Journal of Lie Theory
Volume **18** (2008) 671–696

On the Dimension of the Sheets of a Reductive Lie Algebra

Anne Moreau

Communicated by F. Knop

Abstract. Let $\mathfrak{g}$ be a complex finite dimensional Lie algebra and G its adjoint group. Following a suggestion of A. A. Kirillov, we investigate the dimension of the subset of linear forms $f \in \mathfrak{g}^*$ whose coadjoint orbit has dimension $2m$, for $m \in \mathbb{N}$. In this paper we focus on the reductive case. In this case the problem reduces to the computation of the dimension of the sheets of $\mathfrak{g}$. These sheets are known to be parameterized by the pairs $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$, up to G-conjugacy class, consisting of a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and a rigid nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ in $\mathfrak{l}$. By using this parametrization, we provide the dimension of the above subsets for any m.
Mathematic Subject Classification 2000: 14A10, 14L17, 22E20, 22E46.
Key Words and Phrases: Reductive Lie algebra, coadjoint orbit, sheet, index, Jordan class, induced nilpotent orbit, rigid nilpotent orbit.

1. Introduction

1.1. Motivation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\Bbbk$. The adjoint group G of $\mathfrak{g}$ acts on $\mathfrak{g}$ and on its dual $\mathfrak{g}^*$ via the adjoint and coadjoint actions. The famous "method of orbits", initiated by A. A. Kirillov 40 years ago [14], consists in relating the unitary dual of G to the set $\mathfrak{g}^*/G$ of coadjoint orbits in $\mathfrak{g}^*$. In this context, the understanding of the structure of the set $\mathfrak{g}^*/G$ is crucial. Let us adopt the algebraic geometry point of view. Let us consider the set $\mathfrak{g}^*_m$ of elements $f \in \mathfrak{g}^*$ whose coadjoint orbit has dimension $2m$. For $f \in \mathfrak{g}^*$, denote by B_f the skew-symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$ defined by:

$$B_f(x, y) = f([x, y]),\ \forall x, y \in \mathfrak{g}.$$

As the kernel of B_f is the stabilizer of f in $\mathfrak{g}$ for the coadjoint action, the dimension of the coadjoint orbit of f is equal to the rank of B_f. From these observations, we readily deduce the following lemma:

Lemma 1.1. *The set $\mathfrak{g}^*_m$ is a quasi-affine algebraic variety.*

ISSN 0949–5932 / \$2.50

Proof. Note first that the rank of B_f is even since B_f is a skew-symmetric bilinear form. The set $X_m := \{f \in \mathfrak{g}^*\ ,\ \mathrm{rk}\, B_f \leq 2m\}$ is an algebraic subset of $\mathfrak{g}^*$ defined by the condition that all the minors of order $> 2m$ of the matrix representing B_f vanish. As $\mathfrak{g}^*_m = X_m \setminus X_{m-1}$, the lemma follows. ■

Then, we can show that the set $\mathfrak{g}^*/G$ is a finite disjoint union of quasi-affine algebraic varieties (see [15](§1.2) for a sketch of proof). We turn now to the study of the varieties $\mathfrak{g}^*_m$ for $m \in \mathbb{N}$. Following a suggestion of A. A. Kirillov, a first approach is to investigate the dimension of these varieties. The *index* of $\mathfrak{g}$, as it was defined by J. Dixmier, is the minimal codimension of the coadjoint orbits in $\mathfrak{g}^*$, that is

$$\mathrm{ind}\, \mathfrak{g} := \dim \mathfrak{g} - \max_{f \in \mathfrak{g}^*}(\mathrm{rk}\, B_f).$$

Thereby, the integers m for which $\mathfrak{g}^*_m$ is a nonempty set ranges from 0 to $\mathrm{d}_{\mathfrak{g}}$, where $\mathrm{d}_{\mathfrak{g}}$ is the integer $(\dim \mathfrak{g} - \mathrm{ind}\, \mathfrak{g})/2$. A linear form f is said to be *regular* if $\dim \mathfrak{g} - \mathrm{rk}\, B_f = \mathrm{ind}\, \mathfrak{g}$. Since the set of regular linear forms is an open dense subset of $\mathfrak{g}^*$, we get

$$\dim \mathfrak{g}^*_m = \dim \mathfrak{g}^* = \dim \mathfrak{g}, \quad \text{for } m = \mathrm{d}_{\mathfrak{g}}.$$

What about the smaller integers m? Our discussion shows that this question heavily depends on the problem of computing the index. So far, the index was studied only in very specific cases (see for instance [6], [19], [24], [7], [22], [16], [17] and [11]). As a result, it would be too ambitious to get such integers, as well as the dimension of the sets $\mathfrak{g}^*_m$, for any m and any Lie algebra. In [15], A. A. Kirillov approaches the case where $\mathfrak{g}$ is the Lie algebra of the Lie group of upper triangular matrices with ones on the diagonal over a field $\mathbb{F}_q$ of characteristic q. In this note, we consider the case where $\mathfrak{g}$ is a complex finite dimensional reductive Lie algebra.

