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The Representation Aspect of the Generalized Hydrogen Atoms

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Abstract. Let $D \geq 1$ be an integer. In the Enright-Howe-Wallach classification list of the unitary highest weight modules of $\widetilde{\mathrm{Spin}}(2, D+1)$, the (nontrivial) Wallach representations in Case II, Case III, and the mirror of Case III are special in the sense that they are precisely the ones that can be realized by the Hilbert space of bound states for a generalized hydrogen atom in dimension D. It has been shown recently that each of these special Wallach representations can be realized as the space of L^2-sections of a canonical hermitian bundle over the punctured $\mathbb{R}^D$. Here a simple algebraic characterization of these special Wallach representations is found.
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1. Introduction

The generalized hydrogen atoms, discovered in the late 60s by McIntosh and Cisneros [6] and independently by Zwanziger [12], are hypothetic atoms where the nucleus carries both electric and magnetic charges. Their extension to dimension five were obtained by Iwai [5] in the early 90s, their construction and preliminary analysis in all dimensions higher than or equal to three were given about two years ago by this author [7], and their extension to dimensions one and two will be given in appendix A of this paper.

The main purpose here is to elaborate on the representation theoretical aspect of the generalized hydrogen atoms on the one hand and to give a simple algebraic characterization of a special family of Wallach representations on the other hand. The message I wish to convey to mathematical physicists is that the generalized hydrogen atoms are mathematically beautiful, and the message I wish to convey to mathematicians is that, for the (spin-)conformal group of the (compactified) Minkowski spaces, the Wallach representations in Case II, Case III, and the mirror of Case III from the classification list of Ref. [3] admit a very simple algebraic characterization.

For readers who are only interested in mathematics, theorems 1 and 2 below are our main mathematical results, theorem 3 below can be skipped, and any

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paragraph involving the phrases such as "generalized hydrogen atoms" or "MICZ-Kepler problems" can be ignored; for example, the entire appendix A can be ignored. In other word, this is a mathematical paper which is rigorous by the current mathematical standard, but it is motivated by physical problems and it enhances our understanding of the physical models.

To state the main results, we need to first recall some basic facts and introduce some notations.

1.1. Pseudo-orthogonal groups. Let p, q be nonnegative integers such that $p+q\geq 2$. Denote by x^μ the μ-th standard coordinate for $\mathbb{R}^{p,q}$, and by η the standard indefinite metric tensor whose coordinate matrix $[\eta_{\mu\nu}]$ with respect to the standard basis of $\mathbb{R}^{p,q}$ is $\mathrm{diag}(\underbrace{1,\cdots,1}_{p},\underbrace{-1,\cdots-1}_{q})$. As usual, we use $[\eta^{\mu\nu}]$ to denote the inverse of $[\eta_{\mu\nu}]$, $\mathrm{O}(p,q)$ to denote the set of endomorphisms of $\mathbb{R}^{p,q}$ which preserve the quadratic form η, and $\mathrm{O}^+(p,q)$ to denote the connected component of $\mathrm{O}(p,q)$ containing the identity. Note that it is customary to write $\mathrm{O}(0,q)$ as $\mathrm{O}(q)$ and $\mathrm{O}(p,0)$ as $\mathrm{O}(p)$. The followings are some basic topological facts about the pseudo-orthogonal groups:

Proposition 1.1. *1) $\mathrm{O}(p)$ is compact and has two connected components.*

2) In the case both p and q are nonzero, $\mathrm{O}(p,q)$ is non-compact and has four connected components. In fact, the inclusion map $\mathrm{O}(p)\times\mathrm{O}(q)\to\mathrm{O}(p,q)$ is a homotopic equivalence.

3) The inclusion map $\mathrm{O}^+(p)\times\mathrm{O}^+(q)\to\mathrm{O}^+(p,q)$ is a homotopic equivalence. In fact, $\mathrm{O}^+(p)\times\mathrm{O}^+(q)$ is a maximum compact subgroup of $\mathrm{O}^+(p,q)$.

Let $\mathscr{C}_{p,q}$ be the Clifford algebra over $\mathbb{C}$ generated by X_μ's subject to relations

$$X_\mu X_\nu + X_\nu X_\mu = -2\eta_{\mu\nu}.$$

Let $M_{\mu\nu} := \frac{i}{4}(X_\mu X_\nu - X_\nu X_\mu)$, then one can check that these M's satisfy the following commutation relations:

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i\left(\eta_{\beta\gamma}M_{\alpha\delta} - \eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\beta\delta}M_{\alpha\gamma} + \eta_{\alpha\delta}M_{\beta\gamma}\right).$$

We use $\mathrm{Spin}(p)$ to denote the nontrivial double cover of $\mathrm{SO}(p)$, and $\mathrm{Spin}(2,q)$ to denote the nontrivial double cover of $\mathrm{O}^+(2,q)$ such that the inverse image of $\mathrm{SO}(2)\times\mathrm{SO}(q)$ under the covering map is

$$\mathrm{Spin}(2)\times_{\mathbb{Z}_2}\mathrm{Spin}(q) := \frac{\mathrm{Spin}(2)\times\mathrm{Spin}(q)}{(g_1,g_2)\sim(-g_1,-g_2)}.$$

Note that $\mathrm{Spin}(2,q)$ defined here is connected. We use $\widetilde{\mathrm{Spin}}(2,q)$ to denote the unique double cover of $\mathrm{Spin}(2,q)$ such that the inverse image of $\mathrm{Spin}(2)\times_{\mathbb{Z}_2}\mathrm{Spin}(q)$ under the covering map is $\mathrm{Spin}(2)\times\mathrm{Spin}(q)$.

1.2. Main Mathematical Results. Let G be one of the following real Lie groups: $\mathrm{Spin}(2n)$, $\mathrm{Spin}(2n+1)$, $\mathrm{Spin}(2,2n)$, $\widetilde{\mathrm{Spin}}(2,2n+1)$. We use $\mathfrak{g}_0$ to denote the Lie algebra of G and $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. In case G is non-compact, we use K to denote a maximal compact subgroup of G.

When G is compact, the representations of G are all unitarizable (hence reducible); moreover, an irreducible representation of G is precisely a finite dimensional highest weight modules of $\mathfrak{g}$ with half integral weights.

When G is non-compact, the (continuous) representations of G are not always unitarizable. It is known that a nontrivial unitarizable module of G must be infinite dimensional. By a fundamental theorem of Harish-Chandra[1], the irreducible unitary representations of G are in one-one correspondence with the irreducible unitary $(\mathfrak{g}, K)$-modules. Recall that a representation of G is called a highest weight representation if its underlying $(\mathfrak{g}, K)$-module is a highest weight $\mathfrak{g}$-module. It is known from the definitions and preceding quoted theorem of Harish-Chandra that a highest weight representation of G is an irreducible representation of G. While the unitary highest weight representations of G has been classified in Refs. [10, 11, 3], a classification list for unitary irreducible representations of G is still missing in general. Please note that a representation of G is sometime also called a G-module.

The following problem arises naturally from the construction and analysis of the generalized hydrogen atoms.

Problem 1.2. Classify all unitary highest weight representations of G subject to the following representation relations in the universal enveloping algebra of $\mathfrak{g}_0$: $\{M_{\mu\lambda}, M^\lambda{}_\nu\} = a\eta_{\mu\nu}$, i.e.,

$$\boxed{M_{\mu\lambda}M^\lambda{}_\nu + M^\lambda{}_\nu M_{\mu\lambda} = a\eta_{\mu\nu}} \tag{1.1}$$

where a is a representation-dependent real number, and $M^\lambda{}_\nu = \eta^{\lambda\delta}M_{\delta\nu}$.

It is not hard to see that a is completely determined by the value of the Casimir operator c_2 of $\mathfrak{g}_0$ in a given representation. In the case when G is non-compact, Eq. (1.1) should be understood as an identity for operators on the underlying $(\mathfrak{g}, K)$-module. Hereafter we shall call Eq. (1.1) the (quadratic) representation relations.

