A Product for Harmonic Spinors on
Reductive Homogeneous Spaces

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Abstract.   We define a product for harmonic spinors on reductive homogeneous spaces. We give also some examples where harmonic spinors with coefficients in a module are expressed as a linear combination of products of harmonic spinors with coefficients in two other modules. One such example involves discrete series representations.
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1.   Introduction

Non-compact analogues and spinor analogues of the Borel-Weil-Bott Theorem have provided powerful and elegant tools to study representations of Lie groups. In particular, discrete series representations of non-compact semisimple Lie groups are realized on spaces of square integrable vector-valued harmonic Dirac spinors on non-compact symmetric spaces ([11][13][1]). The introduction of the Dirac operator, replacing the usual $\bar{\partial}$ operator, allows one to treat all non-compact semisimple Lie groups rather than just the ones whose symmetric space is hermitian.

For the introduction, let G be a connected non-compact semisimple real Lie group and H a closed connected subgroup of maximal rank of G with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Some of our results are in fact stated and proved for general connected reductive Lie groups. We assume that the Killing form of $\mathfrak{g}$ restricts to a non-degenerate form on $\mathfrak{h}$ and we write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \quad \mathfrak{q} = \mathfrak{h}^{\perp}$$

for the corresponding orthogonal decomposition of $\mathfrak{g}$. Note that the killing form is non-degenerate (and possibly indefinite) on $\mathfrak{q}$. In particular, we may define the Clifford algebra of $\mathfrak{q}$ and the corresponding spin representation S of $\mathfrak{h}$. Given a finite dimensional representation E of $\mathfrak{h}$ satisfying the property that the tensor product $S \otimes E$ lifts to a representation of H, we have a homogeneous vector bundle

$\mathcal{S} \otimes \mathcal{E} \to G/H$ of finite rank over G/H and a G-invariant differential operator on (smooth) sections:

$$D = D_{G/H}(\mathcal{E}) : C^\infty(G/H, \mathcal{S} \otimes \mathcal{E}) \to C^\infty(G/H, \mathcal{S} \otimes \mathcal{E})$$

known as Kostant's cubic Dirac operator. As we shall see in the next section, D is the sum of a first order term (analogous to the usual Dirac operator on riemannian symmetric spaces) and a zeroth order term defined by some degree three element of the Clifford algebra of $\mathfrak{q}$ ([6]). This zeroth order term vanishes whenever the homogeneous space G/H is symmetric (see Section 2). When H is a maximal compact subgroup K (of maximal rank) of G, the kernel of the Dirac operator $D_{G/K}(\mathcal{E})$ on L^2-sections is an irreducible unitary representation of G in the discrete series of G, and every discrete series representation of G occurs as the L^2-kernel of $D_{G/K}(\mathcal{E})$ for some bundle $\mathcal{E}$ ([11][13][1]). In the case where $H \neq K$, the homogeneous space need not be symmetric, the cubic Dirac operator is not elliptic and traditional L^2-techniques are not available. However, more recently, analogous results have been proved by constructing explicitly a non zero interwining operator from principal series representations of G into the kernel of $D_{G/H}(\mathcal{E})$ for some bundle $\mathcal{E}$ ([9][10]). In particular, a complex structure on G/H is not needed for the construction of interesting representations of G.

On the other hand, when G/H is equipped with an invariant complex structure, one can show that the Dirac operator $D_{G/H}(\mathcal{E})$ reduces to the Dolbeault operator $\partial + \overline{\partial}$ (upto scaling). For this reason 'harmonic spinors', requiring only a spin structure rather than a complex structure, has long been regarded as a substitute for holomorphic forms. Since the product of holomorphic forms (whenever it makes sense) is holomorphic, it is then natural to consider a 'multiplication' for harmonic spinors. Our first result provides various instances of such a phenomenon in the context of Kostant's cubic Dirac operators for general connected reductive Lie groups. The precise statement is a bit technical, however it can be put as follows:

Theorem 1 (Theorem 3.1): *Let E_1 and E_2 be two finite dimensional representations of H. The product of a harmonic spinor in $Ker(D_{G/H}(\mathcal{E}_1))$ and a harmonic spinor in a related '$\mathcal{H}om$' bundle is a harmonic spinor in $Ker(D_{G/H}(\mathcal{E}_2))$.*

Even though the Borel-Weil-Bott Theorem, along with its non-compact analogues and spinor analogues, realizes representations as harmonic spinors, it is better to have a mechanism where this picture can be viewed coherently when the representation parameters change in a coherent way.

