A Satake type Theorem for Super Automorphic Forms

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Abstract.   The aim of this article is a Satake type theorem for super automorphic forms on a complex bounded symmetric super domain $\mathcal{B}$ of rank 1 with respect to a lattice Γ. 'Super' means: additional odd (anticommuting) coordinates on an ordinary complex bounded symmetric domain B (the so-called body of $\mathcal{B}$) of rank 1. Satake's theorem says that for large weight k all spaces

$$sM_k(\Gamma) \cap L_k^s\left(\Gamma\backslash\mathcal{B}\right),$$

$s \in [1,\infty]$ coincide, where $sM_k(\Gamma)$ denotes the space of super automorphic forms for Γ with respect to the weight k, and $L_k^s\left(\Gamma\backslash\mathcal{B}\right)$ denotes the space of s-intergrable functions with respect to a certain measure on the quotient $\Gamma\backslash\mathcal{B}$ depending on k. So all these spaces are equal to the space
$sS_k(\Gamma) := sM_k(\Gamma) \cap L_k^2\left(\Gamma\backslash\mathcal{B}\right)$ of super cusp forms for Γ to the weight k.
As it is already well known for automorphic forms on ordinary complex bounded symmetric domains, we will give a proof of this theorem using an unbounded realization $\mathcal{H}$ of $\mathcal{B}$ and Fourier decomposition at the cusps of the quotient $\Gamma\backslash B$ mapped to ∞ via a partial Cayley transformation.
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1.   Introduction

Automorphic and cusp forms on an ordinary complex bounded symmetric domain B are a classical field of research. Let us give a general definition:

Definition 1.1.   [Automorphic and cusp forms in general] Suppose $B \subset \mathbb{C}^n$ is a bounded symmetric domain and G a semisimple Lie group acting transitively and holomorphically on B. Let $j \in \mathcal{C}^\infty(G \times B, \mathbb{C})$ be a cocycle, this means j is a smooth function on $G \times B$, holomorphic in the second entry, such that

$$j(gh, z) = j(g, hz)j(h, z)$$

for all $g, h \in G$ and $z \in B$. Let $k \in \mathbb{Z}$ and $\Gamma \sqsubset G$ be a discrete subgroup.

(i) A holomorphic function $f \in \mathcal{O}(B)$ on B is called an automorphic form of weight k with respect to Γ if and only if $f = f|_{\gamma}$ for all $\gamma \in \Gamma$, where $f|_g(\mathbf{z}) := f(g\mathbf{z}) j(g, \mathbf{z})^k$ for all $\mathbf{z} \in B$ and $g \in G$, or equivalently the lift $\widetilde{f} \in \mathcal{C}^{\infty}(G)$ is left-Γ-invariant, where $\widetilde{f}(g) := f|_g(\mathbf{0})$ for all $g \in G$. The space of automorphic forms of weight k with respect to Γ is denoted by $M_k(\Gamma)$.

(ii) An automorphic form $f \in M_k(\Gamma)$ is called a cusp form of weight k with respect to Γ if and only if $\widetilde{f} \in L^2(\Gamma \backslash G)$. The Hilbert space of cusp forms of weight k with respect to Γ is denoted by $S_k(\Gamma)$.

In the simplest case, where $B \subset \mathbb{C}$ is just the unit disc, $G = SU(1,1)$ acting on B via Möbius transformations,

$$j(g, z) = \frac{1}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1),$$

and $\Gamma \sqsubset G$ is a lattice, this means a discrete subgroup with finite covolume, one needs a more restrictive definition for automorphic and cusp forms. It is well known that after adding the cusps of $\Gamma \backslash B$ in ∂B, which are always finitely many, the quotient $\Gamma \backslash B$ is compact. Having fixed a cusp $z_0 \in \partial B$ of $\Gamma \backslash B$ there exists a Cayley transform R mapping biholomorphically the unit disc B onto the upper half plane $H \subset \mathbb{C}$ and z_0 to $i\infty$. Since Γ is a lattice there exists an element $\gamma \in \Gamma$ such that

$$R\gamma R^{-1} = \begin{pmatrix} 1 & \lambda_0 \\ 0 & 1 \end{pmatrix},$$

$\lambda_0 \backslash \{0\}$, acting on H as translation $w \mapsto w + \lambda_0$. If a function $f \in \mathcal{O}(B)$ fulfills $f|_{\gamma} = f$ then $f|_{R^{-1}} \in \mathcal{O}(H)$ fulfills

$$f|_{R^{-1}}(w) = f|_{R^{-1}}|_{R\gamma R^{-1}}(w) = f|_{R^{-1}}(w + \lambda_0),$$

and so it has a Fourier decomposition

$$f|_{R^{-1}}(w) = \sum_{m \in \frac{1}{\lambda_0} \mathbb{Z}} c_m e^{2\pi i m w}. \tag{1}$$

Definition 1.2.   [Automorphic and cusp forms on the unit disc B]

(i) A holomorphic function $f \in \mathcal{O}(B)$ is called an automorphic form of weight k for Γ if and only if $f|_{\gamma} = f$ for all $\gamma \in \Gamma$ and for each cusp $z_0 \in \partial B$ of $\Gamma \backslash B$ it has a positive Fourier decomposition, this means precisely $c_m = 0$ in (1) for all $m < 0$, or equivalently $f|_{R^{-1}}(w)$ is bounded for $\operatorname{Im} w \rightsquigarrow \infty$.

(ii) An automorphic form $f \in M_k(\Gamma)$ is called a cusp form if and only if it has a *strictly* positive Fourier decomposition for each cusp $z_0 \in \partial B$ of $\Gamma \backslash B$, which means $c_m = 0$ in (1) for all $m \leq 0$, or equivalently $f|_{R^{-1}}(w) \rightsquigarrow 0$ for $\operatorname{Im} w \rightsquigarrow \infty$.

However, in contrast to the one dimensional case, for higher dimension $n \geq 2$, when $B \subset \mathbb{C}^n$ is the unit ball, $G = SU(n, 1)$ acting on B via Möbius transformations,

$$j(g, z) = \frac{1}{\mathbf{c}\mathbf{z} + d}, \quad g = \left(\begin{array}{c|c} A & \mathbf{b} \\ \hline \mathbf{c} & d \end{array} \right) \begin{array}{l} \}n \\ \leftarrow n+1 \end{array} \in SU(n, 1),$$

and $\Gamma \sqsubset G$ is a lattice, the situation is different: Then again one has partial Cayley transforms R mapping B onto an unbounded realization H of B, which traditionally is a generalization of the right half plane instead of the upper half plane, but a holomorphic function $f \in \mathcal{O}(B)$ fulfilling $f|_{\gamma} = f$ for all $\gamma \in \Gamma$ automatically has a 'positive' Fourier decomposition at each cusp, and therefore the general definition 1.1 is considered to be the right one. This is known as Köcher's principle, see for example in section 11.5 of [1]. Futhermore Satake's theorem says that in this case for weight $k \geq 2n$ all spaces

$$M_k(\Gamma) \cap L_k^s(\Gamma \backslash G) \ ,$$

$s \in [1, \infty]$, coincide, and therefore are equal to $S_k(\Gamma) = M_k(\Gamma) \cap L_k^2(\Gamma \backslash G) \ ,$ where

$$L_k^s(\Gamma \backslash G) := \left\{ f \in \mathbb{C}^B \ \middle| \ \widetilde{f} \in L^s(\Gamma \backslash G) \right\} \ .$$

The crucial argument is that for any function $f \in M_k(\Gamma)$, $k \geq 2n$ and $s \in [1, \infty]$ the following are equivalent:

(i) $f \in L_k^s(\Gamma \backslash G)$

(ii) f has a '*strictly* positive' Fourier decomposition at each cusp.

In [1] one can find this theory in more generality.