Remark 1.3. In the compact case, the representation relations appear first in the preliminary study of the dynamical symmetry of the generalized MICZ-Kepler problems [7]; and in the non-compact case, the representation relations appear first in the study of MICZ-Kepler problems [2], and more recently in the refined study of the dynamical symmetry of the generalized MICZ-Kepler problems [9, 8].

Throughout this paper, we adopt this practice in physics: the Lie algebra generators act as hermitian operators in all unitary representations.

The main mathematical results of this paper are summarized in the following two theorems.

Theorem 1. *Let $n > 0$ be an integer.*

1) An irreducible unitary module of $\mathrm{Spin}(2n+1)$ *satisfies Eq. (1.1) $\Leftrightarrow$ it is either the trivial representation or the fundamental spin representation.*

2) An irreducible unitary module of $\mathrm{Spin}(2n)$ *satisfies Eq. (1.1) $\Leftrightarrow$ it is a Young power of a fundamental spin representation.*

[1] See, for example, Theorem 7 on page 71 of Ref. [1]

Theorem 2. *Let $n > 0$ be an integer.*

1) A unitary highest weight module of $\widetilde{\mathrm{Spin}}(2, 2n+1)$ *satisfies Eq. (1.1)* $\Leftrightarrow$ *it is either the trivial one or the one with highest weight*[2]

$$(-(n+\mu-\frac{1}{2}), \mu, \cdots, \mu)$$

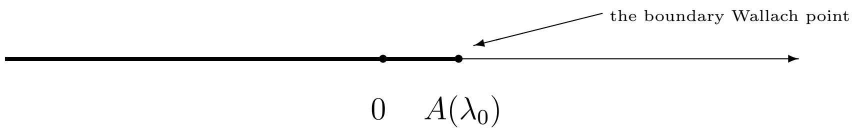
for $\mu = 0$ *or* $1/2$.

2) A unitary highest weight module of $\mathrm{Spin}(2, 2n)$ *satisfies Eq. (1.1)* $\Leftrightarrow$ *it is either the trivial one or the one with highest weight*

$$(-(n+|\mu|-1), |\mu|, \cdots, |\mu|, \mu)$$

for some half integer μ.

Remark 1.4. The representations characterized in part 1) are precisely the Wallach representations in Case II ($\mu = 0$) and Case III ($\mu = 1/2$) on page 128 of Ref. [3]. The representations characterized in part 2) are precisely the Wallach representations in Case II ($\mu = 0$), Case III ($\mu < 0$) and the mirror of Case III ($\mu > 0$) on page 125 of Ref. [3]. In the Enright-Howe-Wallach classification diagram for the unitary highest weight modules, there are two reduction points in Case II and one reduction point in Case III; the nontrivial representations characterized here always sit on the first reduction point, and the trivial representation (in Case II only) always sits on the 2nd reduction point. In other word, the nontrivial representations characterized here are precisely those boundary Wallach points in Case II, Case III and the mirror of Case III (see, page 101, Ref. [3]):



The following subsection is about a corollary of Theorem 2 for the generalized hydrogen atoms and can be safely ignored for readers who are only interested in mathematics.

1.3. Main corollary for the generalized hydrogen atoms.

Let $D \geq 1$ be an integer, μ be a half integer if D is even and be 0 or 1/2 if D is odd. To fix the terminology in this paper, by the *generalized hydrogen atom* in dimension D with magnetic charge μ we mean the hypothetic atom in dimension D whose coulomb problem is the D-dimensional (quantum) MICZ-Kepler problem with magnetic charge μ in the sense of Ref. [7].

For the convenience of the readers, here we will give a quick review of the D-dimensional (quantum) MICZ-Kepler problems. We assume $D \geq 3$ and leave the case D = 1 or 2 to appendix A.

Let $\mathbb{R}^D_*$ be the punctured D-space (i.e., $\mathbb{R}^D$ with the origin removed), S^{D-1} be the unit sphere: $\{\vec{r} \mid |\vec{r}| = 1\}$. As we know, there is a canonical principal $\mathrm{Spin}(D-1)$-bundle $\mathrm{Spin}(D) \to \mathrm{S}^{D-1}$ with a canonical connection[3]. Via the natural

[2]Unlike the case in part 2) of this theorem, a representation here cannot descend to a representation of $\mathrm{Spin}(2, 2n+1)$.

[3]The connection form is $\mathrm{Pr}_{\mathfrak{so}(D-1)}(g^{-1}\, dg)$, where $g^{-1}\, dg$ is the Maurer-Cartan form and $\mathrm{Pr}_{\mathfrak{so}(D-1)}$ is the orthogonal projection from $\mathfrak{so}(D)$ onto $\mathfrak{so}(D-1)$.

retraction map $\mathbb{R}^D_* \to \mathrm{S}^{D-1}$, we get a canonical principal $\mathrm{Spin}(D-1)$-bundle with a canonical connection over $\mathbb{R}^D_*$. By choosing the representation of $\mathfrak{so}(D-1)$ with highest weight $(|\mu|, \cdots, |\mu|, \mu)$, we get an associated hermitian vector bundle with an hermitian connection on Riemannian manifold $(\mathbb{R}^D_*; dx_1^2 + \cdots + dx_D^2)$.

This bundle is denoted by $\mathcal{S}^{2\mu}$ and it is our analogue of the Dirac monopole with magnetic charge μ. By definition, the *D-dimensional MICZ-Kepler problem with magnetic charge μ* is defined to be the quantum mechanical system on $\mathbb{R}^D_*$ for which the wave-functions are sections of $\mathcal{S}^{2\mu}$ and the hamiltonian is

$$H = \begin{cases} -\frac{1}{2}\Delta_\mu + \frac{\mu^2+(n-1)|\mu|}{2r^2} - \frac{1}{r} & \text{if } D = 2n+1 \\ \\ -\frac{1}{2}\Delta_\mu + \frac{(n-1)\mu}{2r^2} - \frac{1}{r} & \text{if } D = 2n \end{cases} \tag{1.2}$$

where Δ_μ is the standard Laplace operator $\partial_1^2 + \cdots + \partial_D^2$ twisted by $\mathcal{S}^{2\mu}$.

Physically it is interesting to find all square integrable eigen-sections of H. It has been shown in Refs. [9, 8] that the linear span of the square integrable eigen-sections of H is a unitary highest weight Harish-Chandra module with highest weight

$$(-\left(\frac{D-1}{2} + |\mu|\right), |\mu|, \cdots, |\mu|, \mu).$$

Recall that, the Hilbert space completion of this linear span is called *the Hilbert space of bound states of the D-dimensional generalized hydrogen atom with magnetic charge μ*; so, in view of the fundamental theorem of Harish-Chandra we quoted earlier, it is a nontrivial unitary highest weight representation of $\widetilde{\mathrm{Spin}}(2, D+1)$.

It has been shown in Refs. [9, 8] that such a unitary highest weight representation of $\widetilde{\mathrm{Spin}}(2, D+1)$ has a very explicit geometric realization. To describe it, we let d^Dx be the Lebesgue measure on $\mathbb{R}^D$. The Hilbert space of square integrable (with respect to d^Dx) sections of $\mathcal{S}^{2\mu}$ (denoted by $L^2(\mathcal{S}^{2\mu})$), being identified with *the twisted Hilbert space of bound states of the D-dimensional generalized hydrogen atom with magnetic charge μ*, turns out to be the representation space. To describe the unitary action of $\widetilde{\mathrm{Spin}}(2, D+1)$ on $L^2(\mathcal{S}^{2\mu})$, we just need to describe the infinitesimal action on $C^\infty(\mathcal{S}^{2\mu})$; for that purpose, it suffices to describe how $M_{\alpha,0}$ $(1 \leq \alpha \leq D)$, $M_{D+1,0}$ and $M_{-1,0}$ act as differential operators: they act as $i\sqrt{r}\nabla_\alpha\sqrt{r}$, $\frac{1}{2}\left(\sqrt{r}\Delta_\mu\sqrt{r} + r - \frac{c}{r}\right)$ and $\frac{1}{2}\left(\sqrt{r}\Delta_\mu\sqrt{r} - r - \frac{c}{r}\right)$ respectively. For example, for $\psi \in C^\infty(\mathcal{S}^{2\mu})$, we have

$$(M_{\alpha,0} \cdot \psi)(r, \Omega) \;=\; i\sqrt{r}\nabla_\alpha\left(\sqrt{r}\psi(r, \Omega)\right).$$

Therefore, together with the results in appendix A, we have the following corollary of Theorem 2 for the generalized hydrogen atoms.