1.2. Description of the paper. From now on, $\Bbbk = \mathbb{C}$ and $\mathfrak{g}$ is supposed to be reductive. The Lie algebra $\mathfrak{g}$ can be identified to $\mathfrak{g}^*$ through a nondegenerate G-invariant bilinear form on $\mathfrak{g}$. In particular the index of $\mathfrak{g}$ is equal to its rank $\mathrm{rk}\, \mathfrak{g}$. Moreover $\mathfrak{g}^*_m$ is identified to the subset $\mathfrak{g}^{(m)}$ of elements $x \in \mathfrak{g}$ whose adjoint orbit $G.x$ has dimension $2m$. The irreducible components of $\mathfrak{g}^{(m)}$ are, by definition, the *sheets* of $\mathfrak{g}$. The notion of sheets was first introduced in [5] for determining polarizable elements and was generalized in [3] and [2]. The study of sheets was motivated by connections with primitive ideals in enveloping algebras. Obviously, $\mathfrak{g}$ is the finite union of its sheets. As a result, for our purpose, it suffices to compute the dimension of all the sheets of $\mathfrak{g}$. Concerning the sheets, many results are already known. To start with, by works of Borho [2], the sheets of $\mathfrak{g}$ are parameterized by the pairs $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$, up to G-conjugacy class, consisting of a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and a *rigid* nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ of $\mathfrak{l}$. Next, G. Kempken gave in [13] a description of the rigid nilpotent orbits in the classical cases in terms of their partitions. For the exceptional cases, we have explicit computations for the rigid nilpotent orbits due to A. G. Elashvili. These computations are collected in [20](Appendix of Chap. II) and a complete version was published later in [8]. Let us mention besides that P. I. Katsylo gave in [12] a construction of a geometric quotient for the sheets. An alternative proof is given in the Ph.D thesis of A. Im Hof [10]. A. Im Hof also proves that the sheets of the classical Lie algebras are smooth.

In this note we give, for any simple Lie algebra, the dimension of its sheets. This answers the initial problem and we obtain the dimension of all the varieties $\mathfrak{g}^{(m)}$. The reductive case easily deduces from the case where $\mathfrak{g}$ is simple. Thanks to the results recalled above, we have already at our disposal almost all the necessary ingredients. The main work here is to obtain explicit formulas in the classical case (see Theorems 3.3 and 3.13). On the other hand, this piece of work is a good opportunity to put together the known information about the sheets.

In Section 2, we collect known results about G-Jordan classes, sheets, induced nilpotent orbits and rigid nilpotent orbits and we describe the links between these different notions. By a Veldkamp's result [23], the set of non-regular elements of $\mathfrak{g}$ has codimension 3 in $\mathfrak{g}$. As a by-product, we will precise this result by showing that $\mathfrak{g}^{(\mathrm{d}_\mathfrak{g}-1)}$ is equidimensional of codimension 3 (see Theorem 2.14). For the smaller integers, there is no visible general rules, as our explicit computations will show up. In particular, the subsets $\mathfrak{g}^{(m)}$ are not always equidimensional. For example in $\mathbf{F}_4$, the set $\mathfrak{g}^{(20)}$ has three irreducible components of dimensions 41, 41 and 42 (see Table 7). Likewise in $\mathfrak{so}_{12}$, the set $\mathfrak{g}^{(26)}$ has two irreducible components of dimensions 54 and 55 (see Table 5).

Section 3 concerns the classical Lie algebras. We recall the characterization of rigid nilpotent orbits in term of partitions, following [13]. Next, we give a formula for the dimension of $\mathfrak{g}^{(m)}$ for any $m \in \mathbb{N}$ (Theorems 3.3 and 3.13). We present in Tables 1, 3, 4 and 5 theses dimensions for $\mathfrak{sl}_6$, $\mathfrak{so}_7$, $\mathfrak{sp}_6$ and $\mathfrak{so}_{12}$. Any sheet contains a unique nilpotent orbit (see Proposition 2.11). However a nilpotent orbit may belong to different sheets. The results of this section specify that, in the classical case, the dimension of a sheet containing a given nilpotent orbit does not depend on the choice of a sheet containing it (see Proposition 3.11). This remarkable fact does not hold for the exceptional case anymore. For example in $\mathbf{E}_6$, the only nilpotent orbit of dimension 66 belongs to two sheets of dimensions 69 and 70 (see Table 8).

Section 4 is devoted to the exceptional case. We compute for each exceptional simple Lie algebra, the dimension of its sheets and then the dimension of the varieties $\mathfrak{g}^{(m)}$ (cf. Tables 6 – 12 and 13 – 17). These Tables give in addition the numbers of sheets in $\mathfrak{g}^{(m)}$, for any m.

1.3. Additional notations. If $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$, we denote by $\mathfrak{z}_\mathfrak{g}(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and by $\mathfrak{z}(\mathfrak{a})$ its center.

For $x \in \mathfrak{g}$, we denote by $\mathfrak{g}^x$ its centralizer in $\mathfrak{g}$. Thus $\dim G.x = \dim \mathfrak{g} - \mathfrak{g}^x$ and x is regular if and only if $\dim \mathfrak{g}^x = \mathrm{rk}\ \mathfrak{g}$.

For $x \in \mathfrak{g}$, we respectively denote by x_s and x_n the semisimple and nilpotent components of x in $\mathfrak{g}$.

For any G-invariant subset Y of $\mathfrak{g}$ we denote by Y^{reg} the set of regular elements of Y, that is those whose G-orbit has maximal dimension.

Denote by $\mathbf{N}_{\mathfrak{g}}$ the set of the dimensions of the nilpotent orbits in $\mathfrak{g}$ and by $\mathbf{N}_{\mathfrak{g}}^{\mathrm{rig}}$ the set of the dimensions of the rigid nilpotent orbits of $\mathfrak{g}$ (The notion of *rigid nilpotent orbit* is recalled in Section 2, §2.2).

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let Π be a basis of the root system Δ associated to the couple $(\mathfrak{g},\mathfrak{h})$. Denote by Δ_+ the positive root system corresponding to Π. For $\alpha\in\Delta$, denote by $\mathfrak{g}^\alpha$ the root subspace associated to α. For S any subset of Π, we denote by Δ^S the root subsystem of Δ generated by S, and by Δ_+^S the intersection $\Delta_+\cap\Delta^S$. Set

$$\mathfrak{l}_S := \mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta^S}\mathfrak{g}^\alpha.$$

Thus $\mathfrak{l}_S$ is a Levi subalgebra of $\mathfrak{g}$ and it is well-known (see for instance [4], Lemma 3.8.1) that every Levi subalgebra of $\mathfrak{g}$ is G-conjugated to $\mathfrak{l}_S$, for some S in Π. We shall denote simply by $\mathbf{N}_S$ and $\mathbf{N}_S^{\mathrm{rig}}$ the sets $\mathbf{N}_{\mathfrak{l}_S}$ and $\mathbf{N}_{\mathfrak{l}_S}^{\mathrm{rig}}$ respectively.