In order to see how the above result yields such a 'mechanism', we illustrate in the following context: let $F(\nu)$ be the finite dimensional representation of $\mathfrak{g}^{\mathbf{C}}$ (the complexification of $\mathfrak{g}$) with highest weight ν, with respect to some positive system (see Section 4). Let $\{\pi(\mu)\}_{\mu \in \Lambda}$ be a coherent family of (virtual) representations of G (see [2]). Typically in a positive cone contained in the parameter space Λ, this arises via the Zuckerman translation functor

$$F(\nu)^\star \otimes \pi(\mu + \nu) \longrightarrow \pi(\mu)$$

where $F(\nu)^\star$ denotes the contragredient representation of $F(\nu)$. Now assume that:

$\pi(\mu)$ is realized as harmonics in some bundle $\mathcal{S} \otimes \mathcal{E}_\mu$,

$\pi(\mu + \nu)$ is realized as harmonics in some bundle $\mathcal{S} \otimes \mathcal{E}_{\mu+\nu}$,

$F(\nu)^{\star}$ is realized as harmonics in some bundle $\mathcal{H}om^{+}(S \otimes E_{\mu+\nu}, S \otimes E_{\mu})$,

where the '$\mathcal{H}om$' bundle $\mathcal{H}om^{+}(S \otimes E_{\mu+\nu}, S \otimes E_{\mu})$ over G/H induced by a H-submodule $\text{Hom}^{+}(S \otimes E_{\mu+\nu}, S \otimes E_{\mu})$ of $\text{Hom}_{\mathbf{C}}(S \otimes E_{\mu+\nu}, S \otimes E_{\mu})$ defined in Section 3. This gives rise to a setup as in Theorem 3.1.

On the other hand, there has been lot of interest in the representation theoretic Dirac cohomology with the proof by Huang and Pandžić of a conjecture of Vogan ([5]), extended by Kostant to the cubic Dirac operator ([7]). In this context, our second result (Theorem 4.2) illustrates the 'mechanism' of Theorem 3.1 in the case of discrete series representations for semisimple Lie groups. More precisely, when G has a compact Cartan subgroup, it is known that discrete series representations of G arise as a particular case of Enright-Varadarajan $(\mathfrak{g}^{\mathbf{C}}, K)$-modules ([3][12]), where K still denotes a maximal compact subgroup of maximal rank of G. Moreover, given a $(\mathfrak{g}^{\mathbf{C}}, K)$-module $(\pi, \mathcal{H})$, there is a Dirac operator

$$\mathcal{D}_{\pi} : \mathcal{H} \otimes S \to \mathcal{H} \otimes S,$$

defined by

$$\mathcal{D}_{\pi} = \sum_{j} \pi(X_j) \otimes \gamma(X_j)$$

where $\{X_j\}$ is an orthonormal basis of $\mathfrak{q}$ and γ is the Clifford multiplication. Now assume that $(\pi(\mu), \mathcal{H}(\mu))$ and $(\pi(\mu + \nu), \mathcal{H}(\mu + \nu))$ are two discrete series representations of G, regarded as $(\mathfrak{g}^{\mathbf{C}}, K)$-modules, where μ and ν are some dominant regular weights with respect to some positive system in $\mathfrak{g}^{\mathbf{C}}$ (see Section 4). Standard inclusions of Verma modules for $\mathfrak{g}^{\mathbf{C}}$ give rise to an inclusion

$$\varphi : \mathcal{H}(\mu) \otimes S \hookrightarrow \mathcal{H}(\mu + \nu) \otimes F(\nu) \otimes S.$$

Moreover, we have a map

$$\beta : (\mathcal{H}(\mu + \nu) \otimes S) \otimes (F(\nu) \otimes S) \otimes S^{\star} \longrightarrow \mathcal{H}(\mu + \nu) \otimes F(\nu) \otimes S$$

by contracting the second factor S and the fifth factor $S^{\star}$. Our second result can now be stated:

Theorem 2 (Theorem 4.2):

$$\varphi(\text{Ker}(\mathcal{D}_{\pi(\mu)})) \subseteq \beta(\text{Ker}(\mathcal{D}_{\pi(\mu+\nu)}) \otimes \text{Ker}(\mathcal{D}_{F(\nu)}) \otimes S^{\star}).$$

In other words, one can relate Dirac spinors for an irreducible representation, a finite dimensional irreducible representation and a third representation which is related to the first two via a Zuckerman translation. We have used the Enright-Varadarajan construction in the proof of this result for a description of the discrete series representations.

While our first result deals with cubic Dirac operators on general homogeneous space G/H (not necessarily symmetric), it should be noted that our second theorem deals with the case where $H = K$ for which the cubic Dirac operator reduces to the usual Dirac operator on the symmetric space G/K. Indeed, outside of the realm of symmetric spaces (where the cubic term vanishes), not much

is available in the literature about non-zero solutions for the cubic Dirac (except for the above mentioned result of Kostant, and a result of Landweber [8], which anyway account for finite dimensional representations). For infinite dimensional representations and non-compact homogeneous spaces (not necessarily symmetric) a beginning is made in [10]. However we do not know any analogue of the Enright-Varadarajan construction for these modules. In a work in progress we are trying to do things in the reverse order: constructing representations in a manner similar to the Enright-Varadarajan method where we hope to be able to deal with cubic Dirac solutions.

This paper is organized as follows: Section 2 contains the main notations and the definition of the cubic Dirac operator. Section 3 includes the definition of the 'product' for harmonic spinors and the proof of our first result (Theorem 3.1). Section 4 is devoted to the proof of the second result (Theorem 4.2), along with a quick review of the Enright-Varadarajan modules. For the convenience of the reader, let us mention that Sections 3 and 4 are actually independant. Finally, in the last section, we address two open questions.