Theorem 3. *Let $D \geq 1$ be an integer.*

1) The Hilbert space of bound states of a D-dimensional generalized hydrogen atom always forms a nontrivial unitary highest weight representation of $\widetilde{\mathrm{Spin}}(2, D+1)$.

2) A nontrivial unitary highest weight representation of $\widetilde{\mathrm{Spin}}(2, D+1)$ *can be realized by the Hilbert space of bound states of a D-dimensional generalized hydrogen atom $\Leftrightarrow$ it satisfies the quadratic representation relations.*

Therefore, the Hilbert spaces of bound states for D-dimensional generalized hydrogen atoms realize precisely the nontrivial Wallach points for $\widetilde{\mathrm{Spin}}(2, D+1)$ listed in Case II, Case III and the mirror of Case III on page 127 (when D is odd), and on page 125 (when D is even) in Ref. [3]. Note that when D is odd, the mirror of Case III is Case I_p with $p = \frac{D+1}{2}$; when D is odd, Case III = the mirror of Case III.

We end this subsection with the following concluding remark.

Remark 1.5. The interesting families of representations of $\widetilde{\mathrm{Spin}}(2, D+1)$ form the following descending chain:

$\{$*admissible irreps*$\} \supset \{$*unirreps*$\} \supset \{$*H.WT. unitary reps*$\} \supset \{$*Wallach reps*$\} \supset \{$*nontrivial Wallach reps of type II, type III or mirror of type III*$\}$.

For the bottom family of representations in this chain, combining the results from Refs. [9, 8], one can reach the following conclusions:

1) The members of this family can be precisely realized as the Hilbert space of bound states for generalized hydrogen atoms in dimension D;

2) Each member of this family can be realized as the Hilbert space of L^2-sections of a canonical hermitian bundle over $\mathbb{R}^D_*$ equipped with a canonical hermitian connection;

3) This family can be characterized by a canonical finite set of quadratic relations among the infinitesimal generators of $\widetilde{\mathrm{Spin}}(2, D+1)$.

1.4. Outline of the paper. As a warm up, we will first give a proof of Theorem 1 in section 2, the idea is essentially taken from the appendix of Ref. [7] and the arguments are purely algebraic. Then we prove Theorem 2 by similar arguments in section 3. I would like to thank Qi You for simplifying the proof of part 2) of Theorem 1 and the referee for his/her careful reading of the manuscript.

2. Proof of Theorem 1

We will follow the approach in the appendix of Ref. [7]. The idea is to find a convenient Cartan basis and then rewrite the representation relations in terms of these Cartan basis elements. We start with the proof of part 1) because it is technically simpler. The proof of part 2) is similar, but technically is a bit more involved.

2.1. Part 1). We assume that $n \geq 1$. To continue, a digression on Lie algebra $\mathfrak{so}(2n+1)$ is needed. Recall that the root space of $\mathfrak{so}(2n+1)$ is $\mathbb{R}^n$. Let e^i be the vector in $\mathbb{R}^n$ whose i-th entry is 1 and all other entries are zero. The positive roots are $e^i \pm e^j$ with $1 \leq i < j \leq n$ and e^k with $1 \leq k \leq n$. Following Ref. [4], we choose the following Cartan basis for $\mathfrak{so}(2n+1)$:

$$\left\{\begin{array}{rcl} H_i & = & M_{2i-1,2i} \qquad 1 \leq i \leq n \\ E_{\eta e^j + \eta' e^k} & = & \frac{1}{2}\left(M_{2j-1,2k-1} + i\eta M_{2j,2k-1} + i\eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k}\right) \\ & & \quad \text{for } j < k \\ E_{\eta e^j} & = & \frac{1}{\sqrt{2}}\left(M_{2j-1,2n+1} + i\eta M_{2j,2n+1}\right) \text{ for } j \leq n \end{array}\right.$$

where $\eta, \eta' \in \{1, -1\}$. For convenience, we also use the same expression above to define $E_{\eta e^j + \eta' e^k}$ when $j > k$, then we have

$$E_{\eta e^j + \eta' e^k} = -E_{\eta' e^k + \eta e^j}$$

for $j \neq k$.

We are interested in unitary representations, i.e., representations such that each M_{ij} acts as an hermitian operator, or equivalently, each H_i act as an hermitian operator, and

$$(E_\alpha)^\dagger = E_{-\alpha}.$$

Let $|\Omega\rangle = |\lambda_1 \cdots \lambda_n\rangle$ be the highest weight state of a unitary representation for which the representation relations hold. So $H_i|\Omega\rangle = \lambda_i|\Omega\rangle$ and $E_\alpha|\Omega\rangle = 0$ if α is a positive root. Since

$$[E_{e^i-e^j}, E_{-e^i+e^j}] = H_i - H_j, \quad [E_{e^i}, E_{-e^i}] = H_i, \tag{2.1}$$

by the unitarity, we conclude that

$$\lambda_1 \geq \cdots \geq \lambda_n \geq 0. \tag{2.2}$$

Since $\{E_{e^i}, E_{-e^i}, H_i\}$ span the Lie algebra of $\mathfrak{su}(2)$, and the orbit of $|\Omega\rangle$ under the action of the universal enveloping algebra of this $\mathfrak{su}(2)$ is a highest weight representation with $|\Omega\rangle$ as its highest weight state, we conclude that λ_i is a half integer. A similar argument shows that $\lambda_i - \lambda_j$ is always an integer.

$\Rightarrow$: The representation relations say that, for $1 \leq j \leq n+1$, we have

$$\langle\Omega|\sum_k (M_{2j-1,k})^2|\Omega\rangle = c, \tag{2.3}$$

where c is a constant independent of j. Since

$$\begin{cases} \sum_k (M_{2j-1,k})^2 &= H_j^2 + \frac{H_j}{2} + \frac{1}{2}\sum_{i\neq j}\left(\{E_{-e^j-e^i}, E_{e^j+e^i}\} + \{E_{-e^j+e^i}, E_{e^j-e^i}\}\right) \\ & \quad +\frac{1}{2}((E_{-e^j})^2 + (E_{e^j})^2) + E_{-e^j}E_{e^j} \\ & \quad +\sum_{i\neq j}\left(E_{-e^j-e^i}E_{-e^j+e^i} + E_{e^j+e^i}E_{e^j-e^i}\right) \quad \text{for } 1 \leq j \leq n, \\ \sum_k (M_{2n+1,k})^2 &= \sum_i \{E_{e^i}, E_{-e^i}\}, \end{cases}$$

We have

$$\begin{cases} \lambda_1^2 + (n-\frac{1}{2})\lambda_1 &= c \\ \lambda_2^2 + (n-\frac{1}{2})\lambda_2 + (\lambda_1 - \lambda_2) &= c \\ \lambda_3^2 + (n-\frac{1}{2})\lambda_3 + (\lambda_1 + \lambda_2 - 2\lambda_3) &= c \\ & \vdots \\ \lambda_n^2 + (n-\frac{1}{2})\lambda_n + (\lambda_1 + \cdots + \lambda_{n-1} - (n-1)\lambda_n) &= c \\ \sum \lambda_i &= c. \end{cases} \tag{2.4}$$

Subtracting 2nd identity from the 1st identity, we have

$$(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 + n - \frac{3}{2}) = 0.$$

So $\lambda_1 = \lambda_2 = \lambda$ if $n \geq 2$. Assume $n \geq 3$, subtracting the 3rd identity from the 1st identity, we have

$$(\lambda - \lambda_3)(\lambda + \lambda_2 + n - \frac{5}{2}) = 0.$$

So $\lambda_3 = \lambda$ if $n \geq 3$. By repeating this argument $(n-1)$ times, we get $\lambda_1 = \cdots = \lambda_n = \lambda$. Then $|\Omega\rangle = |\lambda \cdots \lambda\rangle$.

