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Matsuki's Double Coset Decomposition via Gradient Maps

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Abstract. Let G be a real-reductive Lie group and let G_1 and G_2 be two subgroups given by involutions. We show how the technique of gradient maps can be used in order to obtain a new proof of Matsuki's parametrization of the closed double cosets $G_1\backslash G/G_2$ by Cartan subsets. We also describe the elements sitting in non-closed double cosets.
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Introduction

Let G be a real-reductive Lie group equipped with two involutive automorphisms σ_1 and σ_2 which both commute with a Cartan involution of G. We write G^{σ_j} for the group of σ_j–fixed points and let G_j be an open subgroup of G^{σ_j}. The subject of this paper is to describe how Matsuki's description of the double cosets $G_1\backslash G/G_2$ ([22]) can be proved in a geometric way by using gradient maps and exploiting slice representations.

Let us outline the main results. The product group $G_1 \times G_2$ acts on G by left and right multiplication, i. e. by $(g_1, g_2) \cdot x := g_1xg_2^{-1}$, and the set of double cosets $G_1\backslash G/G_2$ coincides with the orbit space of this action. We will see that the $(G_1 \times G_2)$–orbits in G are generically closed, i. e. that there is a dense open subset G_{sr} of G consisting of closed orbits whose dimension is maximal among all orbits. In [22] the notion of fundamental and standard Cartan subsets is introduced and it is proven via a Jordan-decomposition for elements in G which takes the involutions σ_1 and σ_2 into account that these Cartan subsets are cross sections for the closed $(G_1 \times G_2)$–orbits. We will give a geometric proof of this fact and show that Matsuki's cross sections actually are geometric slices at closed orbits of maximal dimension. Moreover, we will see that locally G_{sr} has the structure of a trivial fiber bundle over a domain in Matsuki's cross sections whose fiber is the closed $(G_1 \times G_2)$–orbit through a point of this domain.

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Included in this setup is the case that G is complex semi-simple and that $\sigma_1 = \sigma_2 =: \sigma$ is anti-holomorphic, i. e. that $G_{\mathbb{R}} := G^\sigma$ is a real form of G. The orbit structure of the closed $(G_{\mathbb{R}} \times G_{\mathbb{R}})$–orbits in G is studied in [4] and [28]. In [5] the set of non-closed orbits is investigated, too. Their analysis is based on the real-algebraic quotient theory available for algebraic actions of complex-reductive groups on affine varieties which are defined over $\mathbb{R}$. In particular, they make use of a good quotient $G/\!\!/(G_{\mathbb{R}} \times G_{\mathbb{R}})$ which parametrizes the closed $(G_{\mathbb{R}} \times G_{\mathbb{R}})$–orbits in G and from which they obtain a stratification of G.

The case that G is a connected reductive algebraic group defined over an algebraically closed field of characteristic not equal to 2 and that σ_1 and σ_2 are commuting regular involutions is studied in [14] with the help of étale slice theorems, stratifications and the categorical quotient. Moreover, they also consider the situation where G is complex reductive and defined over $\mathbb{R}$ such that σ_1, σ_2 are likewise defined over $\mathbb{R}$. Very recently, they have also used the Cartan decomposition of the momentum map in order to describe double coset decompositions of a real form of a complex reductive group ([15]).

In this paper we explain how the presence of a natural $(G_1 \times G_2)$–gradient map on G can be used as a substitute for the methods from the theory of algebraic transformation groups. In particular we obtain the existence of a good quotient $G/\!\!/(G_1 \times G_2)$ and of an isotropy-type stratification from [12] and [29] which provides us from the outset with a lot of information about the set of closed orbits. Afterwards, we analyze the fine structure of the $(G_1 \times G_2)$–action with the help of the isotropy representation on transversal slices in the Lie algebra of G and transfer this infinitesimal information via the Slice Theorem to the group level. It turns out that the isotropy representation on the slice coincides with the adjoint H^σ–representation on $\mathfrak{h}^{-\sigma}$ where (H, σ) is a symmetric reductive Lie group. Therefore we will apply results from [7], [19], [25], and [26] where these representations are investigated.

This paper is organized as follows. In the first section we review the notions of compatible subgroups of complex-reductive groups and gradient maps together with their main properties. In Section 2 we describe the gradient map we use for the $(G_1 \times G_2)$–action on G and investigate in detail the slice representations for this action. Since the slice representations are equivalent to the isotropy representations of reductive symmetric spaces, we investigate these in the third section via a natural gradient map. In Section 4 we use these results to give a geometric proof of the main result in [22] which describes the orbit structure of the closed $(G_1 \times G_2)$–orbits in G. We also describe the non-closed $(G_1 \times G_2)$–orbits. In the last section we consider some examples in order to illustrate our methods and results.

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1. Compatible subgroups and gradient maps

In this section we collect the facts from the theory of gradient maps with main emphasis on the Slice Theorem, the Quotient Theorem and isotropy-type stratifications. Further details and complete proofs can be found in [11], [12] and [29].

Compatible subgroups of complex-reductive groups. Let U be a connected compact Lie group. It is known ([6]) that U carries the structure of a real linear-algebraic group. Let $U^{\mathbb{C}}$ be the corresponding complex-algebraic group. Then $U^{\mathbb{C}}$ is complex-reductive and the inclusion $U \hookrightarrow U^{\mathbb{C}}$ is the universal complexification of U in the sense of [16].

The map $U \times i\mathfrak{u} \to U^{\mathbb{C}}$, $(u,\xi) \mapsto u\exp(\xi)$, is a diffeomorphism, whose inverse is called the Cartan decomposition of $U^{\mathbb{C}}$. Furthermore, the map $\theta\colon U^{\mathbb{C}} \to U^{\mathbb{C}}$, $\theta\bigl(u\exp(\xi)\bigr) := u\exp(-\xi)$, is an anti-holomorphic involutive automorphism of $U^{\mathbb{C}}$ with $U = \mathrm{Fix}(\theta)$, called the Cartan involution of $U^{\mathbb{C}}$ corresponding to the compact real form U. Proofs of these facts can be found for example in [18].

A subgroup G of $U^{\mathbb{C}}$ is called compatible (with the Cartan decomposition of $U^{\mathbb{C}}$) if $G = K\exp(\mathfrak{p})$ for $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$. If G is compatible, then the map $K \times \mathfrak{p} \to G$, $(k,\xi) \mapsto k\exp(\xi)$, is a diffeomorphism. It follows directly from the definition that every compatible subgroup of $U^{\mathbb{C}}$ is invariant under the Cartan involution θ. An open subgroup of a compatible subgroup is again compatible. Moreover, a compatible subgroup $G = K\exp(\mathfrak{p})$ is closed if and only if K is compact, and in this case K is a maximal compact subgroup of G. In this paper a real-reductive Lie group is by definition a closed compatible subgroup of some $U^{\mathbb{C}}$.

Remark 1.1. If a real-reductive group $G = K\exp(\mathfrak{p}) \subset U^{\mathbb{C}}$ is a complex subgroup of $U^{\mathbb{C}}$, then G is automatically complex-reductive with maximal compact subgroup K. Hence, we have $G = K^{\mathbb{C}}$ and $\mathfrak{p} = i\mathfrak{k}$ in this case.

Gradient maps and their properties. Let M be a Riemannian manifold. If $f \in \mathcal{C}^\infty(M)$, then we write ∇f for the gradient vector field of f with respect to the Riemannian metric of M, i. e. $\nabla f \in \mathcal{C}^\infty(M,TM)$ is given by

$$\bigl\langle \nabla f(x), v\bigr\rangle_x = df(x)v$$

for all $x \in M$ and $v \in T_xM$.

Let $G = K\exp(\mathfrak{p})$ be a real-reductive Lie group acting differentiably on M such that the compact group K acts by isometries. Following [24] we call a smooth map $\Phi\colon M \to \mathfrak{p}^*$ a gradient map for the G–action on M if

$$\nabla\Phi^\xi = \xi_M$$

holds for all $\xi \in \mathfrak{p}$. Here, $\Phi^\xi \in \mathcal{C}^\infty(M)$ is defined by $\Phi^\xi(x) := \Phi(x)\xi$, and $\xi_M \in \mathcal{C}^\infty(M,TM)$ is the fundamental vector field induced by $\xi \in \mathfrak{p}$. If such a gradient map exists, we call the G–action on M a gradient action. We will only consider gradient maps Φ which are equivariant with respect to the K–action on M and the co-adjoint K–representation on $\mathfrak{p}^*$.

Example 1.2. Our main example for a gradient action is the following. Let G be realized as a closed compatible subgroup of the complex-reductive group $U^{\mathbb{C}}$. Let Z be a Kähler manifold endowed with a holomorphic action of $U^{\mathbb{C}}$ such that the U–action is Hamiltonian with U–equivariant momentum map $\mu\colon Z \to \mathfrak{u}^*$. Let $M \subset Z$ be a closed G–stable submanifold. If we equip M with the restriction of the Riemannian metric $\langle\cdot,\cdot\rangle$ which corresponds to the Kähler metric on Z, then the G–action on M is a gradient action with K–equivariant gradient map $\Phi := \iota^* \circ (\mu|_M)$, where ι^* is the linear map dual to $\iota\colon \mathfrak{p} \to \mathfrak{u}$, $\xi \mapsto -i\xi$. This can be seen as follows. Since $\mu\colon Z \to \mathfrak{u}^*$ is a U–momentum map, we have for every $\xi \in \mathfrak{u}$ the identity $d\mu^\xi = \omega(\xi_Z,\cdot)$ where ω is the Kähler form of Z and $\mu^\xi \in \mathcal{C}^\infty(Z)$ is given by $\mu^\xi(z) = \mu(z)\xi$. In particular, if $\xi \in \mathfrak{p}$, then we obtain

$$d\mu^{-i\xi} = \omega(-J\xi_Z,\cdot) = \langle\cdot,\xi_Z\rangle,$$

where J denotes the complex structure of Z. Restricting $\mu^{-i\xi}$ to M the claim follows.

Example 1.3. Let V be a finite-dimensional complex vector space with a holomorphic representation $\rho\colon U^{\mathbb{C}} \to \mathrm{GL}(V)$ and let $\langle\cdot,\cdot\rangle$ be a U–invariant Hermitian inner product on V. Then the U–action on V is Hamiltonian with U–equivariant momentum map $\mu\colon V \to \mathfrak{u}^*$, $\mu^\xi(v) := \mu(v)\xi = i\big\langle \rho_*(\xi)v, v\big\rangle$, where ρ_* is the induced representation of $\mathfrak{u}$ on V. If $G = K\exp(\mathfrak{p})$ is a closed compatible subgroup of $U^{\mathbb{C}}$ and if W is a real G–invariant subspace of V, then the map

$$\Phi\colon W \to \mathfrak{p}^*, \quad \Phi^\xi(w) = i\big\langle \rho_*(-i\xi)w, w\big\rangle,$$

is a K–equivariant gradient map with respect to $\mathrm{Re}\langle\cdot,\cdot\rangle|_{W\times W}$ for the G–action on W. This map Φ is called the standard gradient map for the G–representation on W.

Let M be a real G–stable submanifold of a complex Kähler manifold Z endowed with a holomorphic action of $U^{\mathbb{C}}$. We assume that there exists a U–equivariant momentum map $\mu\colon Z \to \mathfrak{u}^*$ and let $\Phi\colon M \to \mathfrak{p}^*$ be the induced K–equivariant gradient map where K acts on $\mathfrak{p}^*$ via the co-adjoint representation. Associated to this map we have its zero fiber $\Phi^{-1}(0)$ and the set of semi-stable points

$$\mathcal{S}_G\big(\Phi^{-1}(0)\big) := \big\{x \in M;\ \overline{G\cdot x} \cap \Phi^{-1}(0) \neq \emptyset\big\}.$$

We will use the following facts from [12].

Proposition 1.4. *If $x \in \mathcal{S}_G\big(\Phi^{-1}(0)\big)$, then $G\cdot x$ is closed in $\mathcal{S}_G\big(\Phi^{-1}(0)\big)$ if and only if $G\cdot x$ intersects $\Phi^{-1}(0)$ non-trivially. If $x \in \Phi^{-1}(0)$, then*

1. *$G\cdot x \cap \Phi^{-1}(0) = K\cdot x$;*
2. *the isotropy subgroup of G at x is compatible, i. e. we have $G_x = K_x\exp(\mathfrak{p}_x)$ with $\mathfrak{p}_x := \big\{\xi \in \mathfrak{p};\ \xi_M(x) = 0\big\}$;*
3. *the isotropy representation of G_x on T_xM is completely reducible.*

By the last statement of Proposition 1.4 there exists a G_x–invariant decomposition $T_xM = \mathfrak{g}\cdot x \oplus W$ where $\mathfrak{g}\cdot x := \{\xi_M(x);\ \xi \in \mathfrak{g}\big\} = T_x(G\cdot x)$. The next theorem gives the existence of a geometric G–slice at points of $\Phi^{-1}(0)$. For its formulation we introduce the following notation. For any subgroup $H \subset G$ and any H–manifold N we write $G\times_H N$ for the quotient manifold of $G\times N$ by the H–action $h\cdot(g,x) := (gh^{-1}, h\cdot x)$. The H–orbit through $(g,x) \in G\times N$ is denoted by $[g,x] \in G\times_H N$.