2. Preliminaries

In this section, we recall some notation and basic definitions following [10]. Let G be a connected non-compact real reductive Lie group with Lie algebra $\mathfrak{g}_0$, i.e. $\mathfrak{g}_0 = \mathfrak{z}_0 \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$, where $\mathfrak{z}_0$ denotes the Lie algebra of the center Z of G. We do not assume that Z is finite. In the sequel the subscript 0 will mean that the base field is the field of real numbers, and we will drop this subscript for the complexification.

If K/Z is a maximal compact subgroup of G/Z then K is the set of fixed points of some Cartan involution θ of G. We write the corresponding Cartan decomposition of $\mathfrak{g}_0$ as $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$. Let $\langle \, , \, \rangle$ be an Ad-invariant non degenerate symmetric bilinear form on $\mathfrak{g}$ which coincides with the Killing form on the semisimple part $[\mathfrak{g}_0, \mathfrak{g}_0]$ of $\mathfrak{g}_0$. We extend linearly both $\langle , \rangle$ and θ to $\mathfrak{g}$, and we shall use the same symbols to denote these extensions. Let H be a closed connected real reductive subgroup of maximal rank of G. We make the following assumption on H:

Assumption 2.1.   *The restriction of $\langle \, , \, \rangle$ to $\mathfrak{h}_0 \times \mathfrak{h}_0$ is non-degenerate.*

There is therefore an orthogonal decomposition of $\mathfrak{g}_0$:

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{q}_0, \quad \mathfrak{q}_0 \stackrel{\text{def}}{=} \mathfrak{h}_0^\perp.$$

In particular $[\mathfrak{h}_0, \mathfrak{q}_0] \subset \mathfrak{q}_0$ and the restriction $\langle , \rangle_{\mathfrak{q}_0}$ of $\langle , \rangle$ to $\mathfrak{q}_0 \times \mathfrak{q}_0$ remains non-degenerate. Let $so(\mathfrak{q}_0)$ be the Lie algebra of the group of orthogonal endomorphisms of $\mathfrak{q}_0$ with respect to $\langle , \rangle_{\mathfrak{q}_0}$ and denote by $(\sigma_{\mathfrak{q}}, S_{\mathfrak{q}})$ the spin representation of $so(\mathfrak{q}_0)$. The spin representation $(s_{\mathfrak{q}}, S_{\mathfrak{q}})$ of $\mathfrak{h}_0$ is defined by:

$$s_{\mathfrak{q}} \stackrel{\text{def}}{=} \sigma_{\mathfrak{q}} \circ \mathrm{ad}.$$

In general this representation does not integrate to a representation of H. So let (τ, E) be a finite dimensional representation of $\mathfrak{h}_0$ such that the tensor product

$S_{\mathfrak{q}} \otimes E$ lifts to an H-representation β. This defines a vector bundle $\mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}$ over G/H whose space of smooth sections $C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E})$ may be identified with the vector space $\left(C^{\infty}(G) \otimes (S_{\mathfrak{q}} \otimes E) \right)^{H}$ of H-invariants of $C^{\infty}(G) \otimes (S_{\mathfrak{q}} \otimes E)$, for the H-action $R \otimes \beta$. Here R denotes the action by right translations on $C^{\infty}(G)$. The space $\left(C^{\infty}(G) \otimes (S_{\mathfrak{q}} \otimes E) \right)^{H}$ carries a natural structure of G-module given by the left translations on $C^{\infty}(G)$. On the other hand, the image, under the Chevalley isomorphism, of the alternating 3-form on $\mathfrak{q}$ defined by:

$$(X, Y, Z) \mapsto \langle X, [Y, Z] \rangle,$$

is a degree 3 element c in the Clifford algebra $Cl(\mathfrak{q})$ of $\mathfrak{q}$. In other words, suppose $\{X_j\}$ is a basis of $\mathfrak{q}_0$ satisfying:

$$\langle X_j, X_k \rangle_{\mathfrak{q}_0} = a_j \delta_{jk}, \quad a_j = \pm 1 \tag{2.2}$$

then

$$c = \sum_{j < k < \ell} a_j a_k a_\ell \langle X_j, [X_k, X_\ell] \rangle X_j X_k X_\ell. \tag{2.3}$$

The Kostant's "cubic" Dirac operator $D_{\mathcal{E}}$, associated with the vector bundle $\mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}$ over G/H, is the G-invariant differential operator

$$D_{\mathcal{E}} : C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}) \to C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E})$$

defined by:

$$D_{\mathcal{E}} = \sum a_j R(X_j) \otimes \gamma(X_j) \otimes 1 - 1 \otimes \gamma(c) \otimes 1,$$

where γ denotes the Clifford multiplication (keeping the same symbol to denote the differential of R). The cubic Dirac operator does not dependent on the choice of the basis $\{X_j\}$ satisfying (2.2). Observe that if H is fixed pointwise by some involution of G then the cubic term c vanishes, since $[\mathfrak{q}_0, \mathfrak{q}_0] \subset \mathfrak{h}_0$ and $\mathfrak{h}_0 \perp \mathfrak{q}_0$. In particular, $D_{\mathcal{E}}$ reduces to the 'usual' (i.e non cubic) Dirac operator (see [11])

$$\widehat{D}_{\mathcal{E}} : C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}) \to C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E})$$