By equating the 1st identity with the last identity, we have

$$\lambda^2 = \frac{1}{2}\lambda,$$

then $\lambda = 0$ or $1/2$. The case that $\lambda = 0$ corresponds to the trivial representation and the case that $\lambda = 1/2$ corresponds to the fundamental spin representation.

$\Leftarrow$: The representation relations are trivially true in the former case, and can be checked easily by using Clifford algebra in the later case: $M_{jk} \propto X_j X_k$, so

$$\{M_{jk}, M_{kl}\} \propto X_j X_k X_k X_l + X_k X_l X_j X_k \propto -X_j X_l - X_l X_j = 2\delta_{jl}.$$

End of the proof of part 1) of Theorem 1.

2.2. Part 2). It is trivial when $n = 1$. So we assume that $n \geq 2$. To continue, a digression on Lie algebra $\mathfrak{so}(2n)$ is needed. Recall that the root space of $\mathfrak{so}(2n)$ is $\mathbb{R}^n$. Let e^i be the vector in $\mathbb{R}^n$ whose i-th entry is 1 and all other entries are zero. The positive roots are $e^i \pm e^j$ with $1 \leq i < j \leq n$. Following Ref. [4], we choose the following Cartan basis for $\mathfrak{so}(2n)$:

$$\begin{cases} H_i = M_{2i-1,2i} \qquad 1 \leq i \leq n \\ E_{\eta e^j + \eta' e^k} = \frac{1}{2}\left(M_{2j-1,2k-1} + i\eta M_{2j,2k-1} + i\eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k}\right) \\ \qquad \text{for } j < k \end{cases}$$

where $\eta, \eta' \in \{1, -1\}$. For convenience, we also use the same expression above to define $E_{\eta e^j + \eta' e^k}$ for $j > k$, then we have

$$E_{\eta e^j + \eta' e^k} = -E_{\eta' e^k + \eta e^j}$$

for $j \neq k$.

Let $|\Omega\rangle = |\lambda_1 \cdots \lambda_n\rangle$ be the highest weight state of a unitary representation for which the representation relations hold. So $H_i|\Omega\rangle = \lambda_i|\Omega\rangle$ and $E_\alpha|\Omega\rangle = 0$ if α is a positive root.

Since

$$[E_{e^i \pm e^j}, E_{-e^i \mp e^j}] = H_i \pm H_j, \tag{2.5}$$

by the unitarity, we conclude that

$$\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|. \tag{2.6}$$

Since $\{E_{e^i+e^j}, E_{-e^i-e^j}, \frac{1}{2}(H_i + H_j)\}$ span the Lie algebra of $\mathfrak{su}(2)$, we conclude that $\lambda_i - \lambda_j$ is an integer. A similar argument shows that $\lambda_i + \lambda_j$ is an integer. So λ_i's are half integers.

$\Rightarrow$: The representation relations say that, for $1 \leq j \leq n$, we have

$$\langle\Omega| \sum_k (M_{2j-1,k})^2 |\Omega\rangle = c, \tag{2.7}$$

where c is a constant independent of j. Since

$$\sum_k (M_{2j-1,k})^2 = H_j^2 + \frac{1}{2}\sum_{i \neq j}\left(\{E_{-e^j-e^i}, E_{e^j+e^i}\} + \{E_{-e^j+e^i}, E_{e^j-e^i}\}\right)$$

$$+\sum_{i\neq j}\left(E_{-e^j-e^i}E_{-e^j+e^i}+E_{e^j+e^i}E_{e^j-e^i}\right),$$

we have

$$\left\{\begin{array}{rcl}\lambda_1^2+(n-1)\lambda_1 & = & c\\ \lambda_2^2+(n-1)\lambda_2+(\lambda_1-\lambda_2) & = & c\\ \lambda_3^2+(n-1)\lambda_3+(\lambda_1+\lambda_2-2\lambda_3) & = & c\\ & \vdots & \\ \lambda_n^2+(n-1)\lambda_n+(\lambda_1+\cdots+\lambda_{n-1}-(n-1)\lambda_n) & = & c.\end{array}\right. \tag{2.8}$$

Subtracting the 2nd identity from the 1st identity, we have

$$(\lambda_1-\lambda_2)(\lambda_1+\lambda_2+n-2)=0.$$

So if $\lambda_1=|\lambda_2|$ if $n=2$, and $\lambda_1=\lambda_2=\lambda$ if $n>2$. Assume $n\geq 3$, subtracting the 3rd identity from the 1st identity, we have

$$(\lambda-\lambda_3)(\lambda+\lambda_2+n-3)=0.$$

So if $\lambda=|\lambda_3|$ if $n=3$, and $\lambda_3=\lambda$ if $n>3$. By repeating this argument $(n-1)$ times, we get $\lambda_1=\cdots=\lambda_{n-1}=\lambda$ and $\lambda=|\lambda_n|$. Then the representation must be a Young power of a fundamental spin representation.

$\Leftarrow$: We need to prove that the representation relations (i.e., Eq. (1.1)) hold for any Young power of a fundamental spin representation. The proof is broken into three steps, with the last one being significantly simplified by Qi You.

Step one. We may assume the representation is $\mathbf{s}_+^{2\mu}$ for some non-negative half integer μ. That is because there exists a $g\in\mathrm{Pin}(2n)$ such that the action by g on $\mathbf{s}_-^{2\mu}\oplus\mathbf{s}_+^{2\mu}$ produces a vector space isomorphism: $\mathbf{s}_-^{2\mu}\to\mathbf{s}_+^{2\mu}$, moreover, $gM_{1,k}g^{-1}=-M_{1,k}$ and $gM_{j,k}g^{-1}=M_{j,k}$ for $1<j<k$; consequently, the representation relations are invariant under the (adjoint) action by g.

Step two. For any $i<j$, relation

$$\sum_k\{M_{i,k},M_{j,k}\}=0 \tag{2.9}$$

hold for $\mathbf{s}_+^{2\mu}$.

Proof. It suffice to prove the statement in the case $i=1$ and $j=2$; that is because, for any $i'<j'$, there is an element in $g\in\mathrm{Spin}(2n)$ such that

$$g\sum_k\{M_{1,k},M_{2,k}\}g^{-1}=\sum_k\{M_{i',k},M_{j',k}\}.$$

Next we observe that

$$\sum_k\{M_{1,k},M_{2,k}\}=\frac{2}{i}(\mathscr{O}^\dagger-\mathscr{O})$$

where

$$\mathscr{O}=\sum_{i\neq 1}E_{-e^1-e^i}E_{-e^1+e^i}.$$

Consequently, we can finish the proof by showing that

$$\mathscr{O}|\Lambda\rangle = 0 \tag{2.10}$$

for any $|\Lambda\rangle \in \mathbf{s}_+^{2\mu}$. But that is OK because of the following easy facts:

$$\begin{array}{rcl} [\mathscr{O}, E_{-\alpha}] & = & 0 \quad \text{for any positive root } \alpha, \\ E_{-e^1+e^i}|\Omega\rangle & = & 0 \quad \text{where } |\Omega\rangle = |\underbrace{\mu\cdots\mu}_{n}\rangle, \end{array}$$

and the fact that $|\Lambda\rangle$ is a linear combination of the states created from $|\Omega\rangle$ by some $E_{-\alpha}$'s with α being positive roots. ■

Step three. For any j, relation

$$\sum_k (M_{j,k})^2 - \frac{1}{n}c_2 \quad = \quad 0 \tag{2.11}$$

hold for $\mathbf{s}_+^{2\mu}$. In fact, it suffices to show that relation

$$\sum_k (M_{1,k})^2 - \frac{1}{n}c_2 \quad = \quad 0 \tag{2.12}$$

hold for $\mathbf{s}_+^{2\mu}$.

