The group G acts on X by isometries for this distance.

2.3. Cartan decompositions and Cartan projections. If $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$, then there is a maximal compact subgroup K of G such that the Cartan decomposition $G=KA^+K$ holds: for every $g\in G$, there are elements $k_1,k_2\in K$ and a unique $a\in A^+$ such that $g=k_1ak_2$ ([9], Chap. 9, Th. 1.1). Setting $\mu(g)=\log a$ defines a map $\mu: G\rightarrow V^+\simeq\log A^+$, which is continuous, proper, and surjective. It is called the *Cartan projection* relative to the Cartan decomposition $G=KA^+K$.

Now assume $\mathbf{k}$ to be nonarchimedean. Consider the extremal point x_0 of the closed cone V^+, defined by $\langle\alpha_i,x_0\rangle=0$ for all $1\leq i\leq r$, and set $K=K_{x_0}$. Let $Z^+\subset Z$ denote the inverse image of V^+ under ν. By [5] the group G acts transitively on the set of couples $(\mathcal{A},\mathcal{C})$, where $\mathcal{A}$ is an apartment of X and $\mathcal{C}$ is a chamber of X contained in $\mathcal{A}$. This can be translated into algebraic terms by the existence of a *Cartan decomposition* $G=KZ^+K$: for every $g\in G$ there are elements k_1, $k_2\in K$ and $z\in Z^+$ such that $g=k_1zk_2$, and $\nu(z)$ is uniquely defined. Setting $\mu(g)=\nu(z)$ defines a map $\mu: G\rightarrow V^+$, which is continuous and proper; its image $\mu(G)$ is the intersection of V^+ with a lattice of V. The map μ is called the *Cartan projection* relative to the Cartan decomposition $G=KZ^+K$.

2.4. A geometric interpretation. Let X be either the Riemannian symmetric space G/K if $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$, or the Bruhat-Tits building of G if $\mathbf{k}$ is nonarchimedean. We now recall a geometric interpretation of the Cartan projection μ in terms of a distance on X.

Assume that $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$, and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to the Cartan decomposition $G=KA^+K$. The Killing form κ of $\mathfrak{g}$ is definite positive on $\mathfrak{p}$, hence induces a Euclidean norm $\|\cdot\|$ on $\mathfrak{p}$. Let π denote the natural projection of G onto $X=G/K$, and set $x_0=\pi(1)\in X$. The map $d\pi_1$ realizes an isomorphism between $\mathfrak{p}$ and the tangent space of X at x_0; thus $\kappa|_{\mathfrak{p}\times\mathfrak{p}}$ induces a G-invariant Riemannian metric on X. Let d denote the corresponding distance on X. The following result is probably well known; we prove it for the reader's convenience.

Lemma 2.1 ($\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$). *Let $\rho: X\rightarrow V^+$ denote the map sending $x=g\cdot x_0\in X$ to $\mu(g)$. For all $x,x'\in X$,*

$$\|\rho(x)-\rho(x')\|\leq d(x,x').$$

Moreover, the restriction of ρ to $A^+\cdot x_0$ is an isometry.

Proof. We identify V^+ with $\log A^+\subset\mathfrak{a}$. Let $\mathrm{Exp}:\mathfrak{p}\rightarrow X$ denote the exponential diffeomorphism mapping $Y\in\mathfrak{p}$ to $\gamma_Y(1)$, where γ_Y is the unique geodesic in X such that $\gamma_Y(0)=x_0$ and $\gamma'_Y(0)=d\pi_1(Y)$. For every $x\in X$, there exists $k\in K$ such that $x=k\exp(\rho(x))\cdot x_0$; by [9], Chap. 4, Th. 3.3,

$$x=\mathrm{Exp}\big((\mathrm{Ad}\,k)(\rho(x))\big). \tag{2.1}$$

Fix $x,x'\in X$ and let $\gamma=(y_t)_{t\in[0,1]}$ be the geodesic segment from $y_0=x$ to $y_1=x'$. By [9], p. 295, and (2.1), the map $t\mapsto\rho(y_t)$ is smooth and there exists a smooth map $t \mapsto k_t$ from $[0,1]$ to K such that $y_t = \mathrm{Exp}((\mathrm{Ad}\, k_t)(\rho(y_t)))$ for all $t \in [0,1]$. Since X has nonpositive sectional curvature ([9], Chap. 5, Th. 3.1), the length of γ in X is not less than the length of $\mathrm{Exp}^{-1}(\gamma)$ in $\mathfrak{p}$ ([9], Chap. 1, Th. 13.1), namely,

$$d(x,x') \geq \int_0^1 \left\| \frac{\mathrm{d}\big((\mathrm{Ad}\, k_t)(\rho(y_t))\big)}{\mathrm{d}t}(t') \right\| \mathrm{d}t'. \tag{2.2}$$

Now for all $t' \in [0,1]$,

$$\frac{\mathrm{d}\big((\mathrm{Ad}\, k_t)(\rho(y_t))\big)}{\mathrm{d}t}(t') = (\mathrm{Ad}\, k_{t'})\Big(\frac{\mathrm{d}(\rho(y_t))}{\mathrm{d}t}(t')\Big) + \Big(\frac{\mathrm{d}(\mathrm{Ad}\, k_t)}{\mathrm{d}t}(t')\Big)\big(\rho(y_{t'})\big),$$

where

$$(\mathrm{Ad}\, k_{t'})\Big(\frac{\mathrm{d}(\rho(y_t))}{\mathrm{d}t}(t')\Big) \in (\mathrm{Ad}\, k_{t'})(\mathfrak{a})$$

and

$$\begin{aligned}\Big(\frac{\mathrm{d}(\mathrm{Ad}\, k_t)}{\mathrm{d}t}(t')\Big)\big(\rho(y_{t'})\big) &= (\mathrm{Ad}\, k_{t'})\bigg(\mathrm{ad}\Big(\frac{\mathrm{d}(k_{t'}^{-1}k_{t'+t})}{\mathrm{d}t}(0)\Big)\big(\rho(y_{t'})\big)\bigg)\\ &\in (\mathrm{Ad}\, k_{t'})\big([\mathfrak{k},\mathfrak{a}]\big).\end{aligned}$$