Theorem 1.5. (Slice Theorem) *For each $x \in \Phi^{-1}(0)$ there exist a G_x–stable open neighborhood S of $0 \in W$, a G–stable open neighborhood Ω of $x \in M$, and a G–equivariant diffeomorphism $G \times_{G_x} S \to \Omega$ with $[e,0] \mapsto x$.*

By abuse of notation we will identify $S \cong [e,S] \subset G \times_{G_x} S$ with its image under the map $G \times_{G_x} S \to \Omega$ and hence obtain $\Omega = G \cdot S$. The map $G \times_{G_x} S \to G \cdot S$ is called a geometric G–slice. The representation of G_x on W is called the slice representation.

In closing we introduce the notion of a topological Hilbert quotient. We call two points $x, y \in M$ equivalent if and only if

$$\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$$

holds. If this relation is an equivalence relation, we denote the corresponding quotient by $\pi \colon M \to M/\!\!/G$ and call it the topological Hilbert quotient of M by the action of G.

Theorem 1.6. (Quotient Theorem) *Suppose that $M = \mathcal{S}_G\big(\Phi^{-1}(0)\big)$. Then the topological Hilbert quotient $\pi \colon M \to M/\!\!/G$ exists and has the following properties.*

1. *Every fiber of π contains a unique closed G–orbit, and every other orbit in the fiber has strictly larger dimension.*
2. *The closure of every G–orbit in a fiber of π contains the closed G–orbit.*
3. *The inclusion $\Phi^{-1}(0) \hookrightarrow M$ induces a homeomorphism $\Phi^{-1}(0)/K \cong M/\!\!/G$.*

Isotropy-type stratifications. Let $G = K \exp(\mathfrak{p}) \subset U^{\mathbb{C}}$ be a reductive Lie group and let M be a G–manifold together with a G–gradient map $\Phi \colon M \to \mathfrak{p}^*$. As above we suppose that M is embedded into a Kähler manifold Z endowed with a holomorphic $U^{\mathbb{C}}$–action such that Φ is induced by a U–equivariant momentum map $\mu \colon Z \to \mathfrak{u}^*$. Moreover, we assume $M = \mathcal{S}_G\big(\Phi^{-1}(0)\big)$ and denote the corresponding quotient by $\pi \colon M \to M/\!\!/G$.

Definition 1.7. For any subgroup $H \subset G$ we define

$$M^{\langle H \rangle} := \big\{x \in M;\ G \cdot x \text{ is closed and } G_x = H\big\}.$$

The saturation $I_H := \pi^{-1}\big(\pi(M^{\langle H \rangle})\big)$ of $M^{\langle H \rangle}$ with respect to π is called the H–isotropy stratum in M.

We collect some properties for later use. The proof of the following theorem can be found in [29].

Theorem 1.8. (Isotropy Stratification Theorem) *1. The manifold M is a disjoint union of the non-empty isotropy strata I_H, and this union is locally finite.*

2. *If $\overline{I_H} \cap I_{H'} \neq \emptyset$ and $I_H \neq I_{H'}$, then there exists a $g \in G$ such that $gHg^{-1} \subsetneq H'$ holds.*
3. *Each stratum I_H is open in its closure $\overline{I_H}$.*

4. *Let $G \times_{G_x} S \to G \cdot S$ be a geometric G–slice at $x \in \Phi^{-1}(0)$, and let $\mathcal{N} := \{w \in W;\ 0 \in \overline{G_x \cdot w}\} \subset W$ be the null cone of the slice representation of G_x. Then we have*
$$I_{G_x} \cap G \cdot S \cong G \times_{G_x} \big(S \cap (W^{G_x} + \mathcal{N})\big).$$
Note that we view S as a G_x–invariant open neighborhood of $0 \in W$ when writing $G \times_{G_x} \big(S \cap (W^{G_x} + \mathcal{N})\big)$.

Remark 1.9. Since the null cone $\mathcal{N}$ is real algebraic in W (see [10]), it makes sense to speak of smooth points of the stratum I_H by Theorem 1.8(4). Moreover, we see that the set of smooth points is open and dense in I_H.

2. Compatible subgroups given by involutive automorphisms and their actions

Regular and strongly regular elements. From now on we fix a closed compatible subgroup $G = K \exp(\mathfrak{p})$ in the complex-reductive group $U^{\mathbb{C}}$. Let θ be the Cartan involution of $U^{\mathbb{C}}$ which defines its compact real form U. Let σ_1 and σ_2 be involutive automorphisms of G which both commute with $\theta|_G$ (but not necessarily with each other).

Remark 2.1. If G is semi-simple, then there exist elements $g_1, g_2 \in G$ such that the Cartan involution $\theta|_G$ commutes with $\sigma_j' := \mathrm{Int}(g_j)\sigma_j \mathrm{Int}(g_j^{-1})$ where $\mathrm{Int}(g_j)$ denotes conjugation by g_j (compare the remark in Section 4.3 in [22]). In the general case let us consider the decomposition $G = G' \cdot Z$, where G' is the semi-simple part of G and Z is the connected component of the neutral element in the center of G. Since all maximal compact subgroups of Z are conjugate, we conclude that $\theta|_Z$ is the unique Cartan involution of Z. Therefore the Cartan involution $\sigma_j'\theta|_Z\sigma_j'$ must coincide with $\theta|_Z$, i. e. σ_j' and $\theta|_Z$ commute. Since $G_1 \backslash G/G_2$ and $g_1G_1g_1^{-1} \backslash G/g_2G_2g_2^{-1}$ are isomorphic, we may assume without loss of generality that $\theta|_G$ commutes with σ_j (see also Corollary 2.2 in [15]).

The composition $\tau := \sigma_2\sigma_1$ is an (in general not involutive) automorphism of G. We only consider involutions for which the restriction of τ to the center of $\mathfrak{g}$ is semi-simple with eigenvalues in the unit circle S^1, i. e. for which $\tau \in \mathrm{Aut}(\mathfrak{g})$ is semi-simple and generates a compact subgroup.

Remark 2.2. If the Lie algebra $\mathfrak{g}$ is semi-simple, then τ is automatically semi-simple with eigenvalues in S^1. This follows from the fact that τ is an isometry of the inner product $\langle \xi_1, \xi_2 \rangle = -B\big(\xi_1, \theta(\xi_2)\big)$ where B is the Killing form of $\mathfrak{g}$.

Let G^{σ_j} be the fixed point sets of σ_j for $j = 1, 2$ and let G_j be an open subgroup of G^{σ_j}, i. e. let us assume that $(G^{\sigma_j})^0 \subset G_j \subset G^{\sigma_j}$ holds. Then the product group $G_1 \times G_2$ act on G by left and right multiplication, i. e. we define
$$(g_1, g_2) \cdot x := g_1 x g_2^{-1}.$$
The arguments presented at the end of Section 2 in [22] allow us to assume that $G = G_1 G^0 G_2$ holds.

Since σ_j is assumed to commute with the Cartan involution $\theta|_G$, the group $G_j = K_j \exp(\mathfrak{p}^{\sigma_j})$ is a closed compatible subgroup of G. In particular, $G_1 \times G_2$ is a closed compatible subgroup of $U^{\mathbb{C}} \times U^{\mathbb{C}}$.

Remark 2.3. It follows that G^{σ_j} has only finitely many connected components. If G is simply-connected, then G^{σ_j} is connected (compare [20]).

Definition 2.4. We say that the element $x \in G$ is regular (with respect to $G_1 \times G_2$) if the orbit $(G_1 \times G_2) \cdot x$ has maximal dimension. We call x strongly regular if it is regular and if $(G_1 \times G_2) \cdot x$ is closed in G. We write G_r and G_{sr} for the sets of regular and strongly regular elements in G, respectively. The orbit $(G_1 \times G_2) \cdot x$ is called generic if x is strongly regular.

Remark 2.5. The sets G_r and G_{sr} are invariant under $G_1 \times G_2$. We will see (Theorem 4.8) that G_{sr} is open and dense in G. This justifies the terminology "generic orbit".

An explicit gradient map. We fix from now on an embedding of U into a unitary group $\mathrm{U}(N)$ and consider the standard Hermitian inner product $(A, B) \mapsto \mathrm{Tr}(A\overline{B}^t)$ on the space of complex $(N \times N)$–matrices. Its real part $\langle \cdot, \cdot \rangle$ defines a scalar product on $\mathfrak{g}$.

Remark 2.6. With respect to this scalar product $\langle \cdot, \cdot \rangle$ the operator $\mathrm{Ad}(k)$, where $k \in K$, is orthogonal, while $\mathrm{Ad}\big(\exp(\xi)\big)$, where $\xi \in \mathfrak{p}$, is symmetric.

By virtue of the Cartan decomposition $U^{\mathbb{C}} = U \exp(i\mathfrak{u})$ we can define a function $\rho\colon U^{\mathbb{C}} \to \mathbb{R}^{\geq 0}$ by

$$\rho\big(u \exp(i\xi)\big) := \frac{1}{2} \mathrm{Tr}(\xi\overline{\xi}^t).$$

Using [2] one verifies that the $(U \times U)$–invariant smooth function ρ is strictly plurisubharmonic. Consequently, the $(1,1)$–form $\omega := \frac{i}{2}\partial\overline{\partial}\rho$ is a $(U \times U)$–invariant Kähler form on $U^{\mathbb{C}}$. It follows from Lemma 9.1(2) in [11] that the U–action on $U^{\mathbb{C}}$ by right multiplication is Hamiltonian with momentum map

$$\mu\colon U^{\mathbb{C}} \to \mathfrak{u}, \quad u \exp(\xi) \mapsto i\xi,$$

where we identify $\mathfrak{u}$ with its dual $\mathfrak{u}^*$ via the standard Hermitian inner product. Since the map $U^{\mathbb{C}} \to U^{\mathbb{C}}$, $g \mapsto g^{-1}$, is biholomorphic and interchanges right and left multiplication, we conclude that the U–action on $U^{\mathbb{C}}$ given by left multiplication is also Hamiltonian and has momentum map

$$\mu\colon U^{\mathbb{C}} \to \mathfrak{u}, \quad u \exp(\xi) \mapsto -i\,\mathrm{Ad}(u)\xi.$$

By restriction we obtain a $(K_1 \times K_2)$–equivariant gradient map $\Phi\colon G \to \mathfrak{p}^{\sigma_1} \oplus \mathfrak{p}^{\sigma_2}$ for the $(G_1 \times G_2)$–action on G with respect to the Riemannian metric induced by $\langle \cdot, \cdot \rangle$. Explicitely, we have

$$\Phi\big(k \exp(\xi)\big) = \Big(\mathrm{Ad}(k)\xi + \sigma_1\big(\mathrm{Ad}(k)\xi\big), -\big(\xi + \sigma_2(\xi)\big)\Big).$$

Hence, the zero fiber of Φ is given by

$$\begin{aligned}\Phi^{-1}(0) &= K \exp(\mathfrak{p}^{-\sigma_2}) \cap \exp(\mathfrak{p}^{-\sigma_1})K \\ &= \big\{k \exp(\xi) \in G;\ \xi \in \mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}\big\},\end{aligned} \tag{1}$$

where $\mathfrak{p}^{-\sigma_j} := \{\xi \in \mathfrak{p};\ \sigma_j(\xi) = -\xi\}$ for $j = 1, 2$.