defined by:

$$\widehat{D}_{\mathcal{E}} = \sum a_j R(X_j) \otimes \gamma(X_j) \otimes 1.$$

3.   A product

In a similar way, we define the cubic Dirac operator

$$D_{\mathcal{E}'} : C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}') \to C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}'),$$

for a finite dimensional representation (τ', E') of $\mathfrak{h}_0$ such that the tensor product $S_{\mathfrak{q}} \otimes E'$ lifts to a representation β' of H. Let $\mathrm{Hom}_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$ be the space of complex homomorphisms $S_{\mathfrak{q}} \otimes E \to S_{\mathfrak{q}} \otimes E'$ equipped with the structure of H-module given by:

$$h \cdot T \stackrel{\mathrm{def}}{=} \beta'(h) \circ T \circ \beta(h)^{-1}.$$

This defines a homogeneous vector bundle $\mathcal{H}om_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$ over G/H whose space of smooth sections $C^{\infty}(G/H, \mathcal{H}om_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E'))$ may be identified with:

$$\left\{ \phi : G \xrightarrow{C^{\infty}} \mathrm{Hom}_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E') \mid \phi(gh) = \beta'(h) \circ \phi(g) \circ \beta(h)^{-1}, \ \forall g \in G, \ \forall h \in H \right\}$$

$$\simeq \left(C^{\infty}(G) \otimes \mathrm{Hom}_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E') \right)^{H}.$$

On the other hand, the canonical embedding $\mathfrak{q} \hookrightarrow Cl(\mathfrak{q})$ extends uniquely to an anti-automorphism

$$Cl(\mathfrak{q}) \to Cl(\mathfrak{q}), \ A \mapsto A^{0}$$

of the Clifford algebra. In particular, we may equip $\mathrm{Hom}_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$ with a structure of a Clifford-module as follows:

$$A \cdot T \stackrel{\mathrm{def}}{=} T \circ (\gamma(A^{0}) \otimes 1), \ \ \forall A \in Cl(\mathfrak{q}).$$

Therefore there is a well defined cubic Dirac operator $D_{\mathcal{E}, \mathcal{E}'}$ associated with the homogeneous vector bundle $\mathcal{H}om_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$:

$$D_{\mathcal{E}, \mathcal{E}'} : C^{\infty}(G/H, \mathcal{H}om_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')) \to C^{\infty}(G/H, \mathcal{H}om_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E'))$$

given by:

$$D_{\mathcal{E}, \mathcal{E}'}(f \otimes T) \stackrel{\mathrm{def}}{=} \sum a_{j}(R(X_{j})f) \otimes (X_{j} \cdot T) - f \otimes c \cdot T.$$

Note that since $c^{0} = -c$, the above equation becomes

$$D_{\mathcal{E}, \mathcal{E}'}(f \otimes T) = \sum a_{j}(R(X_{j})f) \otimes (X_{j} \cdot T) + f \otimes T \circ (\gamma(c) \otimes 1).$$

Let us now define the product map

$\Psi : C^{\infty}(G/H, \mathcal{H}om_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')) \times C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}) \to C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}')$,

$\Psi(f \otimes T, \phi \otimes s \otimes v) = f\phi \otimes T(s \otimes v)$,

with the following properties:

(i) Ψ is G-equivariant,

(ii) $\Psi(1, B) = B$,

(iii) $\Psi(A_{1} A_{2}, B) = \Psi(A_{1}, \Psi(A_{2}, B))$ when $E = E'$.

Let $\mathrm{Hom}^{+}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$ be the H-submodule of $\mathrm{Hom}_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$ given by: $\mathrm{Hom}^{+}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$

$$\stackrel{\mathrm{def}}{=} \left\{ T \in \mathrm{Hom}_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E') \mid (\gamma(X) \otimes 1) \circ T = T \circ (\gamma(X) \otimes 1), \ \forall X \in \mathfrak{q}_{0} \right\}.$$

Note that if $t \in \mathrm{Hom}_{\mathbf{C}}(E, E')$, then $1 \otimes t \in \mathrm{Hom}^{+}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$. It should be noted that the vector subspace $\mathrm{Hom}^{+}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$ is not a Clifford-submodule of $\mathrm{Hom}_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$. Finally, consider the following G-submodules:

$$\mathrm{Ker}(D_{\mathcal{E}}) = \mathrm{Ker} \ \left(D_{\mathcal{E}} : C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}) \to C^{\infty}(G/H, \mathcal{S}_{\mathfrak{q}} \otimes \mathcal{E}) \right),$$

$$\mathrm{Ker}(D_{\mathcal{E}, \mathcal{E}'}) = \mathrm{Ker} \ \left(D_{\mathcal{E}, \mathcal{E}'} : C^{\infty}(G/H, \mathcal{H}om_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')) \right.$$

$$\left. \longrightarrow C^{\infty}(G/H, \mathcal{H}om_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')) \right),$$

$$\mathrm{Ker}^{+}(D_{\mathcal{E}, \mathcal{E}'}) = \mathrm{Ker} \ (D_{\mathcal{E}, \mathcal{E}'}) \cap C^{\infty}(G/H, \mathcal{H}om^{+}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')).$$

We define $\mathrm{Ker}(\widehat{D}_{\mathcal{E}'})$ and $\mathrm{Ker}^+(\widehat{D}_{\mathcal{E},\mathcal{E}'})$ in a similar way. Since G and H are assumed to have the same complex rank, then, by Theorem 4.6 of [10], we may choose bundles $\mathcal{E}$ and $\mathcal{E}'$ such that both $\mathrm{Ker}(D_{\mathcal{E}})$ and $\mathrm{Ker}(D_{\mathcal{E}'})$ are not reduced to $\{0\}$.