Since in recent time super symmetry has become an important field of research for mathematics and physics, one is also interested in super automorphic resp. super cusp forms on complex bounded symmetric super domains with even (commuting) and odd (anticommuting) coordinates, and this article generalizes Köcher's principle and Satake's theorem for super automorphic forms on the complex super unit ball $\mathcal{B}$ with the usual unit ball $B \in \mathbb{C}^n$, $n \geq 2$, as body, see theorems 3.4 (ii) and 3.1. *Acknowledgement:* The present paper is part of my PhD thesis. So I would like to thank my doctoral advisor Professor H. Upmeier for many helpful comments and mentoring and all the other persons who accompanied me during the time I spent in Marburg.

2.   The general setting

Let $n \in \mathbb{N}$, $n \geq 2$, $r \in \mathbb{N}$ and $\mathcal{B} := B^{n|r}$ be the unique complex (n, r)-dimensional super domain with the unit ball

$$B := B^n := \left\{ \mathbf{z} \in \mathbb{C}^n \mid \mathbf{z}^* \mathbf{z} < 1 \right\} \subset \mathbb{C}^n$$

as body, holomorphic even (commuting) coordinate functions $z_1, \ldots, z_n$ and holomorphic odd (anticommuting) coordinate functions $\zeta_1, \ldots, \zeta_r$. Let us denote the space of (smooth) super functions (with values in $\mathbb{C}$) on $\mathcal{B}$ by $\mathcal{D}(\mathcal{B})$ and the space of super holomorphic functions on $\mathcal{B}$ by $\mathcal{O}(\mathcal{B}) \sqsubset \mathcal{D}(\mathcal{B})$. Let $\wp(r) := \wp(\{1, \ldots, r\})$. Then one can decompose every $f \in \mathcal{D}(\mathcal{B})$ uniquely as

$$f = \sum_{I, J \in \wp(r)} f_{IJ} \zeta^I \overline{\zeta}^J \ ,$$

all $f_{IJ} \in \mathcal{C}^\infty(B, \mathbb{C})$, $I, J \in \wp(r)$, where $\zeta^I := \zeta_{i_1} \cdots \zeta_{i_\rho}$, $I = \{i_1, \ldots, i_\rho\} \in \wp(r)$, $i_1 < \cdots < i_\rho$, and every $f \in \mathcal{O}(\mathcal{B})$ uniquely as

$$f = \sum_{I \in \wp(r)} f_I \zeta^I \,,$$

where all $f_I \in \mathcal{O}(B)$. So

$$\mathcal{D}(\mathcal{B}) \simeq \mathcal{C}^\infty(B, \mathbb{C}) \otimes \bigwedge(\mathbb{C}^r) \boxtimes \bigwedge(\mathbb{C}^r) = \mathcal{C}^\infty(B, \mathbb{C}) \otimes \bigwedge(\mathbb{C}^{2r})$$

and

$$\mathcal{O}(\mathcal{B}) \simeq \mathcal{O}(B) \otimes \bigwedge(\mathbb{C}^r) \,.$$

Define

$$G := sS\left(U(n, 1) \times U(r)\right)$$
$$:= \left\{ \left(\begin{array}{c|c} g' & 0 \\ \hline 0 & E \end{array} \right) \in U(n, 1) \times U(r) \,\middle|\, \det g' = \det E \right\} \,,$$

which is a real $((n+1)^2 + r^2 - 1)$ -dimensional Lie group. Then we have a holomorphic action of G on $\mathcal{B}$ given by super fractional linear (Möbius) transformations

$$g\left(\frac{\mathbf{z}}{\zeta} \right) := \left(\frac{(A\mathbf{z} + \mathbf{b})(\mathbf{cz} + d)^{-1}}{E\zeta(\mathbf{cz} + d)^{-1}} \right) \,,$$

where we split

$$g := \left(\begin{array}{c|c|c} A & \mathbf{b} & 0 \\ \hline \mathbf{c} & d & \\ \hline 0 & E \end{array} \right) \begin{array}{l} \}n \\ \leftarrow n+1 \\ \}r \end{array} \,.$$

The stabilizer subgroup of $\mathbf{0}$ in G is

$$K := sS\left((U(n) \times U(1)) \times U(r)\right)$$
$$= \left\{ \left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & d & \\ \hline 0 & E \end{array} \right) \in U(n) \times U(1) \times U(r) \,\middle|\, \det A\, d = \det E \right\} \,,$$

which is a maximal compact subgroup of G. On $G \times B$ we define the cocycle $j \in \mathcal{C}^\infty(G \times B, \mathbb{C})$ as $j(g, \mathbf{z}) := (\mathbf{cz} + d)^{-1}$ for all $g \in G$ and $\mathbf{z} \in \mathcal{B}$. It is holomorphic in the second entry. Let $k \in \mathbb{Z}$ be fixed. Then we have a right-representation of G on $\mathcal{D}(\mathcal{B})$ given by

$$|_g : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B}) \,, \quad f|_g\left(\frac{\mathbf{z}}{\zeta} \right) := f\left(g\left(\frac{\mathbf{z}}{\zeta} \right) \right) j(g, \mathbf{z})^k$$

for all $g \in G$, which is holomorphic, more precisely if $f \in \mathcal{O}(\mathcal{B})$ and $g \in G$ then $f|_g \in \mathcal{O}(\mathcal{B})$. Finally let Γ be a discrete subgroup of G.

Definition 2.1. [Super automorphic forms] Let $f \in \mathcal{O}(\mathcal{B})$. Then f is called a super automorphic form for Γ of weight k if and only if $f|_\gamma = f$ for all $\gamma \in \Gamma$. We denote the space of super automorphic forms for Γ of weight k by $sM_k(\Gamma)$.

Let $\mathbb{C}^{0|r}$ be the purely odd complex super domain with one point $\{0\}$ as body and odd coordinate functions $\eta_1, \ldots, \eta_r$. Then $\mathcal{D}\left(\mathbb{C}^{0|r}\right) \simeq \bigwedge\left(\mathbb{C}^r\right) \boxtimes \bigwedge\left(\mathbb{C}^r\right) \simeq \bigwedge\left(\mathbb{C}^{2r}\right)$. Let us define a lift:

$$\begin{aligned} \widetilde{}: \mathcal{D}(\mathcal{B}) &\rightarrow \mathcal{C}^\infty(G, \mathbb{C}) \otimes \mathcal{D}\left(\mathbb{C}^{0|r}\right) \simeq \mathcal{C}^\infty(G, \mathbb{C}) \otimes \bigwedge\left(\mathbb{C}^r\right) \boxtimes \bigwedge\left(\mathbb{C}^r\right), \\ f &\mapsto \widetilde{f}, \end{aligned}$$

where

$$\widetilde{f}(g) := f|_g\left(\frac{\mathbf{0}}{\eta}\right) = f\left(g\left(\frac{\mathbf{0}}{\eta}\right)\right) j\left(g, \mathbf{0}\right)^k$$

for all $f \in \mathcal{D}(\mathcal{B})$ and $g \in G$. Let $f \in \mathcal{O}(\mathcal{B})$. Then clearly $\widetilde{f} \in \mathcal{C}^\infty(G, \mathbb{C}) \otimes \mathcal{O}\left(\mathbb{C}^{0|r}\right)$ and $f \in sM_k(\Gamma) \Leftrightarrow \widetilde{f} \in \mathcal{C}^\infty\left(\Gamma\backslash G, \mathbb{C}\right) \otimes \mathcal{O}\left(\mathbb{C}^{0|r}\right)$ since for all $g \in G$

$$\begin{array}{ccc} \mathcal{C}^\infty(G) \otimes \mathcal{D}\left(\mathbb{C}^{0|r}\right) & \xrightarrow{l_g} & \mathcal{C}^\infty(G) \otimes \mathcal{D}\left(\mathbb{C}^{0|r}\right) \\ \uparrow{\scriptstyle\sim} & \% & \uparrow{\scriptstyle\sim} \\ \mathcal{D}(\mathcal{B}) & \xrightarrow[|_g]{} & \mathcal{D}(\mathcal{B}) \end{array},$$

where $l_g : \mathcal{C}^\infty(G) \otimes \mathcal{D}\left(\mathbb{C}^{0|r}\right) \rightarrow \mathcal{C}^\infty(G) \otimes \mathcal{D}\left(\mathbb{C}^{0|r}\right)$, $l_g(f)(h) := f(gh)$ simply denotes the left translation with the element $g \in G$.