The subspaces $\mathfrak{a}$ and $[\mathfrak{k},\mathfrak{a}]$ are orthogonal with respect to κ. Indeed, the decomposition of $\mathfrak{g}$ into eigenspaces under the adjoint action of $\mathfrak{a}$ is orthogonal with respect to κ ([9], Chap. 3, Th. 4.2); in particular, $\mathfrak{a}$ is orthogonal to the sum $[\mathfrak{g},\mathfrak{a}]$ of the root spaces of $\mathfrak{g}$. Since κ is invariant under $\mathrm{Ad}\, G$ ([9], p. 131), the subspaces $(\mathrm{Ad}\, k_{t'})(\mathfrak{a})$ and $(\mathrm{Ad}\, k_{t'})([\mathfrak{k},\mathfrak{a}])$ are orthogonal with respect to κ and

$$\begin{aligned}\left\|\frac{\mathrm{d}\big((\mathrm{Ad}\, k_t)(\rho(y_t))\big)}{\mathrm{d}t}(t')\right\| &\geq \left\|(\mathrm{Ad}\, k_{t'})\Big(\frac{\mathrm{d}(\rho(y_t))}{\mathrm{d}t}(t')\Big)\right\|\\ &= \left\|\frac{\mathrm{d}(\rho(y_t))}{\mathrm{d}t}(t')\right\|.\end{aligned} \tag{2.3}$$

Thus

$$d(x,x') \;\geq\; \int_0^1 \left\|\frac{\mathrm{d}(\rho(y_t))}{\mathrm{d}t}(t')\right\| \mathrm{d}t' \;=\; \|\rho(x)-\rho(x')\|.$$

If $x,x' \in A^+ \cdot x_0$, then $k_t = 1$ for all $t \in [0,1]$; hence (2.3) is an equality. Moreover, in this case (2.2) is also an equality since the geodesic submanifold $A \cdot x_0 = \mathrm{Exp}(\mathfrak{a})$ has zero sectional curvature ([9], Chap. 5, §3, Rem. 2). This implies $d(x,x') = \|\rho(x)-\rho(x')\|$. ∎

Since K fixes x_0 and since G acts on X by isometries, Lemma 2.1 implies that for every $a \in A^+$ and every $g \in KaK$,

$$d(g\cdot x_0, x_0) = d(a\cdot x_0, x_0) = \|\rho(a\cdot x_0)-\rho(x_0)\| = \|\mu(g)\|. \tag{2.4}$$

Now assume $\mathbf{k}$ to be nonarchimedean and let X denote the Bruhat-Tits building of G, endowed with the distance d defined in Section 2.2. Recall that $K = K_{x_0}$ is the stabilizer of the point $x_0 \in V$ defined by $\langle \alpha_i, x_0\rangle = 0$ for all

$1 \leq i \leq r$. Since G acts on X by isometries and since V is isometrically embedded as an apartment in X, for every $z \in Z^+$ and every $g \in KzK$,

$$d(g \cdot x_0, x_0) = d(z \cdot x_0, x_0) = d(\mu(g), x_0) = \|\mu(g)\|, \tag{2.5}$$

where $\| \cdot \|$ is the Euclidean norm on V. Lemma 2.1 also holds in this setting.

Lemma 2.2 (k nonarchimedean). *Let $\rho : X \to V^+$ denote the map sending $x = g \cdot x_0 \in X$ to $\mu(g)$. For all $x, x' \in X$,*

$$\|\rho(x) - \rho(x')\| \leq d(x, x').$$

Proof. Let $\mathcal{C}$ denote the unique chamber in V^+ containing x_0. We first recall the construction of a retraction $\rho_{V,\mathcal{C}} : X \to V$, as defined in [5], §2.3. For every $x \in X$, there is an apartment $\mathcal{A}$ containing both x and $\mathcal{C}$ ([5], Prop. 2.3.1), and there is an element $k \in K$ fixing $\mathcal{C}$ pointwise and mapping $\mathcal{A}$ to V ([5], Prop. 2.3.2). The point $k \cdot x \in V$ does not depend on the choice of $\mathcal{A}$ and k. Setting $\rho_{V,\mathcal{C}}(x) = k \cdot x$ defines a map $\rho_{V,\mathcal{C}} : X \to V$ such that for all $x, x' \in X$,

$$\|\rho_{V,\mathcal{C}}(x) - \rho_{V,\mathcal{C}}(x')\| \ \leq \ d(x, x')$$

([5], Prop. 2.5.3). We claim that for all $x, x' \in X$,

$$\|\rho(x) - \rho(x')\| \ \leq \ \|\rho_{V,\mathcal{C}}(x) - \rho_{V,\mathcal{C}}(x')\|. \tag{2.6}$$