Lemma 2.7. *We have $\mathcal{S}_{G_1\times G_2}\big(\Phi^{-1}(0)\big) = G$.*

Proof. Since the Kähler form ω has global potential ρ, we obtain

$$\mathcal{S}_{U^{\mathbb{C}}\times U^{\mathbb{C}}}\big(\mu^{-1}(0)\big) = U^{\mathbb{C}}.$$

By Proposition 11.2 in [11] this means $\mathcal{S}_{G_1\times G_2}\big(\mu^{-1}_{\mathfrak{u}^{-\sigma_1}\oplus\mathfrak{u}^{-\sigma_2}}(0)\big) = U^{\mathbb{C}}$, which proves the claim. ∎

The isotropy representation. In this paragraph we will study the isotropy representation ρ of $(G_1\times G_2)_x$ on T_xG. Since $(\xi_1,\xi_2)\in\mathfrak{g}^{\sigma_1}\oplus\mathfrak{g}^{\sigma_2}$ induces the tangent vector

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\exp(t\xi_1)x\exp(-t\xi_2) &= \frac{d}{dt}\bigg|_{t=0} x\exp(t\operatorname{Ad}(x^{-1})\xi_1)\exp(-t\xi_2)\\ &= (\ell_x)_*\big(\operatorname{Ad}(x^{-1})\xi_1-\xi_2\big)\in T_xG,\end{aligned}$$

where ℓ_x denotes left multiplication with $x\in G$, we obtain $T_x(G_1\times G_2)\cdot x = (\mathfrak{g}^{\sigma_1}\oplus\mathfrak{g}^{\sigma_2})\cdot x = \big\{(\ell)_x\xi;\ \xi\in\mathfrak{g}^{\sigma_2}+\operatorname{Ad}(x^{-1})\mathfrak{g}^{\sigma_1}\big\}$. Moreover, one checks directly that the isotropy group at $x\in G$ is given by

$$(G_1\times G_2)_x = \big\{(xgx^{-1},g);\ g\in G_2\cap x^{-1}G_1x\big\}.$$

Consequently, we may identify $(G_1\times G_2)_x$ with $G_2\cap x^{-1}G_1x$ via the isomorphism $\varphi\colon G_2\cap x^{-1}G_1x\to(G_1\times G_2)_x$, $g\mapsto(xgx^{-1},g)$. Similarly, we will identify the tangent space $T_x(G_1\times G_2)\cdot x$ with $\mathfrak{g}^{\sigma_2}+\operatorname{Ad}(x^{-1})\mathfrak{g}^{\sigma_1}$ via $(\ell_x)_*$. We conclude from

$$\begin{aligned}\rho\big(\varphi(g)\big)(\ell_x)_*\xi &= \frac{d}{dt}\bigg|_{t=0}(xgx^{-1},g)\cdot x\exp(t\xi)\\ &= \frac{d}{dt}\bigg|_{t=0} xg\exp(t\xi)g^{-1} = \frac{d}{dt}\bigg|_{t=0} x\exp\big(t\operatorname{Ad}(g)\xi\big) = (\ell_x)_*\operatorname{Ad}(g)\xi\end{aligned}$$

that the map $(\ell_x)_*$ intertwines the adjoint representation of $G_2\cap x^{-1}G_1x$ on $\mathfrak{g}$ with the isotropy representation of $(G_1\times G_2)_x$ on T_xG modulo φ. We summarize our considerations in the following

Lemma 2.8. *Modulo the isomorphism φ the isotropy representation of $(G_1\times G_2)_x$ on T_xG is equivalent to the adjoint representation of $G_2\cap x^{-1}G_1x$ on $\mathfrak{g}$.*

For any $x\in G$ we define the automorphism $\tau_x := \sigma_2\operatorname{Int}(x^{-1})\sigma_1\operatorname{Int}(x)\in\operatorname{Aut}(G)$ which induces the automorphism $\tau_x=\sigma_2\operatorname{Ad}(x^{-1})\sigma_1\operatorname{Ad}(x)$ of $\mathfrak{g}$. Note that $\tau_e=\sigma_2\sigma_1=\tau$ is not necessarily involutive since we do not assume that σ_1 and σ_2 commute. We need the following technical

Lemma 2.9. *Let $x\in\Phi^{-1}(0)$ be given.*

1. *The automorphism $\tau_x=\sigma_2\operatorname{Ad}(x^{-1})\sigma_1\operatorname{Ad}(x)$ is semi-simple. In particular, the automorphism τ_k with $k\in K$ is semi-simple with eigenvalues in the unit circle $S^1=\{z\in\mathbb{C};\ |z|=1\}$.*

2. *The subalgebra $\mathfrak{g}^{\tau_x} \subset \mathfrak{g}$ is invariant under θ and σ_2. In particular, $\mathfrak{g}^{\tau_x}$ is reductive.*

3. *The eigenspace decomposition of $\mathfrak{g}^{\tau_x}$ with respect to σ_2 is given by*

$$\mathfrak{g}^{\tau_x} = \left(\mathfrak{g}^{\sigma_2} \cap \operatorname{Ad}(x^{-1})\mathfrak{g}^{\sigma_1}\right) \oplus \left(\mathfrak{g}^{-\sigma_2} \cap \operatorname{Ad}(x^{-1})\mathfrak{g}^{-\sigma_1}\right).$$

Proof. Let $x = k\exp(\xi)$ be the Cartan decomposition of $x \in \Phi^{-1}(0)$. It follows that $\xi \in \mathfrak{p}^{-\sigma_2} \cap \operatorname{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}$, which implies $\sigma_2(\xi) = -\xi = \operatorname{Ad}(k^{-1})\sigma_1 \operatorname{Ad}(k)\xi$. Using these identities we obtain

$$\tau_x = \operatorname{Ad}\bigl(\exp(2\xi)\bigr)\tau_k = \tau_k \operatorname{Ad}\bigl(\exp(2\xi)\bigr).$$

Since τ is assumed to be semi-simple and $k \in K$, we conclude that τ_k is semi-simple with eigenvalues in S^1. Since for $\xi \in \mathfrak{p}$ the operator $\operatorname{Ad}\bigl(\exp(2\xi)\bigr)$ is symmetric, it is semi-simple, too. Since both automorphisms commute, we conclude that τ_x is semi-simple, which proves the first claim.

If $\eta \in \mathfrak{g}^{\tau_x}$, then

$$\operatorname{Ad}\bigl(\exp(-2\xi)\bigr)\eta = \tau_k(\eta).$$

Since τ_k has only eigenvalues in S^1 while the eigenvalues of $\operatorname{Ad}\bigl(\exp(-2\xi)\bigr)$ are real, we obtain $\tau_k(\eta) = \eta$ as well as $\operatorname{Ad}\bigl(\exp(-2\xi)\bigr)\eta = \eta$ for all $\eta \in \mathfrak{g}^{\tau_x}$. Together with $\tau_x\theta = \operatorname{Ad}\bigl(\exp(2\xi)\bigr)\tau_k\theta = \theta \operatorname{Ad}\bigl(\exp(-2\xi)\bigr)\tau_k$ and $\tau_x\sigma_2 = \sigma_2\tau_x^{-1}$ this observation implies the second claim. The last assertion is elementary to check. ■

In order to simplify notation we put

$$\mathfrak{h}^x := \mathfrak{g}^{\sigma_2} \cap \operatorname{Ad}(x^{-1})\mathfrak{g}^{\sigma_1} \quad \text{and} \quad \mathfrak{q}^x := \mathfrak{g}^{-\sigma_2} \cap \operatorname{Ad}(x^{-1})\mathfrak{g}^{-\sigma_1}.$$

Consequently, for $x \in \Phi^{-1}(0)$ the Lie algebra $\mathfrak{g}^{\tau_x} = \mathfrak{h}^x \oplus \mathfrak{q}^x$ is reductive and symmetric with respect to $\sigma_2|_{\mathfrak{g}^{\tau_x}} = \operatorname{Ad}(x^{-1})\sigma_1 \operatorname{Ad}(x)|_{\mathfrak{g}^{\tau_x}}$. The set $H^x := G_2 \cap x^{-1}G_1x$ is a closed subgroup of G^{τ_x} with Lie algebra $\mathfrak{h}^x$ and is isomorphic to the isotropy group $(G_1 \times G_2)_x$.

Lemma 2.10. *For $x \in \Phi^{-1}(0)$ we have the H^x–invariant decomposition*

$$\mathfrak{g} = \left(\mathfrak{g}^{\sigma_2} + \operatorname{Ad}(x^{-1})\mathfrak{g}^{\sigma_1}\right) \oplus \mathfrak{q}^x. \tag{2}$$

Consequently, $(\ell_x)_*\mathfrak{q}^x$ is a $(G_1 \times G_2)_x$–invariant complement to $T_x\bigl((G_1 \times G_2)\cdot x\bigr)$ in T_xG.

Proof. Let $x \in \Phi^{-1}(0)$ be given. By Lemma 2.9 the automorphism τ_x is semi-simple, and hence Lemma 1(i) from [22] applies to prove (2). The last assertion follows from Lemma 2.8. ■

As a corollary we obtain the following

Theorem 2.11. *For every $x \in \Phi^{-1}(0)$ there exists a neighborhood N of $0 \in \mathfrak{q}^x$ such that $S_x := x\exp(iN)$ is a geometric $(G_1 \times G_2)$–slice at x. The slice representation of $(G_1 \times G_2)_x$ on T_xS_x is isomorphic to the adjoint representation of H^x on $\mathfrak{q}^x$.*

Remark 2.12. If the group G is complex and if σ_1 and σ_2 are antiholomorphic, then G_1 and G_2 are real forms of G. In this case it turns out that the Lie algebra $\mathfrak{g}^{\tau_x}$ is complex and that $\mathfrak{q}^x = i\mathfrak{h}^x$ holds, i. e. that $\mathfrak{h}^x$ is a real form of $\mathfrak{g}^{\tau_x}$. Hence, the slice representation is isomorphic to the adjoint representation of the real-reductive group H^x on its Lie algebra $\mathfrak{h}^x$.

3. The isotropy representation of reductive symmetric spaces

Since we have seen that the slice representation of $(G_1 \times G_2)_x$ at $x \in \Phi^{-1}(0)$ is isomorphic to the isotropy representation of a reductive symmetric space, we investigate this representation in some detail using again natural gradient maps. We think of the results obtained in this section as the infinitesimal version of the $(G_1 \times G_2)$–orbit structure in G. Later we will make use of the Slice Theorem in order to transfer the results to the group level.

Closed orbits. Let $G = K\exp(\mathfrak{p}) \subset U^{\mathbb{C}}$ be a real-reductive Lie group with Cartan involution θ. We may assume that U is embedded in some unitary group $\mathrm{U}(N)$ and hence obtain the associated scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$.

Let $\sigma \in \mathrm{Aut}(G)$ be any involutive automorphism commuting with θ. In this case the set $H = G^\sigma$ is a θ–stable closed subgroup and consequently $H = K^\sigma \exp(\mathfrak{p}^\sigma)$ is real-reductive. The homogeneous space $X := G/H$ is called a reductive symmetric space. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into σ–eigenspaces. The group H acts on $\mathfrak{q}$ via the adjoint representation, and this representation is isomorphic to the isotropy representation of H on $T_{eH}X$. We refer the reader to [26] for more details on this topic.

Since the Lie algebra $\mathfrak{g}$ is reductive, there is a notion of Jordan-Chevalley decomposition for elements in $\mathfrak{g}$ which goes as follows. Every element $\xi \in \mathfrak{g}$ can be uniquely written as $\xi = \xi_s + \xi_n$ such that ξ_s is semi-simple, ξ_n is nilpotent, and $[\xi_s, \xi_n] = 0$. If $\xi \in \mathfrak{q}$, then we have

$$\sigma(\xi_s) + \sigma(\xi_n) = \sigma(\xi) = -\xi = -\xi_s - \xi_n.$$

From the uniqueness of the Jordan-Chevalley decomposition we conclude that ξ_s and ξ_n are again contained in $\mathfrak{q}$. Hence, it makes sense to speak of semi-simple and nilpotent elements in $\mathfrak{q}$. Moreover, the set of semi-simple elements in $\mathfrak{q}$ is open and dense in $\mathfrak{q}$.

Remark 3.1. In [22] the map $\Psi\colon G \to \mathrm{Aut}(G)$, $x \mapsto \sigma_2 \mathrm{Ad}(x^{-1})\sigma_1 \mathrm{Ad}(x)$, is used in order to define a notion of Jordan decomposition in G as follows. Since $\mathrm{Aut}(\mathfrak{g})$ is an algebraic group, one can decompose $\Psi(x)$ as $\Psi(x) = su = us$ where $s \in \mathrm{Aut}(G)$ is semi-simple and $u \in \mathrm{Aut}(G)$ is unipotent. It is proven (Proposition 2 in [22]) that this decomposition can be lifted to G and thus yields a kind of Jordan decomposition of elements in G which takes the involutions σ_1 and σ_2 into account. By Theorem 2.11 every $x \in G$ is of the form $x = x_0 \exp(\xi)$, where $x_0 \in \Phi^{-1}(0)$ is a point of the unique closed orbit in the closure of $(G_1 \times G_2) \cdot x$ and ξ lies in the null cone of $\mathfrak{q}^{x_0}$ and hence is nilpotent. It is not hard to see that $\Psi(x_0) = s$ and $\mathrm{Ad}\big(\exp(2\xi)\big) = u$ hold. If furthermore $x = \exp(\xi)$ lies in $\exp(\mathfrak{q}^y)$ for some $y \in \Phi^{-1}(0)$, then we can decompose ξ as $\xi = \xi_s + \xi_n$ and obtain $\Psi\big(\exp(\xi_s)\big) = s$ as well as $\mathrm{Ad}\big(\exp(2\xi_n)\big) = u$ (see Remark 4.1 in [9]).