1.4. Acknowledgment. I would like to thank A. A. Kirillov for having provided the motivation for this work. I am also very grateful to Karin Baur for her many comments and suggestions. In particular, she brought the thesis of Andreas Emanuel Im Hof to my attention.

2. Jordan Classes, sheets and induced nilpotent orbits

2.1. Jordan classes. The results of this subsection are mostly due to W. Borho and H. Kraft [3]. We also refer to [21] for a review.

Definition 2.1. Let $x,y\in\mathfrak{g}$. We say that x and y are *G-Jordan equivalent* if there exists $g\in G$ such that: $\mathfrak{g}^{y_s}=g(\mathfrak{g}^{x_s})$, $y_n=g(x_n)$. This defines an equivalence relation on $\mathfrak{g}$. The equivalence class of x is denoted by $J_G(x)$ and called the *G-Jordan class of x* in $\mathfrak{g}$.

The group G acts in a obvious way on the set of the pairs $(\mathfrak{l},\mathcal{O}_{\mathfrak{l}})$ consisting of a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and a nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ of $\mathfrak{l}$. The following lemma is proven for instance in [21](Propositions 39.1.5 and 39.2.9):

Lemma 2.2. *Let $J_G(x)$ be a G-Jordan class. Then*
i) $J_G(x)=G.(\mathfrak{z}(\mathfrak{g}^{x_s})^{\mathrm{reg}}+x_n)$,
ii) $\dim J_G(x)=\dim(G.x)+\dim\mathfrak{z}(\mathfrak{g}^{x_s})$.

Proposition 2.3. *There is a 1-1 correspondence between G-Jordan classes and the set of pairs $(\mathfrak{l},\mathcal{O}_{\mathfrak{l}})$, up to G-conjugacy, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathcal{O}_{\mathfrak{l}}$ a nilpotent orbit of $\mathfrak{l}$.*

We recall here the proof of Proposition 2.3 to precise how the 1-1 correspondence is obtained.

Proof. Let $J_G(x)$ be a G-Jordan class of $\mathfrak{g}$. Since x_s is semisimple, the algebra $\mathfrak{l}:=\mathfrak{g}^{x_s}$ is a Levi subalgebra of $\mathfrak{g}$ which contains x_n. Denote by $\mathcal{O}$ the nilpotent orbit of x_n in $\mathfrak{l}$. Then, the G-orbit of the pair $(\mathfrak{l}, \mathcal{O})$ only depends on $J_G(x)$ and not on the choice of a representative in $J_G(x)$. This defines a first map.

Conversely, let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$ and let $\mathcal{O}_\mathfrak{l}$ be a nilpotent orbit in $\mathfrak{l}$. Fix $z \in \mathfrak{z}_\mathfrak{g}(\mathfrak{l})^{\rm reg}$, $y \in \mathcal{O}_\mathfrak{l}$ and set $x = y + z$. As z belongs to the center of $\mathfrak{l}$, $[z, y] = 0$. This forces $x_s = z$ and $x_n = y$. Then $\mathfrak{g}^{x_s} = \mathfrak{g}^z = \mathfrak{l}$, because $z \in \mathfrak{z}_\mathfrak{g}(\mathfrak{l})^{\rm reg}$. Since $J_G(x) = G \cdot (\mathfrak{z}(\mathfrak{g}^{x_s})^{\rm reg} + x_n)$ by Lemma 2.2, (i), the G-Jordan class of x depends neither on the choices of $z \in \mathfrak{z}(\mathfrak{g}^{x_s})^{\rm reg}$ nor on the choice of $y \in \mathcal{O}_\mathfrak{l}$. Furthermore, it is clear that $J_G(x)$ only depends on the G-orbit of $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ and not on the choice of one of its representatives. This provides the other map. These two maps are clearly inverse each other, whence the 1-1 correspondence. ∎

If J is a G-Jordan class associated via the previous correspondence to a pair $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathcal{O}_\mathfrak{l}$ a nilpotent orbit of $\mathfrak{l}$, we shall say that J *has data* $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$. It follows from the classification of Levi subalgebras of $\mathfrak{g}$, as well as the one of nilpotent orbits in reductive Lie algebras, that there are only finitely many G-Jordan classes. Therefore, $\mathfrak{g}$ is a finite disjoint union of its G-Jordan classes. By Lemma 2.2, a G-Jordan class is a G-stable irreducible subset of $\mathfrak{g}$ contained in $\mathfrak{g}^{(m)}$, for some $m \in \mathbb{N}$. Hence, we easily deduce (see [21](Proposition 39.3.3)):

Proposition 2.4. *Let X be a sheet of $\mathfrak{g}$. Then, there is an unique G-Jordan class J such that $J \subset X$ and $\overline{X} = \overline{J}$. Moreover, $X = \left(\overline{J}\right)^{\rm reg}$.*

Next, we intend to determine which G-Jordan classes are dense in a sheet, what we will do in Subsection 2.3

2.2. Induced nilpotent orbits. We recall in this subsection some facts about *induced nilpotent orbits*. We refer to [4](§7.1) for a survey on this topic.