Proof. Observe that[4]

$$\begin{array}{rcl} [M_{ab}, \sum_k (M_{1,k})^2 - \frac{1}{n}c_2] & = & -i\eta_{b1}\sum_k \{M_{ak}, M_{1k}\} + i\eta_{a1}\sum_k \{M_{bk}, M_{1k}\} \\ & = & 0 \quad \text{on } \mathbf{s}_+^{2\mu} \text{ by step two above.} \end{array}$$

Therefore, it suffices to show that

$$\left(\sum_k (M_{1,k})^2 - \frac{1}{n}c_2\right)|\Omega\rangle = 0. \tag{2.13}$$

But that is not hard, because

$$\begin{array}{rcl} \sum_k (M_{1,k})^2 - \frac{1}{n}c_2 & = & \mathscr{O}_1 + \mathscr{O}^\dagger + \mathscr{O} \\ & = & \mathscr{O}_1 \quad \text{on } \mathbf{s}_+^{2\mu} \text{ by Eq. (2.10).} \\ & = & 0 \quad \text{on } |\Omega\rangle \text{ by a straight forward calculation,} \end{array}$$

where

$$\mathscr{O}_1 \quad = \quad H_1^2 - \frac{c_2}{n} + \frac{1}{2}\sum_{i\neq 1}\left(\{E_{-e^1-e^i}, E_{e^1+e^i}\} + \{E_{-e^1+e^i}, E_{e^1-e^i}\}\right).$$

■

Steps two and three together say that the representation relations hold in $\mathbf{s}_+^{2\mu}$, hence also hold in $\mathbf{s}_-^{2\mu}$ by step one.

End of the proof of part 2) of Theorem 1.

[4]The much simplified proof presented here is due to this key observation by Qi You.

3. Proof of Theorem 2

The proof of theorem 2 is similar to that of theorem 1, but technically more involved. Again, we start with the proof of part 1). Although a straightforward proof does exist, to make the proof shorter, we use results from both Refs. [9, 8] and appendix A.

3.1. Part 1). We assume that $n \geq 1$. To continue, a digression on Lie algebra $\mathfrak{so}(2, 2n+1)$ is needed. Recall that the root space of $\mathfrak{so}(2, 2n+1)$ is $\mathbb{R}^{n+1}$. Let e^i be the vector in $\mathbb{R}^{n+1}$ whose i-th entry is 1 and all other entries are zero. The positive roots are $e^i \pm e^j$ with $0 \leq i < j \leq n$ and e^k with $0 \leq k \leq n$.

Following Ref. [4], we choose the following Cartan basis for $\mathfrak{so}(2, 2n+1)$:

$$\left\{\begin{array}{rcl} H_0 & = & M_{-1,0}, \\ H_i & = & -M_{2i-1,2i} \qquad 1 \leq i \leq n, \\ E_{\eta e^j + \eta' e^k} & = & \frac{1}{2}\left(M_{2j-1,2k-1} + i\eta M_{2j,2k-1} + i\eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k}\right) \\ & & \quad \text{for } 0 \leq j < k \leq n, \\ E_{\eta e^j} & = & \frac{1}{\sqrt{2}}\left(M_{2j-1,2n+1} + i\eta M_{2j,2n+1}\right) \quad \text{for } 0 \leq j \leq n, \end{array}\right.$$

where $\eta, \eta' \in \{1, -1\}$. For convenience, we also use the same expression above to define $E_{\eta e^j + \eta' e^k}$ for $j > k$, then we have

$$E_{\eta e^j + \eta' e^k} = -E_{\eta' e^k + \eta e^j}$$

for $j \neq k$.

Let $|\Omega\rangle = |\lambda_0 \lambda_1 \cdots \lambda_n\rangle$ be the highest weigh state of a unitary representation for which the representation relations hold. So $H_i|\Omega\rangle = \lambda_i|\Omega\rangle$ and $E_\alpha|\Omega\rangle = 0$ if α is a positive root.

Since

$$\left\{\begin{array}{r} [E_{e^0+e^i}, E_{-e^0-e^i}] = -H_0 - H_i, \quad [E_{e^0-e^i}, E_{-e^0+e^i}] = -H_0 + H_i, \\ [E_{e^i+e^j}, E_{-e^i-e^j}] = H_i + H_j, \quad [E_{e^i-e^j}, E_{-e^i+e^j}] = H_i - H_j, \\ [E_{\eta e^i}, E_{\eta' e^j}] = -iE_{\eta e^i + \eta' e^j}, \ [E_{e^i}, E_{-e^i}] = H_i, \quad [E_{e^0}, E_{-e^0}] = -H_0, \end{array}\right.$$

by unitarity, we conclude that

$$-\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0. \tag{3.1}$$

For $i \neq 0$, $\{E_{e^i}, E_{-e^i}, H_i\}$ span the Lie algebra of $\mathfrak{su}(2)$, then λ_i must be a half integer. A similar argument shows that $\lambda_i - \lambda_j$ is an integer for $0 < i < j \leq n$.

$\Rightarrow$: The representation relations say that

$$\left\{\begin{array}{rcl} \langle\Omega| - \sum M_{-1,k} M^k{}_{-1}|\Omega\rangle & = & c, \\ \\ \langle\Omega| \sum M_{2j-1,k} M^k{}_{2j-1}|\Omega\rangle & = & c \quad \text{for } j = 1, 2, \cdots, n+1, \end{array}\right. \tag{3.2}$$

where c is a constant. Since

$$\begin{cases} \sum M_{-1,k}M^k{}_{-1} &= -H_0^2-\frac{H_0}{2}+\frac{1}{2}\sum_{i\neq 0}\left(\{E_{-e^0-e^i},E_{e^0+e^i}\}+\{E_{-e^0+e^i},E_{e^0-e^i}\}\right)\\ &\quad +\frac{1}{2}((E_{-e^0})^2+(E_{e^0})^2)+E_{-e^0}E_{e^0}\\ &\quad +\sum_{i\neq 0}\left(E_{-e^0-e^i}E_{-e^0+e^i}+E_{e^0+e^i}E_{e^0-e^i}\right),\\ \sum M_{2j-1,k}M^k{}_{2j-1} &= H_j^2+\frac{H_j}{2}+\frac{1}{2}\sum_{i\neq 0,j}\left(\{E_{-e^j-e^i},E_{e^j+e^i}\}+\{E_{-e^j+e^i},E_{e^j-e^i}\}\right)\\ &\quad +\frac{1}{2}((E_{-e^j})^2+(E_{e^j})^2)+E_{-e^j}E_{e^j}\\ &\quad +\sum_{i\neq 0,j}\left(E_{-e^j-e^i}E_{-e^j+e^i}+E_{e^j+e^i}E_{e^j-e^i}\right)\\ &\quad -\frac{1}{2}\left(\{E_{-e^0-e^j},E_{e^0+e^j}\}+\{E_{-e^0+e^j},E_{e^0-e^j}\}\right)\\ &\quad -\left(E_{-e^0-e^j}E_{-e^0+e^j}+E_{e^0+e^j}E_{e^0-e^j}\right),\\ \sum M_{2n+1,k}M^k{}_{2n+1} &= -\{E_{e^0},E_{-e^0}\}+\sum_{i>0}\{E_{e^i},E_{-e^i}\}, \end{cases}$$

we have

$$\begin{cases} \lambda_0^2+(n+\frac{1}{2})\lambda_0 &= c\\ \lambda_1^2+(n-\frac{1}{2})\lambda_1+\lambda_0 &= c\\ \lambda_2^2+(n-\frac{1}{2})\lambda_2+(\lambda_1-\lambda_2)+\lambda_0 &= c\\ &\vdots\\ \lambda_n^2+(n-\frac{1}{2})\lambda_n+(\lambda_1+\cdots+\lambda_{n-1}-(n-1)\lambda_n)+\lambda_0 &= c\\ \sum\lambda_i &= c. \end{cases} \tag{3.3}$$