It is known that the adjoint G–orbit through $\xi \in \mathfrak{g}$ is closed if and only if ξ is semi-simple. In the next proposition we obtain a similar characterization of closed H–orbits in $\mathfrak{q}$.

Proposition 3.2. *Let $\xi \in \mathfrak{q}$. Then* $\mathrm{Ad}(G)\xi$ *is closed in* $\mathfrak{g}$ *if and only if* $\mathrm{Ad}(H)\xi$ *is closed in* $\mathfrak{q}$. *Consequently,* $\mathrm{Ad}(H)\xi$ *is closed in* $\mathfrak{q}$ *if and only if* ξ *is semi-simple. Hence, there is a dense open subset of* $\xi \in \mathfrak{q}$ *such that* $\mathrm{Ad}(H)\xi$ *is closed.*

Remark 3.3.

1. The fact that $\mathrm{Ad}(H)\xi$ is closed precisely for semi-simple ξ, is proven by a different method in [7]. It can also be deduced from [19] and [25]. In order to illustrate the method we show how the standard gradient map may be used to prove Proposition 3.2.

2. Note that Proposition 3.2 can be viewed as a generalization of Corollary 5.3 in [3].

Proof. We first assume that $\mathrm{Ad}(G)\xi$ is closed in $\mathfrak{g}$. Since the differential of the map $H \times \mathfrak{q} \to G$, $(h, \xi) \mapsto h\exp(\xi)$, at $(e, 0)$ is given by $(\eta, \xi) \mapsto \eta + \xi$, we conclude that there is an open neighborhood N of $0 \in \mathfrak{q}$ such that $V := H\exp(N)$ is open in G and diffeomorphic to $H \times N$. Since

$$\left.\frac{d}{dt}\right|_0 \mathrm{Ad}\big(\exp(t\eta)\big)\xi' = [\eta, \xi']$$

holds for all $\eta, \xi' \in \mathfrak{g}$ and since $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$, we conclude that for all $\xi' \in \mathrm{Ad}(G)\xi \cap \mathfrak{q}$ the orbit $\mathrm{Ad}(G)\xi'$ intersects $\mathfrak{q}$ locally in $\mathrm{Ad}(H)\xi'$. Consequently, every H–orbit in $\mathrm{Ad}(G)\xi \cap \mathfrak{q}$ is open and hence also closed in $\mathrm{Ad}(G)\xi \cap \mathfrak{q}$. This shows that $\mathrm{Ad}(H)\xi$ is closed if $\mathrm{Ad}(G)\xi$ is closed.

In order to prove the converse, we consider the standard gradient map

$$\Phi_G \colon \mathfrak{g} \to \mathfrak{p}^*, \quad \Phi_G^\eta(\xi) = \big\langle [\eta, \xi], \xi \big\rangle,$$

for the adjoint G–action on $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$. Elementary computations show that

$$\big\langle [\eta, \xi], \xi \big\rangle = \big\langle \eta, [\xi_{\mathfrak{k}}, \xi_{\mathfrak{p}}] \big\rangle,$$

where $\xi = \xi_{\mathfrak{k}} + \xi_{\mathfrak{p}}$ is the Cartan decomposition of $\xi \in \mathfrak{g}$. Since $[\xi_{\mathfrak{k}}, \xi_{\mathfrak{p}}] \in \mathfrak{p}$, the zero fiber of this map is given by $\Phi_G^{-1}(0) = \big\{\xi \in \mathfrak{g};\ [\xi_{\mathfrak{k}}, \xi_{\mathfrak{p}}] = 0\big\}$. Furthermore, the restriction $\Phi_H \colon \mathfrak{q} \to (\mathfrak{p}^\sigma)^*$ of Φ_G is a gradient map for the adjoint H–action on $\mathfrak{q}$. Since we have $[\xi_{\mathfrak{k}}, \xi_{\mathfrak{p}}] \in \mathfrak{p}^\sigma$ for all $\xi \in \mathfrak{q}$, we obtain $\Phi_H^{-1}(0) = \Phi_G^{-1}(0) \cap \mathfrak{q}$. Since one directly checks $\mathcal{S}_G\big(\Phi_G^{-1}(0)\big) = \mathfrak{g}$ and $\mathcal{S}_H\big(\Phi_H^{-1}(0)\big) = \mathfrak{q}$, the claim follows from Proposition 1.4. ■

In closing we describe the connection to Cartan subspaces of $\mathfrak{q}$.

Definition 3.4. A Cartan subspace of $\mathfrak{q}$ is a maximal Abelian subspace $\mathfrak{c} \subset \mathfrak{q}$ which consists of semi-simple elements.

Proposition 3.5. *Every closed H–orbit in $\mathfrak{q}$ intersects some θ–stable Cartan subspace non-trivially. Conversely, if ξ lies in a θ–stable Cartan subspace of $\mathfrak{q}$, then* $\mathrm{Ad}(H)\xi$ *is closed.*

Proof. If the orbit $\mathrm{Ad}(H)\xi \subset \mathfrak{q}$ is closed, we may assume by Proposition 1.4 that ξ lies in $\Phi_H^{-1}(0)$ and hence that $[\xi_{\mathfrak{k}}, \xi_{\mathfrak{p}}] = 0$ holds. Therefore there exists a θ–stable maximal Abelian subspace of $\mathfrak{q}$ which contains ξ. By θ–invariance this maximal Abelian subspace consists of semi-simple elements and thus is a Cartan subspace.

Since every element ξ in a θ–stable Cartan subspace is mapped to zero under Φ_H, the H–orbit through ξ is closed. This finishes the proof. ■

Nonclosed orbits. The following statement is taken from [7] (see Theorem 23).

Proposition 3.6. (van Dijk) *There are only finitely many nilpotent H–orbits in $\mathfrak{q}$. It follows that the dimension of the null cone $\mathcal{N} \subset \mathfrak{q}$ coincides with the dimension of an H–orbit which is open in $\mathcal{N}$.*

In order to describe the non-closed orbits, we will make use of the weight space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathcal{Z}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{c}) \oplus \mathfrak{c}^{\mathbb{C}} \oplus \bigoplus_{\lambda\in\Lambda} \mathfrak{g}^{\mathbb{C}}_{\lambda}$$

of $\mathfrak{g}^{\mathbb{C}}$ with respect to a Cartan subspace $\mathfrak{c}$ of $\mathfrak{q}$.

Remark 3.7. The set Λ of weights fulfills the axioms of an abstract root system, since it coincides with the set of restricted roots for the symmetric space $\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d$ where $(\mathfrak{g}^d, \mathfrak{h}^d)$ is the dual of $(\mathfrak{g}, \mathfrak{h})$ (compare [26]). In particular, it makes sense to speak of a subsystem $\Lambda^+ \subset \Lambda$ of positive roots.

In the following we extend the involution σ by $\mathbb{C}$–linearity to $\mathfrak{g}^{\mathbb{C}}$. Since $\mathfrak{c}$ is contained in $\mathfrak{q}$, we have $\sigma(\mathfrak{g}^{\mathbb{C}}_{\lambda}) = \mathfrak{g}^{\mathbb{C}}_{-\lambda}$ for all $\lambda \in \Lambda$. Consequently, elements of $\mathfrak{q}$ can be written as

$$\xi = \xi_0 + \sum_{\lambda\in\Lambda^+} \big(\xi_\lambda - \sigma(\xi_\lambda)\big)$$

where ξ_0 lies in $\mathcal{Z}_{\mathfrak{q}}(\mathfrak{c}) = \mathfrak{c}$. Elements of $\mathfrak{h}$ can be described in a similar way.

The first goal in this subsection is to find a geometric H–slice at a point $\eta_0 \in \mathfrak{c}$. We identify the tangent space of $H \cdot \eta_0$ at η_0 with $[\mathfrak{h}, \eta_0]$.

Lemma 3.8. *Let $\Lambda(\eta_0) := \big\{\lambda \in \Lambda;\ \lambda(\eta_0) = 0\big\}$. Then we have*

$$\mathfrak{q} = [\mathfrak{h}, \eta_0] \oplus \mathfrak{c} \oplus \left(\mathfrak{q} \cap \bigoplus_{\lambda\in\Lambda(\eta_0)} \mathfrak{g}^{\mathbb{C}}_{\lambda}\right).$$

Proof. Since we have $\mathfrak{q} = \mathfrak{c} \oplus \big(\mathfrak{q} \cap \bigoplus_{\lambda\in\Lambda} \mathfrak{g}^{\mathbb{C}}_{\lambda}\big)$, it is enough to show that $[\mathfrak{h}, \eta_0] = \mathfrak{q} \cap \bigoplus_{\lambda\notin\Lambda(\eta_0)} \mathfrak{g}^{\mathbb{C}}_{\lambda}$ holds. If $\xi = [\xi', \eta_0]$ with $\xi' \in \mathfrak{h}$ is given, then we decompose $\xi' = \sum_{\lambda\in\Lambda} \xi'_\lambda$ and obtain $\xi = -\sum_{\lambda\in\Lambda} \lambda(\eta_0)\xi'_\lambda = -\sum_{\lambda\notin\Lambda(\eta_0)} \lambda(\eta_0)\xi'_\lambda \in \mathfrak{q} \cap \bigoplus_{\lambda\notin\Lambda(\eta_0)} \mathfrak{g}^{\mathbb{C}}_{\lambda}$ which was to be shown.

In order to prove the converse let $\xi \in \mathfrak{q} \cap \bigoplus_{\lambda\notin\Lambda(\eta_0)} \mathfrak{g}^{\mathbb{C}}_{\lambda}$ be given. It follows from the discussion above that ξ has a representation $\xi = \sum_{\lambda\notin\Lambda^+(\eta_0)} \big(\xi_\lambda - \sigma(\xi_\lambda)\big)$. Defining

$$\xi' := \sum_{\lambda\notin\Lambda^+(\eta_0)} \big(\tfrac{1}{\lambda(\eta_0)}\xi_\lambda + \sigma(\tfrac{1}{\lambda(\eta_0)}\xi_\lambda)\big) \in \mathfrak{h} \cap \bigoplus_{\lambda\notin\Lambda(\eta_0)} \mathfrak{g}^{\mathbb{C}}_{\lambda}$$

one checks directly that $[\eta_0, \xi'] = \xi$ holds. Hence, the lemma is proven. ■

As a consequence we obtain the following description of non-closed orbits.

Proposition 3.9. *Let $\xi \in \mathfrak{q}$ be point on a non-closed H–orbit and let $\eta_0 \in \overline{H \cdot \xi}$ be an element lying in the unique closed H–orbit in $\overline{H \cdot \xi}$. We assume that η_0 is contained in the Cartan subspace $\mathfrak{c}$. Then ξ is contained in the null cone of the $\mathcal{Z}_H(\eta_0)$–representation on $\mathfrak{c} \oplus \left(\mathfrak{q} \cap \bigoplus_{\lambda \in \Lambda(\eta_0)} \mathfrak{g}^{\mathbb{C}}_\lambda\right)$.*

4. Structure of the set of closed orbits

In this section we will state and proof the first main result, namely Matsuki's parametrization of the set of closed $(G_1 \times G_2)$–orbits via Cartan subsets. We will present a constructive proof in some detail since this explains how one has to deal with concrete examples. In the case of the $(G_{\mathbb{R}} \times G_{\mathbb{R}})$–action on a semi-simple complex group G, the proof reproduces the Cayley transform of Cartan subalgebras in semi-simple real Lie algebras.

Cartan subsets. We review the notion of fundamental and standard Cartan subsets from [22].

Definition 4.1. Let $\mathfrak{t}_0$ be a maximal Abelian subspace in $\mathfrak{k}^{-\sigma_2} \cap \mathfrak{k}^{-\sigma_1}$ and let $\mathfrak{a}_0$ be an Abelian subspace of $\mathfrak{p}^{-\sigma_2} \cap \mathfrak{p}^{-\sigma_1}$ such that $\mathfrak{c}_0 := \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a maximal Abelian subspace of $\mathfrak{g}^{-\sigma_2} \cap \mathfrak{g}^{-\sigma_1}$. Then the set $C_0 := \exp(\mathfrak{c}_0) \subset G$ is called a fundamental Cartan subset.

Remark 4.2.

1. By maximality of $\mathfrak{t}_0$ the set $T_0 := \exp(\mathfrak{t}_0)$ is a torus in K. In general T_0 is not a maximal torus as we will see in Example 5.4.
2. The subalgebra $\mathfrak{c}_0$ consists by construction of semi-simple elements.