Theorem 2.5. *Let $\mathfrak{l}$ be the reductive part of a parabolic Lie algebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ of $\mathfrak{g}$ with nilradical $\mathfrak{n}$. Then there is a unique nilpotent orbit $\mathcal{O}_\mathfrak{g}$ in $\mathfrak{g}$ meeting $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ in an open dense subset. We have* $\dim \mathcal{O}_\mathfrak{g} = \dim \mathcal{O}_\mathfrak{l} + 2 \dim \mathfrak{n}$ *and the orbit $\mathcal{O}_\mathfrak{g}$ is the unique nilpotent orbit in $\mathfrak{g}$ of this dimension which meets $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$.*

We say that the orbit $\mathcal{O}_\mathfrak{g}$ is *induced from* $\mathcal{O}_\mathfrak{l}$ and we denote it by ${\rm Ind}_\mathfrak{p}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$. If $\mathcal{O}_\mathfrak{l} = 0$, we say that $\mathcal{O}_\mathfrak{g}$ is *a Richardson orbit.* The nilpotent orbit $\mathcal{O}_\mathfrak{g}$ does not depend on the parabolic subalgebra $\mathfrak{p}$ having $\mathfrak{l}$ as Levi factor, see [4](Theorem 7.1.3). As a result, we can equally use the notation ${\rm Ind}_\mathfrak{l}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$ or ${\rm Ind}_\mathfrak{p}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$.

Proposition 2.6. *Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be the Levi decomposition of a parabolic subalgebra in $\mathfrak{g}$ and $\mathcal{O}_\mathfrak{l}$ a nilpotent orbit in $\mathfrak{l}$.*

i) ${\rm codim}_\mathfrak{l}(\mathcal{O}_\mathfrak{l}) = {\rm codim}_\mathfrak{g}({\rm Ind}_\mathfrak{l}^\mathfrak{g}(\mathcal{O}_\mathfrak{l}))$.

ii) *Let $\mathfrak{l}_1$ and $\mathfrak{l}_2$ be two Levi subalgebras of $\mathfrak{g}$ with $\mathfrak{l}_1 \subset \mathfrak{l}_2$. Then* ${\rm Ind}_{\mathfrak{l}_2}^\mathfrak{g}({\rm Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_1})) = {\rm Ind}_{\mathfrak{l}_1}^\mathfrak{g}(\mathcal{O}_{\mathfrak{l}_1})$.

2.3. Sheets. We list in this subsection the main steps useful to reach Theorem 2.10 [2](§4.4). Some of these intermediate results will be needed in the sequel. The most of results we set out here are due to Borho and are presented in the thesis of Andreas im Hof [10]. We omit here the proofs and refer to [2](§3.1, Satz a, §3.6, §4.2 and §4.4).

Proposition 2.7. *Let J be a G-Jordan class with data $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$. Then $\left(\overline{J}\right)^{\mathrm{reg}}$ contains a unique nilpotent orbit, which is* $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$.

Corollary 2.8. *Let J and J' be two G-Jordan classes with data $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ and $(\mathfrak{l}', \mathcal{O}_{\mathfrak{l}'})$ respectively such that $\mathfrak{l}$ contains $\mathfrak{l}'$. Then $\overline{J}$ is contained in $\overline{J'}$ if and only if $\mathcal{O}_{\mathfrak{l}}$ is $\mathrm{Ind}_{\mathfrak{l}'}^{\mathfrak{l}}(\mathcal{O}_{\mathfrak{l}'})$ up to G-conjugacy class.*

By a dimension argument, it follows from Proposition 2.6, (i), that not every nilpotent orbit is induced from another. A nilpotent orbit in $\mathfrak{g}$ which is not induced from any proper Levi subalgebra is called *rigid*.

Corollary 2.9. *A G-Jordan class with data $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ is dense in a sheet if and only if $\mathcal{O}_{\mathfrak{l}}$ is rigid in $\mathfrak{l}$.*

By combining Corollary 2.9 and Proposition 2.4, we obtain the expected classification for the sheets of $\mathfrak{g}$:

Theorem 2.10. *There is a 1-1 correspondence between the set of pairs $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$, up to G-conjugacy class, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathcal{O}_{\mathfrak{l}}$ a rigid nilpotent orbit in $\mathfrak{l}$, and the set of sheets of $\mathfrak{g}$.*

We shall say, in compliance with §1, that a sheet X *has data* $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ if X corresponds to a pair $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ via the correspondence established in Theorem 2.10.

2.4. Additional facts. Let X be a sheet of $\mathfrak{g}^{(m)}$, for $m \in \mathbb{N}$, with data $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ and let $J_G(x)$ be the G-Jordan class with the same data $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$. By Lemma 2.2, we have:

$$\mathfrak{l} = \mathfrak{g}^{x_s}\ , \quad \dim J_G(x) = \dim G.x + \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}).$$

In addition, by Propositions 2.4 and 2.7, $\overline{J_G(x)} = \overline{X}$ and $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$ is the unique nilpotent orbit contained in X. Moreover, since $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$ is contained in $X \subseteq \mathfrak{g}^{(m)}$, we have $\dim(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})) = \dim G.x = 2m$. To summarize, using in addition Proposition 2.6, (i), we have obtain:

Proposition 2.11. *Let X with data $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$. Then $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$ is the unique nilpotent orbit contained in X. Moreover:*

$$\begin{aligned} \dim X &= \dim(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})) + \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}) \\ &= \dim \mathfrak{g} - \dim \mathfrak{l} + \dim \mathcal{O}_{\mathfrak{l}} + \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}). \end{aligned}$$

Recall that $\mathbf{N}_{\mathfrak{g}}$ denotes the set of the dimensions of the nilpotent orbits in $\mathfrak{g}$ (see Introduction, §1.3). Obviously, if $2m \in \mathbf{N}_{\mathfrak{g}}$, then $\mathfrak{g}^{(m)}$ is a nonempty set. Proposition 2.11 says that the converse holds, too. Thereby, we can claim:

Proposition 2.12. *The set $\mathfrak{g}^{(m)}$ is nonempty if and only if $2m \in \mathbf{N}_{\mathfrak{g}}$.*

We conclude the section with general properties and remarks about the sheets and the varieties $\mathfrak{g}^{(m)}$.