Indeed, it follows from the definitions of ρ and $\rho_{V,\mathcal{C}}$ that $\rho_{V,\mathcal{C}}(x) \in W \cdot \rho(x)$ for all $x \in X$. Since the norm $\| \cdot \|$ is W-invariant, it is enough to show that

$$\|\rho(x) - \rho(x')\| \ \leq \ \|\rho(x) - w \cdot \rho(x')\| \tag{2.7}$$

for all $x, x' \in X$ and all $w \in W$. Recall that W is generated by the set S of orthogonal reflections in the hyperplanes $\big\{x \in V,\ \langle \alpha_i, x \rangle = 0\big\}$, where $1 \leq i \leq r$. Write $w = s_m \dots s_1$, where $s_j \in S$ for all j. We argue by induction on m. If $(s_m \dots s_1) \cdot \rho(x') \in V^+$, then $s_m \dots s_1 = 1$ and (2.7) is obvious. Otherwise, the points $\rho(x)$ and $(s_m \dots s_1) \cdot \rho(x')$ lie in two distinct connected components of $V \setminus \mathcal{H}$, where $\mathcal{H}$ denotes the hyperplane of fixed points of s_m. Let y be the intersection point of $\mathcal{H}$ with the line segment $[\rho(x), (s_m \dots s_1) \cdot \rho(x')]$. Since s_m is an orthogonal reflection,

$$\begin{aligned} \|\rho(x) - (s_m \dots s_1) \cdot \rho(x')\| &= \|\rho(x) - y\| + \|y - (s_m \dots s_1) \cdot \rho(x')\| \\ &= \|\rho(x) - y\| + \|y - (s_{m-1} \dots s_1) \cdot \rho(x')\| \\ &\geq \|\rho(x) - (s_{m-1} \dots s_1) \cdot \rho(x')\|. \end{aligned}$$

By the induction assumption, $\|\rho(x) - (s_m \dots s_1) \cdot \rho(x')\| \geq \|\rho(x) - \rho(x')\|$. This proves (2.6) and completes the proof of Lemma 2.2. ∎

The following result will be needed in the proof of Theorem 1.2.

Lemma 2.3. *Let* $\mathbf{k}$ *be a local field,* G *the set of* $\mathbf{k}$*-points of a connected semisimple algebraic* $\mathbf{k}$*-group, and* $\mu : G \to V^+$ *a Cartan projection. For all* $g, g' \in G$*, the following two inequalities hold:*

$$\|\mu(gg') - \mu(g)\| \leq \|\mu(g')\|, \tag{2.8}$$
$$\|\mu(gg') - \mu(g')\| \leq \|\mu(g)\|. \tag{2.9}$$

Proof. Since G acts on X by isometries, (2.8) follows immediately from Lemmas 2.1 and 2.2, together with Formulas (2.4) and (2.5). We claim that (2.8) implies (2.9). Indeed, if $w \in W$ denotes the "longest" element of W, such that $w \cdot z^{-1} \in Z^+$ for all $z \in Z^+$, then $\mu(g^{-1}) = w \cdot (-\mu(g))$ for all $g \in G$. Since the norm $\| \cdot \|$ on V is W-invariant, the opposition involution $\iota : \mu(G) \to \mu(G)$, which maps $\mu(g)$ to $\mu(g^{-1})$ for all $g \in G$, is an isometry. Together with (2.8), this implies

$$\|\mu(gg') - \mu(g')\| = \|\mu(g'^{-1}g^{-1}) - \mu(g'^{-1})\| \leq \|\mu(g^{-1})\| = \|\mu(g)\|. \qquad \blacksquare$$

3. Proper actions on G/H in the corank-one case

In this section we give a proof of Theorem 1.2 and we discuss the assumption that Γ is not a torsion group.

3.1. Proof of Theorem 1.2. With the notation of Section 2, let $\mathbf{H}$ be a connected reductive algebraic $\mathbf{k}$-subgroup of $\mathbf{G}$ with $\mathrm{rank}_{\mathbf{k}}(\mathbf{H}) = \mathrm{rank}_{\mathbf{k}}(\mathbf{G}) - 1$. Fix a maximal $\mathbf{k}$-split $\mathbf{k}$-torus $\mathbf{A_H}$ of $\mathbf{H}$. After conjugating $\mathbf{H}$ by an element of G, we may assume that $\mathbf{A_H} \subset \mathbf{A}$ ([4], Th. 4.21). Recall that $\mathbf{H}$ is the almost product of a central torus and of its derived group, which is semisimple ([4], Prop. 2.2). Therefore H admits a Cartan decomposition $H = K_H Z_H^+ K_H$, where $\mathbf{Z_H}$ is the centralizer of $\mathbf{A_H}$ in $\mathbf{H}$ and K_H is some maximal compact subgroup of H. We now use a result proved by Mostow [17] and Karpelevich [10] in the real case, and by Landvogt [16] in the nonarchimedean case: after conjugating $\mathbf{H}$ by an element of G, we may assume that $K_H \subset K$. Thus $\mu(H) = \mu(Z_H)$ and the convex hull C_H of $\mu(H)$ in V^+ is the intersection of V^+ with a finite union of hyperplanes of V parametrized by the Weyl group W. The opposition involution $\iota : \mu(G) \to \mu(G)$, which maps $\mu(g)$ to $\mu(g^{-1})$ for all $g \in G$, extends to an isometry of V^+, still denoted by ι. It preserves $\mu(H)$, hence C_H, and permutes the connected components of $V^+ \setminus C_H$.

Our proof of Theorem 1.2 is based on the *properness criterion* of Benoist ([2], Cor. 5.2) and Kobayashi ([14], Th. 1.1), which states that a subgroup Γ of G acts properly discontinuously on G/H if and only if the set $\mu(\Gamma) \cap (\mu(H) + C')$ is bounded for every compact subset C' of V. This condition is equivalent to the boundedness of $\mu(\Gamma) \cap (C_H + C')$ for every compact subset C' of V.

Our proof is also based on the following observation $(*)$: if $(x_n)_{n \in \mathbb{N}}$ is a sequence of points of V^+ whose distance to C_H is larger than a given $R > 0$, and if $\|x_{n+1} - x_n\| \leq R$ for all $n \in \mathbb{N}$, then all elements x_n belong to the same connected component of $V^+ \setminus C_H$.

