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Lie Algebras of Hamiltonian Vector Fields and Symplectic Manifolds

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Abstract. We construct a local characteristic map to a symplectic manifold M via certain cohomology groups of Hamiltonian vector fields. For each $p \in M$, the Leibniz cohomology of the Hamiltonian vector fields on $\mathbf{R}^{2n}$ maps to the Leibniz cohomology of all Hamiltonian vector fields on M. For a particular extension $\mathfrak{g}_n$ of the symplectic Lie algebra, the Leibniz cohomology of $\mathfrak{g}_n$ is shown to be an exterior algebra on the canonical symplectic two-form. The Leibniz cohomology of this extension is then a direct summand of the Leibniz cohomology of all Hamiltonian vector fields on $\mathbf{R}^{2n}$.
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1. Introduction

We construct a local characteristic map to a symplectic manifold M via certain cohomology groups of Hamiltonian vector fields. Recall that the group of affine symplectomorphisms, i.e., the affine symplectic group ASp_n, is given by all transformations $\psi : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ of the form

$$\psi(z) = z_0 + Az,$$

where A is a $2n\times 2n$ symplectic matrix and z_0 a fixed element of $\mathbf{R}^{2n}$ [7, p. 55]. Let $\mathfrak{g}_n$ denote the Lie algebra of ASp_n, referred to as the affine symplectic Lie algebra. Then $\mathfrak{g}_n$ is the largest finite dimensional Lie subalgebra of the Hamiltonian vector fields on $\mathbf{R}^{2n}$, and serves as our point of departure for calculations. Particular attention is devoted to the Leibniz homology of $\mathfrak{g}_n$, i.e., $HL_*(\mathfrak{g}_n; \mathbf{R})$, and proven is that

$$HL_*(\mathfrak{g}_n; \mathbf{R}) \simeq \Lambda^*(\omega_n),$$

where $\omega_n = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i}$ and Λ^* denotes the exterior algebra. Dually, for cohomology,

$$HL^*(\mathfrak{g}_n; \mathbf{R}) \simeq \Lambda^*(\omega_n^*),$$

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where $\omega_n^* = \sum_{i=1}^n dx^i \wedge dy^i$.

For $p \in M$, the local characteristic map factors through

$$\iota^* \circ \rho_p : HL^*(\mathcal{X}_H(\mathbf{R}^{2n});\, \mathbf{R}) \to HL^*(\mathcal{X}_H(M);\, C^\infty(M)),$$

where $\mathcal{X}_H$ denotes the Lie algebra of Hamiltonian vector fields, and $C^\infty(M)$ is the ring of C^∞ real-valued functions on M. The maps ι^* and ρ_p are defined in §5. Using previous work of the author [6], there is a natural map

$$H^*_{dR}(M;\, \mathbf{R}) \to HL^*(\mathcal{X}(M);\, C^\infty(M)),$$

where H^*_{dR} denotes deRham cohomology. Composing with

$$HL^*(\mathcal{X}(M);\, C^\infty(M)) \to HL^*(\mathcal{X}_H(M);\, C^\infty(M)),$$

we have

$$H^*_{dR}(M;\, \mathbf{R}) \to HL^*(\mathcal{X}_H(M);\, C^\infty(M)).$$

The local characteristic map acquires the form

$$\begin{array}{ccc} & & \Lambda^*(\omega_n^*) \simeq HL^*(\mathfrak{g}_n;\, \mathbf{R}) \\ & & \big\downarrow \mu_p \\ H^*_{dR}(M;\, \mathbf{R}) & \longrightarrow & HL^*(\mathcal{X}_H(M);\, C^\infty(M)) \end{array}$$

for each $p \in M$.

The calculational tools for $HL_*(\mathfrak{g}_n)$ include the Hochschild-Serre spectral sequence for Lie-algebra (co)homology, the Pirashvili spectral sequence for Leibniz homology, and the identification of certain symplectic invariants of $\mathfrak{g}_n$ which appear in the appendix.

2. The Affine Symplectic Lie Algebra

As a point of departure, consider a C^∞ Hamiltonian function $H : \mathbf{R}^{2n} \to \mathbf{R}$ with the associated Hamiltonian vector field

$$X_H = \sum_{i=1}^n \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y^i} - \sum_{i=1}^n \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x^i},$$

where $\mathbf{R}^{2n}$ is given coordinates

$$(x_1,\, x_2,\, \ldots,\, x_n,\, y_1,\, y_2,\, \ldots,\, y_n),$$

and $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial y^i}$ are the unit vector fields parallel to the x_i and y_i axes respectively. The vector field X_H is then tangent to the level curves (or hyper-surfaces) of H. Restricting H to a quadratic function (with no linear terms) in

$$\{x_1,\, x_2,\, \ldots,\, x_n,\, y_1,\, y_2,\, \ldots,\, y_n\},$$

yields a family of vector fields isomorphic to the real symplectic Lie algebra $\mathfrak{sp}_n$. An $\mathbf{R}$-vector space basis, $\mathcal{B}_1$, for $\mathfrak{sp}_n$ is given by the families:

(1) $x_k \frac{\partial}{\partial y^k}$, $k = 1, 2, 3, \ldots, n$,

(2) $y_k \frac{\partial}{\partial x^k}$, $k = 1, 2, 3, \ldots, n$,

(3) $x_i \frac{\partial}{\partial y^j} + x_j \frac{\partial}{\partial y^i}$, $1 \le i < j \le n$,

(4) $y_i \frac{\partial}{\partial x^j} + y_j \frac{\partial}{\partial x^i}$, $1 \le i < j \le n$,

(5) $y_j \frac{\partial}{\partial y^i} - x_i \frac{\partial}{\partial x^j}$, $i = 1, 2, 3, \ldots, n$, $j = 1, 2, 3, \ldots, n$.

It follows that $\dim_{\mathbf{R}}(\mathfrak{sp}_n) = 2n^2 + n$.

Let I_n denote the abelian Lie algebra of Hamiltonian vector fields arising from the linear (affine) functions $H : \mathbf{R}^{2n} \to \mathbf{R}$. Then I_n has an $\mathbf{R}$-vector space basis given by

$$\mathcal{B}_2 = \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \ldots, \frac{\partial}{\partial y^n} \right\}.$$

The affine symplectic Lie algebra, $\mathfrak{g}_n$, has an $\mathbf{R}$-vector space basis $\mathcal{B}_1 \cup \mathcal{B}_2$. There is a short exact sequence of Lie algebras

$$0 \longrightarrow I_n \xrightarrow{\ i\ } \mathfrak{g}_n \xrightarrow{\ \pi\ } \mathfrak{sp}_n \longrightarrow 0,$$

where i is the inclusion map and π is the projection

$$\mathfrak{g}_n \to (\mathfrak{g}_n / I_n) \simeq \mathfrak{sp}_n.$$

In fact, I_n is an abelian ideal of $\mathfrak{g}_n$ with I_n acting on $\mathfrak{g}_n$ via the bracket of vector fields.

Let $\mathfrak{H}_n$ denote the Lie algebra of formal Hamiltonian vector fields

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y^i} - \sum_{i=1}^{n} \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x^i},$$

where $H \in R = \mathbf{R}[[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]]$. As usual, endow $\mathfrak{H}_n$ with the $\mathcal{M}$-adic topology, where $\mathcal{M}$ is the maximal ideal of R generated by $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Let $H^*_{\mathrm{Lie}}(\mathfrak{H}_n; \mathbf{R})$ and $HL^*(\mathfrak{H}_n; \mathbf{R})$ denote continuous Lie-algebra and continuous Leibniz cohomology respectively, computed using continuous cochains. For any $H \in C^\infty(\mathbf{R}^{2n})$, the Taylor series expansion of H about the origin induces a morphism of Lie algebras

$$T : \mathcal{X}_H(\mathbf{R}^{2n}) \to \mathfrak{H}_n,$$

as well as maps on cohomology

$$T^* : H^*_{\mathrm{Lie}}(\mathfrak{H}_n; \mathbf{R}) \to H^*_{\mathrm{Lie}}(\mathcal{X}_H(\mathbf{R}^{2n}); \mathbf{R}),$$
$$T^* : HL^*(\mathfrak{H}_n; \mathbf{R}) \to HL^*(\mathcal{X}_H(\mathbf{R}^{2n}); \mathbf{R}),$$

where $\mathcal{X}_H(\mathbf{R}^{2n})$ is given the strong C^∞-topology. See [1] and [6] for further properties of T^*. Also, let $\mathfrak{H}_n^{\mathrm{Poly}}$ denote the Lie algebra of polynomial Hamiltonian vector fields on $\mathbf{R}^{2n}$. For continuous cohomology, we have

$$H^*_{\mathrm{Lie}}(\mathfrak{H}_n; \mathbf{R}) \simeq \mathrm{Hom}(H_*^{\mathrm{Lie}}(\mathfrak{H}_n^{\mathrm{Poly}}; \mathbf{R}), \mathbf{R}),$$
$$HL^*(\mathfrak{H}_n; \mathbf{R}) \simeq \mathrm{Hom}(HL_*(\mathfrak{H}_n^{\mathrm{Poly}}; \mathbf{R}), \mathbf{R}).$$

Also, there are natural inclusions of Lie algebras $\mathfrak{g}_n \hookrightarrow \mathfrak{H}_n^{\mathrm{Poly}} \hookrightarrow \mathfrak{H}_n$.