Lemma 2.13. *Suppose that the semisimple part of $\mathfrak{g}$ has dimension strictly bigger than 3. Then the subregular nilpotent orbit of $\mathfrak{g}$ is not rigid.*

Proof. We adopt the notations of the Introduction, §1.3. For $\alpha \in \Pi$, the nilpotent orbit of $\mathfrak{g}$ induced from the zero orbit in $\mathfrak{l}_S$ has dimension,

$$\dim \mathfrak{g} - \dim \mathfrak{l}_S = \dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g} - 2,$$

by Proposition 2.6, (i). Hence the subregular nilpotent orbit is $\mathrm{Ind}_{\mathfrak{l}_S}^{\mathfrak{g}}(\mathcal{O}_0)$ since it is the only nilpotent orbit of $\mathfrak{g}$ of dimension $\dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g} - 2$. As $\mathfrak{g}$ is different from $\mathfrak{l}$ by hypothesis, we deduce that the subregular nilpotent orbit is not rigid. ■

Recall that $\mathrm{d}_{\mathfrak{g}}$ is the integer $(\dim \mathfrak{g} - \mathrm{ind}\, \mathfrak{g})/2$.

Theorem 2.14. *i) If $m > \mathrm{d}_{\mathfrak{g}}$, then $\mathfrak{g}^{(m)}$ is an empty set, and $\mathfrak{g}^{(\mathrm{d}_{\mathfrak{g}})}$ is an irreducible subset of $\mathfrak{g}$ of dimension* $\dim \mathfrak{g}$.

(ii) The subset $\mathfrak{g}^{(\mathrm{d}_{\mathfrak{g}}-1)}$ is equidimensional of dimension $\dim \mathfrak{g} - 3$.

Proof. i) has already been noticed in the Introduction.

iii) Let X be a sheet of $\mathfrak{g}^{(\mathrm{d}_{\mathfrak{g}}-1)}$ with data $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$. By Proposition 2.6, (i), the codimension in $\mathfrak{l}$ of $\mathcal{O}_{\mathfrak{l}}$ is $\mathrm{rk}\, \mathfrak{l} + 2 = \mathrm{rk}\, \mathfrak{g} + 2$. Therefore, $\mathcal{O}_{\mathfrak{l}}$ is the subregular nilpotent orbit of $\mathfrak{l}$. As $\mathcal{O}_{\mathfrak{l}}$ is a rigid nilpotent orbit of $\mathfrak{l}$, Lemma 2.13 implies that the semisimple part of $\mathfrak{l}$ has dimension 3. Then $\mathcal{O}_{\mathfrak{l}}$ is the zero orbit of $\mathfrak{l}$ and X has dimension

$$\begin{aligned} \dim \mathfrak{g} - \dim \mathfrak{l} + 0 + \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}) &= (\dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g} - 2) + (\mathrm{rk}\, \mathfrak{g} - 1) \\ &= \dim \mathfrak{g} - 3, \end{aligned}$$

by Proposition 2.11, since the center of $\mathfrak{l}$ has dimension $\mathrm{rk}\, \mathfrak{g} - 1$. ■

Remark 2.15. We cannot expect analogous results for the smaller integers. Indeed, as noticed in Introduction, our computations will show that $\mathfrak{g}^{(m)}$ is not always equidimensional (see examples in $\mathfrak{sl}_5$, $\mathfrak{so}_{12}$, F_4, E_6, E_7 and E_8 in Tables 5, 6 – 12). This phenomenon arises from two problems. Firstly, different orbits can have the same dimension. Secondly, a nilpotent orbit can be induced from a rigid nilpotent orbit in different ways. For example the nilpotent orbit of dimension 44 in $\mathbf{F}_4$ is induced in two different ways (see Table 7). But, if $\dim(\mathrm{Ind}_{\mathfrak{l}_1}(\mathcal{O}_{\mathfrak{l}_1})) = \dim(\mathrm{Ind}_{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_2}))$, there is no reason for that $\dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_1) = \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_1)$, even if $\mathrm{Ind}_{\mathfrak{l}_1}(\mathcal{O}_{\mathfrak{l}_1}) = \mathrm{Ind}_{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_2})$. Surprisingly, in the classical case, we will notice that if $\mathrm{Ind}_{\mathfrak{l}_1}(\mathcal{O}_{\mathfrak{l}_1}) = \mathrm{Ind}_{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_2})$, then $\dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_1) = \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_1)$ (see Theorem 3.11). But for all that, as the first problem still occurs, the varieties $\mathfrak{g}^{(m)}$ are not equidimensional in general in the classical cases too, as the examples of $\mathfrak{sl}_6$ and $\mathfrak{so}_{12}$ show.

Remark 2.16. The previous remark underlines the fact that a nilpotent orbit may belong to different sheets. Again, this comes from the fact that a nilpotent orbit can be induced from a rigid nilpotent orbit in different way. We will detail specific examples of such situations in the sequel. In $\mathfrak{sl}_N$ the situation is slightly simpler since, in this case, there is a 1-1 correspondence between the the set of the sheets and the set of the nilpotent orbits (see Section 3, §3.1).

3. Computations in the classical cases

In this section, we investigate the varieties $\mathfrak{g}^{(m)}$ and the sheets in the case where $\mathfrak{g}$ is a classical simple Lie algebra. Because of the 1-1 correspondence established in Theorem 2.10, we first need of a precise description of rigid nilpotent orbits. Since Levi subalgebras of a simple exceptional Lie algebra may have simple factors of classical type, this piece of work will serve as well Section 4 devoted to the exceptional cases.