Let $\langle\ ,\ \rangle$ be the canonical scalar product on $\mathcal{D}\left(\mathbb{C}^{0|r}\right) \simeq \bigwedge\left(\mathbb{C}^{2r}\right)$ (semi-linear in the second entry). Then for all $a \in \mathcal{D}\left(\mathbb{C}^{0|r}\right)$ we write $|a| := \sqrt{\langle a, a\rangle}$, and $\langle\ ,\ \rangle$ induces a 'scalar product'

$$(f, h)_\Gamma := \int_{\Gamma\backslash G}\left\langle \widetilde{h}, \widetilde{f}\right\rangle$$

for all $f, g \in \mathcal{D}(\mathcal{B})$ such that $\langle \widetilde{h}, \widetilde{f}\rangle \in L^1(\Gamma\backslash G)$ and for all $s \in\ ]0, \infty]$ a 'norm'

$$\|f\|_{s,\Gamma} := \left\|\ \left|\widetilde{f}\right|\ \right\|_{s, \Gamma\backslash G}$$

for all $f \in \mathcal{D}(\mathcal{B})$ such that $\left|\widetilde{f}\right| \in L^s\left(\Gamma\backslash G\right)$. Recall that the scalar product $(\ ,\ )_\Gamma$ and the norm $\|\ \|_{s,\Gamma}$ actually depend on the weight k. Let us define

$$L_k^s(\Gamma\backslash\mathcal{B}) := \left\{ f \in \mathcal{D}(\mathcal{B})\ \middle|\ \widetilde{f} \in \mathcal{C}^\infty(\Gamma\backslash G, \mathbb{C}) \otimes \mathcal{D}\left(\mathbb{C}^{0|r}\right), \|f\|_{s,\Gamma} < \infty \right\}$$

for all $s \in\ ]0, \infty]$.

Definition 2.2. [Super cusp forms] Let $f \in sM_k(\Gamma)$. f is called a super cusp form for Γ of weight k if and only if $f \in L_k^2(\Gamma\backslash\mathcal{B})$. The $\mathbb{C}$- vector space of all super cusp forms for Γ of weight k is denoted by $sS_k(\Gamma)$. It is a Hilbert space.

Observe that $|_g$ respects the splitting

$$\mathcal{O}(\mathcal{B}) = \bigoplus_{\rho=0}^{r} \mathcal{O}^{(\rho)}(\mathcal{B})$$

for all $g \in G$, where $\mathcal{O}^{(\rho)}(\mathcal{B})$ is the space of all $f = \sum_{I \in \wp(r), |I|=\rho} f_I \zeta^I$, all $f_I \in \mathcal{O}(\mathcal{B})$, $I \in \wp(r)$, $|I| = \rho$, $\rho = 0, \ldots, r$, and $\widetilde{}$ maps the space $\mathcal{O}^{(\rho)}(\mathcal{B})$ into $\mathcal{C}^\infty(G, \mathbb{C}) \otimes \mathcal{O}^{(\rho)}(\mathbb{C}^{0|r}) \simeq \mathcal{C}^\infty(G, \mathbb{C}) \bigwedge^{(\rho)}(\mathbb{C}^r)$. Therefore we have splittings

$$sM_k(\Gamma) = \bigoplus_{\rho=0}^r sM_k^{(\rho)}(\Gamma) \quad \text{and} \quad sS_k(\Gamma) = \bigoplus_{\rho=0}^r sS_k^{(\rho)}(\Gamma),$$

where $sM_k^{(\rho)}(\Gamma) := sM_k(\Gamma) \cap \mathcal{O}^{(\rho)}(\mathcal{B})$, $sS_k^{(\rho)}(\Gamma) := sS_k(\Gamma) \cap \mathcal{O}^{(\rho)}(\mathcal{B})$, $\rho = 0, \ldots, r$, and the last sum is orthogonal.

In the following we will use the Jordan triple determinant $\Delta : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by

$$\Delta(\mathbf{z}, \mathbf{w}) := 1 - \mathbf{w}^* \mathbf{z}$$

for all $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$. Let us recall the basic properties:

(i) $|j(g, \mathbf{0})| = \Delta(g\mathbf{0}, g\mathbf{0})^{\frac{1}{2}}$ for all $g \in G$,

(ii) $\Delta(g\mathbf{z}, g\mathbf{w}) = \Delta(\mathbf{z}, \mathbf{w}) j(g, \mathbf{z}) \overline{j(g, \mathbf{w})}$ for all $g \in G$ and $\mathbf{z}, \mathbf{w} \in B$, and

(iii) $\int_B \Delta(\mathbf{z}, \mathbf{z})^\lambda \, dV_{\text{Leb}} < \infty$ if and only if $\lambda > -1$.

Since $\left|\det(\mathbf{z} \mapsto g\mathbf{z})'\right| = |j(g, \mathbf{z})|^{n+1}$ and because of (i) we have the G-invariant volume element $\Delta(\mathbf{z}, \mathbf{z})^{-(n+1)} dV_{\text{Leb}}$ on B.

For all $I \in \wp(r)$, $h \in \mathcal{O}(B)$, $\mathbf{z} \in B$ and $g = \left(\begin{array}{c|c} g' & 0 \\ \hline 0 & E \end{array}\right) \in G$ we have

$$\left(h\zeta^I\right)\big|_g (\mathbf{z}) = h(g'\mathbf{z}) (E\eta)^I j(g, \mathbf{z})^{k+|I|},$$

where $E \in U(r)$. So for all $s \in \,]0, \infty]$, $f = \sum_{I \in \wp(r)} f_I \zeta^I$ and $h = \sum_{I \in \wp(r)} h_I \zeta^I \in \mathcal{O}(\mathcal{B})$ we obtain

$$\|f\|_{s,\Gamma} \equiv \left\| \sqrt{\sum_{I \in \wp(r)} |f_I|^2 \Delta(\mathbf{z}, \mathbf{z})^{k+|I|}} \right\|_{s, \Gamma \backslash B, \Delta(\mathbf{z}, \mathbf{z})^{-(n+1)} dV_{\text{Leb}}},$$

and

$$(f, h)_\Gamma \equiv \sum_{I \in \wp(r)} \int_{\Gamma \backslash B} \overline{f_I} h_I \Delta(\mathbf{z}, \mathbf{z})^{k+|I|-(n+1)} \, dV_{\text{Leb}}$$

if $\langle \widetilde{h}, \widetilde{f} \rangle \in L^1(\Gamma \backslash G)$, where '$\equiv$' means equality up to a constant $\neq 0$ depending on Γ, k and s.

3. Satake's theorem in the super case

We keep the notation of section 1, in particular $n \in \mathbb{N}$, $n \geq 2$. Here now the main theorem of the article, which is the analog to Satake's theorem for super automorphic forms:

Theorem 3.1.   *Let $\rho \in \{0, \ldots, r\}$. Assume $\Gamma \sqsubset G$ is a lattice (discrete such that* vol $(\Gamma \backslash G) < \infty$, $\Gamma \backslash G$ *not necessarily compact). Then*

$$sS_k^{(\rho)}(\Gamma) = sM_k^{(\rho)}(\Gamma) \cap L_k^s(\Gamma \backslash \mathcal{B})$$

for all $s \in [1, \infty]$ and $k \geq 2n - \rho$.

If $\Gamma \backslash G$ is compact then the assertion is trivial. For the non-compact case we will give a proof in the end of this section using the so-called unbounded realization $\mathcal{H}$ of $\mathcal{B}$, which we will develop in the following.

By the way, as for ordinary automorphic forms, theorem 3.1 implies that $sS_k(\Gamma)$ is finite dimensional for $n \geq 2$, $\Gamma \sqsubset G$ being a lattice and $k \geq 2n$ via lemma 12 of [1] section 10. 2, which says the following:

Let (X, μ) be a locally compact measure space, where μ is a positive measure such that $\mu(X) < \infty$. Let $\mathcal{F}$ be a closed subspace of $L^2(X, \mu)$ which is contained in $L^\infty(X, \mu)$. Then

$$\dim \mathcal{F} < \infty.$$

From now on let $\Gamma \backslash G$ be not compact.