3. The Lie Algebra Homology of $\mathfrak{g}_n$

For any Lie algebra $\mathfrak{g}$ over a ring k, the Lie algebra homology of $\mathfrak{g}$, written $H_*^{\mathrm{Lie}}(\mathfrak{g};\, k)$, is the homology of the chain complex $\Lambda^*(\mathfrak{g})$, namely

$$k \xleftarrow{0} \mathfrak{g} \xleftarrow{[\ ,\ ]} \mathfrak{g}^{\wedge 2} \longleftarrow \dots \longleftarrow \mathfrak{g}^{\wedge(n-1)} \xleftarrow{d} \mathfrak{g}^{\wedge n} \longleftarrow \dots,$$

where

$$d(g_1 \wedge g_2 \wedge \dots \wedge g_n) = \sum_{1\le i<j\le n} (-1)^j\, (g_1 \wedge \dots \wedge g_{i-1} \wedge [g_i,\, g_j] \wedge g_{i+1} \wedge \dots \hat{g}_j \dots \wedge g_n).$$

For actual calculations in this paper, $k = \mathbf{R}$. Additionally, Lie algebra homology with coefficients in the adjoint representation, written $H_*^{\mathrm{Lie}}(\mathfrak{g};\, \mathfrak{g})$, is the homology of the chain complex $\mathfrak{g} \otimes \Lambda^*(\mathfrak{g})$, i.e.,

$$\mathfrak{g} \longleftarrow \mathfrak{g}\otimes\mathfrak{g} \longleftarrow \mathfrak{g}\otimes\mathfrak{g}^{\wedge 2} \longleftarrow \dots \longleftarrow \mathfrak{g}\otimes\mathfrak{g}^{\wedge(n-1)} \xleftarrow{d} \mathfrak{g}\otimes\mathfrak{g}^{\wedge n} \longleftarrow \dots,$$

where

$$d(g_1 \otimes g_2 \wedge g_3 \dots \wedge g_{n+1}) = \sum_{i=2}^{n+1} (-1)^i\, ([g_1,\, g_i] \otimes g_2 \wedge \dots \hat{g}_i \dots \wedge g_{n+1})$$
$$+ \sum_{2\le i<j\le n+1} (-1)^j\, (g_1 \otimes g_2 \wedge \dots \wedge g_{i-1} \wedge [g_i,\, g_j] \wedge g_{i+1} \wedge \dots \hat{g}_j \dots \wedge g_{n+1}).$$

The canonical projection $\mathfrak{g}\otimes\Lambda^*(\mathfrak{g}) \to \Lambda^{*+1}(\mathfrak{g})$ given by $\mathfrak{g}\otimes\mathfrak{g}^{\wedge n} \to \mathfrak{g}^{\wedge(n+1)}$ is a map of chain complexes and induces a k-linear map on homology

$$H_n^{\mathrm{Lie}}(\mathfrak{g};\, \mathfrak{g}) \to H_{n+1}^{\mathrm{Lie}}(\mathfrak{g};\, k).$$

Given a (right) $\mathfrak{g}$-module M, the module of invariants $M^{\mathfrak{g}}$ is defined as

$$M^{\mathfrak{g}} = \{m \in M \mid [m,\, g] = 0 \ \ \forall g \in \mathfrak{g}\}.$$

Note that $\mathfrak{sp}_n$ acts on I_n and on the affine symplectic Lie algebra $\mathfrak{g}_n$ via the bracket of vector fields. The action is extended to $I_n^{\wedge k}$ by

$$[\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k,\, X] = \sum_{i=1}^{k} \alpha_1 \wedge \alpha_2 \wedge \dots \wedge [\alpha_i,\, X] \wedge \dots \wedge \alpha_k$$

for $\alpha_i \in I_n$, $X \in \mathfrak{sp}_n$, and similarly for the $\mathfrak{sp}_n$ action on $\mathfrak{g}_n \otimes I_n^{\wedge k}$. The main result of this section is the following.

Lemma 3.1. *There are natural vector space isomorphisms*

$$H_*^{Lie}(\mathfrak{g}_n;\, \mathbf{R}) \simeq H_*^{Lie}(\mathfrak{sp}_n;\, \mathbf{R}) \otimes [\Lambda^*(I_n)]^{\mathfrak{sp}_n}$$
$$H_*^{Lie}(\mathfrak{g}_n;\, \mathfrak{g}_n) \simeq H_*^{Lie}(\mathfrak{sp}_n;\, \mathbf{R}) \otimes [\mathfrak{g}_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n}.$$

Proof. The lemma follows essentially from the Hochschild-Serre spectral sequence [3], the application of which we briefly outline to aid in the identification of representative homology cycles, and to reconcile the lemma with its cohomological version in [3]. Consider the filtration $\mathcal{F}_m$, $m \geq -1$, of the complex $\Lambda^*(\mathfrak{g}_n)$ given by the $\mathbf{R}$-vector spaces:

$$\mathcal{F}_{-1} = \{0\},$$
$$\mathcal{F}_0 = \Lambda^*(I_n), \quad \mathcal{F}_0^k = I_n^{\wedge k}, \quad k = 0,\ 1,\ 2,\ 3,\ \dots,$$
$$\mathcal{F}_m^k = \text{Span of } \{g_1 \wedge \dots \wedge g_{k+m} \in \mathfrak{g}_n^{\wedge(k+m)} \mid \text{at most } m\text{-many } g_i\text{'s} \notin I_n\}.$$

Then each $\mathcal{F}_m$ is a chain complex, and $\mathcal{F}_m$ is a subcomplex of $\mathcal{F}_{m+1}$. For $m \geq 0$, we have

$$E^0_{m,\,k} = \mathcal{F}_m^k / \mathcal{F}_{m-1}^{k+1} \simeq I_n^{\wedge k} \otimes (\mathfrak{g}_n/I_n)^{\wedge m}.$$

Since I_n is abelian and the action of I_n on $\mathfrak{g}_n/I_n$ is trivial, it follows that

$$E^1_{m,\,k} \simeq I_n^{\wedge k} \otimes (\mathfrak{g}_n/I_n)^{\wedge m}.$$

Using the isomorphism $\mathfrak{g}_n/I_n \simeq \mathfrak{sp}_n$, we have

$$E^2_{m,\,k} \simeq H_m(\mathfrak{sp}_n;\ I_n^{\wedge k}).$$

Now, $\mathfrak{sp}_n$ is a simple Lie algebra and as an $\mathfrak{sp}_n$-module

$$I_n^{\wedge k} \simeq (I_n^{\wedge k})^{\mathfrak{sp}_n} \oplus M,$$

where $M \simeq M_1 \oplus M_2 \oplus \dots \oplus M_t$ is a direct sum of simple modules on which $\mathfrak{sp}_n$ acts non-trivially. Hence

$$H_*(\mathfrak{sp}_n;\ I_n^{\wedge k}) \simeq H_*(\mathfrak{sp}_n;\ (I_n^{\wedge k})^{\mathfrak{sp}_n}) \oplus H_*(\mathfrak{sp}_n;\ M).$$

Clearly,

$$H_*(\mathfrak{sp}_n;\ (I_n^{\wedge k})^{\mathfrak{sp}_n}) \simeq H_*(\mathfrak{sp}_n;\ \mathbf{R}) \otimes (I_n^{\wedge k})^{\mathfrak{sp}_n}$$
$$H_*(\mathfrak{sp}_n;\ M) \simeq \sum_{i=1}^{t} H_*(\mathfrak{sp}_n;\ M_i) \simeq 0,$$

where the latter isomorphism holds since each M_i is simple with non-trivial $\mathfrak{sp}_n$ action. See [2, Prop. VII.5.6] for more details.