3.1. Type A.

In this paragraph, $\mathfrak{g}$ is the algebra $\mathfrak{sl}_N$, for $N \geq 2$. First of all, recall that the set of the nilpotent orbits of $\mathfrak{g}$ are in 1-1 correspondence with the set $\mathcal{P}(N)$ of all the partitions of N. Here, by *partition* we mean a sequence $\mathbf{d} = [d_1, \ldots, d_N]$ of nonnegative integers, possibly zero, with $d_1 \geq \cdots \geq d_N$ and $d_1 + \cdots + d_N = N$. If $\mathbf{d} = [d_1, \ldots, d_N] \in \mathcal{P}(N)$, we denote by $\mathcal{O}_{\mathbf{d}}$ the corresponding nilpotent orbit of $\mathfrak{g}$. Let $\mathbf{d} = [d_1, \ldots, d_N]$ be in $\mathcal{P}(N)$. Set $s_i = \#\{j \mid d_j \geq i\}$ the *dual partition* of $\mathbf{d}$. Notice that d_1 is the biggest integer j such that s_j is different from zero. Define now a subset $S^{\mathbf{d}}$ of Π as follows: the connected components $S_1^{\mathbf{d}}, \ldots, S_{d_1}^{\mathbf{d}}$ of $S^{\mathbf{d}}$ have the cardinalities $s_1 - 1, \ldots, s_{d_1} - 1$ respectively. Then we denote by $\mathfrak{l}_{\mathbf{d}}$ the Levi subalgebra $\mathfrak{l}_{S^{\mathbf{d}}}$ in the notations of the Introduction, §1.3. The following result, presented in [4](Theorem 7.2.3), is due to Kraft, Ozeki, Wakimoto:

Proposition 3.1. *The partition associated to* $\mathrm{Ind}_{\mathfrak{l}_{\mathbf{d}}}^{\mathfrak{g}}(\mathcal{O}_0)$ *is* $\mathbf{d}$. *In particular, every nonzero nilpotent orbit in* $\mathfrak{g}$ *is Richardson and the unique rigid nilpotent orbit is the zero orbit.*

Let $\mathbf{d}$ be a partition of N. The following formula is very standard, see [4](Corollary 6.1.4):

$$\dim \mathcal{O}_{\mathbf{d}} = 2m(\mathbf{d}) \ , \ \text{ where } \ m(\mathbf{d}) := (N^2 - \sum_{i=1}^{d_1} s_i^2)/2. \tag{1}$$

Lemma 3.2. *Let* $\mathbf{d}$ *be in* $\mathcal{P}(N)$ *and let* X *be a sheet with data* $(\mathfrak{l}_{\mathbf{d}}, \mathcal{O}_0)$. *Then* $\dim X = 2m(\mathbf{d}) + d_1 - 1$.

Proof. By Proposition 3.1 and Relation (1), $\dim(\mathrm{Ind}_{\mathfrak{l}_{\mathbf{d}}}^{\mathfrak{g}}(\mathcal{O}_0)) = 2m(\mathbf{d})$. In addition, Proposition 2.11 gives: $\dim X = 2m(\mathbf{d}) + d_1 - 1$, since the center of $\mathfrak{l}_{\mathbf{d}}$ has dimension: $\mathrm{rk}\,\mathfrak{g} - \#S^{\mathbf{d}} = N - 1 - \sum\limits_{i=1}^{d_1}(s_i - 1) = d_1 - 1$. ■

Notice that we can also deduce (1) from Propositions 3.1 and 2.6, (i), since $\mathfrak{l}_{\mathbf{d}}$ has dimension: $\sum\limits_{i=1}^{d_1}(s_i^2 - 1) + (d_1 - 1) = \sum\limits_{i=1}^{d_1} s_i^2 - 1$.

Theorem 3.3. *For* $m \in \mathbb{N}$, *we have:* $\dim \mathfrak{g}^{(m)} = \max\limits_{\substack{\mathbf{d} \in \mathcal{P}(N) \\ m(\mathbf{d}) = m}} (2m + (d_1 - 1))$, *where, by convention,* $\max\limits_{\emptyset}(2m + (d_1 - 1)) = 0$ *and* $\dim \emptyset = 0$.

Proof. Let m be in $\mathbb{N}$ and let X be a sheet of $\mathfrak{g}^{(m)}$. This forces $2m \in \mathbf{N}_{\mathfrak{g}}$, by Proposition 2.12. By Theorem 2.10 and Proposition 3.1, X has data $(\mathfrak{l}_{\mathbf{d}}, \mathcal{O}_0)$, for some $\mathbf{d} \in \mathcal{P}(N)$. The assertion is now an easy consequence of Lemma 3.2. ■

We present in Table 1 the dimension of the sheets and the varieties $\mathfrak{g}^{(m)}$ for $N = 6$. In this table we give for each $\mathbf{d} \in \mathcal{P}(6)$ the dimension $2m(\mathbf{d})$ of $\mathcal{O}_{\mathbf{d}}$ and the quantity $d_1 - 1$. Next, we give the dimension of the sheets of $\mathfrak{g}^{(m(\mathbf{d}))}$ and then, the dimension of $\mathfrak{g}^{(m(\mathbf{d}))}$, according to Theorem 3.3. In Table 1, $X_{\mathbf{d}}$ denotes the sheet whose data is $(\mathfrak{l}_{\mathbf{d}}, \mathcal{O}_0)$, for $\mathbf{d} \in \mathcal{P}(6)$. As two different orbits can have the same dimension, we need to compute $d_1 - 1$, for each $\mathbf{d} \in \mathcal{P}(N)$ such that $m(\mathbf{d}) = m$ to get the dimension of $\dim \mathfrak{g}^{(m)}$ (e.g. there are two nilpotent orbits of dimension 18).

$2m \in \mathbf{N}_{\mathfrak{g}}$	$\mathbf{d} \in \mathcal{P}(6)$, $m(\mathbf{d}) = m$	$d_1 - 1$, $m(\mathbf{d}) = m$	$\dim X_{\mathbf{d}}$, $m(\mathbf{d}) = m$	$\dim \mathfrak{g}^{(m)}$
30	$[6]$	5	35	35
28	$[5, 1]$	4	32	32
26	$[4, 2]$	3	29	29
24	$[4, 1^2]$ / $[3^2]$	3 / 2	27 / 26	27
22	$[3, 2, 1]$	2	24	24
18	$[3, 1^3]$ / $[2^3]$	2 / 1	20 / 19	20
16	$[2^2, 1^2]$	1	17	17
10	$[2, 1^4]$	1	11	11
0	$[1^6]$	0	0	0

Table 1: Dimensions of the sets $\mathfrak{g}^{(m)}$ for $\mathfrak{g} = \mathfrak{sl}_6$.