We now give a proof of Theorem 1.2. Let $C_1, \ldots, C_s$ be the connected components of $V^+ \setminus C_H$ and let Γ be a discrete subgroup of G acting properly discontinuously on G/H. The set $\mu(\Gamma)$ is invariant under the opposition involution ι.

Assume that Γ is not a torsion group and fix an element $\gamma \in \Gamma$ of infinite order. Since Γ is discrete and since μ is a proper map, the sequence $(\|\mu(\gamma^n)\|)_{n\in\mathbb{Z}}$ tends to infinity as n tends to $\pm\infty$. Let F be the set of elements $\gamma' \in \Gamma$ such that the distance of $\mu(\gamma')$ to C_H is $\leq \|\mu(\gamma)\|$. ¿From the discreteness of Γ, the properness of μ, and the properness criterion, we deduce that F is finite. Moreover, by Lemma 2.3,

$$\left\|\mu(\gamma^{n+1}) - \mu(\gamma^n)\right\| \leq \|\mu(\gamma)\|$$

for all $n \in \mathbb{Z}$. By the observation $(*)$ above, there are integers $1 \leq i, j \leq s$ such that $\mu(\gamma^n) \in C_i$ (resp. $\mu(\gamma^{-n}) \in C_j$) for almost all $n \in \mathbb{N}$. The opposition involution ι interchanges C_i and C_j.

Note that for every $\gamma' \in \Gamma$, Lemma 2.3 implies

$$\left\|\mu(\gamma'\gamma^n) - \mu(\gamma^n)\right\| \leq \|\mu(\gamma')\|$$

for all $n \in \mathbb{Z}$. By the properness criterion, $\mu(\gamma'\gamma^n) \in C_i$ and $\mu(\gamma'\gamma^{-n}) \in C_j$ for almost all $n \in \mathbb{N}$.

First consider the case $i = j$. Let F' be the set of elements $\gamma' \in \Gamma$ such that $\mu(\gamma') \notin C_i$. We claim that F' is finite. Indeed, let $\gamma' \in F'$. By Lemma 2.3,

$$\left\|\mu(\gamma'\gamma^{n+1}) - \mu(\gamma'\gamma^n)\right\| \leq \|\mu(\gamma)\|$$

for all $n \in \mathbb{Z}$. Moreover, $\mu(\gamma') \notin C_i$, and we have just seen that $\mu(\gamma'\gamma^n) \in C_i$ for almost all $n \in \mathbb{Z}$. By the observation $(*)$ above, there is an integer $n \in \mathbb{Z}$ such that $\gamma'\gamma^n \in F$. Therefore, $F' \subset F\gamma^{\mathbb{Z}}$. Since F is finite and since for every $f \in F$ the element $f\gamma^n$ belongs to C_i for almost all $n \in \mathbb{Z}$, the set F' is finite. This proves the claim.

Now consider the case $i \neq j$. We claim that the subgroup $\gamma^{\mathbb{Z}}$ has finite index in Γ. Indeed, let $\gamma' \in \Gamma$. By Lemma 2.3,

$$\left\|\mu(\gamma'\gamma^{n+1}) - \mu(\gamma'\gamma^n)\right\| \leq \|\mu(\gamma)\|$$

for all $n \in \mathbb{Z}$. Moreover, we have seen that $\mu(\gamma'\gamma^n) \in C_i$ and $\mu(\gamma'\gamma^{-n}) \in C_j$ for almost all $n \in \mathbb{N}$. By the observation $(*)$ above, there is an integer $n \in \mathbb{Z}$ such that $\gamma'\gamma^n \in F$. Therefore, $\Gamma = F\gamma^{\mathbb{Z}}$. Since F is finite, $\gamma^{\mathbb{Z}}$ has finite index in Γ. This proves the claim and completes the proof of Theorem 1.2.

3.2. Discrete torsion groups in characteristic zero. In this subsection we show that when $\mathbf{k}$ has characteristic zero, the assumption that Γ is not a torsion group may be removed from Theorem 1.2. When Γ is known to be finitely generated, this follows from Selberg's lemma ([21], Lem. 8). In general it is also true, based on the following lemma, which is probably well known.

Lemma 3.1. *Let $\mathbf{k}$ be a local field of characteristic zero and $\mathbf{G}$ a linear algebraic $\mathbf{k}$-group. If $\mathbf{k}$ is a p-adic field, then every torsion subgroup of G is finite. If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, then every discrete torsion subgroup of G is finite.*

Proof. Embed $\mathbf{G}$ in $\mathbf{GL}_n$ for some $n \geq 1$. Let Γ be a torsion subgroup of G. By a result of Schur ([8], Th. 36.14), Γ contains a finite-index abelian subgroup Γ' whose elements are all semisimple. To show that Γ is finite, it is enough to prove the finiteness of Γ'.

Assume that $\mathbf{k}$ is a p-adic field. The elements of Γ' are diagonalizable in a common basis over an algebraic closure of $\mathbf{k}$. For every $\gamma \in \Gamma'$ the eigenvalues of γ are roots of unity; they generate a cyclotomic extension $\mathbf{k}_\gamma$ of $\mathbf{k}$, and $[\mathbf{k}_\gamma : \mathbf{k}] \leq n$ since the characteristic polynomial of γ has degree n. Now there are only finitely many cyclotomic extensions of $\mathbf{k}$ of degree $\leq n$ ([18], Chap. 2, Th. 7.12 & Prop. 7.13). Therefore the field generated by all extensions $\mathbf{k}_\gamma$, $\gamma \in \Gamma'$, has finite degree over $\mathbf{k}$, hence contains only finitely many roots of unity ([18], Chap. 2, Prop. 5.7). This implies the finiteness of Γ'.