Let θ be a cycle in $\Lambda^m(\mathfrak{sp}_n)$ representing an element of $H_m(\mathfrak{sp}_n;\ \mathbf{R})$, and let $z \in (I_n^{\wedge k})^{\mathfrak{sp}_n}$. Then $z \wedge \theta \in \mathfrak{g}_n^{\wedge(m+k)}$ represents an absolute cycle in $\Lambda^*(\mathfrak{g}_n)$, since, if θ is a sum of elements of the form $s_1 \wedge s_2 \wedge \dots \wedge s_m$, then $[z,\ s_i] = 0$ for each $s_i \in \mathfrak{sp}_n$. Thus, $E^2_{m,\,k} \simeq E^\infty_{m,\,k}$, and

$$H_*(\mathfrak{g}_n;\ \mathbf{R}) \simeq H_*(\mathfrak{sp}_n;\ \mathbf{R}) \otimes [\Lambda^*(I_n)]^{\mathfrak{sp}_n}.$$

By an application of the Hochschild-Serre spectral sequence to the subalgebra $\mathfrak{sp}_n$ of $\mathfrak{g}_n$, we have

$$H_*(\mathfrak{g}_n;\ \mathfrak{g}_n) \simeq H_*(\mathfrak{sp}_n;\ \mathbf{R}) \otimes [\mathfrak{g}_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n}. \qquad \blacksquare$$

Let $\omega_n = \sum_{i=1}^n \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i} \in I_n^{\wedge 2}$. One checks that $\omega_n \in (I_n^{\wedge 2})^{\mathfrak{sp}_n}$ against the basis for $\mathfrak{sp}_n$ given in §2. It follows that

$$\omega_n^{\wedge k} \in [I_n^{\wedge 2k}]^{\mathfrak{sp}_n}.$$

Letting $\Lambda^*(\omega_n)$ denote the exterior algebra generated by ω_n, we prove in the appendix that

Lemma 3.2. *There are isomorphisms*

$$[\Lambda^*(I_n)]^{\mathfrak{sp}_n} \simeq \Lambda^*(\omega_n) := \sum_{k\geq 0} \Lambda^k(\omega_n)$$
$$[\mathfrak{g}_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n} \simeq \bar{\Lambda}^*(\omega_n) := \sum_{k\geq 1} \Lambda^k(\omega_n),$$

where the first is an isomorphism of algebras, and the second is an isomorphism of vector spaces.

Combining this with Lemma (3.1), we have

Lemma 3.3. *There are vector space isomorphisms*

$$H_*^{Lie}(\mathfrak{g}_n;\, \mathbf{R}) \simeq H_*(\mathfrak{sp}_n;\, \mathbf{R}) \otimes \Lambda^*(\omega_n)$$
$$H_*^{Lie}(\mathfrak{g}_n;\, \mathfrak{g}_n) \simeq H_*(\mathfrak{sp}_n;\, \mathbf{R}) \otimes \bar{\Lambda}^*(\omega_n).$$

It is known that for cohomology,

$$H^*_{\mathrm{Lie}}(\mathfrak{sp}_n;\, \mathbf{R}) \simeq \Lambda^*(u_3,\, u_7,\, u_{11},\, \ldots,\, u_{4n-1}),$$

where u_i is a class in dimension i. Also,

$$H_k^{\mathrm{Lie}}(\mathfrak{sp}_n;\, \mathbf{R}) \simeq H^k_{\mathrm{Lie}}(\mathfrak{sp}_n;\, \mathbf{R}).$$

See the reference [10, p. 343] for the homology of the symplectic Lie group.

4. The Leibniz Homology of $\mathfrak{g}_n$

Recall that for a Lie algebra $\mathfrak{g}$ over a ring k, and more generally for a Leibniz algebra $\mathfrak{g}$ [4], the Leibniz homology of $\mathfrak{g}$, written $HL_*(\mathfrak{g};\, k)$, is the homology of the chain complex $T(\mathfrak{g})$:

$$k \xleftarrow{\;0\;} \mathfrak{g} \xleftarrow{[\;,\;]} \mathfrak{g}^{\otimes 2} \longleftarrow \ldots \longleftarrow \mathfrak{g}^{\otimes(n-1)} \xleftarrow{\;d\;} \mathfrak{g}^{\otimes n} \longleftarrow \ldots,$$

where

$$d(g_1,\, g_2,\, \ldots,\, g_n) = \sum_{1\leq i<j\leq n} (-1)^j\, (g_1,\, g_2,\, \ldots,\, g_{i-1},\, [g_i,\, g_j],\, g_{i+1},\, \ldots\, \hat{g}_j\, \ldots,\, g_n),$$

and $(g_1,\, g_2,\, \ldots,\, g_n)$ denotes the element $g_1 \otimes g_2 \otimes \ldots \otimes g_n \in \mathfrak{g}^{\otimes n}$.

The canonical projection $\pi_1 : \mathfrak{g}^{\otimes n} \to \mathfrak{g}^{\wedge n}$, $n \geq 0$, is a map of chain complexes, $T(\mathfrak{g}) \to \Lambda^*(\mathfrak{g})$, and induces a k-linear map on homology

$$HL_*(\mathfrak{g};\, k) \to H_*^{\mathrm{Lie}}(\mathfrak{g};\, k).$$

Letting

$$(\ker \pi_1)_n[2] = \ker [\mathfrak{g}^{\otimes(n+2)} \to \mathfrak{g}^{\wedge(n+2)}], \quad n \geq 0,$$

Pirashvili [9] defines the relative theory $H^{\mathrm{rel}}(\mathfrak{g})$ as the homology of the complex

$$C_n^{\mathrm{rel}}(\mathfrak{g}) = (\ker \pi_1)_n[2],$$

and studies the resulting long exact sequence relating Lie and Leibniz homology:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & H^{\text{rel}}_{n-2}(\mathfrak{g}) & \longrightarrow & HL_n(\mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_n(\mathfrak{g}) & \xrightarrow{\partial} & H^{\text{rel}}_{n-3}(\mathfrak{g}) \longrightarrow \\
\cdots & \xrightarrow{\partial} & H^{\text{rel}}_0(\mathfrak{g}) & \longrightarrow & HL_2(\mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_2(\mathfrak{g}) & \longrightarrow & 0 \\
 & & 0 & \longrightarrow & HL_1(\mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_1(\mathfrak{g}) & \longrightarrow & 0 \\
 & & 0 & \longrightarrow & HL_0(\mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_0(\mathfrak{g}) & \longrightarrow & 0.
\end{array}$$

An additional exact sequence is required for calculations of HL_*. Consider the projection

$$\pi_2 : \mathfrak{g} \otimes \mathfrak{g}^{\wedge n} \to \mathfrak{g}^{\wedge(n+1)}, \quad n \geq 0,$$

and the resulting chain map

$$\pi_2 : \mathfrak{g} \otimes \Lambda^*(\mathfrak{g}) \to \Lambda^{*+1}(\mathfrak{g}).$$

Let $HR_n(\mathfrak{g})$ denote the homology of the complex

$$CR_n(\mathfrak{g}) = (\ker \pi_2)_n[1] = \ker [\mathfrak{g} \otimes \mathfrak{g}^{\wedge(n+1)} \to \mathfrak{g}^{\wedge(n+2)}], \quad n \geq 0.$$

There is a resulting long exact sequence

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & HR_{n-1}(\mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_n(\mathfrak{g};\, \mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_{n+1}(\mathfrak{g}) & \xrightarrow{\partial} & \\
\cdots & \xrightarrow{\partial} & HR_0(\mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_1(\mathfrak{g};\, \mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_2(\mathfrak{g}) & \xrightarrow{\partial} & \\
 & & 0 & \longrightarrow & H^{\text{Lie}}_0(\mathfrak{g};\, \mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_1(\mathfrak{g}) & \longrightarrow & 0.
\end{array}$$

The projection $\pi_1 : \mathfrak{g}^{\otimes(n+1)} \to \mathfrak{g}^{\wedge(n+1)}$ can be written as the composition of projections

$$\mathfrak{g}^{\otimes(n+1)} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}^{\wedge n} \longrightarrow \mathfrak{g}^{\wedge(n+1)},$$

which leads to a natural map between exact sequences

$$\begin{array}{ccccccc}
H^{\text{rel}}_{n-1}(\mathfrak{g}) & \longrightarrow & HL_{n+1}(\mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_{n+1}(\mathfrak{g}) & \xrightarrow{\partial} & H^{\text{rel}}_{n-2}(\mathfrak{g}) \\
\downarrow & & \downarrow & & \mathbf{1}\downarrow & & \downarrow \\
HR_{n-1}(\mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_n(\mathfrak{g};\, \mathfrak{g}) & \longrightarrow & H^{\text{Lie}}_{n+1}(\mathfrak{g}) & \xrightarrow{\partial} & HR_{n-2}(\mathfrak{g})
\end{array}$$

and an articulation of their respective boundary maps ∂.