Let $\mathfrak{g}' = \mathfrak{su}(n, 1)$ be the Lie algebra of $G' := SU(n, 1)$,

$$G' \hookrightarrow G, \, g' \mapsto \left(\begin{array}{c|c} g' & 0 \\ \hline 0 & 1 \end{array} \right),$$

and let $\mathfrak{a} \sqsubset \mathfrak{g}'$ be the standard Cartan sub Lie algebra of $\mathfrak{g}'$. Then $A := \exp_G \mathfrak{a}$ is the common standard maximal split Abelian subgroup of G' and G, it is the image of the Lie group embedding

$$\mathbb{R} \hookrightarrow G', \, t \mapsto a_t := \left(\begin{array}{c|c|c} \cosh t & 0 & \sinh t \\ \hline 0 & 1 & 0 \\ \hline \sinh t & 0 & \cosh t \end{array} \right) \begin{array}{l} \leftarrow 1 \\ \}n - 1 \\ \leftarrow n + 1 \end{array}.$$

Let $\mathfrak{n} \sqsubset \mathfrak{g}'$ be the standard maximal nilpotent sub Lie algebra, which is at the same time the direct sum of all root spaces of $\mathfrak{g}'$ of positive roots with respect to $\mathfrak{a}$. Let $N := \exp \mathfrak{n}$. Then we have an Iwasawa decomposition

$$G = NAK,$$

N is 2-step nilpotent, and so $N' := [N, N]$ is at the same time the center of N.

Now we transform the whole problem to the unbounded realization via the standard partial Cayley transformation

$$R := \left(\begin{array}{c|c|c} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \hline 0 & 1 & 0 \\ \hline -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array} \right) \begin{array}{l} \leftarrow 1 \\ \}n - 1 \\ \leftarrow n + 1 \end{array} \in G'^{\mathbb{C}} = SL(n + 1, \mathbb{C})$$

mapping B via Möbius transformation biholomorphically onto the unbounded domain

$$H := \left\{ \mathbf{w} = \left(\begin{array}{c} w_1 \\ \mathbf{w}_2 \end{array} \right) \begin{array}{l} \leftarrow 1 \\ \}n - 1 \end{array} \in \mathbb{C}^n \, \middle| \, \text{Re } w_1 > \frac{1}{2} \mathbf{w}_2^* \mathbf{w}_2 \right\},$$

which is a generalized right half plane, and $\mathbf{e}_1$ to ∞. We see that

$$RG'R^{-1} \sqsubset G'^{\mathbb{C}} = SL(n+1, \mathbb{C}) \hookrightarrow GL(n+1, \mathbb{C}) \times GL(r, \mathbb{C})$$

acts holomorphically and transitively on H via fractional linear transformations, and explicit calculations show that

$$a'_t := Ra_t R^{-1} = \left(\begin{array}{c|c|c} e^t & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & e^{-t} \end{array}\right) \begin{array}{l} \leftarrow 1 \\ \}n-1 \\ \leftarrow n+1 \end{array}$$

for all $t \in \mathbb{R}$, and RNR^{-1} is the image of

$$\mathbb{R} \times \mathbb{C}^{n-1} \to RG'R^{-1}, \ (\lambda, \mathbf{u}) \mapsto n'_{\lambda, \mathbf{u}} := \left(\begin{array}{c|c|c} 1 & \mathbf{u}^* & i\lambda + \frac{1}{2}\mathbf{u}^*\mathbf{u} \\ \hline 0 & 1 & \mathbf{u} \\ \hline 0 & 0 & 1 \end{array}\right),$$

which is a smooth diffeomorphism onto its image, with the multiplication rule

$$n'_{\lambda, \mathbf{u}} n'_{\mu, \mathbf{v}} = n'_{\lambda + \mu + \mathrm{Im}\,(\mathbf{u}^*\mathbf{v}), \mathbf{u} + \mathbf{v}}$$

for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{n-1}$ and acting on H as pseudo translations

$$\mathbf{w} \mapsto \left(\begin{array}{c} w_1 + \mathbf{u}^*\mathbf{w}_2 + i\lambda + \frac{1}{2}\mathbf{u}^*\mathbf{u} \\ \mathbf{w}_2 + \mathbf{u} \end{array}\right).$$

Define $j(R, \mathbf{z}) = \frac{\sqrt{2}}{1-z_1} \in \mathcal{O}(B)$,
$j(R^{-1}, \mathbf{w}) := j(R, R^{-1}\mathbf{w})^{-1} = \frac{\sqrt{2}}{1+w_1} \in \mathcal{O}(H)$, and for all

$$g = \left(\begin{array}{c|c|c} A & \mathbf{b} & \\ \hline \mathbf{c} & d & 0 \\ \hline & 0 & E \end{array}\right) \in RGR^{-1}$$

define

$$j(g, \mathbf{w}) = j(R, R^{-1}g\mathbf{w})\, j(R^{-1}gR, R^{-1}\mathbf{w})\, j(R^{-1}, \mathbf{w}) = \frac{1}{\mathbf{c}\mathbf{w} + d}.$$

Let $\mathcal{H}$ be the unique (n, r)-dimensional complex super domain with body H, holomorphic even coordinate functions $w_1, \ldots, w_n$ and holomorphic odd coordinate functions $\vartheta_1, \ldots, \vartheta_r$. R commutes with all $g \in Z(G')$, where

$$Z(G') = \left\{ \left(\begin{array}{c|c} \varepsilon 1 & 0 \\ \hline 0 & E \end{array}\right) \begin{array}{l} \}n+1 \\ \}r \end{array} \ \middle| \ \varepsilon \in U(1), E \in U(r), \varepsilon^{n+1} = \det E \right\} \sqsubset K$$

denotes the centralizer of G' in G, and we have a right-representation of the group RGR^{-1} on $\mathcal{D}(\mathcal{H})$ given by

$$|_g : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}), \ f|_g\left(\frac{\mathbf{w}}{\vartheta}\right) := f\left(g\left(\frac{\mathbf{w}}{\vartheta}\right)\right) j(g, \mathbf{w})^k$$

for all $g \in RGR^{-1}$, which is again holomorphic. If we define

$$|_R : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{B}), \ f|_R\left(\frac{\mathbf{z}}{\zeta}\right) := f\left(R\left(\frac{\mathbf{z}}{\zeta}\right)\right) j(R, \mathbf{z})^k$$

and

$$|_{R^{-1}} : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{H}) \, , \, f|_{R^{-1}} \left(\frac{\mathbf{w}}{\vartheta} \right) := f \left(R^{-1} \left(\frac{\mathbf{w}}{\vartheta} \right) \right) j \left(R^{-1}, \mathbf{w} \right)^k ,$$

then we see that again if $f \in \mathcal{O}(\mathcal{H})$ then $f|_R \in \mathcal{O}(\mathcal{B})$, and if $f \in \mathcal{O}(\mathcal{B})$ then $f|_{R^{-1}} \in \mathcal{O}(\mathcal{H})$, and

$$
\begin{array}{ccc}
\mathcal{D}(\mathcal{H}) & \overset{|_{RgR^{-1}}}{\longrightarrow} & \mathcal{D}(\mathcal{H}) \\
|_R \downarrow & \% & \downarrow \, |_R \\
\mathcal{D}(\mathcal{B}) & \underset{|_g}{\longrightarrow} & \mathcal{D}(\mathcal{B})
\end{array} .
$$