Subtracting the 3rd identity from the 2nd identity, we have

$$(\lambda_1-\lambda_2)(\lambda_1+\lambda_2+n-\frac{3}{2})=0.$$

So $\lambda_1=\lambda_2=\lambda$ if $n\geq 2$. Assume $n\geq 3$, subtracting the 4th identity from the 3rd identity, we have

$$(\lambda-\lambda_3)(\lambda+\lambda_2+n-\frac{5}{2})=0.$$

So $\lambda_3=\lambda$ if $n\geq 3$. By repeating this argument $(n-1)$ times, we get $\lambda_1=\cdots=\lambda_n=\lambda$. Then $|\Omega\rangle=|\lambda_0\lambda\cdots\lambda\rangle$.

By comparing the 2nd with the last identities, we get

$$\lambda^2=\frac{1}{2}\lambda,$$

so $\lambda=0$ or $1/2$.

By comparing the first two identities, we get

$$(\lambda-\lambda_0)(\lambda+\lambda_0+n-\frac{1}{2})=0.$$

So either $\lambda_0=\lambda$ or $\lambda_0=-(\lambda+n-\frac{1}{2})$. In view of the fact that $-\lambda_0\geq\lambda$, we conclude that (λ_0,λ) must be one of the following three pairs: $(0,0)$, $(-n+1/2,0)$, $(-n,1/2)$. Consequently, the unitary highest weight representation, if it exists, must be one of the following three cases: 1) the trivial one, 2) the one with highest weight $(-n+1/2,0,\ldots,0)$, 3) the one with highest weight $(-n,1/2,\ldots,1/2)$.

$\Leftarrow$: The remaining question we must answer is this: such representations do exist and satisfy the representation relations. This is certainly clear in the trivial case.

The existence of such representations in the nontrivial case is clear from the classification result of Refs. [10, 11, 3]. As a matter of fact, in view of Theorem 1 in Ref. [8], the one with highest weight $(-n+1/2, 0, \ldots, 0)$ can be realized by the $2n$-dimensional generalized Kepler problem with magnetic charge 0, and the one with highest weight $(-n, 1/2, \ldots, 1/2)$ can be realized by the $2n$-dimensional generalized Kepler problem with magnetic charge $1/2$. Moreover, in view of part 2) of Theorem 2 in Ref. [8], these representations indeed satisfy the representation relations. The only problem with this argument is that the case $n=1$ is not covered; however, the results in Refs. [7, 8] can be pushed down to the case $n=1$, see subsection A in appendix A.

End of the proof of part 1) of Theorem 2.

We would like to remark that, by following the argument in the proof of part 2) of Theorem 1, one can also verify the representation relations directly. Since this argument is a bit long, we choose to skip it.

3.2. Part 2). We assume that $n \geq 1$. To continue, a digression on Lie algebra $\mathfrak{so}(2,2n)$ is needed. Recall that the root space of $\mathfrak{so}(2,2n)$ is $\mathbb{R}^{n+1}$. Let e^i be the vector in $\mathbb{R}^{n+1}$ whose i-th entry is 1 and all other entries are zero. The positive roots are $e^i \pm e^j$ with $0 \leq i < j \leq n$.

Following Ref. [4], we choose the following Cartan basis for $\mathfrak{so}(2,2n)$:

$$\left\{\begin{array}{rcl} H_0 & = & M_{-1,0}, \\ H_i & = & -M_{2i-1,2i} \qquad 1 \leq i \leq n, \\ E_{\eta e^j+\eta' e^k} & = & \frac{1}{2}\left(M_{2j-1,2k-1} + i\eta M_{2j,2k-1} + i\eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k}\right) \\ & & \qquad\qquad \text{for } 0 \leq j < k \leq n. \end{array}\right.$$

Here $\eta, \eta' \in \{1,-1\}$. For convenience, we also use the same expression above to define $E_{\eta e^j+\eta' e^k}$ for $j > k$, then we have

$$E_{\eta e^j+\eta' e^k} = -E_{\eta' e^k+\eta e^j}$$

for $j \neq k$.

Let $|\Omega\rangle = |\lambda_0\lambda_1\cdots\lambda_n\rangle$ be the highest weigh state of a representation for which the representation relations hold. So $H_i|\Omega\rangle = \lambda_i|\Omega\rangle$ and $E_\alpha|\Omega\rangle = 0$ if α is a positive root.

Since

$$\left\{\begin{array}{l} [E_{e^0+e^i}, E_{-e^0-e^i}] = -H_0 - H_i, \quad [E_{e^0-e^i}, E_{-e^0+e^i}] = -H_0 + H_i, \\ \qquad [E_{e^i+e^j}, E_{-e^i-e^j}] = H_i + H_j, \quad [E_{e^i-e^j}, E_{-e^i+e^j}] = H_i - H_j, \end{array}\right.$$

by unitarity, we conclude that

$$-\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|. \tag{3.4}$$

Just as before, one can show that each λ_i with $i > 0$ is a half integer and each $\lambda_i - \lambda_j$ with $0 < i < j \leq n$ is an integer.

$\Rightarrow$: The representation relations say that

$$\left\{\begin{array}{rcl} \langle\Omega| - \sum M_{-1,k}{M^k}_{-1}|\Omega\rangle & = & c, \\ \langle\Omega| \sum M_{2j-1,k}{M^k}_{2j-1}|\Omega\rangle & = & c \quad \text{for } j = 1, 2, \cdots, n, \end{array}\right. \tag{3.5}$$

where c is a constant. Since

$$\begin{cases} \sum M_{-1,k}M^k{}_{-1} & = & -H_0^2+\frac{1}{2}\sum_{i\neq 0}\left(\{E_{-e^0-e^i},E_{e^0+e^i}\}+\{E_{-e^0+e^i},E_{e^0-e^i}\}\right) \\ & & +\sum_{i\neq 0}\left(E_{-e^0-e^i}E_{-e^0+e^i}+E_{e^0+e^i}E_{e^0-e^i}\right), \\ \sum M_{2j-1,k}M^k{}_{2j-1} & = & H_j^2+\frac{1}{2}\sum_{i\neq 0,j}\left(\{E_{-e^j-e^i},E_{e^j+e^i}\}+\{E_{-e^j+e^i},E_{e^j-e^i}\}\right) \\ & & +\sum_{i\neq 0,j}\left(E_{-e^j-e^i}E_{-e^j+e^i}+E_{e^j+e^i}E_{e^j-e^i}\right) \\ & & -\frac{1}{2}\left(\{E_{-e^0-e^j},E_{e^0+e^j}\}+\{E_{-e^0+e^j},E_{e^0-e^j}\}\right) \\ & & -\left(E_{-e^0-e^j}E_{-e^0+e^j}+E_{e^0+e^j}E_{e^0-e^j}\right), \end{cases}$$

we have

$$\begin{cases} \lambda_0^2+n\lambda_0 & = & c \\ \lambda_1^2+(n-1)\lambda_1+\lambda_0 & = & c \\ \lambda_2^2+(n-1)\lambda_2+(\lambda_1-\lambda_2)+\lambda_0 & = & c \\ & \vdots & \\ \lambda_n^2+(n-1)\lambda_n+(\lambda_1+\cdots+\lambda_{n-1}-(n-1)\lambda_n)+\lambda_0 & = & c. \end{cases} \tag{3.6}$$

Subtracting the 3rd identity from the 2nd identity, we have

$$(\lambda_1-\lambda_2)(\lambda_1+\lambda_2+n-2)=0.$$

So $\lambda_1=\lambda_2=\lambda$ if $n>2$ and $\lambda_2=|\lambda_1|$ if $n=2$. Assume $n\geq 3$, subtracting the 4th identity from the 3rd identity, we have

$$(\lambda-\lambda_3)(\lambda+\lambda_2+n-3)=0.$$

So $\lambda_1=\lambda_2=\lambda_3$ if $n>3$ and $\lambda_1=\lambda_2=|\lambda_3|$ if $n=3$. By repeating this argument $(n-1)$ times, we get $\lambda_1=\cdots=\lambda_{n-1}=|\lambda_n|=\lambda$. Therefore, for $n\geq 1$, we have $|\Omega\rangle=|\lambda_0\underbrace{\lambda\cdots\lambda}_{n-1}(\pm\lambda)\rangle$.