Lemma 4.1. *For the affine symplectic Lie algebra $\mathfrak{g}_n$, there is a natural isomorphism*

$$H_k(\mathfrak{sp}_n;\, \mathbf{R}) \xrightarrow{\simeq} HR_{k-3}(\mathfrak{g}_n;\, \mathbf{R}), \quad k \geq 3,$$

that factors as the composition

$$H^{Lie}_k(\mathfrak{sp}_n;\, \mathbf{R}) \underset{\partial}{\xrightarrow{\simeq}} HR_{k-3}(\mathfrak{sp}_n;\, \mathbf{R}) \xrightarrow{\simeq} HR_{k-3}(\mathfrak{g}_n; \mathbf{R}),$$

and the latter isomorphism is induced by the inclusion $\mathfrak{sp}_n \hookrightarrow \mathfrak{g}_n$.

Proof. Since $\mathfrak{sp}_n$ is a simple Lie algebra, from [2, Prop. VII.5.6] we have

$$H_k^{\text{Lie}}(\mathfrak{sp}_n;\, \mathfrak{sp}_n) = 0, \quad k \geq 0.$$

¿From the long exact sequence

$$\cdots \longrightarrow HR_{k-1}(\mathfrak{sp}_n;\, \mathbf{R}) \longrightarrow H_k^{\text{Lie}}(\mathfrak{sp}_n;\, \mathfrak{sp}_n) \longrightarrow H_{k+1}^{\text{Lie}}(\mathfrak{sp}_n;\, \mathbf{R}) \xrightarrow{\partial} \cdots,$$

it follows that $\partial : H_k^{\text{Lie}}(\mathfrak{sp}_n;\, \mathbf{R}) \to HR_{k-3}(\mathfrak{sp}_n;\, \mathbf{R})$ is an isomorphism for $k \geq 3$. The inclusion of Lie algebras $\mathfrak{sp}_n \hookrightarrow \mathfrak{g}_n$ induces a map of exact sequences

$$\begin{array}{ccccccc}
\longrightarrow & HR_{k-1}(\mathfrak{sp}_n;\, \mathbf{R}) & \longrightarrow & H_k^{\text{Lie}}(\mathfrak{sp}_n;\, \mathfrak{sp}_n) & \longrightarrow & H_{k+1}^{\text{Lie}}(\mathfrak{sp}_n;\, \mathbf{R}) & \xrightarrow{\partial} \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & HR_{k-1}(\mathfrak{g}_n;\, \mathbf{R}) & \longrightarrow & H_k^{\text{Lie}}(\mathfrak{g}_n;\, \mathfrak{g}_n) & \longrightarrow & H_{k+1}^{\text{Lie}}(\mathfrak{g}_n;\, \mathbf{R}) & \xrightarrow{\partial}
\end{array}$$

¿From Lemma (3.3)

$$H_*^{\text{Lie}}(\mathfrak{g}_n;\, \mathbf{R}) \simeq H_*(\mathfrak{sp}_n;\, \mathbf{R}) \otimes \Lambda^*(\omega_n)$$
$$H_*^{\text{Lie}}(\mathfrak{g}_n;\, \mathfrak{g}_n) \simeq H_*(\mathfrak{sp}_n;\, \mathbf{R}) \otimes \bar{\Lambda}^*(\omega_n).$$

The map $H_*^{\text{Lie}}(\mathfrak{g}_n;\, \mathfrak{g}_n) \to H_{*+1}^{\text{Lie}}(\mathfrak{g}_n;\, \mathbf{R})$ is an inclusion on homology with cokernel $H_{*+1}^{\text{Lie}}(\mathfrak{sp}_n;\, \mathbf{R})$. The result now follows from the map between exact sequences and a knowledge of the generators of $H_*^{\text{Lie}}(\mathfrak{g}_n;\, \mathbf{R})$ gleaned from Lemma (3.1). ■

Theorem 4.2. *There is an isomorphism of vector spaces*

$$HL_*(\mathfrak{g}_n;\, \mathbf{R}) \simeq \Lambda^*(\omega_n)$$

and an algebra isomorphism

$$HL^*(\mathfrak{g}_n;\, \mathbf{R}) \simeq \Lambda^*(\omega_n^*), \quad \omega_n^* = \sum_{i=1}^{n} dx^i \wedge dy^i,$$

where HL^* is afforded the shuffle algebra.

Proof. Consider the Pirashvili filtration [9] of the complex

$$C_n^{\text{rel}}(\mathfrak{g}) = \ker(\mathfrak{g}^{\otimes(n+2)} \to \mathfrak{g}^{\wedge(n+2)}), \quad n \geq 0,$$

given by

$$\mathcal{F}_m^k(\mathfrak{g}) = \mathfrak{g}^{\otimes k} \otimes \ker(\mathfrak{g}^{\otimes(m+2)} \to \mathfrak{g}^{\wedge(m+2)}), \quad m \geq 0,\ k \geq 0.$$

Then $\mathcal{F}_m^*$ is a subcomplex of $\mathcal{F}_{m+1}^*$ and the resulting spectral sequence converges to $H_*^{\text{rel}}(\mathfrak{g})$. ¿From [9] we have

$$E_{m,k}^2 \simeq HL_k(\mathfrak{g}) \otimes HR_m(\mathfrak{g}), \quad m \geq 0,\ k \geq 0.$$

¿From the proof of Lemma (4.1), there is an isomorphism

$$\partial : H_3^{\text{Lie}}(\mathfrak{g}_n;\, \mathbf{R}) \xrightarrow{\simeq} HR_0(\mathfrak{g}_n;\, \mathbf{R}) \simeq \mathbf{R}.$$

¿From the long exact sequence relating Lie and Leibniz homology, it follows that $HL_2(\mathfrak{g}_n;\, \mathbf{R}) \to H_2^{\mathrm{Lie}}(\mathfrak{g}_n;\, \mathbf{R})$ is an isomorphism. Since

$$\tilde{\omega}_n = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial y^i} - \frac{\partial}{\partial y^i}\otimes\frac{\partial}{\partial x^i}\right)$$

is a cycle in the Leibniz complex that maps to ω_n in the Lie algebra complex, it follows that $\tilde{\omega}_n$ generates $HL_2(\mathfrak{g}_n;\, \mathbf{R})$.

We claim that all elements in $HL_0(\mathfrak{g}_n)\otimes HR_*(\mathfrak{g}_n)$ are absolute cycles. The inclusion $\mathfrak{sp}_n \hookrightarrow \mathfrak{g}_n$ induces a map between exact sequences

$$\begin{array}{ccccccc}
HL_k(\mathfrak{sp}_n) & \longrightarrow & H_k^{\mathrm{Lie}}(\mathfrak{sp}_n) & \xrightarrow{\partial} & H_{k-3}^{\mathrm{rel}}(\mathfrak{sp}_n) & \longrightarrow & HL_{k-1}(\mathfrak{sp}_n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
HL_k(\mathfrak{g}_n) & \longrightarrow & H_k^{\mathrm{Lie}}(\mathfrak{g}_n) & \xrightarrow{\partial} & H_{k-3}^{\mathrm{rel}}(\mathfrak{g}_n) & \longrightarrow & HL_{k-1}(\mathfrak{g}_n)
\end{array}$$

Since $\mathfrak{sp}_n$ is a simple Lie algebra, $HL_k(\mathfrak{sp}_n;\, \mathbf{R}) = 0,\ k \geq 1$ [8]. Thus, $\partial : H_k^{\mathrm{Lie}}(\mathfrak{sp}_n) \to H_{k-3}^{\mathrm{rel}}(\mathfrak{sp}_n)$ is an isomorphism for $k \geq 3$. The inclusion $\mathcal{F}_m^*(\mathfrak{sp}_n) \hookrightarrow \mathcal{F}_m^*(\mathfrak{g}_n)$ induces a map of spectral sequences, and hence a map

$$HL_0(\mathfrak{sp}_n)\otimes HR_*(\mathfrak{sp}_n) \longrightarrow HL_0(\mathfrak{g}_n)\otimes HR_*(\mathfrak{g}_n).$$