Now define the Jordan triple determinant Δ' on $H \times H$, which is again holomorphic in the first and antiholomorphic in the second variable, as

$$\Delta' (\mathbf{z}, \mathbf{w}) := \Delta \left(R^{-1}\mathbf{z}, R^{-1}\mathbf{w} \right) j \left(R^{-1}, \mathbf{z} \right)^{-1} \overline{j \left(R^{-1}, \mathbf{w} \right)}^{-1} = z_1 + \overline{w_1} - \mathbf{w}_2^* \, \mathbf{z}_2$$

for all $\mathbf{z}, \mathbf{w} \in H$. Clearly again $\left| \det \left(\mathbf{w} \mapsto g\mathbf{w} \right)' \right| = \left| j \left(g, \mathbf{w} \right) \right|^{n+1}$ and $\left| j \left(g, \mathbf{e}_1 \right) \right| = \Delta' \left(g\mathbf{e}_1, g\mathbf{e}_1 \right)^{\frac{1}{2}}$ for all $g \in RGR^{-1}$, and so $\Delta' \left(\mathbf{w}, \mathbf{w} \right)^{-(n+1)} dV_{\text{Leb}}$ is the RGR^{-1}-invariant volume element on H. If $f = \sum_{I \in \wp(r)} f_I \zeta^I \in \mathcal{O}(\mathcal{B})$, all $f_I \in \mathcal{O}(B)$, $I \in \wp(r)$, then

$$f|_{R^{-1}} \left(\frac{\mathbf{w}}{\vartheta} \right) = \sum_{I \in \wp(r)} f_I \left(R^{-1}\mathbf{w} \right) j \left(R^{-1}, \mathbf{w} \right)^{k+|I|} \vartheta^I \in \mathcal{O}(\mathcal{H}),$$

and if $f = \sum_{I \in \wp(r)} f_I \vartheta^I \in \mathcal{O}(\mathcal{H})$, all $f_I \in \mathcal{O}(H)$, $I \in \wp(r)$, and $g = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E \end{array} \right) \in RGR^{-1}$, $E \in U(r)$, then

$$f|_g \left(\frac{\mathbf{w}}{\vartheta} \right) = \sum_{I \in \wp(r)} f_I \left(g\mathbf{w} \right) j \left(g, \mathbf{w} \right)^{k+|I|} \left(E\vartheta \right)^I \in \mathcal{O}(\mathcal{H}) .$$

Let $\partial H = \left\{ \mathbf{w} \in \mathbb{C}^n \, \middle| \, \text{Re } w_1 = \frac{1}{2}\mathbf{w}_2^*\mathbf{w} \right\}$ be the boundary of H in $\mathbb{C}^n$. Then Δ' and ∂H are RNR^{-1}-invariant, and RNR^{-1} acts transitively on ∂H and on each

$$\left\{ \mathbf{w} \in H \, \middle| \, \Delta' \left(\mathbf{w}, \mathbf{w} \right) = e^{2t} \right\} = RNa_t\mathbf{0} \, ,$$

$t \in \mathbb{R}$.

Figure 1: the geometry of H.

For all $t \in \mathbb{R}$ define the rays $A_{<t} := \{ a_\tau \mid \tau < t \} \subset A$ and $A_{>t} := \{ a_\tau \mid \tau > t \} \subset A$.

Theorem 3.2.   [a 'fundamental domain' for $\Gamma \backslash G$] *There exist $\eta \subset N$ open and relatively compact, $t_0 \in \mathbb{R}$ and $\Xi \subset G'$ finite such that if we define*

$$\Omega := \bigcup_{g \in \Xi} g \eta A_{>t_0} K$$

then
(i) $g^{-1} \Gamma g \cap NZ(G') \sqsubset NZ(G')$ and $g^{-1} \Gamma g \cap N'Z(G') \sqsubset N'Z(G')$ are lattices, and

$$NZ(G') = \left(g^{-1} \Gamma g \cap NZ(G') \right) \eta Z(G')$$

for all $g \in \Xi$,
(ii) $G = \Gamma \Omega$,
(iii) the set $\{ \gamma \in \Gamma \mid \gamma \Omega \cap \Omega \neq \varnothing \}$ is finite.

Proof.   We use theorem 0.6 (i) - (iii) of [4], which says the following:

Let $\Gamma' \subset G'$ be an admissible discrete subgroup of G'. Then there exists $t_0' > 0$, an open, relatively compact subset $\eta_0 \subset N^+$, a finite set $\Xi \subset G'$, and an open, relatively compact subset Ω' of G' (Ξ being empty if G'/Γ' is compact, and Ω' being empty if G'/Γ' is non-compact) such that

(i) For all $b \in \Xi$, $\Gamma \cap b^{-1} N^+ b$ is a lattice in $b^{-1} N^+ b$.

(ii) For all $t > t'_0$ and for all open, relatively compact subsets η of N^+ such that $\eta \supset \eta_0$, if

$$\Omega'_{t,\eta} = \Omega' \cup \left(\bigcup_{b \in \Xi} \sigma_{t,\eta} b \right),$$

then $\Omega'_{t,\eta} \Gamma' = G'$, and

(iii) the set $\left\{ \gamma' \in \Gamma', \ \Omega'_{t,\eta} \gamma' \cap \Omega'_{t,\eta} \neq \varnothing \right\}$ is finite.

Hereby G' is a connected semisimple Lie group of real rank 1, $N^+ = N$ is the standard nilpotent sub Lie group of G' and $\sigma_{t,\eta} := K' A_{<t} \eta$ for all $t > 0$ and $\eta \subset N^+$ open and relatively compact, where A denotes the standard maximal non-compact abelian and K' the standard maximal compact sub Lie group of G'. Admissibility is a geometric property of the quotient $\Gamma' \backslash G' / K'$, roughly speaking Γ' is called admissible if and only if $\Gamma' \backslash G' / K'$ has only finitely many cusps.

Let us apply theorem 0.6 (i) - (iii) of [4] with $G' = SU(n,1) \hookrightarrow G$, $K' := K \cap G' = S\left(U(n) \times U(1)\right)$ and

$$\Gamma' := \left\{ \gamma' \in G' \mid \text{ there exists } w \in Z\left(G'\right) \text{ such that } \gamma' w \in \Gamma \right\} \sqsubset G',$$

which is of course again a lattice such that $\Gamma' \backslash G'$ is not compact and so it is admissible in the sense of [4] by theorem 0.7 of [4]. By lemma 3.18 of [4] $g^{-1} \Gamma' g \cap N' \sqsubset N'$ is a lattice, and lemma 3.16 of [4] applied with any $\rho \in \Gamma' \cap N' \backslash \{1\}$ tells us that $\left(g^{-1} \Gamma' g \cap N\right) \backslash N$ is compact. So we see that there exist $t_0 \in \mathbb{R}$, $\eta \subset N$ open and relatively compact and $\Xi \subset G'$ finite such that for all $g \in \Xi$

$$\Gamma' \cap gNg^{-1} \sqsubset gNg^{-1}$$

is a lattice, $\Gamma' \Omega' = G'$ if we define $\Omega' = \bigcup_{b \in \Xi} b\eta A_{<t_0} K'$ and

$$\Delta := \left\{ \gamma' \in \Gamma' \mid \gamma' \Omega' \cap \Omega' \neq \varnothing \right\}$$

is finite.

(i) and (ii) : now trivial by definition of $\Gamma' \sqsubset G'$.

(iii) : Let $\gamma = \gamma' w \in \Gamma$, $\gamma' \in \Gamma'$, $w \in Z\left(G'\right)$, such that $\gamma \Omega \cap \Omega \neq \varnothing$. Then

$$\gamma' \Omega' Z\left(G'\right) \cap \Omega' Z\left(G'\right) \neq \varnothing.$$

Since $Z\left(G'\right) \cap G' \sqsubset K'$ and Ω' is right-K'-invariant we have $\gamma' \Omega' \cap \Omega' \neq \varnothing$ as well and therefore $\gamma' \in \Delta$. Conversely $\gamma' Z\left(G'\right)$ is compact and therefore $\Gamma \cap \gamma' Z\left(G'\right)$ is finite for all $\gamma' \in \Gamma'$. ∎

From the fundamental domain $\Omega := \bigcup_{g \in \Xi} g\eta A_{>t_0} K$ one can really deduce the position of the cusps of $\Gamma \backslash B$ in ∂B : they are up to the action of Γ on ∂B the limit points

$$\lim_{t \to +\infty} ga_t \mathbf{0} = g\mathbf{e}_1,$$

$g \in \Xi$, where the limits are taken with respect to the Euclidian metric on $\mathbb{C}^n$. Their number is bounded above by $|\Xi|$ and is therefore finite, as expected.