Theorem 3.1.   *We have:*

$$\text{I. } \Psi(Ker^+(\widehat{D}_{\mathcal{E},\mathcal{E}'}), Ker(\widehat{D}_{\mathcal{E}})) \subseteq Ker(\widehat{D}_{\mathcal{E}'}).$$

$$\text{II. } \Psi(Ker^+(\widehat{D}_{\mathcal{E},\mathcal{E}'}), Ker(D_{\mathcal{E}})) \subseteq Ker(D_{\mathcal{E}'}).$$

$$\text{III. } \Psi(Ker^+(D_{\mathcal{E},\mathcal{E}'}), Ker(D_{\mathcal{E}})) \subseteq Ker(\widehat{D}_{\mathcal{E}'}).$$

In other words, the "product" of two usual harmonic spinors is a usual harmonic spinor, whereas the "product" of a usual harmonic spinor and a cubic harmonic spinor is a cubic harmonic spinor. Moreover the "product" of two cubic harmonic spinors is a usual harmonic spinor.

Proof.   For all $f \otimes T \in \left(C^\infty(G) \otimes \mathrm{Hom}_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E') \right)^H$,

$\phi \otimes s \otimes v \in \left(C^\infty(G) \otimes (S_{\mathfrak{q}} \otimes E) \right)^H$ and $T \in \mathrm{Hom}_{\mathbf{C}}(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$, we have:

$$
\begin{aligned}
& D_{\mathcal{E}'}(\Psi(f \otimes T, \phi \otimes s \otimes v)) \\
=\ & \Psi(D_{\mathcal{E},\mathcal{E}'}(f \otimes T), \phi \otimes s \otimes v) + \Psi(f \otimes T, D_{\mathcal{E}}(\phi \otimes s \otimes v)) \\
+\ & \sum a_j (R(X_j)(f\phi)) \otimes \Big((\gamma(X_j) \otimes 1)(T(s \otimes v)) - T((\gamma(X_j) \otimes 1)(s \otimes v)) \Big) \\
-\ & f\phi \otimes (\gamma(c) \otimes 1)(T(s \otimes v)).
\end{aligned}
$$

In particular, if $T \in \mathrm{Hom}^+(S_{\mathfrak{q}} \otimes E, S_{\mathfrak{q}} \otimes E')$, this formula can be rewritten in different ways:

$$\widehat{D}_{\mathcal{E}'}(\Psi(f \otimes T, \phi \otimes s \otimes v)) = \Psi(\widehat{D}_{\mathcal{E},\mathcal{E}'}(f \otimes T), \phi \otimes s \otimes v) + \Psi(f \otimes T, \widehat{D}_{\mathcal{E}}(\phi \otimes s \otimes v)),$$

$$D_{\mathcal{E}'}(\Psi(f \otimes T, \phi \otimes s \otimes v)) = \Psi(\widehat{D}_{\mathcal{E},\mathcal{E}'}(f \otimes T), \phi \otimes s \otimes v) + \Psi(f \otimes T, D_{\mathcal{E}}(\phi \otimes s \otimes v)),$$

$$\widehat{D}_{\mathcal{E}'}(\Psi(f \otimes T, \phi \otimes s \otimes v)) = \Psi(D_{\mathcal{E},\mathcal{E}'}(f \otimes T), \phi \otimes s \otimes v) + \Psi(f \otimes T, D_{\mathcal{E}}(\phi \otimes s \otimes v)).$$

∎

4.   The discrete series and the Enright-Varadarajan modules

At the outset, we state that while the contents of this section may seem to be unrelated to the main theorem in the previous section, the purpose of this section is to illustrate that theorem by resorting to an algebraic construction of discrete series representations. Let $\{\pi(\mu)\}_{\mu \in \Lambda}$ be a coherent family of (virtual) representations (see [2]). Typically in a positive cone contained in the parameter space Λ, this arises via the Zuckerman translation functor

$$F(\nu)^\star \otimes \pi(\mu + \nu) \longrightarrow \pi(\mu).$$

where $F(\nu)^\star$ is the contragredient of the finite dimensional $\mathfrak{g}$-representation $F(\nu)$ with highest weight ν, with respect to some positive system. Suppose that:

$\pi(\mu)$ is realized as harmonics in some bundle $\mathcal{S} \otimes \mathcal{E}_\mu$,

$\pi(\mu + \nu)$ is realized as harmonics in some bundle $\mathcal{S} \otimes \mathcal{E}_{\mu+\nu}$,

$F(\nu)^\star$ is realized as harmonics in some bundle $\mathcal{H}om^+(S \otimes E_{\mu+\nu}, S \otimes E_\mu)$.