Since $HR_*(\mathfrak{sp}_n) \simeq H_*^{\mathrm{rel}}(\mathfrak{sp}_n)$, all classes in $HL_0(\mathfrak{sp}_n)\otimes HR_*(\mathfrak{sp}_n)$ are absolute cycles. Now, $HR_*(\mathfrak{sp}_n)$ maps isomorphically to $HR_*(\mathfrak{g}_n)$, and by naturality, all classes in $HL_0(\mathfrak{g}_n)\otimes HR_*(\mathfrak{g}_n)$ are absolute cycles. Moreover,

$$\partial : H_*^{\mathrm{Lie}}(\mathfrak{g}_n) \to H_{*-3}^{\mathrm{rel}}(\mathfrak{g}_n)$$

maps the classes in $\bar{H}_*^{\mathrm{Lie}}(\mathfrak{sp}_n)$ injectively to $H_{*-3}^{\mathrm{rel}}(\mathfrak{g}_n)$ in the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_*^{\mathrm{Lie}}(\mathfrak{sp}_n) & \xrightarrow{\partial} & H_{*-3}^{\mathrm{rel}}(\mathfrak{sp}_n) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_*^{\mathrm{Lie}}(\mathfrak{g}_n) & \xrightarrow{\partial} & H_{*-3}^{\mathrm{rel}}(\mathfrak{g}_n) & \longrightarrow & \cdots
\end{array}$$

where the vertical arrows are inclusions.

We claim that all elements in $HL_2(\mathfrak{g}_n)\otimes HR_*(\mathfrak{g}_n)$ are absolute cycles as well. Let $[\theta] \in HR_m(\mathfrak{g}_n)$ be represented by the sum

$$\theta = \sum_{j=1}^{n} X_{1,j}\otimes X_{2,j}\wedge X_{3,j}\wedge\ldots\wedge X_{m+1,j}\ ,$$

where each $X_{i,j} \in \mathfrak{sp}_n$ and $d\theta = 0$. By invariance,

$$[\tilde{\omega}_n,\, X_{i,j}] = 0 \ \text{ for each } \ X_{i,j}.$$

It follows that $d(\tilde{\omega}_n\otimes\theta) = d(\tilde{\omega}_n)\otimes\theta + \tilde{\omega}_n\otimes d\theta = 0$, and $\tilde{\omega}_n\otimes\theta$ represents an absolute cycle in $H_*^{\mathrm{rel}}(\mathfrak{g}_n)$. To compute

$$\partial : H_*^{\mathrm{Lie}}(\mathfrak{g}_n) \to H_{*-3}^{\mathrm{rel}}(\mathfrak{g}_n)$$

on classes of the form $[\omega_n]\otimes \bar{H}^{\mathrm{Lie}}_*(\mathfrak{sp}_n)$, let $[\theta']\in \bar{H}^{\mathrm{Lie}}_*(\mathfrak{sp}_n)$ with $\partial(\theta')=\theta$. By lifting $\omega_n\wedge\theta'$ to $\tilde{\omega}_n\otimes\theta'$ in $T(\mathfrak{g}_n)$ and using invariance, we have

$$\partial(\omega_n\wedge\theta')=\tilde{\omega}_n\otimes\partial(\theta')=\tilde{\omega}_n\otimes\theta.$$

At this point $H^{\mathrm{rel}}_k(\mathfrak{g}_n)$ is completely determined for $k\le 2$. By an examination of $H^{\mathrm{rel}}_1(\mathfrak{g}_n)$,

$$\omega_n^{\wedge 2}\in\ker\partial,\quad \partial: H^{\mathrm{Lie}}_4(\mathfrak{g}_n)\to H^{\mathrm{rel}}_1(\mathfrak{g}_n).$$

Thus, $(\tilde{\omega}_n)^{\wedge 2}$ generates a non-zero class in $HL_4(\mathfrak{g}_n)$ mapping to the class $\omega_n^{\wedge 2}\in H^{\mathrm{Lie}}_4(\mathfrak{g}_n)$. As before, all classes in $HL_4(\mathfrak{g}_n)\otimes HR_*(\mathfrak{g}_n)$ are absolute cycles and in $\operatorname{Im}\partial$. Thus, $H^{\mathrm{rel}}_k(\mathfrak{g}_n)$ is completely determined for $k\le 4$. By induction on k, $(\tilde{\omega_n})^{\wedge k}$ is a non-zero class in $HL_{2k}(\mathfrak{g}_n)$, and

$$\begin{aligned}&H^{\mathrm{rel}}_*(\mathfrak{g}_n)\simeq\Lambda^*(\omega_n)\otimes HR_*(\mathfrak{g}_n)\simeq\Lambda^*(\omega_n)\otimes H^{\mathrm{Lie}}_{*+3}(\mathfrak{sp}_n)\\&HL_*(\mathfrak{g}_n)\simeq\Lambda^*(\omega_n).\end{aligned}$$

For the cohomology isomorphism

$$HL^*(\mathfrak{g}_n;\,\mathbf{R})\simeq\Lambda^*(\omega_n^*),\quad \omega_n^*=\sum_{i=1}^n dx^i\wedge dy^i,$$

where dx^i is the dual of $\frac{\partial}{\partial x^i}$ and dy^i the dual of $\frac{\partial}{\partial y^i}$ with respect to the basis of $\mathfrak{g}_n$ given by $\mathcal{B}_1\cup\mathcal{B}_2$ in §2. Since

$$HL^*(\mathfrak{g}_n;\,\mathbf{R})\simeq\operatorname{Hom}(HL_*(\mathfrak{g}_n;\,\mathbf{R}),\ \mathbf{R}),$$

the result follows by using the full shuffle product on cochains. ■

Recall that $HL^*(\mathfrak{g}_n;\,\mathbf{R})$ carries the structure of a dual Leibniz algebra (Zinbiel algebra) induced on cochains by semi-shuffles [5]. Given $\alpha\in\operatorname{Hom}(\mathfrak{g}_n^{\otimes p},\,\mathbf{R})$ and $\beta\in\operatorname{Hom}(\mathfrak{g}_n^{\otimes q},\,\mathbf{R})$, the semi-shuffle $\alpha\cdot\beta\in\operatorname{Hom}(\mathfrak{g}_n^{\otimes(p+q)},\,\mathbf{R})$ is given by

$$\sum_{\sigma\in\mathrm{Sh}_{p-1,\,q}}(\operatorname{sgn}\sigma)\,\alpha(g_1,\,g_{\sigma^{-1}(2)},\,g_{\sigma^{-1}(3)},\,\ldots,\,g_{\sigma^{-1}(p)})\,\beta(g_{\sigma^{-1}(p+1)},\,\ldots,\,g_{\sigma^{-1}(p+q)}),$$

where the summation is over all $(p-1,\ q)$ shuffles of

$$(2,\ 3,\ 4,\ \ldots,\ p,\ p+1,\ \ldots,\ p+q).$$

The full shuffle product, denoted by $\wedge$, satisfies

$$\alpha\wedge\beta=\alpha\cdot\beta+(-1)^{pq}\beta\cdot\alpha.$$

Note that in $HL^*(\mathfrak{g}_n;\,\mathbf{R})$ the Zinbiel product $\omega_n^*\cdot\omega_n^*$ is completely determined by $\omega_n^*\wedge\omega_n^*$, since

$$\omega_n^*\wedge\omega_n^*=2\omega_n^*\cdot\omega_n^*\quad\text{and}\quad\omega_n^*\cdot\omega_n^*=\frac{1}{2}\,\omega_n^*\wedge\omega_n^*.$$

The skew-symmetry of $\omega_n^*\cdot\omega_n^*$ can be verified by direct calculation with (co)chains as well. For example, if $f\in\operatorname{Hom}(\mathfrak{g}_n^{\otimes 2},\,\mathbf{R})$ generates $HL^2(\mathfrak{g}_n;\,\mathbf{R})$, then the cohomology class of f is determined by

$$f\Big(\sum_{i=1}^n\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial y^i}\Big)=\sum_{i=1}^n f\Big(\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial y^i}\Big).$$

Since $d\big((y_i \frac{\partial}{\partial y^i} - x_i \frac{\partial}{\partial x^i}) \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial y^i}\big) = \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial y^i} + \frac{\partial}{\partial y^i} \otimes \frac{\partial}{\partial x^i}$, it follows that f must be skew-symmetric. Now,

$$(f \cdot f)(g_1,\, g_2,\, g_3,\, g_4) =$$
$$f(g_1,\, g_2)f(g_3,\, g_4) - f(g_1,\, g_3)f(g_2,\, g_4) + f(g_1,\, g_4)f(g_1,\, g_3).$$

Restricting $\{g_1,\, g_2,\, g_3,\, g_4\}$ to $\{\frac{\partial}{\partial x^i},\, \frac{\partial}{\partial y^i},\, \frac{\partial}{\partial x^j},\, \frac{\partial}{\partial y^j}\}$, the reader may check skew-symmetry of $f \cdot f$ by hand.