Corollary 3.3. *Let $t_0 \in \mathbb{R}$, $\eta \subset N$ and $\Xi \subset G$ be given by theorem 3.2. Let $h \in \mathcal{C}(\Gamma \backslash G, \mathbb{C})$ and $s \in]0, \infty]$. Then $h \in L^s(\Gamma \backslash G)$ if and only if $h(g\mathbf{w}) \in L^s(\eta A_{>t_0} K)$ for all $g \in \Xi$.*

Proof. If $s = \infty$ then it is evident since $G = \Gamma\Omega$ by theorem 3.2 (ii). Now assume $s \in]0, \infty[$ and $h \in L^s(\Gamma \backslash G)$.

$$S := |\{\gamma \in \Gamma \,|\, \gamma\Omega \cap \Omega \neq \emptyset\}| < \infty$$

by theorem 3.2 (iii). So for all $g \in \Xi$ we have

$$\int_{\eta A_{>t_0} K} |h(g\mathbf{w})|^s = \int_{g\eta A_{>t_0} K} |h|^s \leq \int_\Omega |h|^s \leq S \int_{\Gamma \backslash G} |h|^s < \infty.$$

Conversely assume $h(g\mathbf{w}) \in L^s(\eta A_{>t_0} K)$ for all $g \in \Xi$. Then since $G = \Gamma\Omega$ by theorem 3.2 (ii) we obtain

$$\int_{\Gamma \backslash G} |h|^s \leq \int_\Omega |h|^s \leq \sum_{g \in \Xi} \int_{\eta A_{>t_0} K} |h(g\mathbf{w})|^s < \infty. \qquad \blacksquare$$

Let $f \in sM_k(\Gamma)$ and $g \in \Xi$. Then we may decompose

$$f|_g|_{R^{-1}} = \sum_{I \in \wp(r)} q_I \vartheta^I \in \mathcal{O}(\mathcal{H}),$$

all $q_I \in \mathcal{O}(H)$, $I \in \wp(r)$, and by theorem 3.2 (i) we know that $g^{-1}\Gamma g \cap N'Z(G') \not\subset Z(G')$. So let $n \in g^{-1}\Gamma g \cap N'Z(G') \setminus Z(G')$,

$$RnR^{-1} = n'_{\lambda_0, \mathbf{0}} \left(\begin{array}{c|c} \varepsilon 1 & 0 \\ \hline 0 & E \end{array} \right),$$

$\lambda_0 \in \mathbb{R} \setminus \{0\}$, $\varepsilon \in U(1)$, $E \in U(r)$, $\varepsilon^{n+1} = \det E$.
$j(RnR^{-1}) := j(RnR^{-1}, \mathbf{w}) = \varepsilon^{-1} \in U(1)$ is independent of $\mathbf{w} \in H$. So there exists $\chi \in \mathbb{R}$ such that $j(RnR^{-1}) = e^{2\pi i \chi}$. Without loss of generality we can assume that E is diagonal, otherwise conjugate n with an appropriate element of $Z(G')$. So there exists $D \in \mathbb{R}^{r \times r}$ diagonal such that $E = \exp(2\pi i D)$. If

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix} \quad \text{and } I \in \wp(r) \text{ then we define } \operatorname{tr}_I D := \sum_{j \in I} d_j.$$

Theorem 3.4. *Fourier expansion of $f|_g|_{R^{-1}}$*
(i) *There exist unique $c_{I,m} \in \mathcal{O}(\mathbb{C}^{n-1})$, $I \in \wp(r)$, $m \in \frac{1}{\lambda_0}(\mathbb{Z} - \operatorname{tr}_I D - (k + |I|)\chi)$, such that*

$$q_I(\mathbf{w}) = \sum_{m \in \frac{1}{\lambda_0}(\mathbb{Z} - \operatorname{tr}_I D - (k+|I|)\chi)} c_{I,m}(\mathbf{w}_2) e^{2\pi m w_1}$$

for all $\mathbf{w} \in H$ and $I \in \wp(r)$, and so

$$f|_g|_{R^{-1}}(\mathbf{w}) = \sum_{I \in \wp(r)} \sum_{m \in \frac{1}{\lambda_0}(\mathbb{Z} - \operatorname{tr}_I D - (k+|I|)\chi)} c_{I,m}(\mathbf{w}_2) e^{2\pi m w_1} \vartheta^I$$

for all $\mathbf{w} = \left(\dfrac{w_1}{\mathbf{w}_2} \right) \begin{array}{l} \leftarrow 1 \\ \}n-1 \end{array} \in H$, where the convergence is absolute and compact.

(ii) $c_{I,m} = 0$ *for all* $I \in \wp(r)$ *and* $m > 0$, *and if*
$\text{tr}_I D + (k + |I|) \chi \equiv 0 \mod \mathbb{Z}$ *in the group* $(\mathbb{R}, +)$ *then* $c_{I,0}$ *is a constant.*

> This is the super analog for Köcher's principle. (See section 11.5 of [1].) The condition $m > 0$ instead of $m < 0$ in definition 1.2 comes from the fact that $\mathcal{H}$ is a generalized right half plane instead of the upper half plane.

(iii) *Let* $I \in \wp(r)$ *and* $s \in [1, \infty]$. *If* $\text{tr}_I D + (k + |I|) \chi \not\equiv 0 \mod \mathbb{Z}$ *then*

$$q_I \Delta' (\mathbf{w}, \mathbf{w})^{\frac{k+|I|}{2}} \in L^s (R\eta A_{>t_0} \mathbf{0})$$

with respect to the RGR^{-1} *-invariant measure* $\Delta' (\mathbf{w}, \mathbf{w})^{-(n+1)} dV_{\text{Leb}}$ *on* H. *If* $\text{tr}_I D + (k + |I|) \chi \equiv 0 \mod \mathbb{Z}$ *and* $k \geq 2n - |I|$ *then*

$$q_I \Delta' (\mathbf{w}, \mathbf{w})^{\frac{k+|I|}{2}} \in L^s (R\eta A_{>t_0} \mathbf{0})$$

with respect to the RGR^{-1} *-invariant measure on* H *if and only if* $c_{I,0} = 0$.

Proof. (i) $f|_g$ is $g^{-1} \Gamma g$ invariant, so we see that for all $\mathbf{w} \in H$

$$
\begin{aligned}
\sum_{I \in \wp(r)} q_I (\mathbf{w}) \vartheta^I &= f|_g|_{R^{-1}} (\mathbf{w}) \\
&= f|_g|_n|_{R^{-1}} (\mathbf{w}) \\
&= \sum_{I \in \wp(r)} q_I (\mathbf{w} + i\lambda_0 \mathbf{e}_1) \left(E\vartheta j \left(RnR^{-1} \right) \right)^I j \left(RnR^{-1} \right)^k \\
&= \sum_{I \in \wp(r)} q_I (\mathbf{w} + i\lambda_0 \mathbf{e}_1) e^{2\pi i (\text{tr}_I D + (k+|I|)\chi)} \vartheta^I .
\end{aligned}
$$

Therefore for all $\mathbf{w} \in H$ and $I \in \wp(r)$

$$q_I (\mathbf{w}) = q_I (\mathbf{w} + i\lambda_0 \mathbf{e}_1) e^{2\pi i (\text{tr}_I D + (k+|I|)\chi)} .$$