Definition 4.3. A subset $C := n \exp(\mathfrak{c}) \subset G$ is called a standard Cartan subset, if n lies in T_0 and $\mathfrak{c}$ is a θ–stable Abelian subspace of $\mathfrak{q}^n = \mathfrak{g}^{-\sigma_2} \cap \mathrm{Ad}(n^{-1})\mathfrak{g}^{-\sigma_1}$ with decomposition $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ such that $\mathfrak{t} \subset \mathfrak{t}_0$, $\mathfrak{a} \supset \mathfrak{a}_0$ and $\dim \mathfrak{c} = \dim \mathfrak{c}_0$ hold.

Remark 4.4. The subspace $\mathfrak{c}$ is a Cartan subspace of $\mathfrak{q}^n$.

Lemma 4.5. *Each standard Cartan subset C is contained in the zero-fiber $\Phi^{-1}(0)$.*

Proof. Let $C = n \exp(\mathfrak{c})$ be a standard Cartan subset and let $z = n \exp(\eta)$ for some $\eta \in \mathfrak{c}$. According to the decomposition $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ we write $\eta = \eta_{\mathfrak{t}} + \eta_{\mathfrak{a}}$. Since $\mathfrak{c}$ is Abelian, we obtain $z = n \exp(\eta_{\mathfrak{t}}) \exp(\eta_{\mathfrak{a}})$ where $n \exp(\eta_{\mathfrak{t}}) \in K$ and $\exp(\eta_{\mathfrak{a}}) \in \exp(\mathfrak{p})$ hold. Therefore we can compute as follows:

$$\begin{aligned}\Phi(z) &= \Big(\mathrm{Ad}\big(n \exp(\eta_{\mathfrak{t}})\big)\eta_{\mathfrak{a}} + \sigma_1 \left(\mathrm{Ad}\big(n \exp(\eta_{\mathfrak{t}})\big)\eta_{\mathfrak{a}}\right), -\big(\eta_{\mathfrak{a}} + \sigma_2(\eta_{\mathfrak{a}})\big)\Big) \\ &= \Big(\mathrm{Ad}(n)\eta_{\mathfrak{a}} + \sigma_1\big(\mathrm{Ad}(n)\eta_{\mathfrak{a}}\big), 0\Big) = (0,0),\end{aligned}$$

where we have used that $\mathfrak{c}$ is Abelian and contained in $\mathfrak{q}^n$. ∎

We shall regard two standard Cartan subsets as equivalent, if there is a generic orbit which intersects both of them non-trivially. We will see later that there are only finitely many equivalence classes of Cartan subsets. Let $\{C_j\}_{j\in J}$ be a complete set of representatives. Then we can state the first main result.

Theorem 4.6. (Matsuki) *Each closed $(G_1\times G_2)$–orbit intersects one of the C_j, i. e. we have*

$$\{z\in G;\ (G_1\times G_2)\cdot z \text{ is closed}\} = \bigcup_j G_1C_jG_2. \tag{3}$$

Each generic $(G_1\times G_2)$–orbit intersects exactly one C_j in a finite number of points. Hence, the set of strongly regular elements in G coincides with the disjoint union of the open sets $\Omega_j := G_1(C_j\cap G_{rs})G_2$.

Remark 4.7. Let $G_j = K_j\exp(\mathfrak{p}^{\sigma_j})$, $j=1,2$, denote the Cartan decomposition. The only point missing for a proof of Matsuki's theorem is that the $(K_1\times K_2)$–orbit through every element in $\Phi^{-1}(0)$ intersects some standard Cartan subset. This fact can be deduced from [25], where it is shown that any maximal Abelian subspace of $\mathfrak{g}^{-\sigma_2}\cap \mathrm{Ad}(k^{-1})\mathfrak{g}^{-\sigma_1}$ is conjugate under $\mathrm{Ad}(G_2\times k^{-1}G_1k)$ to a θ–stable maximal Abelian subspace. We will give another proof of this fact which is organized in a way such that it becomes clear how to deal with concrete examples.

We conclude from Proposition 3.2 the following

Theorem 4.8. *The set G_{sr} of strongly regular elements is open and dense in G. For each $x\in G_{sr}$ the subspace $\mathfrak{q}^x$ lies in the center of $\mathfrak{g}^{\tau_x}$. In particular, in this case $\mathfrak{q}^x$ is Abelian and consists of semi-simple elements.*

Proof. Since $\mathcal{S}_{G_1\times G_2}\big(\Phi^{-1}(0)\big) = G$, every orbit contains a unique closed orbit in its closure and we have a geometric slice at every closed $(G_1\times G_2)$–orbit. Moreover, by Theorem 2.11 the slice representation is equivalent to the isotropy representation of a reductive symmetric space as considered in Section 3. Hence, Proposition 3.2 implies that G_{sr} is open and dense in G.

For dimensional reasons the effective part of the slice representation of a generic orbit must be that of a finite group, which implies that for $x\in G_{sr}$ the adjoint representation of $\mathfrak{h}^x$ on $\mathfrak{q}^x$ is trivial. We claim that this implies $\mathfrak{q}^x\subset \mathcal{Z}(\mathfrak{g}^{\tau_x})$. In order to prove this claim, we decompose the reductive Lie algebra $\mathfrak{g}^{\tau_x}$ into its center $\mathfrak{z}$ and its semi-simple part $\mathfrak{s}$. Since every automorphism of $\mathfrak{g}^{\tau_x}$ leaves its center and semi-simple part invariant, we obtain the decompositions $\mathfrak{s} = (\mathfrak{s}\cap\mathfrak{h}^x)\oplus(\mathfrak{s}\cap\mathfrak{q}^x)$ as well as $\mathfrak{q}^x = (\mathfrak{q}^x\cap\mathfrak{s})\oplus(\mathfrak{q}^x\cap\mathfrak{z})$. Hence, we have to show that $\mathfrak{s}\cap\mathfrak{q}^x = \{0\}$. We conclude from the semi-simplicity of $\mathfrak{s}$ and from $[\mathfrak{h}^x,\mathfrak{q}^x] = \{0\}$ that

$$\mathfrak{s} = [\mathfrak{s},\mathfrak{s}] = [\mathfrak{s}\cap\mathfrak{h}^x,\mathfrak{s}\cap\mathfrak{h}^x] + [\mathfrak{s}\cap\mathfrak{q}^x,\mathfrak{s}\cap\mathfrak{q}^x] \subset \mathfrak{s}\cap\mathfrak{h}^x,$$

i. e. we have $\mathfrak{s} = \mathfrak{s}\cap\mathfrak{h}^x$ which yields the claim. Since $\mathfrak{q}^x$ is θ–stable and Abelian, it consists of semi-simple elements. ∎

Corollary 4.9. *Let $C = n\exp(\mathfrak{c})$ be a standard Cartan subset. For $x \in C \cap G_{sr}$ we have $\mathfrak{q}^x = \mathfrak{c}$. In particular, if $x \in C \cap G_{sr}$, then the connected component of $C \cap G_{sr}$ which contains x defines a geometric slice to $(G_1 \times G_2) \cdot x$.*

Proof. This follows from Theorem 4.8 since $\mathfrak{c}$ is a Cartan subspace of $\mathfrak{q}^x$ which is Abelian and consists of semi-simple elements for strongly regular elements. ■

The $(K_1 \times K_2)$–action on $\Phi^{-1}(0)$. In this subsection we review Theorem 1 from [22]. Let $\mathfrak{t}_0 \subset \mathfrak{k}^{-\sigma_2} \cap \mathfrak{k}^{-\sigma_1}$ be a maximal Abelian subspace and let $T_0 := \exp(\mathfrak{t}_0)$ be the corresponding torus in K.

Proposition 4.10. (Matsuki) *Each $(K_1 \times K_2)$–orbit in K intersects the torus T_0.*

To understand the intersection of the $(K_1 \times K_2)$–orbits with T_0, we introduce the groups

$$\mathcal{N}_{K_1\times K_2}(T_0) := \left\{(k_1,k_2) \in K_1 \times K_2;\ k_1 T_0 k_2^{-1} = T_0\right\}$$

as well as

$$\mathcal{Z}_{K_1\times K_2}(T_0) := \left\{(k_1,k_2) \in K_1 \times K_2;\ k_1 \exp(\eta) k_2^{-1} = \exp(\eta) \text{ for all } \eta \in \mathfrak{t}_0\right\}$$

and $W_{K_1\times K_2}(T_0) := \mathcal{N}_{K_1\times K_2}(T_0)/\mathcal{Z}_{K_1\times K_2}(T_0)$.

Remark 4.11. The group $W_{K_1\times K_2}(T_0)$ is finite (see Lemma 2.2.6 in [23]).

Proposition 4.12. (Matsuki) *Every $(K_1 \times K_2)$–orbit in K intersects T_0 in a $W_{K_1\times K_2}(T_0)$–orbit. Hence, the inclusion $T_0 \hookrightarrow K$ induces a homeomorphism $T_0/W_{K_1\times K_2}(T_0) \cong K_1\backslash K/K_2$.*

Remark 4.13. In the special case $\sigma_1 = \sigma_2$ this statement can be found in [13], while for commuting involutions σ_1 and σ_2 it is proven in [17].

Consequently, after applying an element of $K_1 \times K_2$, we can assume that $k \in K$ is of the form $k = \exp(\eta)$ for some $\eta \in \mathfrak{t}_0$ which is unique up to the action of $W := W_{K_1\times K_2}(T_0)$.

The extended weight decomposition. Since the maximal Abelian subspace $\mathfrak{t}_0 \subset \mathfrak{k}^{-\sigma_2} \cap \mathfrak{k}^{-\sigma_1}$ consists of semi-simple elements, we may form the weight space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}} = \bigoplus_{\lambda\in\Lambda_{\mathfrak{k}}} \mathfrak{k}^{\mathbb{C}}_\lambda \oplus \bigoplus_{\lambda\in\Lambda_{\mathfrak{p}}} \mathfrak{p}^{\mathbb{C}}_\lambda$$

with respect to $\mathfrak{t}_0$.

Remark 4.14.

1. If G is complex-reductive, then K is a compact real form of $G = K^{\mathbb{C}}$ and $\mathfrak{p} = i\mathfrak{k}$. In this case we identify $\mathfrak{p}^{\mathbb{C}}$ with $\mathfrak{k}^{\mathbb{C}} = \mathfrak{g}$. Hence, $\Lambda_{\mathfrak{k}}$ and $\Lambda_{\mathfrak{p}}$ are essentially the same, and we do not have to complexify $\mathfrak{g}$ in order to consider the weight decompositions when $\mathfrak{g}$ is already complex.

2. It is proven in [22] that the set of non-zero weights in $\Lambda_{\mathfrak{k}}$ fulfills the axioms of an abstract root system.

We extend the involutions σ_1 and σ_2 as $\mathbb{C}$–linear maps to $\mathfrak{g}^{\mathbb{C}}$. Since the semi-simple automorphism $\tau = \sigma_2\sigma_1$ leaves each weight space invariant, we obtain the finer decomposition

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{(\lambda,a)\in\widetilde{\Lambda}_{\mathfrak{k}}} \mathfrak{k}^{\mathbb{C}}_{\lambda,a} \oplus \bigoplus_{(\lambda,a)\in\widetilde{\Lambda}_{\mathfrak{p}}} \mathfrak{p}^{\mathbb{C}}_{\lambda,a}, \tag{4}$$

where $\mathfrak{k}^{\mathbb{C}}_{\lambda,a} := \big\{\xi \in \mathfrak{k}^{\mathbb{C}}_{\lambda};\ \tau(\xi) = a\xi\big\}$ and $\widetilde{\Lambda}_{\mathfrak{k}} := \big\{(\lambda, a) \in \Lambda_{\mathfrak{k}} \times S^1;\ \mathfrak{k}^{\mathbb{C}}_{\lambda,a} \neq \{0\}\big\}$. The sets $\mathfrak{p}^{\mathbb{C}}_{\lambda,a}$ and $\widetilde{\Lambda}_{\mathfrak{p}}$ are defined similarly. We call the decomposition (4) the extended weight space decomposition of $\mathfrak{g}^{\mathbb{C}}$. For $\eta \in \mathfrak{t}_0$ we define

$$\widetilde{\Lambda}_{\mathfrak{k}}(\eta) := \big\{(\lambda, a) \in \widetilde{\Lambda}_{\mathfrak{k}};\ ae^{2\lambda(\eta)} = 1\big\}$$

and analogously $\widetilde{\Lambda}_{\mathfrak{p}}(\eta)$.