We now prove that $HL^*(\mathfrak{g}_n;\, \mathbf{R})$ is a direct summand of $HL^*(\mathfrak{H}_n;\, \mathbf{R})$. We begin with the Lie algebra homology groups $H_*^{\mathrm{Lie}}(\mathfrak{H}_n^{\mathrm{Poly}};\, \mathbf{R})$.

Lemma 4.3. *The vector space $H_*^{Lie}(\mathfrak{H}_n^{Poly};\, \mathbf{R})$ contains $\Lambda^*(\omega_n)$ as a direct summand.*

Proof. Apply the Hochschild-Serre spectral sequence to the subalgebra $\mathfrak{sp}_n$ of $\mathfrak{H}_n^{\mathrm{Poly}}$. Then

$$E^2_{m,\,k} \simeq H_k^{\mathrm{Lie}}(\mathfrak{sp}_n;\, \mathbf{R}) \otimes H_m((\mathfrak{H}_n^{\mathrm{Poly}}/\mathfrak{sp}_n)^{\mathfrak{sp}_n};\, \mathbf{R}).$$

As before, $\Lambda^*(\omega_n) \subseteq (\mathfrak{H}_n^{\mathrm{Poly}}/\mathfrak{sp}_n)^{\mathfrak{sp}_n}$. Since $d(\omega_n) = d(\sum_{i=1}^n \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i}) = 0$, ω_n and $\Lambda^*(\omega_n)$ are infinite cycles. Since the elements $\Lambda^*(\omega_n)$ occur along the horizontal axis ($k = 0$) in a first-quadrant spectral sequence, these elements are not boundaries. Thus, $\Lambda^*(\omega_n)$ is a subvector space of $H_*^{\mathrm{Lie}}(\mathfrak{H}_n^{\mathrm{Poly}};\, \mathbf{R})$ induced by the morphism of Lie algebras $\mathfrak{g}_n \to \mathfrak{H}_n^{\mathrm{Poly}}$. ■

Lemma 4.4. *The vector space $HL_*(\mathfrak{H}_n^{Poly};\, \mathbf{R})$ contains $\Lambda^*(\omega_n)$ as a direct summand.*

Proof. The elements $\Lambda^*(\omega_n)$ are cycles in the Leibniz complex that map to $\Lambda^*(\omega_n)$ under the canonical morphism

$$HL_*(\mathfrak{H}_n^{\mathrm{Poly}};\, \mathbf{R}) \to H_*^{\mathrm{Lie}}(\mathfrak{H}_n^{\mathrm{Poly}};\, \mathbf{R}).$$ ■

Thus, the map on homology

$$HL_*(\mathfrak{g}_n; \mathbf{R}) \to HL_*(\mathfrak{H}_n^{\mathrm{Poly}};\, \mathbf{R})$$

induced by $\mathfrak{g}_n \to \mathfrak{H}_n^{\mathrm{Poly}}$ is injective.

Lemma 4.5. *The vector space $HL^*(\mathfrak{H}_n;\, \mathbf{R})$ contains $\Lambda^*(\omega_n^*)$ as a direct summand.*

Proof. The proof follows from the isomorphism

$$HL^*(\mathfrak{H}_n;\, \mathbf{R}) \simeq \mathrm{Hom}(HL_*(\mathfrak{H}_n^{\mathrm{Poly}};\, \mathbf{R}),\ \mathbf{R}),$$

using continuous cohomology. ■

5. A Characteristic Map

Let M be a symplectic manifold, $\mathcal{X}(M)$ the Lie algebra of C^∞ vector fields on M, and $\mathcal{X}_H(M)$ the Lie algebra of Hamiltonian vector fields [7, p. 85], both considered in the strong C^∞-topology. The functor HL^* denotes continuous Leibniz cohomology when applied to a topological Lie algebra. From [6], there is a natural map

$$H^*_{dR}(M;\, \mathbf{R}) \longrightarrow HL^*(\mathcal{X}(M);\, C^\infty(M)),$$

where $H^*_{dR}(M)$ denotes deRham cohomology, and $C^\infty(M)$ is also given the strong C^∞-topology. The inclusion of Lie algebras $\mathcal{X}_H(M) \hookrightarrow \mathcal{X}(M)$ induces a (contravariant) map

$$HL^*(\mathcal{X}(M);\, C^\infty(M)) \longrightarrow HL^*(\mathcal{X}_H(M);\, C^\infty(M))$$

on cohomology, while the inclusion of coefficients $\iota : \mathbf{R} \to C^\infty(M)$ induces a (covariant) map

$$\iota^* : HL^*(\mathcal{X}_H(M);\, \mathbf{R}) \longrightarrow HL^*(\mathcal{X}_H(M);\, C^\infty(M)).$$

Let $p \in M$ and let U be an open neighborhood of p homeomorphic to R^{2n} in the atlas of charts for M. There is a natural morphism of Lie algebras $\mathcal{X}_H(M) \to \mathcal{X}_H(U)$ given by the restriction of vector fields from M to U, and resulting linear maps

$$HL^*(\mathcal{X}_H(U);\, \mathbf{R}) \to HL^*(\mathcal{X}_H(M); \mathbf{R}) \xrightarrow{\iota^*} HL^*(\mathcal{X}_H(M);\, C^\infty(M)).$$

Now, $\mathcal{X}_H(U) \simeq \mathcal{X}_H(\mathbf{R}^{2n})$ as Lie algebras, and thus there are local maps

$$\rho_p : HL^*(\mathcal{X}_H(\mathbf{R}^{2n});\, \mathbf{R}) \to HL^*(\mathcal{X}_H(M);\, \mathbf{R})$$

for each $p \in M$. Note:

$$HL^*(\mathfrak{g}_n;\, \mathbf{R}) \simeq \Lambda^*(\omega_n^*) \subseteq HL^*(\mathfrak{H}_n;\, \mathbf{R}),$$

$$HL^*(\mathfrak{H}_n;\, \mathbf{R}) \xrightarrow{T^*} HL^*(\mathcal{X}_H(\mathbf{R}^{2n});\, \mathbf{R}) \xrightarrow{\iota^* \circ \rho_p} HL^*(\mathcal{X}_H(M);\, C^\infty(M)),$$

where T^* is induced by the Taylor series expansion. Let μ_p be the composition from $HL^*(\mathfrak{g}_n;\, \mathbf{R})$ to $HL^*(\mathcal{X}_H(M);\, C^\infty(M))$, $p \in M$. The local characteristic map is expressed as:

$$\begin{array}{ccc} & & \Lambda^*(\omega_n^*) \simeq HL^*(\mathfrak{g}_n;\, \mathbf{R}) \\ & & \big\downarrow \mu_p \\ H^*_{dR}(M;\, \mathbf{R}) & \longrightarrow & HL^*(\mathcal{X}_H(M);\, C^\infty(M)), \end{array}$$

where $p \in M$. The image of μ_p appears to depend on p.

6. Appendix

The goal of the appendix is to establish Lemma (3.2), namely the vector space isomorphisms

$$[\Lambda^*(I_n)]^{\mathfrak{sp}_n} \simeq \Lambda^*(\omega_n) := \sum_{k \geq 0} \Lambda^k(\omega_n) \tag{1}$$

$$[\mathfrak{g}_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n} \simeq \bar{\Lambda}^*(\omega_n) := \sum_{k \geq 1} \Lambda^k(\omega_n), \tag{2}$$

where the former is also an algebra isomorphism. First, note that as an $\mathfrak{sp}_n$-module, $\mathfrak{g}_n \simeq I_n \oplus \mathfrak{sp}_n$, and

$$[\mathfrak{g}_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n} \simeq [I_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n} \oplus [\mathfrak{sp}_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n}.$$

Thus, line (2) would follow from the vector space isomorphisms

$$[I_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n} \simeq \bar{\Lambda}^*(\omega_n)$$
$$[\mathfrak{sp}_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n} = \{0\}.$$

We first demonstrate isomorphism (1) in the following lemma.