By comparing the first two identities, we get

$$(\lambda-\lambda_0)(\lambda+\lambda_0+n-1)=0.$$

In view of the fact that $-\lambda_0\geq\lambda$, we conclude that (λ_0,λ) must be one of following pairs: 1) (0,0), 2) $(-n-\lambda+1,\lambda)$ where $\lambda\geq 0$ is a half integer. Consequently, when $n\geq 1$, the unitary highest weight representation, if it exists, must be one of the following cases: 1) the trivial one, 2) the one with highest weight

$$(-(n-1+|\mu|),\underbrace{|\mu|,\ldots,|\mu|}_{n-1},\mu)$$

for a half integer μ.

$\Leftarrow$: The remaining question we must answer is this: such representations do exist and satisfy the representation relations. This is certainly clear in the trivial case.

The existence of such representations in the nontrivial case is clear from the classification result of Refs. [10, 11, 3]. As a matter of fact, in view of Theorem 1 in Ref. [9], the one with highest weight

$$(-(n-1+|\mu|),\underbrace{|\mu|,\ldots,|\mu|}_{n-1},\mu)$$

can be realized by the $(2n-1)$-dimensional generalized Kepler problem with magnetic charge μ. Moreover, in view of part 2) of Theorem 2 in Ref. [9], these representations indeed satisfy the representation relations. The only problem with this argument is that the case $n=1$ is not covered; however, the results in Refs. [7, 9] can be pushed down to the case $n=1$, see subsection A in appendix A.

End of the proof of part 2) of Theorem 2.

We would like to remark that, by following the argument in the proof of part 2) of Theorem 1, one can also verify the representation relations directly. Since this argument is a bit long, we choose to skip it.

A. MICZ-Kepler problems in dimensions one or two

In Ref. [7], the generalized MICZ-Kepler problems are introduced in dimension three or higher. Here we introduce their limits in dimension one and dimension two. Since the arguments given in Refs. [7, 9, 8] are still valid for these limiting cases, the theorems listed below are stated without detailed proof.

A.1. MICZ-Kepler problems in dimension one.

Definition A.1. Let μ a half integer and $|\mu|\geq 1/2$. Let $\mathbb{R}_\mu$ be $\mathbb{R}_+$ if $\mu<0$ and be $\mathbb{R}_-$ if $\mu>0$. The 1-dimensional MICZ-Kepler problem with magnetic charge μ is defined to be the quantum mechanical system on $\mathbb{R}_\mu$ for which the wave-functions are complex-valued functions on $\mathbb{R}_\mu$ and the hamiltonian is

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+\frac{\mu^2-|\mu|}{2x^2}-\frac{1}{|x|}. \tag{A.1}$$

Let $c=\mu^2-|\mu|$ and $p=-i\frac{d}{dx}$. Define the dynamical symmetry operators as follows:

$$\left\{\begin{array}{rcl} A &=& -\frac{1}{2}\left(xp^2+x+\frac{c}{x}\right),\\ M &=& -\frac{1}{2}\left(xp^2-x+\frac{c}{x}\right)\\ T &=& xp,\\ \Gamma &=& |x|p,\\ \Gamma_{-1} &=& \frac{1}{2}\left(|x|p^2+|x|+\frac{c}{|x|}\right),\\ \Gamma_2 &=& \frac{1}{2}\left(|x|p^2-|x|+\frac{c}{|x|}\right). \end{array}\right. \tag{A.2}$$

Let the capital Latin letters A, B run from -1 to 2. Introduce J_{AB} as follows:

$$J_{AB}=\left\{\begin{array}{ll} A & \text{if } A=1,\, B=2\\ M & \text{if } A=1,\, B=-1\\ \Gamma & \text{if } A=1,\, B=0\\ T & \text{if } A=2,\, B=-1\\ \Gamma_2 & \text{if } A=2,\, B=0\\ \Gamma_{-1} & \text{if } A=-1,\, B=0\\ -J_{BA} & \text{if } A>B\\ 0 & \text{if } A=B. \end{array}\right. \tag{A.3}$$

The following theorem can be proved by direct computation:

Theorem A.2. *Let $C^\infty(\mathbb{R}_\mu)$ be the space of smooth complex-valued functions on $\mathbb{R}_\mu$. Let J_{AB} be defined by (A.3).*

1) As operators on $C^\infty(\mathbb{R}_\mu)$, J_{AB}'s satisfy the following commutation relation:

$$[J_{AB}, J_{A'B'}] = -i\eta_{AA'}J_{BB'} - i\eta_{BB'}J_{AA'} + i\eta_{AB'}J_{BA'} + i\eta_{BA'}J_{AB'}$$

where the indefinite metric tensor η is $\mathrm{diag}\{++--\}$ relative to the following order: -1, 0, 1, 2 for the indices.

2) As operators on $C^\infty(\mathbb{R}_\mu)$,

$$\{J_{AB}, J^A{}_C\} := J_{AB}J^A{}_C + J^A{}_C J_{AB} = 2c\eta_{BC}.$$

Consequently, one can obtain the following two theorems:

Theorem A.3. *For the 1-dimensional MICZ-Kepler problem with magnetic charge μ, the following statements are true:*

1) The negative energy spectrum is

$$E_I = -\frac{1/2}{(I+|\mu|)^2}$$

where $I = 0, 1, 2, \dots$;

2) The Hilbert space $\mathscr{H}$ of negative-energy states admits a linear $\mathrm{Spin}(2)$-action under which there is a decomposition

$$\mathscr{H} = \widehat{\bigoplus}_{I=0}^{\infty} \mathscr{H}_I$$

where $\mathscr{H}_I$ is the irreducible $\mathrm{Spin}(2)$-representation witht weight $(I+|\mu|)sign(\mu)$;

3) $\mathrm{Spin}(1,1)$ acts linearly on the positive-energy states and $\mathbb{R}^1$ acts linearly on the zero-energy states;

4) $\mathscr{H}_I$ in part 2) is the energy eigenspace with eigenvalue E_I in part 1).

Theorem A.4. *Let $\mathscr{H}(\mu)$ be the Hilbert space of bound states for the 1-dimensional generalized MICZ-Kepler problem with magnetic charge μ.*

1) There is a natural unitary action of $\mathrm{Spin}(2,2)$ on $\mathscr{H}(\mu)$. In fact, $\mathscr{H}(\mu)$ is the unitary highest weight module of $\mathrm{Spin}(2,2)$ with highest weight $(-|\mu|, \mu)$; consequently, it occurs at the unique reduction point of the Enright-Howe-Wallach classification diagram[5] for the unitary highest weight modules, so it is a non-discrete series representation.