This gives rise to a setup as in Theorem 3.1. Putting these ideas in a valid framework involves:

(a) checking whether it works algebraically (without bothering about geometry like homogeneous spaces, homogeneous bundles and the geometric Dirac operator) working purely in the context of the representation theoretic Dirac operator.

(b) (having accomplished (a)). We try to make a transition from the purely algebraic considerations to its geometric counterpart. In this endeavour we are guided by many instances of such a relationship.

   1. Kostant's study of $\mathfrak{n}$-cohomology of finite dimensional modules led to his proof of the Borel-Weil-Bott Theorem.

   2. The geometric result: certain spaces of square integrable harmonic spinors (twisted by a homogeneous bundle) on a symmetric space vanish. The underlying algebraic observation: Dirac Inequality for unitarizability of a $(\mathfrak{g}, K)$-module.

   3. Algebraic result (Gross, Kostant, Ramond and Sternberg [4]): certain multiplets of representations of a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ associated to a $\mathfrak{g}$-representation occur as the kernel of Kostant's (algebraic) cubic Dirac operator. Geometric result (Landweber [8]): the above irreducible G-representation arises as the kernel of a geometric cubic Dirac operator twisted by a homogeneous bundle arising from one of the above H-multiplets. Landweber figuratively refers to this construction as "putting the algebraic consideration on its head" or "effectively inverting the algebraic construction".

   We did not really pursue the lead (b) above as it involves an elaborate lay-out which, in our opinion, may not be worth the time demanded of a prospective reader at present. Instead we stopped after following lead (a) as we felt it was sufficiently interesting for its own sake. Outside of the realm of symmetric spaces (where the cubic term vanishes) not much is available in the literature about non-zero solutions for the cubic Dirac (except for the above mentioned results of Kostant, Landweber - which anyway account for finite dimensional representations). For infinite dimensional representations and non-compact homogeneous spaces a beginning is made in [10]. Since we do not know any analogue of the Enright-Varadarajan construction for these modules we did not attempt to write this section in the context of the cubic Dirac operator.

   In this section, we assume that G is semisimple with finite center and $H = K$. In particular, the assumption (2.1) is satisfied and the cubic term (2.3) vanishes. Moreover the form $\langle , \rangle_{\mathfrak{q}_0}$ (restriction of the Killing form to $\mathfrak{q}_0 \times \mathfrak{q}_0$) is positive definite on $\mathfrak{q}_0 = \mathfrak{s}_0$, so that the basis (2.2) is now an orthonormal basis of

$\mathfrak{s}_0$ with $a_j = 1$ for all j. We shall simply write (s, S) for the spin representation of $\mathfrak{k}_0$. In particular, a $(\mathfrak{g}, K)$-module $(\pi, \mathcal{H})$ defines a Dirac operator:

$$\mathcal{D}_\pi : \mathcal{H} \otimes S \to \mathcal{H} \otimes S,$$
$$\mathcal{D}_\pi \overset{\text{def}}{=} \sum_j \pi(X_j) \otimes \gamma(X_j). \tag{4.1}$$

As we have already mentioned in the introduction, when G has a compact Cartan subgroup, discrete series representations arise as a particular case of Enright-Varadarajan modules ([3][12]). More precisely, let $\mathfrak{r}$ be a θ-stable Borel subalgebra of $\mathfrak{g}$. Fix a θ-stable Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ such that $\mathfrak{c} \subset \mathfrak{r}$ and $\mathfrak{b} \overset{\text{def}}{=} \mathfrak{c} \cap \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$. Let Π be the corresponding (θ-stable) system of positive roots. We fix a Borel subalgebra $\mathfrak{r}_\mathfrak{k}$ of $\mathfrak{k}$ by $\mathfrak{r}_\mathfrak{k} \overset{\text{def}}{=} \mathfrak{r} \cap \mathfrak{k}$. The corresponding positive system of roots of $\mathfrak{k}$ with respect to $\mathfrak{b}$ is denoted $\Pi_\mathfrak{k}$. We shall denote by δ and $\delta_\mathfrak{k}$ the half-sums of Π and $\Pi_\mathfrak{k}$ respectively. Let μ be a regular integral weight which is dominant with respect to Π. Consider the Verma module $V_{\mathfrak{g}, \Pi, -\mu-\delta}$ for $\mathfrak{g}$ with highest weight $-\mu-\delta$ with respect to Π and the Verma modules $V_{\mathfrak{k}, \Pi_\mathfrak{k}, -\mu-\delta}$ for $\mathfrak{k}$ with $\Pi_\mathfrak{k}$-highest weight given by the restriction of $-\mu-\delta$ to $\mathfrak{b}$. Evidently, $V_{\mathfrak{k}, \Pi_\mathfrak{k}, -\mu-\delta}$ can be canonically identified with the $\mathcal{U}(\mathfrak{k})$-module generated by the highest weight vector of $V_{\mathfrak{g}, \Pi, -\mu-\delta}$. There is a unique $\Pi_\mathfrak{k}$-dominant integral weight η such that $V_{\mathfrak{k}, \Pi_\mathfrak{k}, -\mu-\delta} \subseteq V_{\mathfrak{k}, \Pi_\mathfrak{k}, \eta}$. The $\mathfrak{k}$-module $V_{\mathfrak{k}, \Pi_\mathfrak{k}, \eta}$ and the $\mathfrak{g}$-module $V_{\mathfrak{g}, \Pi, -\mu-\delta}$ can both be simultaneously imbedded in a $\mathfrak{g}$-module $W_{\mathfrak{r}, \mu}$, compatible with the prolongement $V_{\mathfrak{k}, \Pi_\mathfrak{k}, -\mu-\delta} \subset V_{\mathfrak{k}, \Pi_\mathfrak{k}, \eta}$ and having nice properties. Some of the important properties of the inclusions $V_{\mathfrak{g}, \Pi, -\mu-\delta} \subseteq W_{\mathfrak{r}, \mu}$ and $V_{\mathfrak{k}, \Pi_\mathfrak{k}, \eta} \subseteq W_{\mathfrak{r}, \mu}$ are the following (see [3]):