Lemma 6.1.

$$[\Lambda^*(I_n)]^{\mathfrak{sp}_n} \simeq \Lambda^*(\omega_n).$$

Proof. We proceed by induction on n. For $n = 1$,

$$I_1 = \left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1} \right\rangle$$
$$\mathfrak{sp}_1 = \left\langle x_1 \frac{\partial}{\partial y^1}, y_1 \frac{\partial}{\partial x^1}, y_1 \frac{\partial}{\partial y^1} - x_1 \frac{\partial}{\partial x^1} \right\rangle.$$

By direct calculation, $(I_1)^{\mathfrak{sp}_1} = \{0\}$, and $(I_1^{\wedge 2})^{\mathfrak{sp}_1} = \langle \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial y^1} \rangle$.

By the inductive hypothesis, suppose

$$[\Lambda^*(I_{n-1})]^{\mathfrak{sp}_{n-1}} = \Lambda^*(\omega_{n-1}).$$

Consider then two cases for $I_n^{\wedge k}$, k odd, and k even. For k odd, let $z \in I_n^{\wedge k}$ and consider

$$z = z_1 + z_2 \wedge \frac{\partial}{\partial x^n} + z_3 \wedge \frac{\partial}{\partial y^n} + z_4 \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n},$$

where $z_1 \in I_{n-1}^{\wedge k}$, z_2, $z_3 \in I_{n-1}^{\wedge(k-1)}$, and $z_4 \in I_{n-1}^{\wedge(k-2)}$. Note that

$$[z, \; y_n \frac{\partial}{\partial y^n} - x_n \frac{\partial}{\partial x^n}] = -z_2 \wedge \frac{\partial}{\partial x^n} + z_3 \frac{\partial}{\partial y^n}.$$

For $z \in (I_n^{\wedge k})^{\mathfrak{sp}_n}$, $[z, \; y_n \frac{\partial}{\partial y^n} - x_n \frac{\partial}{\partial x^n}] = 0$, and

$$z = z_1 + z_4 \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial x^n}.$$

For any $X \in \mathfrak{sp}_{n-1} \subseteq \mathfrak{sp}_n$, we have

$$0 = [z, \; X] = [z_1, \; X] + [z_4, \; X] \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n}.$$

If non-zero, the terms $[z_1, \; X]$ and $[z_4, \; X] \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n}$ are linearly independent and would not sum to zero. Thus,

$$z_1 \in (I_{n-1}^{\wedge k})^{\mathfrak{sp}_{n-1}} = \{0\}, \quad z_4 \in (I_{n-1}^{\wedge(k-2)})^{\mathfrak{sp}_{n-1}} = \{0\}.$$

It follows that $(I_n^{\wedge k})^{\mathfrak{sp}_n} = \{0\}$ for k odd.

For k even, let $k = 2q$, $z \in (I_n^{\wedge 2q})^{\mathfrak{sp}_n}$, and repeat the above argument to the point

$$z_1 \in (I_{n-1}^{\wedge 2q})^{\mathfrak{sp}_{n-1}} = \langle \omega_{n-1}^{\wedge q} \rangle$$
$$z_4 \in (I_{n-1}^{\wedge 2(q-1)})^{\mathfrak{sp}_{n-1}} = \langle \omega_{n-1}^{\wedge (q-1)} \rangle$$

Thus, $z = c_1 \omega_{n-1}^{\wedge q} + c_2 \omega_{n-1}^{\wedge (q-1)} \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n}$, $c_1,\ c_2 \in \mathbf{R}$. Bracketing with $X = x_1 \frac{\partial}{\partial y^n} + x_n \frac{\partial}{\partial y^1}$ yields

$$0 = [z,\ X] = (c_2 - qc_1)\omega_{n-1}^{\wedge (q-1)} \wedge \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^n}.$$

Hence, z is a real multiple of

$$\omega_{n-1}^{\wedge q} + q\omega_{n-1}^{\wedge (q-1)} \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n} = \left(\omega_{n-1} + \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n} \right)^{\wedge q} = \omega_n^{\wedge q}. \quad \blacksquare$$

Lemma 6.2.

$$[I_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n} \simeq \bar{\Lambda}^*(\omega_n).$$

Proof. The proof proceeds by induction on n. For $n = 1$, a direct verification yields

$$(I_1)^{\mathfrak{sp}_1} = \{0\}, \quad (I_1 \otimes I_1)^{\mathfrak{sp}_1} = \left\langle \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial y^1} \right\rangle,$$

where $\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial y^1} = \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^1} \otimes \frac{\partial}{\partial x^1}$. Also, $(I_1 \otimes I_1^{\wedge 2})^{\mathfrak{sp}_1} = \{0\}$ by direct calculation. The inductive hypothesis states

$$[I_{n-1} \otimes \Lambda^*(I_{n-1})]^{\mathfrak{sp}_{n-1}} \simeq \bar{\Lambda}^*(\omega_{n-1}).$$

Let $v \in I_n \otimes I_n^{\wedge k}$, $v = u_1 + u_2$, where

$$u_1 \in I_{n-1} \otimes I_{n-1}^{\wedge k}, \quad u_2 \in (I_n \otimes I_n^{\wedge k})/(I_{n-1} \otimes I_{n-1}^{\wedge k}).$$

A vector space basis of $(I_n \otimes I_n^{\wedge k})/(I_{n-1} \otimes I_{n-1}^{\wedge k})$ is given by the families of elements:

(1) $\frac{\partial}{\partial x^n} \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-2}}$

(2) $\frac{\partial}{\partial y^n} \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-2}}$

(3) $\frac{\partial}{\partial x^n} \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}}$

(4) $\frac{\partial}{\partial x^n} \otimes \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}}$

(5) $\frac{\partial}{\partial y^n} \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}}$

(6) $\frac{\partial}{\partial y^n} \otimes \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}}$

(7) $\frac{\partial}{\partial z^1} \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}}$

(8) $\frac{\partial}{\partial z^1} \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial z^2} \wedge \frac{\partial}{\partial z^3} \wedge \ldots \wedge \frac{\partial}{\partial z^k}$

(9) $\frac{\partial}{\partial z^1} \otimes \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^2} \wedge \frac{\partial}{\partial z^3} \wedge \ldots \wedge \frac{\partial}{\partial z^k}$,

where, for each family, the z^i's are elements of

$$\{x^1,\ x^2,\ \ldots,\ x^{n-1},\ y^1,\ y^2,\ \ldots,\ y^{n-1}\}.$$

Let $v \in (I_n \otimes I_n^{\wedge k})^{\mathfrak{sp}_n}$ and $X = y_n \frac{\partial}{\partial y^n} - x_n \frac{\partial}{\partial x^n}$. Then

$$0 = [v,\ X] = [u_1 + u_2,\ X] = [u_2,\ X].$$

To compute the $\mathfrak{sp}_n$-invariants, consider $u_2 \in \ker(\mathrm{ad}_X)$, where $\mathrm{ad}_X(w) = [w,\ X]$. The families (4), (5) and (7) above fall into $\ker(\mathrm{ad}_X)$. Now consider $X = x_n \frac{\partial}{\partial y^n}$. Family (7) along with

$$\frac{\partial}{\partial x^n} \otimes \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^1} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}} - \frac{\partial}{\partial y^n} \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial z^1} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}}$$

are elements of $\ker(\mathrm{ad}_X)$, $X = x_n \frac{\partial}{\partial y^n}$. Then $v = u_1 + s_1 + s_2$,

$$s_1 = \sum_{z^1,\ldots,z^{k-1}} c_{1,*}\left(\frac{\partial}{\partial x^n} \otimes \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^1} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}} - \frac{\partial}{\partial y^n} \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial z^1} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}}\right)$$

$$s_2 = \sum_{z^1,\ldots,z^{k-1}} c_{2,*}\left(\frac{\partial}{\partial z^1} \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}}\right)$$

For $X \in \mathfrak{sp}_{n-1}$,

$$0 = [v,\ X] = [u_1,\ X] + [s_1,\ X] + [s_2,\ X].$$

Note that

$$[u_1,\ X] \in I_{n-1} \otimes I_{n-1}^{\wedge k}, \quad [s_1,\ X] \notin I_{n-1} \otimes I_{n-1}^{\wedge k}, \quad [s_2,\ X] \notin I_{n-1} \otimes I_{n-1}^{\wedge k}.$$

If non-zero, the summands of $[s_1,\ X]$ and $[s_2,\ X]$ would be linearly independent. Thus, $[s_1,\ X] = 0$, $[s_2,\ X] = 0$, and $u_1 \in (I_{n-1} \otimes I_{n-1}^{\wedge k})^{\mathfrak{sp}_{n-1}}$. For k even, $(I_{n-1} \otimes I_{n-1}^{\wedge k})^{\mathfrak{sp}_{n-1}} = \{0\}$, $u_1 = 0$,

$$[s_2,\ X] = \sum_{z^1,\ldots,z^{k-1}} c_{2,*}[\frac{\partial}{\partial z^1} \otimes \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}},\ X] \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n},$$

$$\sum_{z^1,\ldots,z^{k-1}} c_{2,*}\frac{\partial}{\partial z^1} \otimes \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}} \in (I_{n-1} \otimes I_{n-1}^{\wedge(k-2)})^{\mathfrak{sp}_{n-1}} = \{0\}.$$

Thus, $v = s_1$. From

$$0 = [s_1,\ x_n\frac{\partial}{\partial y^i} + x_i\frac{\partial}{\partial y^n}], \quad 0 = [s_1,\ y_i\frac{\partial}{\partial x^n} + y_n\frac{\partial}{\partial x^i}],$$

for $1 \le i \le n-1$, it follows that $s_1 = 0$.