Assume that $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ and that in addition Γ is discrete in G. The elements of Γ' are diagonalizable in a common basis over $\mathbb{C}$, and their eigenvalues are roots of unity. Since the group $\mathbb{U}$ of complex numbers of modulus one is compact, every discrete subgroup of $\mathbb{U}^n$ is finite. This implies the finiteness of Γ'. ■

When $\mathbf{k}$ has positive characteristic, there exist infinite discrete torsion subgroups in G. They all have a unipotent subgroup of finite index (this follows from [22], Prop. 2.8, for instance). Some of them do not satisfy the conclusions of Theorem 1.2 : we will give an example of such a group in Section 5.2.

4. An application to $\mathbf{SL}_n(\mathbf{k})/\mathbf{SL}_{n-1}(\mathbf{k})$

In this section we discuss the case of $G = \mathrm{SL}_n(\mathbf{k})$ and $H = \mathrm{SL}_{n-1}(\mathbf{k})$. We show how Theorem 1.2 implies Corollary 1.4.

Let $\mathbf{G} = \mathbf{SL}_n$ for some integer $n \geq 2$. The group $\mathbf{A}$ of diagonal matrices of determinant one is a maximal $\mathbf{k}$-split $\mathbf{k}$-torus of $\mathbf{G}$, which is its own centralizer, *i.e.*, $\mathbf{Z} = \mathbf{A}$. The corresponding roots are the linear forms $\varepsilon_i - \varepsilon_j$, $1 \leq i \neq j \leq n$, where

$$\varepsilon_i\big(\mathrm{diag}(a_1, \ldots, a_n)\big) = a_i.$$

A basis of the root system of $\mathbf{A}$ in $\mathbf{G}$ is given by the roots $\varepsilon_i - \varepsilon_{i+1}$, where $1 \leq i \leq n-1$. If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ (resp. if $\mathbf{k}$ is nonarchimedean), the corresponding positive Weyl chamber is

$$\begin{aligned} A^+ &= \big\{\mathrm{diag}(a_1, \ldots, a_n) \in A,\ \ a_i \in]0, +\infty[\ \forall i \text{ and } a_1 \geq \ldots \geq a_n\big\} \\ \big(\text{resp. } A^+ &= \big\{\mathrm{diag}(a_1, \ldots, a_n) \in A,\ |a_1| \geq \ldots \geq |a_n|\big\}\big). \end{aligned}$$

Set $K = \mathrm{SO}(n)$ (resp. $K = \mathrm{SU}(n)$, resp. $K = \mathrm{SL}_n(\mathcal{O})$) if $\mathbf{k} = \mathbb{R}$ (resp. if $\mathbf{k} = \mathbb{C}$, resp. if $\mathbf{k}$ is nonarchimedean). The Cartan decomposition $G = KA^+K$ holds. If $\mathbf{k} = \mathbb{R}$ (resp. if $\mathbf{k} = \mathbb{C}$), it follows from the polar decomposition in $\mathrm{GL}_n(\mathbb{R})$ (resp. in $\mathrm{GL}_n(\mathbb{C})$) and from the reduction of symmetric (resp. Hermitian) matrices; if $\mathbf{k}$ is nonarchimedean, it follows from the structure theorem for finitely generated modules over a principal ideal domain. The real vector space

$$V = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n,\ x_1 + \ldots + x_n = 0\big\} \ \simeq\ \mathbb{R}^{n-1}$$

and its closed convex cone

$$V^+ = \big\{(x_1, \dots, x_n) \in V,\ x_1 \geq \dots \geq x_n\big\}$$

do not depend on $\mathbf{k}$. Let $\mu : G \to V^+$ denote the Cartan projection relative to the Cartan decomposition $G = KA^+K$. If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, then $\mu(g) = (x_1, \dots, x_n)$, where e^{2x_i} is the i-th eigenvalue of ${}^t\overline{g}g$.

Let $\mathbf{H} = \mathbf{SL}_{n-1}$, which we consider as a subgroup of $\mathbf{G}$ by embedding $(n-1) \times (n-1)$ matrices in the upper left corner of $n \times n$ matrices. Then

$$C_H = \bigcup_{1 \leq i \leq n} \big\{(x_1, \dots, x_n) \in V^+,\ x_i = 0\big\}$$

and the connected components of $V^+ \setminus C_H$ are the sets

$$C_i = \big\{(x_1, \dots, x_n) \in V^+,\ x_i > 0 > x_{i+1}\big\},$$

where $1 \leq i \leq n-1$. The opposition involution $\iota : V^+ \to V^+$ is given by

$$\iota(x_1, \dots, x_n) = (-x_n, \dots, -x_1)\,;$$

it maps C_i to C_{n-i} for all $1 \leq i \leq n-1$. Here is a reformulation of Theorem 1.2 in the present situation.

Proposition 4.1. *Let Γ be a discrete subgroup of $\mathrm{SL}_n(\mathbf{k})$ that acts properly discontinuously on $\mathrm{SL}_n(\mathbf{k})/\mathrm{SL}_{n-1}(\mathbf{k})$ and that is not a torsion group. There exists an integer $1 \leq i \leq n-1$ such that $\mu(\gamma) \in C_i \cup C_{n-i}$ for almost all $\gamma \in \Gamma$. If Γ is not virtually cyclic, then $C_i = C_{n-i}$.*

Note that if n is odd, then $C_i \neq C_{n-i}$ for all $1 \leq i \leq n-1$, which implies Corollary 1.4. Another consequence of Proposition 4.1 is the following.