(i) $W_{\mathfrak{r}, \mu}$ has a unique irreducible quotient $\mathfrak{g}$-module $D_{\mathfrak{r}, \mu}$ which is $\mathfrak{k}$-finite,

(ii) the irreducible finite dimensional $\mathfrak{k}$-module $F_{\mathfrak{k}, \eta}$ with $\Pi_\mathfrak{k}$- highest weight η occurs with multiplicity one in $D_{\mathfrak{r}, \mu}$,

(iii) if $\chi_{\Pi, -\mu}$ denotes the algebra homomorphism from $U(\mathfrak{g})^\mathfrak{k}$ into $\mathbf{C}$ defining the scalar by which $u \in \mathcal{U}(\mathfrak{g})^\mathfrak{k}$ acts on the highest weight vector of $V_{\mathfrak{g}, \Pi, -\mu-\delta}$, then the same homomorphism defines the action of $\mathcal{U}(\mathfrak{g})^\mathfrak{k}$ on $F_{\mathfrak{k}, \eta} \subseteq D_{\mathfrak{r}, \mu}$.

When G has a compact Cartan subgroup, which we assume to hold, it is a result due to Wallach ([12]) that the $(\mathfrak{g}, K)$-module $D_{\mathfrak{r}, \mu}$ is isomorphic to the space of $\mathfrak{k}$-finite vectors in a discrete class representation of G.

Let ν be Π-dominant integral and $F(\nu)$ the finite dimensional irreducible representation for $\mathfrak{g}$ with highest weight ν. Assume that $\mu + \nu$ is Π-dominant and regular, so that we have the irreducible $(\mathfrak{g}, K)$-modules $D_{\mathfrak{r}, \mu}$ and $D_{\mathfrak{r}, \mu+\nu}$ as above both belonging to the discrete series. We have a canonical inclusion

$$\varphi : D_{\mathfrak{r}, \mu} \hookrightarrow D_{\mathfrak{r}, \mu+\nu} \otimes F(\nu)$$

which is a consequence of the inclusion of Verma modules for $\mathfrak{g}$:

$$V_{\mathfrak{g}, \Pi, -\mu-\delta} \hookrightarrow V_{\mathfrak{g}, \Pi, -\mu-\nu-\delta} \otimes F(\nu).$$

In turn this gives rise to an inclusion

$$\varphi_S : D_{\mathfrak{r}, \mu} \otimes S \hookrightarrow D_{\mathfrak{r}, \mu+\nu} \otimes F(\nu) \otimes S.$$

Moreover, we have a map

$$\beta : (D_{\mathfrak{r},\mu+\nu} \otimes S) \otimes (F(\nu) \otimes S) \otimes S^{\star} \to D_{\mathfrak{r},\mu+\nu} \otimes F(\nu) \otimes S$$

by contracting the second factor S and the fifth factor $S^{\star}$. Now, we can state our result.

Theorem 4.2. *Denote by W_1, W_2 and W_3 the kernel of the Dirac operator (4.1) associated with $D_{\mathfrak{r},\mu+\nu}$, $F(\nu)$ and $D_{\mathfrak{r},\mu}$ respectively. We have:*

$$\varphi_S(W_3) \subseteq \beta(W_1 \otimes W_2 \otimes S^{\star}). \tag{4.3}$$

Proof. To show this we first study the kernel of the Dirac operator acting on $V_{\mathfrak{g},\Pi,-\mu-\nu-\delta} \otimes S$, $V_{\mathfrak{g},\Pi,-\mu-\delta} \otimes S$ and $F(\nu) \otimes S$.

It is easy to describe the kernel of the Dirac operator acting on $F(\nu) \otimes S$. This kernel is the $\mathcal{U}(\mathfrak{k})$-span of $x \otimes s$ where x is a weight vector of $F(\nu)$ of weight $w \cdot \nu$ in the Weyl group orbit of ν and is $\Pi_{\mathfrak{k}}$-dominant and s is a $\Pi_{\mathfrak{k}}$-highest weight vector of an irreducible component of S of weight $-\delta_{\mathfrak{k}} + w \cdot \delta$.