For k odd, let $k = 2q-1$. Then

$$\begin{aligned}
&u_1 \in (I_{n-1} \otimes I_{n-1}^{\wedge k})^{\mathfrak{sp}_{n-1}} = \langle \omega_{n-1}^{\wedge q} \rangle \\
&\theta := \sum_{z^1,\ldots,z^{k-1}} c_{2,*} \frac{\partial}{\partial z^1} \otimes \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}} \in (I_{n-1} \otimes I_{n-1}^{\wedge(k-2)})^{\mathfrak{sp}_{n-1}} = \langle \omega_{n-1}^{\wedge(q-1)} \rangle \\
&u_1 = \lambda_1 \omega_{n-1}^{\wedge q}, \quad \theta = \lambda_2 \omega_{n-1}^{\wedge(q-1)}, \quad \lambda_1,\ \lambda_2 \in \mathbf{R}.
\end{aligned}$$

Note that

$$\begin{aligned}
&[\lambda_1 \omega_{n-1}^{\wedge q} + \lambda_2 \omega_{n-1}^{\wedge(q-1)} \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n},\ y_i \frac{\partial}{\partial x^n} + y_n \frac{\partial}{\partial x^i}] \\
&= (q\lambda_1 - \lambda_2) \omega_n^{\wedge(q-1)} \wedge \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^n}.
\end{aligned}$$

From

$$0 = [\lambda_1 \omega_{n-1}^{\wedge q} + \lambda_2 \omega_{n-1}^{\wedge(q-1)} \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n} + s_1,\ X]$$

for $X = x_n \frac{\partial}{\partial y^i} + x_i \frac{\partial}{\partial y^n}$, $X = y_i \frac{\partial}{\partial x^n} + y_n \frac{\partial}{\partial x^i}$, $1 \le i \le n-1$, it follows that $s_1 = 0$, and $(q\lambda_1 - \lambda_2) = 0$. Letting $\lambda_1 = 1$, we have $\lambda_2 = q$, and

$$v = \omega_{n-1}^{\wedge q} + q\omega_{n-1}^{\wedge q} \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n} = \omega_n^{\wedge q}. \qquad \blacksquare$$

Lemma 6.3.

$$[\mathfrak{sp}_n \otimes \Lambda^*(I_n)]^{\mathfrak{sp}_n} = \{0\}.$$

Proof. We apply induction on n. For $n = 1$, write a general element of $\mathfrak{sp}_1 \otimes \Lambda^*(I_1)$ as a linear combination of the basis elements given in $\mathcal{B}_1$ and $\mathcal{B}_2$ of §2 $(n=1)$. Then apply ad_X for $X = (y_1 \frac{\partial}{\partial y^1} - x_1 \frac{\partial}{\partial x^1})$. The result $[\mathfrak{sp}_1 \otimes \Lambda^*(I_1)]^{\mathfrak{sp}_1} = \{0\}$ follows from linear algebra.

Suppose that $[\mathfrak{sp}_{n-1} \otimes \Lambda^*(I_{n-1})]^{\mathfrak{sp}_{n-1}} = \{0\}$. Since $\mathfrak{sp}_n$ is a simple Lie algebra, we have $(\mathfrak{sp}_n)^{\mathfrak{sp}_n} = \{0\}$. Let $\mathcal{B}_1$ be the vector space basis for $\mathfrak{sp}_{n-1}$ given in §2, and let

$$\begin{aligned}
S &= \{x^1,\ x^2,\ \ldots,\ x^n,\ y^1,\ y^2,\ \ldots,\ y^n\} \\
S' &= \{x^1,\ x^2,\ \ldots,\ x^{n-1},\ y^1,\ y^2,\ \ldots,\ y^{n-1}\}.
\end{aligned}$$

A vector space basis of $(\mathfrak{sp}_n \otimes I_n^{\wedge k})/(\mathfrak{sp}_{n-1} \otimes I_{n-1}^{\wedge k})$ is given by the families of elements:

(1) $e \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-2}}, \quad e \in \mathcal{B}_1,\ z^i \in S'$

(2) $e \otimes \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}}, \quad e \in \mathcal{B}_1,\ z^i \in S'$

(3) $e \otimes \frac{\partial}{\partial y^n} \wedge \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^{k-1}}, \quad e \in \mathcal{B}_1,\ z^i \in S'$

(4) $(x_n \frac{\partial}{\partial y^n}) \otimes \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^k}, \quad z^i \in S$

(5) $(x_n \frac{\partial}{\partial y^i} + x_i \frac{\partial}{\partial y^n}) \otimes \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^k}, \quad i < n,\ z^j \in S$

(6) $(y_n \frac{\partial}{\partial x^n}) \otimes \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^k}, \quad z^i \in S$

(7) $(y_i \frac{\partial}{\partial x^n} + y_n \frac{\partial}{\partial x^i}) \otimes \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^k}, \quad i < n,\ z^j \in S$

(8) $(y_n \frac{\partial}{\partial y^n} - x_n \frac{\partial}{\partial x^n}) \otimes \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^k}, \quad z^i \in S$

(9) $(y_i \frac{\partial}{\partial y^n} - x_n \frac{\partial}{\partial x^i}) \otimes \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^k}, \quad i < n,\ z^j \in S$

(10) $(y_n \frac{\partial}{\partial y^i} - x_i \frac{\partial}{\partial x^n}) \otimes \frac{\partial}{\partial z^1} \wedge \frac{\partial}{\partial z^2} \wedge \ldots \wedge \frac{\partial}{\partial z^k}, \quad i < n,\ z^j \in S$

Given $w \in (\mathfrak{sp}_n \otimes I_n^{\wedge k})^{\mathfrak{sp}_n}$, let $w = u + v$, where

$$u \in (\mathfrak{sp}_{n-1} \otimes I_{n-1}^{\wedge k}), \quad v \in (\mathfrak{sp}_n \otimes I_n^{\wedge k})/(\mathfrak{sp}_{n-1} \otimes I_{n-1}^{\wedge k}).$$

For all $X \in \mathfrak{sp}_n$, $0 = \mathrm{ad}_X(w) = \mathrm{ad}_X(u) + \mathrm{ad}_X(v)$. Restricting to $X \in \mathfrak{sp}_{n-1}$, notice that if non-zero, the elements $\mathrm{ad}_X(u)$ and $\mathrm{ad}_X(v)$ are linearly independent. Thus, $\mathrm{ad}_X(u) = 0$, and

$$u \in (\mathfrak{sp}_{n-1} \otimes I_{n-1}^{\wedge k})^{\mathfrak{sp}_{n-1}} = \{0\}.$$

Now, v can be written as a linear combination of the elements in families (1)–(10). We prove that $v = 0$ by applying the condition $\mathrm{ad}_X(v) = 0$ for successive choices of $X \in \mathfrak{sp}_n$. First apply $X = (y_n \frac{\partial}{\partial y^n} - x_n \frac{\partial}{\partial x^n})$, then $X \in \mathfrak{sp}_{n-1}$ together with the inductive hypothesis. Third, apply $X = x_n \frac{\partial}{\partial y^n}$, fourth $X = (y_i \frac{\partial}{\partial y^i} - x_i \frac{\partial}{\partial x^i})$, fifth $X = x_i \frac{\partial}{\partial y^i}$, and finally $X = (x_n \frac{\partial}{\partial y^i} + x_i \frac{\partial}{\partial y^n})$, where $1 \le i \le n - 1$. ■

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