3.2. Types $\mathbf{B}, \mathbf{C}, \mathbf{D}$.

Set $\varepsilon = \pm 1$ and consider a nondegenerate bilinear form $\langle \cdot, \cdot \rangle_\varepsilon$ on $\mathbb{C}^N$ such that:

$$\langle a, b \rangle_\varepsilon = \varepsilon \langle a, b \rangle_\varepsilon, \text{ for all } b, a \in \mathbb{C}^N.$$

If $\varepsilon = -1$ (resp. 1), then the form is $\langle \cdot, \cdot \rangle_\varepsilon$ is symplectic (resp. symmetric). Define:

$$\begin{aligned} I(\langle \cdot, \cdot \rangle_\varepsilon) &:= \{g \in \mathrm{GL}_N \mid \langle ga, gb \rangle_\varepsilon = \langle a, b \rangle_\varepsilon, \text{ for all } a, b \in \mathbb{C}^N\}, \\ \mathfrak{g}_\varepsilon &:= \{x \in \mathfrak{sl}_N \mid \langle xa, b \rangle_\varepsilon = -\langle a, xb \rangle_\varepsilon, \text{ for all } a, b \in \mathbb{C}^N\}. \end{aligned}$$

If $\varepsilon = -1$, then $N = 2n$ and $\mathfrak{g}_{-1} \simeq \mathfrak{sp}_{2n}$. If $\varepsilon = 1$, then $\mathfrak{g}_{-1} \simeq \mathfrak{so}_N$. Thus $I(\langle \cdot, \cdot \rangle_\varepsilon)$ is the isotropy group of the form $\langle \cdot, \cdot \rangle_\varepsilon$ on $\mathbb{C}^N$, and $\mathfrak{g}_\varepsilon$ is its Lie algebra. Now, set:

$$\mathcal{P}_\varepsilon(N) := \{[d_1, \ldots, d_N] \in \mathcal{P}(N) \text{ , } \#\{j \mid d_j = i\} \text{ is even for } i \text{ s.t. } (-1)^i = \varepsilon\}.$$

The following theorem is standard and due to Gerstanhaber [9], Springer and Steinberg. We refer to [4](§5.1) for more details.

Theorem 3.4. *Nilpotent $I(\langle \cdot, \cdot \rangle_\varepsilon)$-orbits in $\mathfrak{g}_\varepsilon$ are in 1-1 correspondence with the partition $\mathcal{P}_\varepsilon(N)$ of N, except that if $\varepsilon = 1$, and $N = 2n$, then very even partitions of N (those with only even parts, each having even multiplicity) correspond to two orbits that we label with I and II.*

For the reader's convenience, we will detail in the sequel some results of Kempken [13] and Spaltenstein [20] presented in [4](§7.3) concerning induced nilpotent orbits in the types $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$. We endow the set $\mathcal{P}(N)$ with the classical order which corresponds to the classical order on the set of nilpotent orbits in $\mathfrak{sl}_N$. To begin with, recall a result due to Gerstenhaber which generalizes the transpose operation in the set $\mathcal{P}(N)$, see [4](Lemma 6.3.3):

Proposition 3.5. *Let $\mathbf{p} = [p_1, \ldots, p_{2n+1}]$ be a partition in $\mathcal{P}(2n+1)$. Then there is an unique largest partition $\mathbf{p_B}$ in $\mathcal{P}_1(2n+1)$ dominated by $\mathbf{p}$. The partition may be defined as follows. If $\mathbf{p}$ is not already in $\mathcal{P}_1(2n+1)$, then at least one of its even parts must occur with odd multiplicity; let q be the largest such part. Replace the last occurrence of q in $\mathbf{p}$ by $q-1$ and the first subsequent part r strictly less than $q-1$ by $r+1$; we may have to add a 0 to $\mathbf{p}$ to find such an r. Repeat this process until a partition in $\mathcal{P}_1(2n+1)$ is obtained. Similarly, there are unique largest partitions $\mathbf{q_C}$, $\mathbf{q_D}$ in $\mathcal{P}_{-1}(2n)$, $\mathcal{P}_1(2n)$ dominated by any given partition $\mathbf{q}$ of $2n+1$.*

The partition $\mathbf{p_B}$ is called the *B-collapse* of $\mathbf{p}$. Similarly, the partitions $\mathbf{q_C}$ and $\mathbf{q_D}$ are called the $\mathbf{C}$- and $\mathbf{D}$-*collapses* of $\mathbf{q}$. Their definitions are the obvious analogues of that of $\mathbf{p_B}$. Henceforth, we shall denote by $\mathbf{T}$ the type of $\mathfrak{g}_\varepsilon$, that is to say $\mathbf{B}$, $\mathbf{C}$ or $\mathbf{D}$.

Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$. Then, there are integers $i_1, \ldots, i_S \geq 0$ and R such that,

$$\mathfrak{l} \simeq \mathfrak{gl}_{i_S} \times \cdots \times \mathfrak{gl}_{i_1} \times \mathfrak{m},$$

where $\mathfrak{m}$ has the same type as $\mathfrak{g}_\varepsilon$ and whose standard representation has dimension R. After a possible renumbering, we can suppose that $[i_1, \ldots, i_S]$ belongs to $\mathcal{P}(S)$, with $2S + R = N$. Then we define:

$$\mathcal{P}_\varepsilon^{\mathrm{Levi}} = \{(\mathbf{i}; R) \in \mathcal{P}(S) \times \mathbb{N}_{\geq 0} \mid 2S + R = N, S \geq 0 \text{ and } R \neq 2, \text{ if } \varepsilon = 1\}.$$

Lemma 3.6. *There is a 1-1 correspondence between G-conjugacy classes of Levi subalgebras of $\mathfrak{g}$ and elements of $\mathcal{P}_\varepsilon^{\mathrm{Levi}}$.*

If $\mathfrak{l}$ corresponds to an element $(\mathbf{i}; R)$ of $\mathcal{P}_\varepsilon^{\mathrm{Levi}}$, we shall say that $\mathfrak{l}$ is *of type* $(\mathbf{i}; R)$.