Next, we describe the kernel of the Dirac operator $\mathcal{D}$ acting on $D_{\mathfrak{r},\mu+\nu} \otimes S$ as in (4.1). (Similar remarks for $D_{\mathfrak{r},\mu} \otimes S$ will hold.) Since $D_{\mathfrak{r},\mu+\nu} \otimes S$ is a discrete class module, the kernel of the Dirac operator $\mathcal{D}$ acting on $D_{\mathfrak{r},\mu+\nu} \otimes S$ is an irreducible $\mathfrak{k}$-module. We describe its highest weight.

Let y' be a $\Pi_{\mathfrak{k}}$-highest weight vector of weight γ' in $V_{\mathfrak{g},\Pi,-\mu-\nu-\delta} \otimes S$ annihilated by the Dirac operator. Assume that $\gamma' + \delta_{\mathfrak{k}}$ is $-\Pi_{\mathfrak{k}}$-dominant and non-singular. Let γ be a $\Pi_{\mathfrak{k}}$-dominant integral weight such that $\gamma + \delta_{\mathfrak{k}} \in W_{\mathfrak{k}} \cdot (\gamma' + \delta_{\mathfrak{k}})$. Then there is a $\Pi_{\mathfrak{k}}$-highest weight vector $\tilde{y}$ of $W_{\mathfrak{r},\mu+\nu} \otimes S$ of weight γ such that $\mathcal{U}(\mathfrak{k}) \cdot y' \subseteq \mathcal{U}(\mathfrak{k}) \cdot \tilde{y}$. It is not difficult to show that $\tilde{y}$ (hence, also its image y in $D_{\mathfrak{r},\mu+\nu} \otimes S$) is in the kernel of the Dirac operator. We apply these observations by making the simplest choice for y'. Namely, $y' = x \otimes s$, where x is the Π-highest weight vector of $V_{\mathfrak{r},\Pi,-\mu-\nu-\delta}$ and s is the $\Pi_{\mathfrak{k}}$-highest weight vector of S of weight $\delta - \delta_{\mathfrak{k}}$.

Let $\overline{y}'$ be a $\Pi_{\mathfrak{k}}$-highest weight vector of weight $\overline{\gamma}'$ in $V_{\mathfrak{g},\Pi,-\mu-\delta} \otimes S$ annihilated by the Dirac operator. Assume that $\overline{\gamma}' + \delta_{\mathfrak{k}}$ is $-\Pi_{\mathfrak{k}}$-dominant and non-singular. Let $\overline{\gamma}$ be a $\Pi_{\mathfrak{k}}$-dominant integral weight such that $\overline{\gamma} + \delta_{\mathfrak{k}} \in W_{\mathfrak{k}} \cdot (\overline{\gamma}' + \delta_{\mathfrak{k}})$. Then there is a $\Pi_{\mathfrak{k}}$-highest weight vector $\tilde{\overline{y}}$ of $W_{\mathfrak{r},\mu} \otimes S$ of weight $\overline{\gamma}$ such that $\mathcal{U}(\mathfrak{k}) \cdot \overline{y}' \subseteq \mathcal{U}(\mathfrak{k}) \cdot \tilde{\overline{y}}$. One can show that $\tilde{\overline{y}}$ (hence, also its image $\overline{y}$ in $D_{\mathfrak{r},\mu} \otimes S$) is in the kernel of the Dirac operator. Apply these observations by choosing $\overline{y}' = \overline{x} \otimes s$, where $\overline{x}$ is the Π-highest weight vector of $V_{\mathfrak{r},\Pi,-\mu-\delta}$ and s is the $\Pi_{\mathfrak{k}}$-highest weight vector of S of weight $\delta - \delta_{\mathfrak{k}}$.

The statement analogous to result stated in (4.3) relating

$$(V_{\mathfrak{g},\Pi,-\mu-\nu-\delta} \otimes S) \otimes (F(\nu) \otimes S) \otimes S^{\star}$$

and $(V_{\mathfrak{g},\Pi,-\mu-\delta} \otimes S)$, namely the fact that $\varphi_S(\overline{y}' \otimes s) \in \beta(y' \otimes W_2 \otimes S^{\star})$, is evident and this can be traced to statements relating $(W_{\mathfrak{r},\mu+\nu} \otimes S) \otimes (F(\nu) \otimes S) \otimes S^{\star}$ and $(W_{\mathfrak{r},\mu} \otimes S)$ and these in turn can finally be related to $(D_{\mathfrak{r},\mu+\nu} \otimes S) \otimes (F(\nu) \otimes S) \otimes S^{\star}$ and $(D_{\mathfrak{r},\mu} \otimes S)$. ∎

5. Open questions

In this section, we let G be a connected non-compact real reductive Lie group. As in section 2, we do not assume that the center of G is finite.

Question 5.1.   *Under which condition(s), do we have $Ker^+(D_{\mathcal{E},\mathcal{E}'}) \neq \{0\}$?*
Note that if the cubic term c vanishes (as is the case when G/H is a symmetric space) then $Ker^+(D_{\mathcal{E},\mathcal{E}}) \neq \{0\}$.

Question 5.2.   *Prove the most general results analogous to Theorem 4.2 for any coherent family of virtual modules, for the Zuckerman translation functor which goes forward from μ to $\mu + \nu$ and for the other which goes backward from $\mu + \nu$ to μ.*

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