Let $I \in \wp(r)$. Then $h \in \mathcal{O}(H)$ given by

$$h (\mathbf{w}) := q_I (\mathbf{w}) e^{-2\pi i \frac{1}{\lambda_0} (\text{tr}_I D + (k+|I|)\chi) w_1}$$

for all $\mathbf{w} \in H$ is $i\lambda_0 \mathbf{e}_1$ periodic, and therefore there exists $\widehat{h}$ holomorphic on

$$\widehat{H} := \left\{ \mathbf{z} = \begin{pmatrix} z_1 \\ \mathbf{z}_2 \end{pmatrix} \begin{matrix} \leftarrow 1 \\ \}n-1 \end{matrix} \ \middle| \ |z_1| > e^{\frac{\pi}{\lambda_0} \mathbf{z}_2^* \mathbf{z}_2} \right\}$$

such that for all $\mathbf{w} \in H$

$$h (\mathbf{w}) = \widehat{h} \left(\frac{e^{\frac{2\pi}{\lambda} w_1}}{\mathbf{w}_2} \right) .$$

Laurent expansion now tells us that there exist $a_{m',\mathbf{l}} \in \mathbb{C}$, $m' \in \mathbb{Z}$, $\mathbf{l} \in \mathbb{N}^{n-1}$, such that

$$\widehat{h} (\mathbf{z}) = \sum_{m' \in \mathbb{Z}} \sum_{\mathbf{l} \in \mathbb{N}^{n-1}} a_{m',\mathbf{l}} z_1^{m'} \mathbf{z}_2^{\mathbf{l}}$$

for all $\mathbf{z} = \begin{pmatrix} z_1 \\ \mathbf{z}_2 \end{pmatrix} \begin{matrix} \leftarrow 1 \\ \}n-1 \end{matrix} \in \widehat{H}$, where the convergence is absolute and compact. Now let us define $d_{m'} \in \mathcal{O}(\mathbb{C}^{n-1})$ as

$$d_{m'}(\mathbf{z}) := \sum_{\mathbf{l} \in \mathbf{N}^{n-1}} a_{m',\mathbf{l}} \mathbf{z}_2^{\mathbf{l}},$$

$m' \in \mathbb{Z}$. Then for all $\mathbf{w} \in H$

$$q_I(\mathbf{w}) e^{-\frac{2\pi i}{\lambda_0}(\operatorname{tr}_I D + (k+|I|)\chi)w_1} = h(\mathbf{w}) = \sum_{m' \in \mathbb{Z}} d_{m'}(\mathbf{w}_2) e^{\frac{2\pi}{\lambda_0} m' w_1}.$$

So taking $c_m := d_{\lambda_0 m + \operatorname{tr}_I D + (k+|I|)\chi}$, $m \in \frac{1}{\lambda_0}(\mathbb{Z} - \operatorname{tr}_I D - (k+|I|)\chi)$, gives the desired result. Uniqueness follows from standard Fourier theory.

(ii) *Step I Show that all q_I, $I \in \wp(r)$, are bounded on*
$RN\mathbf{0} = \{\mathbf{w} \in H \mid \Delta'(\mathbf{w},\mathbf{w}) = 2\}$.

Obviously all q_I, $I \in \wp(r)$, are bounded on $R\eta\mathbf{0}$ since $R\eta\mathbf{0}$ lies relatively compact in H. Let $C \geq 0$ such that $|q_I| \leq C$ on $R\eta\mathbf{0}$ for all $I \in \wp(r)$. By theorem 3.2

$$RN\mathbf{0} = R\left(g^{-1}\Gamma g \cap NZ(G')\right) \eta\mathbf{0}.$$

So let $Rn'R^{-1} = n'_{\lambda',\mathbf{u}} \left(\begin{array}{c|c} \varepsilon'1 & 0 \\ \hline 0 & E' \end{array} \right) \in g^{-1}\Gamma g \cap NZ(G')$, $\lambda' \in \mathbb{R}$, $\mathbf{u} \in \mathbb{C}^{n-1}$, $\varepsilon' \in U(1)$ and $E' \in (r)$. Then again

$$j\left(Rn'R^{-1}\right) := j\left(Rn'R^{-1},\mathbf{w}\right) = \varepsilon'^{-1} \in U(1)$$

is independent of $\mathbf{w} \in H$. Now if we use that $f \in sM_k(\Gamma)$ we get

$$\sum_{I \in \wp} q_I \vartheta^I \;\; = \;\; f|_g|_{R^{-1}} = f|_g|_{n'}|_{R^{-1}} = \sum_{I \in \wp(r)} q_I\left(Rn'R^{-1}\mathbf{w}\right)(E'\vartheta)^I \varepsilon'^{k+|I|}.$$

$\bigwedge(\mathbb{C}^r) \to \bigwedge(\mathbb{C}^r)$, $\vartheta^I \mapsto (E'\vartheta)^I \varepsilon'^{k+|I|}$ is unitary, therefore

$$|q_I| \leq 2^r \left| q_I\left(Rn'R^{-1}\mathbf{w}\right) \right|.$$

We see that $|q_I| \leq 2^r C$ on the whole $RN\mathbf{0}$.

Step II Show that

$$\left| c_{I,m}(\mathbf{w}_2) e^{2\pi m w_1} \right| \leq ||q_I||_{\infty,RN\mathbf{0}}$$

on $RN\mathbf{0}$ for all $I \in \wp(r)$ and $m \in \frac{1}{\lambda_0}(\mathbb{Z} - \operatorname{tr}_I D - (k+|I|)\chi)$.

Let $I \in \wp(r)$ and $m \in \frac{1}{\lambda_0}(\mathbb{Z} - \operatorname{tr}_I D - (k+|I|)\chi)$. By classical Fourier analysis

$$c_{I,m}(\mathbf{w}_2) e^{2\pi m w_1} = \frac{1}{\lambda_0} \int_0^{\lambda_0} q_I(\mathbf{w} + i\lambda\mathbf{e}_1) e^{-2\pi i m\lambda} d\lambda$$

for all $\mathbf{w} \in H$, and since $\mathbf{w} + i\lambda\mathbf{e}_1 = n'_{\lambda,\mathbf{0}}\mathbf{w} \in RNR^{-1}\mathbf{w}$ the claim follows.

Step III Conclusion.

Let $I \in \wp(r)$ and $m \in \frac{1}{\lambda_0}\left(\mathbb{Z} - \text{tr}_I D - (k + |I|)\chi\right)$. Let $\mathbf{u} \in \mathbb{C}^{n-1}$ be arbitrary. Then

$$\begin{pmatrix} \frac{1 + \frac{1}{2}\mathbf{u}^*\mathbf{u}}{\mathbf{u}} \end{pmatrix} \in RN\mathbf{0},$$

and so

$$|c_{I,m}(\mathbf{u})| \leq \|q_I\|_{\infty, RN\mathbf{0}}\, e^{-\pi m \mathbf{u}^*\mathbf{u}}.$$

Now the assertion follows by Liouville's theorem, *where $n \geq 2$ is of course essential.*

(iii) Let

$$\eta' := \left\{ (iy, \mathbf{u}) \in i\mathbb{R} \oplus \mathbb{C}^{n-1} \,\middle|\, \begin{pmatrix} \frac{1 + \frac{1}{2}\mathbf{u}^*\mathbf{u} + iy}{\mathbf{u}} \end{pmatrix} \in R\eta\mathbf{0} \right\}$$

be the projection of $R\eta\mathbf{0}$ onto $i\mathbb{R} \oplus \mathbb{C}^{n-1}$ in direction of $\text{Re } w_1 \in \mathbb{R}$. Then

$$\Psi : \mathbb{R}_{>e^{2t_0}} \times \eta' \to R\eta A_{>t_0}\mathbf{0}, \ (x, iy, \mathbf{u}) \mapsto \begin{pmatrix} \frac{x + \frac{1}{2}\mathbf{u}^*\mathbf{u} + iy}{\mathbf{u}} \end{pmatrix}$$

is a $\mathcal{C}^\infty$-diffeomorphism with determinant 1, and

$$\Delta'\left(\Psi(x, iy, \mathbf{u}), \Psi(x, iy, \mathbf{u})\right) = 2x$$

for all $(x, iy, \mathbf{u}) \in \mathbb{R}_{>e^{2t_0}} \times \eta'$. So

$$q_I \Delta'(\mathbf{w}, \mathbf{w})^{\frac{k+|I|}{2}} \in L^s\left(R\eta A_{>t_0}\mathbf{0}\right)$$

with respect to the measure $\Delta'(\mathbf{w}, \mathbf{w})^{-(n+1)}\, dV_{\text{Leb}}$ if and only if

$$(q_I \circ \Psi)\, x^{\frac{k+|I|}{2}} \in L^s\left(\mathbb{R}_{>e^{2t_0}} \times \eta'\right)$$

with respect to the measure $x^{-(n+1)} dV_{\text{Leb}}$.