Corollary 4.2. *Assume that $n \geq 4$ is even. Let Γ be a discrete subgroup of $\mathrm{SL}_n(\mathbf{k})$ that acts properly discontinuously on $\mathrm{SL}_n(\mathbf{k})/\mathrm{SL}_{n-1}(\mathbf{k})$ and that is not virtually cyclic. Every element $\gamma \in \Gamma$ of infinite order has $n/2$ eigenvalues t with $|t| > 1$ and $n/2$ eigenvalues t with $|t| < 1$, counting multiplicities.*

The eigenvalues of an element $g \in \mathrm{SL}_n(\mathbf{k})$ belong to some finite extension $\mathbf{k}_g$ of $\mathbf{k}$; in Corollary 4.2 we denote by $|\cdot|$ the unique absolute value on $\mathbf{k}_g$ extending the absolute value $|\cdot|$ on $\mathbf{k}$. As above, replacing $\mathbf{k}$ by $\mathbf{k}_g$, we obtain a Cartan decomposition $\mathrm{SL}_n(\mathbf{k}_g) = K_g A_g^+ K_g$ with $K = K_g \cap \mathrm{SL}_n(\mathbf{k})$ and $A^+ = A_g^+ \cap \mathrm{SL}_n(\mathbf{k})$. The corresponding Cartan projection $\mu_g : \mathrm{SL}_n(\mathbf{k}_g) \to V^+$ extends μ.

Proof of Corollary 4.2. We may assume that Γ is not a torsion group. Since the only connected component of $V^+ \setminus C_H$ that is invariant under ι is $C_{n/2}$, Proposition 4.1 implies that $\mu(\gamma) \in C_{n/2}$ for almost all $\gamma \in \Gamma$. Fix an element $\gamma \in \Gamma$ of infinite order. Since Γ is discrete and since μ is a proper map, $\|\mu(\gamma^m)\| \to +\infty$ as $m \to +\infty$. Therefore

$$\frac{1}{m}\,\mu(\gamma^m) \in C_{n/2}$$

for almost all $m \geq 1$. Let $\lambda : \mathrm{SL}_n(\mathbf{k}) \to V^+$ be the *Lyapunov projection* of $\mathrm{SL}_n(\mathbf{k})$, mapping $g \in \mathrm{SL}_n(\mathbf{k})$ to $\mu_g(a_g)$, where $a_g \in \mathrm{SL}_n(\mathbf{k}_g)$ is any diagonal matrix whose entries are the eigenvalues of g counted with multiplicities. By [3], Cor. 2.5,

$$\lambda(\gamma) = \lim_{m \to +\infty} \frac{1}{m}\, \mu(\gamma^m).$$

Thus $\lambda(\gamma)$ belongs to the closure of $C_{n/2}$ in V^+. We claim that $\lambda(\gamma) \notin C_H$. Indeed, by [3], Lem. 4.6, there is a constant $C_\gamma > 0$ such that for all $m \geq 1$,

$$\|\lambda(\gamma^m) - \mu(\gamma^m)\| \leq C_\gamma. \tag{4.1}$$

If $\lambda(\gamma) \in C_H$, then $\lambda(\gamma^m) = m\lambda(\gamma) \in C_H$ for all $m \geq 1$, so that (4.1) would contradict the properness criterion (see Section 3.1). This proves the claim. Therefore, $\lambda(\gamma) \in C_{n/2}$, which means that γ has $n/2$ eigenvalues t with $|t| > 1$ and $n/2$ eigenvalues t with $|t| < 1$, counting multiplicities. ■

5. An application to $(G \times G)/\Delta_G$ in the rank-one case

In this section we prove Theorem 1.3, we show that the hypothesis that Γ is not a torsion group is necessary in the case of a local field of positive characteristic, and we describe an application to three-dimensional quadrics over a local field.

5.1. Proof of Theorem 1.3. Assume that $\mathrm{rank}_{\mathbf{k}}(\mathbf{G}) = 1$ and let $\mathbf{\Delta_G}$ denote the diagonal of $\mathbf{G} \times \mathbf{G}$. Fix a Cartan projection μ of G and let $\mu_\bullet = \mu \times \mu$ be the corresponding Cartan projection of $G \times G$. We identify the cone V^+ with $\mathbb{R}^+ \times \mathbb{R}^+$, and C_{Δ_G} with the diagonal of $\mathbb{R}^+ \times \mathbb{R}^+$. There are two connected components in $V^+ \setminus C_{\Delta_G}$; let C_+ (resp. C_-) denote the one above (resp. below) the diagonal. The opposition involution ι is the identity.

We now give a proof of Theorem 1.3. Let Γ be a discrete subgroup of $G \times G$ that acts properly discontinuously on $(G \times G)/\Delta_G$ and that is not a torsion group. Since ι is the identity, Theorem 1.2 implies that either $\mu_\bullet(\gamma) \in C_+$ for almost all $\gamma \in \Gamma$, or $\mu_\bullet(\gamma) \in C_-$ for almost all $\gamma \in \Gamma$. Up to switching the factors of $G \times G$, we may assume that $\mu_\bullet(\gamma) \in C_-$ for almost all $\gamma \in \Gamma$.

Let pr_1 (resp. pr_2) denote the projection of Γ on the first (resp. second) factor of $G \times G$. The kernel F of pr_1 is finite. If Γ is residually finite, then Γ contains a normal finite-index subgroup Γ' such that $\Gamma' \cap F$ is trivial. If Γ is torsion-free, then F is already trivial and we set $\Gamma' = \Gamma$. In both cases, if we set $\Gamma_0 = \mathrm{pr}_1(\Gamma)$, then $\varphi = \mathrm{pr}_2 \circ \mathrm{pr}_1^{-1} : \Gamma_0 \to G$ is a group homomorphism and

$$\Gamma' = \{(g, \varphi(g)),\ g \in \Gamma_0\}.$$

Since $\mu(\varphi(g)) < \mu(g)$ for almost all $g \in \Gamma_0$, the group Γ_0 is discrete in G. Indeed, if it were not, then there would be a sequence $(g_n)_{n \in \mathbb{N}}$ of pairwise distinct points of Γ_0 converging to 1. Since Γ is discrete in $G \times G$ and since μ is a proper map, the sequence $(\mu(\varphi(g_n)))_{n \in \mathbb{N}}$ would tend to infinity. Therefore there would be infinitely many elements $(g, \varphi(g)) \in \Gamma$ with $\mu(\varphi(g)) \geq \mu(g)$, contradicting the assumption that $\mu_\bullet(\gamma) \in C_-$ for almost all $\gamma \in \Gamma$. This proves that Γ_0 is discrete in G. Since

$\mu(\varphi(g)) < \mu(g)$ for almost all $g \in \Gamma_0$, the properness criterion (see Section 3.1) ensures that for all $R > 0$, almost all $g \in \Gamma_0$ satisfy $\mu(\varphi(g)) < \mu(g) - R$.