2) As a representation of subgroup $\mathrm{Spin}(2,1)$,

$$\mathscr{H}(\mu) = \mathcal{D}^-_{2|\mu|} \tag{A.4}$$

where $\mathcal{D}^-_{2|\mu|}$ is the anti-holomorphic discrete series representation[6] of $\mathrm{Spin}(2,1)$ with highest weight $-|\mu|$.

[5] Page 101, Ref. [3]. See also Refs. [10, 11].

[6] The case $\mu = \pm 1/2$ is a limit of the discrete series representation.

3) As a representation of the maximal compact subgroup ($=\mathrm{Spin}(2)\times_{\mathbb{Z}_2}\mathrm{Spin}(2)$),

$$\mathscr{H}(\mu)=\widehat{\bigoplus}_{l=0}^{\infty} D(-l-|\mu|)\otimes D((l+|\mu|)sign(\mu)) \tag{A.5}$$

where $D(\lambda)$ denotes the irreducible module of $\mathrm{Spin}(2)$ *with weight λ.*

A.2. MICZ-Kepler problems in dimension two.

Definition A.5. Let $\mu=0$ or $1/2$. The 2-dimensional MICZ-Kepler problem with magnetic charge μ is defined to be the quantum mechanical system on $\mathbb{R}^2_*$ for which the wave-functions are complex-valued functions ψ on

$$\mathbb{R}^2_*=\mathbb{R}_+\times\mathbb{R}/(r,\theta)\sim(r,\theta+2\pi)$$

satisfying identity

$$\psi(r,\theta+2\pi)=(-1)^{2\mu}\psi(r,\theta)\quad\text{for any }(r,\theta)\in\mathbb{R}_+\times\mathbb{R},$$

and the hamiltonian is

$$H=-\frac{1}{2}\left(\frac{1}{r}\partial_r r\partial_r+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)-\frac{1}{r}. \tag{A.6}$$

Let the small Greek letters α, β run from 1 to 2, $x^1:=r\cos\theta$, $x^2:=r\sin\theta$, $x_\alpha:=x^\alpha$, $p_\alpha:=-i\frac{\partial}{\partial x^\alpha}$. Define the dynamical symmetry operators as follows:

$$\left\{\begin{array}{rcl}
J_{12} & = & x_1p_2-x_2p_1=-i\partial_\theta,\\
A_\alpha & = & \frac{1}{2}x_\alpha p^2-p_\alpha(\vec{r}\cdot\vec{p})-\frac{i}{2}p_\alpha-\frac{1}{2}x_\alpha,\\
M_\alpha & = & \frac{1}{2}x_\alpha p^2-p_\alpha(\vec{r}\cdot\vec{p})-\frac{i}{2}p_\alpha+\frac{1}{2}x_\alpha,\\
T & = & \vec{r}\cdot\vec{p}-\frac{i}{2},\\
\Gamma_\alpha & = & rp_\alpha,\\
\Gamma_{-1} & = & \frac{1}{2}\left(rp^2+r\right),\\
\Gamma_3 & = & \frac{1}{2}\left(rp^2-r\right).
\end{array}\right. \tag{A.7}$$

Let the capital Latin letters A, B run from -1 to 3. Introduce J_{AB} as follows:

$$J_{AB}=\left\{\begin{array}{ll}
J_{12} & \text{if } A=1,\ B=2\\
A_\alpha & \text{if } A=\alpha,\ B=3\\
M_\alpha & \text{if } A=\alpha,\ B=-1\\
\Gamma_\alpha & \text{if } A=\alpha,\ B=0\\
T & \text{if } A=3,\ B=-1\\
\Gamma_3 & \text{if } A=3,\ B=0\\
\Gamma_{-1} & \text{if } A=-1,\ B=0\\
-J_{BA} & \text{if } A>B\\
0 & \text{if } A=B.
\end{array}\right. \tag{A.8}$$

The following theorem can be proved by direct computation:

Theorem A.6. *Let $C^\infty(\mathbb{R}^2_*)$ be the space of smooth complex-valued functions on $\mathbb{R}^2_*$. Let J_{AB} be defined by (A.8).*

1) As operators on $C^\infty(\mathbb{R}^2_*)$, J_{AB}'s satisfy the following commutation relation:

$$[J_{AB}, J_{A'B'}] = -i\eta_{AA'}J_{BB'} - i\eta_{BB'}J_{AA'} + i\eta_{AB'}J_{BA'} + i\eta_{BA'}J_{AB'}$$

where the indefinite metric tensor η is $\mathrm{diag}\{++---\}$ relative to the following order: -1, 0, 1, 2, 3 for the indices.

2) As operators on $C^\infty(\mathbb{R}^2_*)$,

$$\{J_{AB}, J^A{}_C\} := J_{AB}J^A{}_C + J^A{}_C J_{AB} = -\eta_{BC}.$$

Consequently, one can obtain the following two theorems:

Theorem A.7. *For the 2-dimensional MICZ-Kepler problem with magnetic charge μ, the following statements are true:*

1) The negative energy spectrum is

$$E_I = -\frac{1/2}{(I + \mu + \frac{1}{2})^2}$$

where $I = 0, 1, 2, \ldots$;

2) The Hilbert space $\mathscr{H}$ of negative-energy states admits a linear $\mathrm{Spin}(3)$-action under which there is a decomposition

$$\mathscr{H} = \widehat{\bigoplus}_{I=0}^{\infty} \mathscr{H}_I$$

where $\mathscr{H}_I$ is the irreducible $\mathrm{Spin}(3)$-representation with highest weight is $I + \mu$;

3) $\mathrm{Spin}(2,1)$ acts linearly on the positive-energy states and $\mathrm{Spin}(2) \rtimes \mathbb{R}^2$ acts linearly on the zero-energy states;

4) The linear action in part 2) extends the manifest linear action of $\mathrm{Spin}(2)$, and $\mathscr{H}_I$ in part 2) is the energy eigenspace with eigenvalue E_I in part 1).

Theorem A.8. *Let $\mathscr{H}(\mu)$ be the Hilbert space of bound states for the 2-dimensional generalized MICZ-Kepler problem with magnetic charge μ.*

1) There is a natural unitary action of $\widetilde{\mathrm{Spin}}(2,3)$ on $\mathscr{H}(\mu)$ which extends the manifest unitary action of $\mathrm{Spin}(2)$. In fact, $\mathscr{H}(\mu)$ is the unitary highest weight module of $\widetilde{\mathrm{Spin}}(2,3)$ with highest weight $(-(\mu + 1/2), \mu)$; consequently, it occurs at the first reduction point of the Enright-Howe-Wallach classification diagram[7] for the unitary highest weight modules, so it is a non-discrete series representation.

2) As a representation of $\mathrm{Spin}(2,1) \times \mathrm{Spin}(2)$,

$$\mathscr{H}(\mu) = \widehat{\bigoplus}_{l=\mu+\mathbb{Z}} \mathcal{D}^-_{2|l|+1} \otimes D(l) \tag{A.9}$$

where $D(l)$ is the irreducible module of $\mathrm{Spin}(2)$ with weight l and $\mathcal{D}^-_{2|l|+1}$ is the anti-holomorphic discrete series representation of $\mathrm{Spin}(2,1)$ with highest weight $-(|l| + 1/2)$.

[7]Page 101, Ref. [3]. While there is a unique reduction point when $\mu = 1/2$, there are two reduction points when $\mu = 0$. See also Refs. [10, 11].

3) As a representation of the maximal compact subgroup $(=\mathrm{Spin}(2)\times\mathrm{Spin}(3))$,

$$\mathscr{H}(\mu)=\widehat{\bigoplus}_{l=0}^{\infty}D(-(l+\mu+1/2))\otimes D^l \tag{A.10}$$

where D^l is the irreducible module of Spin(3) *with highest weight $l+\mu$ and $D(-(l+\mu+1/2))$ is the irreducible module of* Spin(2) *with weight $-(l+\mu+1/2)$.*

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