Now assume either $\text{tr}_I D + (k + |I|)\chi \not\equiv 0 \mod \mathbb{Z}$ or $\text{tr}_I D + (k + |I|)\chi \equiv 0 \mod \mathbb{Z}$ and $c_{I,0} = 0$. Then in both cases by (ii) we can write

$$q_I(\mathbf{w}) = \sum_{m \in \frac{1}{\lambda_0}(\mathbb{Z} - \text{tr}_I D - (k+|I|)\chi) \cap \mathbb{R}_{<0}} c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1}$$

for all $\mathbf{w} \in H$, where the sum converges absolutely and uniformly on compact subsets of H. Let us define

$$M_0 := \max \frac{1}{\lambda_0}\left(\mathbb{Z} - \text{tr}_I D - (k + |I|)\chi\right) \cap \mathbb{R}_{<0} < 0.$$

Then since $R\eta a_{t_0}\mathbf{0} \subset H$ is relatively compact and the Fourier expansion in (i) has compact convergence we can define

$$C'' := e^{-2\pi M_0 e^{2t_0}} \sum_{m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi) \cap \mathbb{R}_{<0}} \left\|c_{I,m}\left(\mathbf{w}_2\right) e^{2\pi m w_1}\right\|_{\infty, R\eta a_{t_0}\mathbf{0}}$$
$$< \infty.$$

So we see that
$$\left|q_I\left(\mathbf{w}\right)\right| \leq C'' e^{\pi M_0 \Delta'(\mathbf{w},\mathbf{w})}$$

for all $\mathbf{w} \in R\eta A_{>t_0}\mathbf{0}$,
$$\left|q_I \circ \Psi\right| \leq C'' e^{2\pi M_0 x},$$

and so $x^{\frac{k+|I|}{2}}\left(q_I \circ \Psi\right) \in L^s\left(\mathbb{R}_{>e^{2t_0}} \times \eta'\right)$ with respect to the measure $x^{-(n+1)}dV_{\mathrm{Leb}}$.

Conversely assume $\mathrm{tr}_I D + (k+|I|)\chi \equiv 0 \mod \mathbb{Z}$, $k \geq 2n - |I|$ and $c_{I,0} \neq 0$. Then as before we have the estimate

$$\left|\sum_{m \in \frac{1}{\lambda_0}\mathbb{Z}_{<0}} c_{I,m}\left(\mathbf{w}_2\right) e^{2\pi m w_1}\right| \leq C'' e^{-\pi \Delta'(\mathbf{w},\mathbf{w})}$$

for all $\mathbf{w} \in R\eta A_{>t_0}\mathbf{0}$ if we define

$$C'' := e^{2\pi e^{2t_0}} \sum_{m \in \frac{1}{\lambda_0}\mathbb{Z}_{<0}} \left\|c_{I,m}\left(\mathbf{w}_2\right) e^{2\pi m w_1}\right\|_{\infty, R\eta a_{t_0}\mathbf{0}} < \infty.$$

Therefore there exists $S \geq 0$ such that

$$\left|\sum_{m \in \frac{1}{\lambda_0}\mathbb{Z}_{<0}} c_{I,m}\left(\mathbf{w}_2\right) e^{2\pi m w_1}\right| \leq \frac{1}{2}\left|c_{I,0}\right|,$$

and so $\left|q_I\left(\mathbf{w}\right)\right| \geq \frac{1}{2}\left|c_{I,0}\right|$ for all $\mathbf{w} \in R\eta A_{>t_0}\mathbf{0}$ having $\Delta'\left(\mathbf{w},\mathbf{w}\right) \geq S$. So $\left|\left(q_I \circ \Phi\right)\left(x, iy, \mathbf{u}\right)\right| \geq \frac{1}{2}\left|c_{I0}\right|$ for all $\left(x, iy, \mathbf{u}\right) \in \mathbb{R}_{\geq S} \times \eta'$, and so definitely $x^{\frac{k+|I|}{2}}\left(q_I \circ \Phi\right) \notin L^s\left(\mathbb{R}_{>e^{2t_0}} \times \eta'\right)$ with respect to the measure $x^{-(n+1)}dV_{\mathrm{Leb}}$. ∎

Now we prove theorem 3.1:

Proof. Let $\rho \in \{0, \ldots, r\}$ and $k \geq 2n - \rho$. Since $\mathrm{vol}\,\Gamma\backslash G < \infty$ it suffices to show that $f \in sM_k^{(\rho)}(\Gamma)$ and $\widetilde{f} \in L^1\left(\Gamma\backslash G\right) \otimes \mathcal{O}\left(\mathbb{C}^{0|r}\right)$ imply $\widetilde{f} \in L^\infty\left(\Gamma\backslash G\right) \otimes \mathcal{O}\left(\mathbb{C}^{0|r}\right)$. So let $f \in sM_k^{(\rho)}(\Gamma)$ such that $\widetilde{f} \in L^1\left(\Gamma\backslash G\right) \otimes \mathcal{O}\left(\mathbb{C}^{0|r}\right)$. Let $g \in \Xi$. By corollary 3.3 it is even enough to show that $l_g\left(\widetilde{f}\right) \in L^\infty\left(\eta A_{>t_0}K\right) \otimes \mathcal{O}\left(\mathbb{C}^{0|r}\right)$, where $l_g\left(\widetilde{f}\right)$ again denotes the left translation of $\widetilde{f}$ by the group element $g \in G$. Let

$$f|_g|_{R^{-1}} = \sum_{I \in \wp(r), |I|=\rho} q_I \vartheta^I,$$

all $q_I \in \mathcal{O}(H)$, $I \in \wp(r)$, $|I| = \rho$. Then

$$f|_g = \sum_{I \in \wp(r),\, |I|=\rho} q_I\, (R\mathbf{w})\, \zeta^I j\, (R, \mathbf{w})^{k+\rho} \ .$$

Since by corollary 3.3 $\widetilde{f} \in L^1\left(\eta A_{>t_0} K\right) \otimes \mathcal{O}\left(\mathbb{C}^{0|r}\right)$ we conclude that

$$q_I\, (R\mathbf{z})\, j\, (R, \mathbf{z})^{k+\rho}\, \Delta\, (\mathbf{z}, \mathbf{z})^{\frac{k+\rho}{2}} \in L^1\left(\eta A_{>t_0}\mathbf{0}\right)$$

with respect to the G-invariant measure on B or equivalently $q_I \Delta'\, (\mathbf{w}, \mathbf{w})^{\frac{k+\rho}{2}} \in L^1\left(R\eta A_{>t_0}\mathbf{0}\right)$ for all $I \in \wp(r)$, $|I| = \rho$, with respect to the RGR^{-1} -invariant measure on H. So by theorem 3.4 (iii) we see that $q_I \Delta'\, (\mathbf{w}, \mathbf{w})^{\frac{k+\rho}{2}} \in L^\infty\left(R\eta A_{>t_0}\mathbf{0}\right)$ as well, or equivalently $q_I\, (R\mathbf{z})\, j\, (R, \mathbf{z})^{k+\rho}\, \Delta\, (\mathbf{z}, \mathbf{z})^{\frac{k+\rho}{2}} \in L^\infty\left(\eta A_{>t_0}\mathbf{0}\right)$ for all $I \in \wp(r)$, $|I| = \rho$. Therefore

$$l_g\left(\widetilde{f}\right) \in L^\infty\left(\eta A_{>t_0} K\right) \otimes \mathcal{O}\left(\mathbb{C}^{0|r}\right) \ . \qquad \blacksquare$$

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