Conversely, if there exist a discrete subgroup Γ_0 of G and a group homomorphism $\varphi : \Gamma_0 \to G$ satisfying the conditions of Theorem 1.3, then Γ acts properly discontinuously on $(G \times G)/\Delta_G$ by the properness criterion.

5.2. Infinite torsion groups in positive characteristic. Take $\mathbf{G} = \mathbf{SL}_2$ over $\mathbf{k} = \mathbb{F}_q((t))$, where $\mathbb{F}_q$ is a finite field of characteristic p. We now give an example of an infinite discrete torsion subgroup of $G \times G$ that acts properly discontinuously on $(G \times G)/\Delta_G$ and nevertheless does not satisfy the conclusions of Theorems 1.2 and 1.3. The Cartan decomposition $G = KA^+K$ holds, where $K = \mathrm{SL}_2(\mathcal{O}) = \mathrm{SL}_2(\mathbb{F}_q[[t]])$ and where A^+ is the set of diagonal matrices $\mathrm{diag}(a_1, a_2)$ of G with $|a_1| \geq |a_2|$ (see Section 4). Let μ be the corresponding Cartan projection. For every $n \in \mathbb{N}$, set

$$g_n = \begin{pmatrix} 1 & t^{-n} \\ 0 & 1 \end{pmatrix}.$$

Note that for $1 \leq r \leq p-1$,

$$\mu(g_n^r) = 2n.$$

This can be seen by expanding g_n^r as follows:

$$g_n^r = \begin{pmatrix} 1 & rt^{-n} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r & 0 \\ t^n & r^{-1} \end{pmatrix} \begin{pmatrix} t^{-n} & 0 \\ 0 & t^n \end{pmatrix} \begin{pmatrix} r^{-1}t^n & 1 \\ -1 & 0 \end{pmatrix}.$$

Let Γ be the subgroup of $G \times G$ generated by the elements (g_n, g_{2n}) and the elements (g_{2n}, g_n), where $n \in \mathbb{N}$. It is an infinite residually finite discrete subgroup of G and each of its nontrivial elements has order p. The group Γ acts properly discontinuously on $(G \times G)/\Delta_G$ by the properness criterion (see Section 3.1). It is not virtually cyclic. However, the two connected components of $V^+ \setminus C_{\Delta_G}$ both contain infinitely many points of the form $\mu(\gamma)$, $\gamma \in \Gamma$.

5.3. An application to three-dimensional quadrics over a local field. As was pointed out in the introduction, one of the motivations for our investigation of $(G \times G)/\Delta_G$ in the rank-one case is its application to three-dimensional quadrics over a local field $\mathbf{k}$. We now discuss this point in more detail.

Let $\mathbf{k}$ be a local field and Q be a quadratic form on $\mathbf{k}^4$. Consider the unit sphere

$$S(Q) = \{x \in \mathbf{k}^4,\ Q(x) = 1\}.$$

By Witt's theorem, it identifies with the homogeneous space $\mathrm{SO}(Q)/H$, where $\mathbf{SO}(Q)$ is the special orthogonal group of Q and $\mathbf{H}$ is an algebraic $\mathbf{k}$-subgroup of $\mathbf{SO}(Q)$ defined as the stabilizer of some point $x \in S(Q)$.

If Q is $\mathbf{k}$-anisotropic, then $\mathrm{SO}(Q)$ is compact ([4], §4.24); thus every discrete subgroup of $\mathrm{SO}(Q)$ is finite and acts properly discontinuously on $S(Q)$.

Assume that Q has Witt index one, *i.e.*, that $\mathrm{rank}_{\mathbf{k}}(\mathbf{SO}(Q)) = 1$. If $\mathbf{H}$ is $\mathbf{k}$-anisotropic, then H is compact, and every discrete subgroup of $\mathrm{SO}(Q)$ acts properly discontinuously on $S(Q)$. On the other hand, if $\mathrm{rank}_{\mathbf{k}}(\mathbf{H}) = 1$, then every discrete subgroup of $\mathrm{SO}(Q)$ acting properly discontinuously on $S(Q)$ is finite: this is the Calabi-Markus phenomenon ([12], Cor. 4.4). For instance, if $\mathbf{k} = \mathbb{R}$, then every quadratic form Q on $\mathbf{k}^4$ of Witt index one is equivalent to $x_1^2 - x_2^2 - x_3^2 - x_4^2$ or to $x_1^2 + x_2^2 + x_3^2 - x_4^2$. In the first case, $\mathrm{SO}(Q)$ (resp. H) is isomorphic to $\mathrm{SO}(1,3)$ (resp. to $\mathrm{SO}(3)$) and every discrete subgroup of $\mathrm{SO}(Q)$ acts properly discontinuously on $S(Q)$. In the second case, $\mathrm{SO}(Q)$ (resp. H) is isomorphic to $\mathrm{SO}(3,1)$ (resp. to $\mathrm{SO}(2,1)$) and every discrete subgroup of $\mathrm{SO}(Q)$ acting properly discontinuously on $S(Q)$ is finite.