Lemma 4.15. *Let $k = \exp(\eta)$ with $\eta \in \mathfrak{t}_0$ be given. Then we have*

$$\big(\mathfrak{k}^{\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{k}^{\sigma_1}\big)^{\mathbb{C}} = (\mathfrak{k}^{\sigma_2})^{\mathbb{C}} \cap \bigoplus_{(\lambda,a)\in\widetilde{\Lambda}_{\mathfrak{k}}(\eta)} \mathfrak{k}^{\mathbb{C}}_{\lambda,a}$$

as well as

$$\big(\mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}\big)^{\mathbb{C}} = (\mathfrak{p}^{-\sigma_2})^{\mathbb{C}} \cap \bigoplus_{(\lambda,a)\in\widetilde{\Lambda}_{\mathfrak{p}}(\eta)} \mathfrak{p}^{\mathbb{C}}_{\lambda,a}.$$

Proof. Since the $\mathbb{C}$–linear automorphism $\tau_k = \sigma_2 \mathrm{Ad}(k^{-1})\sigma_1 \mathrm{Ad}(k)$ of $\mathfrak{g}^{\mathbb{C}}$ commutes with θ and with the complex conjugation κ on $\mathfrak{g}^{\mathbb{C}}$ which defines $\mathfrak{g}$, it follows that τ_k leaves $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^{\mathbb{C}}$ invariant. Moreover, for every $\xi_{\lambda,a} \in \mathfrak{g}^{\mathbb{C}}_{\lambda,a} := \mathfrak{k}^{\mathbb{C}}_{\lambda,a} \oplus \mathfrak{p}^{\mathbb{C}}_{\lambda,a}$ we have

$$\tau_k(\xi_{\lambda,a}) = \tau \mathrm{Ad}(k^2)\xi_{\lambda,a} = ae^{2\lambda(\eta)}\xi_{\lambda,a}.$$

Hence, the fixed point sets of τ_k in $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^{\mathbb{C}}$ are given by

$$\bigoplus_{(\lambda,a)\in\widetilde{\Lambda}_{\mathfrak{k}}(\eta)} \mathfrak{k}^{\mathbb{C}}_{\lambda,a} \qquad \text{and} \qquad \bigoplus_{(\lambda,a)\in\widetilde{\Lambda}_{\mathfrak{p}}(\eta)} \mathfrak{p}^{\mathbb{C}}_{\lambda,a},$$

respectively. Both fixed point sets are invariant under σ_2, and furthermore the subalgebra $\big(\mathfrak{k}^{\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{k}^{\sigma_1}\big)^{\mathbb{C}}$ is the $(+1)$–eigenspace of σ_2 restricted to the fixed point set of τ_k in $\mathfrak{k}^{\mathbb{C}}$ while $\big(\mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}\big)^{\mathbb{C}}$ is the (-1)–eigenspace of σ_2 restricted to the fixed point set of τ_k in $\mathfrak{p}^{\mathbb{C}}$. These observations proof the lemma. ∎

Remark 4.16. Since $\sigma_2(\mathfrak{g}^{\mathbb{C}}_{\lambda,a}) = \mathfrak{g}^{\mathbb{C}}_{-\lambda,a^{-1}}$ and $\kappa(\mathfrak{g}^{\mathbb{C}}_{\lambda,a}) = \mathfrak{g}^{\mathbb{C}}_{-\lambda,a^{-1}}$, Lemma 4.15 enables us to determine $\mathfrak{k}^{\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{k}^{\sigma_1}$ and $\mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}$.

A normal form for elements in $\Phi^{-1}(0)$. In this paragraph we show that for every element $x \in \Phi^{-1}(0)$ there exists a pair $(k_1, k_2) \in K_1 \times K_2$ such that $k_1 x k_2^{-1}$ lies in some standard Cartan subset. This proves that the closed $(G_1 \times G_2)$–orbits in G are precisely those which intersect a standard Cartan subset non-trivially.

For this let $x = k\exp(\xi)$ be an arbitrary element of $\Phi^{-1}(0)$, i.e. let $\xi \in \mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}$. By virtue of Proposition 4.10 the element k is conjugate to an element of the torus T_0 under $K_1 \times K_2$. If (k_1, k_2) is an element of the isotropy group $(K_1 \times K_2)_k$, then $k_2 \in K_2 \cap k^{-1}K_1 k$ and $k_1 = k k_2 k^{-1}$ hold. Consequently, we have

$$(k_1, k_2) \cdot x = k_1 k \exp(\xi) k_2^{-1} = k_1 k k_2^{-1} \exp\bigl(\mathrm{Ad}(k_2)\xi\bigr) = k \exp\bigl(\mathrm{Ad}(k_2)\xi\bigr).$$

Hence, we have to understand the adjoint action of $K_2 \cap k^{-1}K_1 k$ on $\mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}$. Since the set $\bigl(K_2 \cap k^{-1}K_1 k\bigr)\exp\bigl(\mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}\bigr)$ is a closed compatible subgroup of G, we conclude from Proposition 7.29 in [18] that all maximal Abelian subspaces of $\mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}$ are conjugate under $K_2 \cap k^{-1}K_1 k$. Hence, we see that there exists an element $k_2 \in K_2 \cap k^{-1}K_1 k$ such that $\mathrm{Ad}(k_2)\xi \in \mathfrak{a}$ holds for a maximal Abelian subspace $\mathfrak{a}$ which contains $\mathfrak{a}_0$. Let $\mathfrak{t} := \mathcal{Z}_{\mathfrak{t}_0}(\mathfrak{a})$ and $\mathfrak{c} := \mathfrak{t} \oplus \mathfrak{a}$. Moreover, we decompose $\eta \in \mathfrak{t}_0$ as

$$\eta = \eta_1 + \eta_2 \in \mathfrak{t}^{\perp} \oplus \mathfrak{t} = \mathfrak{t}_0$$

and put $n := \exp(\eta_1)$.

Lemma 4.17. *The set $C := n\exp(\mathfrak{c}) \subset G$ is a standard Cartan subset. Hence, every element $x \in \Phi^{-1}(0)$ is conjugate under the group $K_1 \times K_2$ to an element of some standard Cartan subset.*

Proof. It follows directly from the construction that $n \in T_0$ holds and that $\mathfrak{c} \subset \mathfrak{g}^{-\sigma_2} \cap \mathrm{Ad}(n^{-1})\mathfrak{g}^{-\sigma_1}$ is a θ–stable Abelian subalgebra with decomposition $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ such that $\mathfrak{t} \subset \mathfrak{t}_0$ and $\mathfrak{a} \supset \mathfrak{a}_0$ hold.

It remains to show that $\dim \mathfrak{c} = \dim \mathfrak{c}_0$ holds. It follows from the construction that $\mathfrak{c}$ is a Cartan subspace of $\mathfrak{g}^{-\sigma_2} \cap \mathrm{Ad}(n^{-1})\mathfrak{g}^{-\sigma_1}$. Moreover, since $n \in T_0$, we conclude that $\mathfrak{c}_0$ is also a Cartan subspace of $\mathfrak{g}^{-\sigma_2} \cap \mathrm{Ad}(n^{-1})\mathfrak{g}^{-\sigma_1}$, hence that their dimensions coincide. ∎

In the following we will define the appropriate notion of equivalence of standard Cartan slices in order to make considerations independent of the point x.

Definition 4.18. Two standard Cartan subsets $C_1 = T_1\exp(\mathfrak{a}_1)$ and $C_2 = T_2\exp(\mathfrak{a}_2)$ are called equivalent (or conjugate), if there exists an element $(k_1, k_2) \in \mathcal{N}_{K_1 \times K_2}(T_0)$ such that $T_2 = k_1 T_1 k_2^{-1}$ holds.

Lemma 4.19. *If two standard Cartan subsets C_1 and C_2 are equivalent, then there exists an element $(k_1, k_2) \in K_1 \times K_2$ such that $C_2 = k_1 C_1 k_2^{-1}$ holds.*

Proof. This is the content of Lemma 10 in [22]. ∎

Proposition 4.20. *Let $\{C_j\}_{j\in J}$ be a complete set of representatives of equivalence classes of standard Cartan subsets. Then J is finite.*

Proof. Our construction of the standard Cartan subset $C = n\exp(\mathfrak{c})$ reveals that it is completely determined by the maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}$. Since all maximal Abelian subspaces of $\mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}$ are conjugate under $K_2 \cap k^{-1}K_1k$, it is enough to show that there are only finitely many possibilities for the subspace $\mathfrak{p}^{-\sigma_2} \cap \mathrm{Ad}(k^{-1})\mathfrak{p}^{-\sigma_1}$ with $k \in T_0$. Since this fact is a consequence of Lemma 4.15, the claim follows. ∎

Remark 4.21. In [22] standard Cartan subsets are described in terms of orthogonal systems of weight vectors (compare also [25] and [21]).

In the next step we have to understand the intersection of the $(G_1 \times G_2)$–orbits with the standard Cartan sets. For this, we introduce the following groups. Let C_j be one of the standard Cartan subsets and define

$$\mathcal{N}_{K_1\times K_2}(C_j) := \big\{(k_1,k_2) \in K_1 \times K_2;\ k_1C_jk_2^{-1} = C_j\big\},$$
$$\mathcal{Z}_{K_1\times K_2}(C_j) := \big\{(k_1,k_2) \in K_1 \times K_2;\ k_1xk_2^{-1} = x \text{ for all } x \in C_j\big\},$$

and $W_{K_1\times K_2}(C_j) := \mathcal{N}_{K_1\times K_2}(C_j)/\mathcal{Z}_{K_1\times K_2}(C_j)$.

Proposition 4.22. *If $x \in C_j$ is regular, then*

$$(G_1 \times G_2)\cdot x \cap (C_j \cap G_{sr}) = W_{K_1\times K_2}(C_j)\cdot x$$

holds.

Proof. This is Proposition 2.2.28 in [23]. ∎

Corollary 4.23. *The groups $W_{K_1\times K_2}(C_j)$ are finite.*

Proof. Since C_j defines a geometric slice at its regular points, the intersection of a generic $(G_1\times G_2)$–orbit with C_j is zero-dimensional, and since this intersection is given by an orbit of the compact group $K_1 \times K_2$, it is finite. Moreover, by Proposition 4.22 this intersection coincides with an orbit of $W_{K_1\times K_2}(C_j)$. Since this group acts effectively, the claim follows. ∎

Finally we restate and prove the main theorem 4.6.

Theorem 4.24. (Matsuki) *Let $\{C_j\}$ be a complete set of representatives of standard Cartan subsets. Then*

$$G_1\Phi^{-1}(0)G_2 = \bigcup_j G_1C_jG_2 \quad \text{and} \quad G_{sr} = \dot{\bigcup_j} G_1(C_j \cap G_{sr})G_2.$$

Moreover, each generic $(G_1\times G_2)$–orbit intersects $C_j\cap G_{sr}$ in a $W_{K_1\times K_2}(C_j)$–orbit.

Proof. The only claim which has not been proved up to now is that the second union is disjoint. For convenience of the reader we reproduce the argument from the proof of Theorem 3 in [22].

Let $x \in C_j \cap G_{sr}$ and $x' \in C_k \cap G_{sr}$ with $x' \in (G_1 \times G_2) \cdot x$ be given. Since C_j and C_k are contained in $\Phi^{-1}(0)$, there exists an element $(k_1, k_2) \in K_1 \times K_2$ such that $x' = k_1 x k_2^{-1}$ holds. We finish the proof by showing that the element (k_1, k_2) normalizes C_j. For this we write $C_j = n \exp(\mathfrak{c}_j)$. Since $x \in C_j$ holds, we conclude $x^{-1}n \in \exp(\mathfrak{c}_j)$ and thus $C_j = x \exp(\mathfrak{c}_j)$. This implies

$$k_1 C_j k_2^{-1} = k_1\big(x \exp(\mathfrak{c}_j)\big)k_2^{-1} = k_1 x k_2^{-1} \exp\big(\mathrm{Ad}(k_2)\mathfrak{c}_j\big) = x' \exp\big(\mathrm{Ad}(k_2)\mathfrak{c}_j\big).$$

Moreover, since x is assumed to be strongly regular, we obtain $\mathfrak{c}_j = \mathfrak{q}^x$ and therefore

$$\begin{aligned}\mathrm{Ad}(k_2)\mathfrak{c}_j = \mathrm{Ad}(k_2)\mathfrak{q}^x &= \mathrm{Ad}(k_2)\big(\mathfrak{g}^{-\sigma_2} \cap \mathrm{Ad}(x^{-1})\mathfrak{g}^{-\sigma_1}\big)\\ &= \mathfrak{g}^{-\sigma_2} \cap \mathrm{Ad}(k_2 x^{-1} k_1^{-1})\mathfrak{g}^{-\sigma_1} = \mathfrak{q}^{x'}.\end{aligned}$$

Since x' is also strongly regular, we conclude $\mathfrak{q}^{x'} = \mathfrak{c}_k$. Hence, the theorem is proven. ∎

In course of our proof of this theorem we have obtained the following fact.