Now assume that Q has Witt index two, *i.e.*, that $\mathrm{rank}_{\mathbf{k}}(\mathbf{SO}(Q)) = 2$. For instance, if $\mathbf{k} = \mathbb{R}$, then $\mathrm{SO}(Q)$ (resp. H) is isomorphic to $\mathrm{SO}(2,2)$ (resp. to $\mathrm{SO}(1,2)$). We may assume that Q is given by

$$Q(x_1, x_2, x_3, x_4) = x_1 x_4 - x_2 x_3$$

and that $\mathbf{H}$ is the stabilizer of $x = (1,0,0,1) \in S(Q)$. Note that there is a natural transitive action of the group $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$ on $S(Q)$. Indeed, $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$ acts on $\mathrm{M}_2(\mathbf{k})$ by the formula $(g_1, g_2) \cdot u = g_1 u \, g_2^{-1}$ for all $(g_1, g_2) \in \mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$ and all $u \in \mathrm{M}_2(\mathbf{k})$; identifying $\mathrm{M}_2(\mathbf{k})$ with $\mathbf{k}^4$ gives a linear action of $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$ on $\mathbf{k}^4$ that preserves Q and is transitive on $S(Q)$. Since the stabilizer of $x = (1,0,0,1)$ in $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$ is $\Delta_{\mathrm{SL}_2(\mathbf{k})}$, the quadric $S(Q)$ identifies with the homogeneous space $(\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k}))/\Delta_{\mathrm{SL}_2(\mathbf{k})}$. By Theorem 1.3, up to switching the factors of $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$, the torsion-free discrete subgroups Γ of $\mathrm{SL}_2(\mathbf{k}) \times \mathrm{SL}_2(\mathbf{k})$ acting properly discontinuously on $S(Q)$ are exactly the graphs of the form

$$\Gamma = \{(\gamma, \varphi(\gamma)),\ \gamma \in \Gamma_0\},$$

where Γ_0 is a discrete subgroup of $\mathrm{SL}_2(\mathbf{k})$ and $\varphi : \Gamma_0 \to \mathrm{SL}_2(\mathbf{k})$ is a group homomorphism such that for all $R > 0$, almost all $\gamma \in \Gamma_0$ satisfy $\mu(\varphi(\gamma)) < \mu(\gamma) - R$.

References

[1] Alperin, R. C., *An elementary account of Selberg's lemma,* Enseign. Math. (2) **33** (1987), 269–273.

[2] Benoist, Y., *Actions propres sur les espaces homogènes réductifs,* Ann. Math. **144** (1996), 315–347.

[3] —, *Propriétés asymptotiques des groupes linéaires,* Geom. Funct. Anal. **7** (1997), 1–47.

[4] Borel, A., and J. Tits, *Groupes réductifs,* Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150.

[5] Bruhat, F., and J. Tits, *Groupes réductifs sur un corps local : I. Données radicielles valuées,* Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251.

[6] Bruhat, F., and J. Tits, *Groupes réductifs sur un corps local : II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. **60** (1984), 5–184.

[7] Chevalley, C., "Théorie des groupes de Lie : II. Groupes algébriques," Actualités Sci. Ind. **1152**, Hermann & Cie, Paris, 1951.

[8] Curtis, C., and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics **11**, Interscience Publishers, John Wiley & Sons, New York, London, 1962.

[9] Helgason, S., "Differential geometry, Lie groups, and symmetric spaces," corrected reprint of the 1978 original, Graduate Studies in Mathematics **34**, Amer. Math. Soc., Providence, RI, 2001.

[10] Karpelevich, F. I., *Surfaces of transitivity of semisimple groups of motions of a symmetric space*, Soviet Math. Dokl. **93** (1953), 401–404.

[11] Klingler, B., *Complétude des variétés lorentziennes à courbure constante*, Math. Ann. **306** (1996), 353–370.

[12] Kobayashi, T., *Proper action on a homogeneous space of reductive type*, Math. Ann. **285** (1989), 249–263.

[13] —, *On discontinuous groups acting on homogeneous spaces with noncompact isotropy subgroups*, J. Geom. Phys. **12** (1993), 133–144.

[14] —, *Criterion for proper actions on homogeneous spaces of reductive groups*, J. Lie Theory **6** (1996), 147–163.

[15] Kulkarni, R. S., and F. Raymond, *3-dimensional Lorentz space-forms and Seifert fiber spaces*, J. Differential Geom. **21** (1985), 231–268.

[16] Landvogt, E., *Some functorial properties of the Bruhat-Tits building*, J. Reine Angew. Math. **518** (2000), 213–241.

[17] Mostow, G. D., *Some new decomposition theorems for semi-simple groups*, Mem. Amer. Math. Soc. **14** (1955), 31–54.

[18] Neukirch, J., "Algebraic Number Theory," Grundlehren der Mathematischen Wissenschaften **322**, Springer-Verlag, Berlin, 1999.

[19] Rousseau, G., *Euclidean buildings*, in: "Nonpositive curvature geometries, discrete groups and rigidity," Proceedings of the 2004 Summer School at the Joseph Fourier Institute in Grenoble, Séminaires et Congrès **18**, Soc. Math. France, Paris, to appear.

[20] Salein, F., *Variétés anti-de Sitter de dimension 3 exotiques*, Ann. Inst. Fourier **50** (2000), 257–284.

[21] Selberg, A., *On discontinuous groups in higher-dimensional symmetric spaces*, in "Collected papers," vol. **1**, 475–492, Springer-Verlag, Berlin, 1989.

[22] Tits, J., *Free subgroups in linear groups*, J. Algebra **20** (1972), 250–270.

Fanny Kassel
Département de Mathématiques
Bâtiment 425
Faculté des Sciences d'Orsay
Université Paris-Sud 11
91405 Orsay Cedex
France
fanny.kassel@math.u-psud.fr

Received December 19, 2008
and in final form February 3, 2009