Proposition 4.25. *Let $C = n \exp(\mathfrak{c})$ be a standard Cartan subset in G and let x_0 be a point of C such that the slice representation of $(G_1 \times G_2)_{x_0}$ is trivial. Then $x_0 \in C \cap G_{sr}$ and there exists an open neighborhood C^0 of x_0 in $C \cap G_{sr}$ such that $\Omega = G_1 C^0 G_2$ is diffeomorphic to $\big((G_1 \times G_2)/(G_1 \times G_2)_{x_0}\big) \times C^0$.*

Remark 4.26. Since the effective part of the slice representation is that of a finite group there exists an open and dense subset of points in $C \cap G_{sr}$ such that the slice representation at these points is trivial. Moreover, we claim that the isotropy groups of all these points are isomorphic. This can be seen from Theorem 2.11 since if $(G_1 \times G_2)_x$ acts trivially on $\mathfrak{q}^x = \mathfrak{c}$, then $(G_1 \times G_2)_x \cong \mathcal{Z}_{G_1 \times G_2}(C)$ holds.

The maximal region with proper $(G_1 \times G_2)$–action. Let Ω be an open $(G_1 \times G_2)$–invariant subset of G. We assume that every $(G_1 \times G_2)$–orbit in Ω is closed in G, i. e. that $\Omega \subset G_1 \Phi^{-1}(0) G_2$ holds. It follows that every $(G_1 \times G_2)$–orbit in Ω admits a geometric slice and that the quotient $\Omega/\!\!/(G_1 \times G_2) = \Omega/(G_1 \times G_2) \cong \big(\Omega \cap \Phi^{-1}(0)\big)/(K_1 \times K_2)$ is Hausdorff. Therefore, $G_1 \times G_2$ acts properly on $\Omega \subset G_1 \Phi^{-1}(0) G_2$ if and only if the isotropy group $(G_1 \times G_2)_x$ is compact for every $x \in \Omega$ (compare [27]). This discussion leads to the following

Proposition 4.27. *The set*

$$\mathrm{Comp}_{G_1 \times G_2}(G) := \{x \in G;\ (G_1 \times G_2) \cdot x \text{ is closed and } (G_1 \times G_2)_x \text{ is compact}\}$$

is the maximal open subset of $G_1 \Phi^{-1}(0) G_2$ on which $G_1 \times G_2$ acts properly.

Proof. This is immediate from Proposition 14.24 in [11]. ∎

Remark 4.28. 1. The reader should be aware of the fact that the set $\mathrm{Comp}_{G_1\times G_2}(G)$ is in most cases empty.

2. If the group G is complex and the involutions σ_1 and σ_2 are both anti-holomorphic, then a $(G_1 \times G_2)$–orbit with compact isotropy group is automatically closed in G. This can be deduced with the help of the Slice Theorem from the fact that an adjoint $G_{\mathbb{R}}$–orbit in $\mathfrak{g}_{\mathbb{R}}$ with compact isotropy is automatically closed, where $G_{\mathbb{R}}$ is real-reductive. As we will see in the second example in Section 5, this is not the case if one of the involutions is holomorphic.

Example 4.29. Let G/K be a Riemannian symmetric space of non-compact type and let $G/K \hookrightarrow G^{\mathbb{C}}/K^{\mathbb{C}}$ be its complexification. Then $\mathrm{Comp}_{G\times K^{\mathbb{C}}}(G^{\mathbb{C}})$ coincides with the Akhiezer-Gindikin subset $GU^+K^{\mathbb{C}} \subset G^{\mathbb{C}}$ defined in Proposition 4 in [1].

Non-closed $(G_1 \times G_2)$–orbits. We describe the set of regular elements in G which lie in non-closed $(G_1 \times G_2)$–orbits.

Proposition 4.30. *Every element $x \in G_r$ can be written in the form*

$$x = n \exp(\eta) \exp(\xi),$$

where $x_0 := n \exp(\eta)$ lies in the standard Cartan subset $C = n \exp(\mathfrak{c})$ and ξ is a point of the null cone of the H^{x_0}–representation on $\mathfrak{q}^{x_0}$.

Proof. This is a consequence of the Slice Theorem and the description of the isotropy representation. ■

5. Examples

Real forms. Let G be complex semi-simple, and let $\sigma_1 = \sigma_2 =: \sigma$ define the real form $G_{\mathbb{R}}$ of G. In [4] and [28] it is shown that the closed $(G_{\mathbb{R}} \times G_{\mathbb{R}})$–orbits in G are parametrized by the different conjugacy classes of real Cartan subalgebras in $\mathfrak{g}_{\mathbb{R}}$. Moreover, in [5] also the structure of non-closed orbits is investigated in great detail. The case that σ_1 and σ_2 are any (not necessarily commuting) anti-holomorphic involutive automorphisms of G defining the two real forms G_1 and G_2 is considered in [23] where a natural gradient map is used in order to analyze the set of closed $(G_1 \times G_2)$–orbits in the same spirit as in this paper. A special feature when actions of real forms are considered is that the slice representations are equivalent to the adjoint representation of real-reductive Lie groups. These are technically simpler to deal with than the isotropy representations of arbitrary reductive symmetric spaces.

Complexification of semi-simple symmetric spaces. Let G be a linear semi-simple real Lie group with an involutive automorphism σ, and let H be a subgroup of G such that $(G^\sigma)^0 \subset H \subset G^\sigma$ holds. Then we can form the complexification $G/H \hookrightarrow G^{\mathbb{C}}/H^{\mathbb{C}}$ of the semi-simple symmetric space G/H. The first basic question in this situation is how one can understand the orbit structure of the G–action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ or equivalently the orbit structure of the $(G\times H^{\mathbb{C}})$–action on $G^{\mathbb{C}}$. Let us assume that there exists an anti-holomorphic involutive automorphism $\kappa\in \operatorname{Aut}(G^{\mathbb{C}})$ which defines the real form G. The holomorphic extension of $\sigma\in\operatorname{Aut}(G)$ to $G^{\mathbb{C}}$ defines the group $H^{\mathbb{C}}$. Let $\theta\in\operatorname{Aut}(G)$ be a Cartan involution which commutes with σ and induces the decompositions $G=K\exp(\mathfrak{p})$ and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$; for a proof of the existence of such a Cartan involution see e. g. [20]. Then the group U generated by $K\exp(i\mathfrak{p})$ is a compact real form of $G^{\mathbb{C}}$ such that the groups G and $H^{\mathbb{C}}$ are compatible subgroups of $G^{\mathbb{C}}=U^{\mathbb{C}}=U\exp(i\mathfrak{u})$. It follows that the group $U^{\sigma}=U\cap H^{\mathbb{C}}$ is a compact real form of $H^{\mathbb{C}}=U^{\sigma}\exp(i\mathfrak{u}^{\sigma})$.

Let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ be the decomposition of $\mathfrak{g}$ with respect to σ. We conclude from equation (1) that with this notation the zero fiber of our gradient map has the form

$$\begin{aligned}\Phi^{-1}(0)&=U\exp\big((i\mathfrak{u})^{-\sigma}\big)\cap\exp\big((i\mathfrak{u})^{-\kappa}\big)\\&=\big\{u\exp(\xi);\ \xi\in\big(i(\mathfrak{q}\cap\mathfrak{k})\oplus(\mathfrak{q}\cap\mathfrak{p})\big)\cap i\operatorname{Ad}(u^{-1})\mathfrak{k}\big\}.\end{aligned}$$

Moreover, from

$$\mathfrak{u}^{-\sigma}\cap\mathfrak{u}^{-\kappa}=i(\mathfrak{p}\cap\mathfrak{q})\quad\text{and}\quad(i\mathfrak{u})^{-\sigma}\cap(i\mathfrak{u})^{-\kappa}=i(\mathfrak{k}\cap\mathfrak{q})$$

we see that if we choose a maximal torus $\mathfrak{t}_0$ in $i(\mathfrak{p}\cap\mathfrak{q})$ and a maximal Abelian subspace $\mathfrak{a}_0$ of $\mathcal{Z}_{i(\mathfrak{k}\cap\mathfrak{q})}(\mathfrak{t}_0)$, then $C_0:=\exp(\mathfrak{c}_0)$ with $\mathfrak{c}_0:=\mathfrak{t}_0\oplus\mathfrak{a}_0$ is a fundamental Cartan subset of $G^{\mathbb{C}}$.

Remark 5.1. 1. It follows from the construction that $i\mathfrak{c}_0$ is a θ–stable Cartan subspace of $\mathfrak{q}$ whose non-compact factor $i\mathfrak{t}_0$ is maximal.

2. If we form the weight space decomposition $\mathfrak{g}=\mathfrak{g}_0\oplus\bigoplus_{\lambda\in\Lambda}\mathfrak{g}_\lambda$, $\Lambda=\Lambda(\mathfrak{g},i\mathfrak{t}_0)$, of $\mathfrak{g}$ with respect to $i\mathfrak{t}_0$, then the fact that Λ is a (possibly non-reduced) root system is also proven in [26].

Let us form the extended weight space decomposition of $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{C}}$. For this we consider the embedding $\mathfrak{g}^{\mathbb{C}}\hookrightarrow\mathfrak{g}^{\mathbb{C}}\oplus\mathfrak{g}^{\mathbb{C}}$, $\xi\mapsto\big(\xi,\kappa(\xi)\big)$. One checks immediately that the $\mathbb{C}$–linear extensions of σ and κ to $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{C}}\cong\mathfrak{g}^{\mathbb{C}}\oplus\mathfrak{g}^{\mathbb{C}}$ are given by $(\xi,\xi')\mapsto\big(\sigma(\xi),\sigma(\xi')\big)$ and $(\xi,\xi')\mapsto(\xi',\xi)$, respectively. Forming the weight space decomposition of $\mathfrak{g}$ with respect to $i\mathfrak{t}_0$ with weights $\Lambda=\Lambda(\mathfrak{g},i\mathfrak{t}_0)$, it follows for each $\lambda\in\Lambda\cup\{0\}$ that

$$(\mathfrak{g}^{\mathbb{C}}\oplus\mathfrak{g}^{\mathbb{C}})_\lambda=\mathfrak{g}^{\mathbb{C}}_\lambda\oplus\mathfrak{g}^{\mathbb{C}}_{-\lambda}$$

holds. Consequently, the set of extended weights is given by $\widetilde{\Lambda}=\Lambda\times\{\pm1\}$.

Remark 5.2. The set

$$\omega_0:=\big\{i\eta\in\mathfrak{t}_0;\ |\lambda(\eta)|<\tfrac{\pi}{2}\text{ for all }\lambda\in\Lambda(\mathfrak{g},i\mathfrak{t}_0)\big\}$$

can be used to define a generalized Akhiezer-Gindikin domain in $G^{\mathbb{C}}/H^{\mathbb{C}}$ containing G/H (see Proposition 2.3 in [9]). If $\eta\in\omega_0$ and $u=\exp(\eta)$, then we have $\big(i(\mathfrak{q}\cap\mathfrak{k})\oplus(\mathfrak{q}\cap\mathfrak{p})\big)\cap i\operatorname{Ad}(u^{-1})\mathfrak{k}=i(\mathfrak{q}\cap\mathfrak{k})$. Hence, the $(G\times H^{\mathbb{C}})$–orbits in $G\exp(\omega_0\times\mathfrak{a}_0)H^{\mathbb{C}}$ intersect only standard Cartan subsets which are conjugate to the fundamental Cartan subset C_0. Since we have $\widetilde{\Lambda}=\Lambda\times\{\pm1\}$, the set $G\exp(\omega_0\times\mathfrak{a}_0)H^{\mathbb{C}}$ is an open neighborhood of $GH^{\mathbb{C}}=(G\times H^{\mathbb{C}})\cdot e$ in $G^{\mathbb{C}}$ and $GH^{\mathbb{C}}$ is the only non-generic orbit in this neighborhood.

In closing we describe three explicit examples in detail.

Example 5.3. Let G be as above and let θ be a Cartan involution of G. Taking $\sigma = \theta$ we obtain the Riemannian symmetric space G/K. The analysis of the G–action on the complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ has begun in [1]. For a formulation of Matsuki's results in this context we refer the reader to [8].

A simple example for this setup is the complexification of the upper half plane $\mathbb{H}^+ = \bigl\{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\bigr\}$ which can be written as G/K with $G = \mathrm{SL}(2,\mathbb{R})$ and $K = \mathrm{SO}(2,\mathbb{R})$. The complexification $G^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$ has $\mathrm{SU}(2)$ as compact real form, and G and $K^{\mathbb{C}} = \mathrm{SO}(2,\mathbb{C})$ are closed compatible subgroups of $G^{\mathbb{C}} = \mathrm{SU}(2)\exp\bigl(i\mathfrak{su}(2)\bigr)$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to θ. Then a fundamental Cartan subset of $G^{\mathbb{C}}$ is given by $C_0 = \exp(i\mathfrak{a})$ where $\mathfrak{a} \subset \mathfrak{p}$ is a maximal Abelian subspace. From $\dim \mathfrak{a} = 1$ we conclude that generic $(G \times K^{\mathbb{C}})$–orbits are hypersurfaces in $G^{\mathbb{C}}$ and that the $(G \times K^{\mathbb{C}})$–action is generically free.

Here we choose

$$\mathfrak{a} = \left\{ \eta_t := \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix};\ t \in \mathbb{R} \right\}$$

and set $x_t := \exp(i\eta_t) \in C_0$. One checks directly that the Weyl group $W_{K\times K}(C_0)$ is generated by $x_t \mapsto x_{-t}$ and $x_t \mapsto x_{t+\pi}$ and hence is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. It follows that the set $\mathcal{F} := \bigl\{x_t \in C_0;\ t \in [0, \pi/2]\bigr\}$ forms a fundamental domain for the $W_{K\times K}(C_0)$–action on C_0. The only non-generic orbits in C_0 are the ones through $x_0 = e$, $x_{\pi/4}$ and $x_{\pi/2}$. Note that x_0 and $x_{\pi/2}$ have compact isotropy isomorphic to K while $x_{\pi/4}$ has non-compact isotropy isomorphic to $\exp(i\mathfrak{k}) \cong \mathbb{R}$. The slice representation at $x_{\pi/4}$ is isomorphic to the representation $s \mapsto \begin{pmatrix} e^{2s} & 0 \\ 0 & e^{-2s} \end{pmatrix}$ of $\mathbb{R}$ on $\mathbb{R}^2$. We conclude that there are precisely four non-closed $(G \times K^{\mathbb{C}})$–orbits which contain $x_{\pi/4}$ in their closure and that these non-closed orbits form the smooth part of the boundaries of the four connected components of $G^{\mathbb{C}}_{sr}$.

Identifying $G^{\mathbb{C}}/K^{\mathbb{C}}$ with $(\mathbb{P}_1 \times \mathbb{P}_1) \setminus \Delta$ where Δ denotes the diagonal in $\mathbb{P}_1 \times \mathbb{P}_1$, one finds that the four connected components of $G^{\mathbb{C}}_{sr}$ coincide with the preimages of the G–invariant domains $(\mathbb{H}^+ \times \mathbb{H}^+) \setminus \Delta$, $\mathbb{H}^+ \times \mathbb{H}^-$, $\mathbb{H}^- \times \mathbb{H}^+$ and $(\mathbb{H}^- \times \mathbb{H}^-) \setminus \Delta$ under the quotient map $G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}} = (\mathbb{P}_1 \times \mathbb{P}_1) \setminus \Delta$. The Akhiezer-Gindikin domain in this example is the domain $\mathbb{H}^+ \times \mathbb{H}^-$.

Finally, we remark that

$$\mathrm{Comp}_{G\times K^{\mathbb{C}}}(G^{\mathbb{C}}) = G^{\mathbb{C}} \setminus \pi^{-1}\bigl(\pi(x_{\pi/4})\bigr) = G^{\mathbb{C}}_{sr} \cup (G \times K^{\mathbb{C}}) \cdot e \cup (G \times K^{\mathbb{C}}) \cdot x_{\pi/2},$$

where $\pi \colon G^{\mathbb{C}} \to G^{\mathbb{C}}/\!\!/(G \times K^{\mathbb{C}})$ denotes the topological Hilbert quotient.

Example 5.4. We now turn to the example $G := \mathrm{SU}(2,2)$ and $K := \mathrm{S}\bigl(\mathrm{U}(2)\times\mathrm{U}(2)\bigr)$. The group $U := \mathrm{SU}(4)$ is a compact real form of $G^{\mathbb{C}} = \mathrm{SL}(4,\mathbb{C}) = U^{\mathbb{C}}$ such that G and $K^{\mathbb{C}} = \mathrm{S}\bigl(\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})\bigr)$ are closed compatible subgroups of $U^{\mathbb{C}}$. A fundamental Cartan subset of $G^{\mathbb{C}}$ is given by $C_0 = T_0 = \exp(\mathfrak{t}_0)$ where $i\mathfrak{t}_0$ is a maximal Abelian subspace of $\mathfrak{p}$. Since every such space has dimension 2, generic $(G \times K^{\mathbb{C}})$–orbits in $G^{\mathbb{C}}$ are two-codimensional which implies that the isotropy groups have generically dimension 1. In particular T_0 is not a maximal torus in U since every maximal torus in $\mathrm{SU}(4)$ is three-dimensional. Choosing

$$i\mathfrak{t}_0 := \left\{ \eta_{t,s} := \begin{pmatrix} 0 & 0 & 0 & s \\ 0 & 0 & t & 0 \\ 0 & t & 0 & 0 \\ s & 0 & 0 & 0 \end{pmatrix} ; t, s \in \mathbb{R} \right\}$$

one checks directly that the restricted root system $\Lambda = \Lambda(\mathfrak{g}, i\mathfrak{t}_0)$ is given by $\Lambda = \{\pm\lambda_1, \pm\lambda_2, \pm(\lambda_1+\lambda_2), \pm(\lambda_1-\lambda_2)\}$ where $\lambda_1(\eta_{t,s}) = t+s$ and $\lambda_2(\eta_{t,s}) = t-s$ hold. A fundamental domain for the $(K \times K)$–action on U is given by $\exp(\overline{\mathcal{F}})$ with

$$\mathcal{F} := \left\{ i\eta_{t,s} \in \mathfrak{t}_0;\ 0 < t < s < \tfrac{\pi}{4} \right\} \subset \left\{ i\eta_{t,s} \in \mathfrak{t}_0;\ |t|, |s| < \tfrac{\pi}{4} \right\} =: \omega_0.$$

Direct computations give that $\mathfrak{p} \cap i\,\mathrm{Ad}(u^{-1})\mathfrak{k} = \{0\}$ holds for all $u \in \exp(\omega_0) \subset T_0$, i.e. $\Phi^{-1}(0) \cap \exp(\omega_0) \subset U$. Hence, $G \times K^{\mathbb{C}}$ acts properly on the domain $G\exp(\omega_0)K^{\mathbb{C}}$. In fact, one can show that $G\exp(\omega_0)K^{\mathbb{C}}$ is the connected component of $\mathrm{Comp}_{G\times K^{\mathbb{C}}}(G^{\mathbb{C}})$ containing $GK^{\mathbb{C}}$ (see Proposition 7 in [1]).

In the next step we describe the boundary of $G\exp(\omega_0)K^{\mathbb{C}}$ in $G^{\mathbb{C}}$. There are two qualitatively different types of boundary points of ω_0, namely those $\eta_{t,s}$ where $|t| = \frac{\pi}{4}$ and $|s| < \frac{\pi}{4}$ (or vice versa) and those where $|t| = |s| = \frac{\pi}{4}$. To make our considerations explicit, we take the element $\eta_1 := \begin{pmatrix} 0 & 0 & 0 & i\pi/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\pi/4 & 0 & 0 & 0 \end{pmatrix} \in \partial\omega_0$.

Let $u_1 := \exp(\eta_1)$. Since $\mathfrak{p} \cap i\,\mathrm{Ad}(u_1^{-1})\mathfrak{k} = \mathbb{R}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$, we conclude the u_1 is contained in the standard Cartan subset $C_1 = u_1\exp(\mathfrak{c}_1)$ with

$$\begin{aligned} \mathfrak{c}_1 &= \left\{ \begin{pmatrix} 0 & 0 & 0 & is \\ 0 & 0 & it & 0 \\ 0 & it & 0 & 0 \\ -is & 0 & 0 & 0 \end{pmatrix} ;\ t, s \in \mathbb{R} \right\} \\ &= \underbrace{\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & it & 0 \\ 0 & it & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}}_{\mathfrak{t}_1} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & 0 & 0 & is \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -is & 0 & 0 & 0 \end{pmatrix} \right\}}_{\mathfrak{a}_1}. \end{aligned}$$

The isotropy of the point u_1 is isomorphic to

$$K^{\mathbb{C}} \cap u_1^{-1}Gu_1 = \left\{ \begin{pmatrix} z & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & \overline{z}^{-1} \end{pmatrix} ;\ z \in \mathbb{C}^*, a, b \in S^1, z\overline{z}^{-1}ab = 1 \right\}$$

and the tangent space of the geometric slice at u_1 is given by

$$\mathfrak{p}^{\mathbb{C}} \cap i\,\mathrm{Ad}(u_1^{-1})\mathfrak{g} = \left\{ \begin{pmatrix} 0 & 0 & 0 & it \\ 0 & 0 & w & 0 \\ 0 & -\overline{w} & 0 & 0 \\ is & 0 & 0 & 0 \end{pmatrix} ;\ t, s \in \mathbb{R}, w \in \mathbb{C} \right\}.$$

With this information, one sees directly that the (non-closed) $(K^{\mathbb{C}} \cap u_1^{-1}Gu_1)$–orbits through

$$\begin{pmatrix} 0 & 0 & 0 & \pm i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm i & 0 & 0 & 0 \end{pmatrix}$$

form the smooth part of the nullcone in $\mathfrak{p}^{\mathbb{C}} \cap i\,\mathrm{Ad}(u_1^{-1})\mathfrak{g}$. Consequently, these elements lie in the smooth part of a one-codimensional stratum.

Next we consider the point $\eta_2 := \begin{pmatrix} 0 & 0 & 0 & i\pi/4 \\ 0 & 0 & i\pi/4 & 0 \\ 0 & i\pi/4 & 0 & 0 \\ i\pi/4 & 0 & 0 & 0 \end{pmatrix} \in \partial\omega_0$ and put $u_2 := \exp(\eta_2)$. From

$$\mathfrak{p} \cap i\,\mathrm{Ad}(u_2^{-1})\mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 & x & is \\ 0 & 0 & it & -\overline{x} \\ \overline{x} & -it & 0 & 0 \\ -is & -x & 0 & 0 \end{pmatrix} ;\ t, s \in \mathbb{R}, x \in \mathbb{C} \right\}$$

we see that u_2 is contained in the standard Cartan subset $C_2 = u_2 \exp(\mathfrak{c}_2)$ with

$$\mathfrak{c}_2 = \mathfrak{a}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & is \\ 0 & 0 & it & 0 \\ 0 & -it & 0 & 0 \\ -is & 0 & 0 & 0 \end{pmatrix} ;\ t, s \in \mathbb{R} \right\}.$$

Going through the different boundary parts of ω_0 we find all the conjugacy classes of standard Cartan subsets in $G^{\mathbb{C}}$.

Example 5.5. Let $G = \mathrm{SU}(2,2)$ and $\sigma\colon G \to G$, $g \mapsto \overline{g}$, be given. The involution σ defines the group $H := G^{\sigma} = \mathrm{SO}(2,2)$. The groups G and $H^{\mathbb{C}} = \mathrm{SO}(4,\mathbb{C})$ are compatible subgroups of $G^{\mathbb{C}} = \mathrm{SL}(4,\mathbb{C})$ with respect to the compact real form $U = \mathrm{SU}(4)$. As usual we write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ for the decomposition of $\mathfrak{g}$ with respect to σ. Direct computations show that $C_0 = \exp(\mathfrak{c}_0)$ with

$$\mathfrak{c}_0 = \underbrace{\left\{ \begin{pmatrix} 0 & 0 & 0 & s \\ 0 & 0 & t & 0 \\ 0 & -t & 0 & 0 \\ -s & 0 & 0 & 0 \end{pmatrix} ;\ t, s \in \mathbb{R} \right\}}_{=\mathfrak{t}_0} \oplus \underbrace{\left\{ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} ;\ \alpha \in \mathbb{R} \right\}}_{=\mathfrak{a}_0} \subset i\mathfrak{q}$$

is a fundamental Cartan subset. Consequently, the generic $(G \times H^{\mathbb{C}})$–orbits in $G^{\mathbb{C}}$ are three-codimensional. Since $\dim_{\mathbb{R}} G \times H^{\mathbb{C}} = 15 + 12 = 27$, we see that the $(G \times H^{\mathbb{C}})$–isotropy of a regular element is trivial. In particular, there exist non-closed orbits with compact isotropy.

Taking the same fundamental domain $\mathcal{F} \subset \mathfrak{t}_0$ as in the previous example it is possible to find representatives of the standard Cartan subsets in the same way as above. Moreover, computing the slice representations one obtains a description of the elements lying in non-closed orbits.

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