Proposition 3.7. *Let $\mathfrak{l} = \mathfrak{gl}_l \times \mathfrak{m}$ be a maximal Levi subalgebra, where $\mathfrak{m}$ has the same type as $\mathfrak{g}_\varepsilon$ (then $2l + r = N$ if r is the dimension of the standard representation of $\mathfrak{m}$ and $\mathfrak{l}$ is of type $(l; r)$). Let $\mathcal{O}_\mathfrak{l} = \mathcal{O}_0 \times \mathcal{O}_\mathbf{f}$ be a nilpotent orbit in $\mathfrak{l}$ whose component in the $\mathfrak{gl}_l$ factor is the zero orbit and whose component $\mathcal{O}_\mathbf{f}$ in the $\mathfrak{m}$ factor has partition $\mathbf{f}$. Then the partition of $\mathrm{Ind}_\mathfrak{l}^{\mathfrak{g}_\varepsilon}(\mathcal{O}_\mathfrak{l})$ is $\mathbf{p}$, where the partition $\mathbf{p}$ is obtained from $\mathbf{f}$ as follows:*

We add 2 to the first l terms of $\mathbf{f}$, obtaining a partition $\tilde{\mathbf{f}}$ (extending by zero if necessary in $\mathbf{f}$), and then take the $\mathbf{T}$-collapse of $\tilde{\mathbf{f}}$. If the collapse is nontrivial (ie $\tilde{\mathbf{f}}_\mathbf{T} \neq \tilde{\mathbf{f}}$), it is obtained by subtracting 1 from the l^{th} part of $\tilde{\mathbf{f}}$ and adding 1 to its $(l+1)^{th}$ part. If $\mathfrak{g}_\varepsilon = \mathfrak{so}_{4n}$, $r \neq 0$ and the collapsed partition is very even, then $\mathbf{f}$ is also very even and the induced orbit inherits the label I or II of $\mathcal{O}_\mathbf{f}$; if $r = 0$, then the label of the induced orbit is the same as that of $\mathcal{O}_\mathfrak{l}$ if n is even but differs from it if n is odd.

Define $\mathcal{P}^*_\varepsilon(N)$ to be the set of all the partitions $[d_1, \dots, d_N]$ in $\mathcal{P}_\varepsilon(N)$ such that the following two conditions hold:

(i) $0 \leq d_{i+1} \leq d_i \leq d_{i+1} + 1$ for all i,

(ii) $\#\{j \mid d_j = i\} \neq 2$ if $\varepsilon(-1)^i = -1$.

We plan to show that $\mathcal{P}^*_\varepsilon(N)$ encodes rigid nilpotent orbits. Let $\mathbf{d} = [d_1, \dots, d_N]$ be in $\mathcal{P}_\varepsilon(N)$. We wish to construct an element of $\mathcal{P}^*_\varepsilon(N)$ from $\mathbf{d}$ "compatible with the induction operation". Set $i_0 := 0$, $\mathbf{d}^{(0)} := \mathbf{d}$ and $\mathfrak{g}'^{(0)} := \mathfrak{g}_\varepsilon$.

Step 1: if $\mathbf{d} \in \mathcal{P}^*_\varepsilon(N)$, set $\mathbf{d}^{(1)} := \mathbf{d} = \mathbf{d}^{(0)}$. Otherwise, there is $j \in \{1, \dots, N\}$ such that either $d_j \geq d_{j+1} + 2$, or $d_{j-1} > d_j = d_{j+1} > d_{j+2}$ with $\varepsilon(-1)^{d_j} = -1$ (where we set $d_0 = 0$ and $d_j = 0$, for all $j > N$, by convention). Denote by i_1 the smallest integer j such that one of those two situations happens.

(a) In the first situation, set:

$$\mathbf{d}^{(1)} := [d_1 - 2, \dots, d_{i_1} - 2, d_{i_1+1}, \dots, d_N].$$

(b) In the second situation, set:

$$\mathbf{d}^{(1)} := [d_1 - 2, \dots, d_{i_1-1} - 2, d_{i_1} - 1, d_{i_1+1} - 1, d_{i_1+2}, \dots, d_N].$$

In both situations, $\mathbf{d}^{(1)}$ is an element of $\mathcal{P}_\varepsilon(N - 2i_1)$ so that $\mathcal{O}_{\mathbf{d}^{(1)}}$ is a nilpotent orbit of $\mathfrak{m}^{(1)}$, where $\mathfrak{m}^{(1)}$ is a classical simple Lie algebra of the same type as $\mathfrak{g}_\varepsilon$ whose standard representation has dimension $N - 2i_1$. Moreover, by Proposition 3.7,

$$\mathcal{O}_{\mathbf{d}} = \mathrm{Ind}_{\mathfrak{l}^{(1)}}^{\mathfrak{g}_\varepsilon}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(1)}}),$$

where $\mathfrak{l}^{(1)}$ is a Levi subalgebra of $\mathfrak{g}_\varepsilon$ of type $(i_1; N - 2i_1)$.

Step 2: suppose that $i_0, i_1, \dots, i_{p-1}$, $\mathbf{d}^{(0)}, \mathbf{d}^{(1)}, \dots, \mathbf{d}^{(p-1)}$, $\mathfrak{m}^{(0)}, \mathfrak{m}^{(1)}, \dots, \mathfrak{m}^{(p-1)}$ and $\mathfrak{l}^{(1)}, \dots, \mathfrak{l}^{(p-1)}$ have been defined for some $p \in \{1, \dots, N\}$ so that:

($\mathbf{c_1}$) for all $k \in \{0, \dots, p-1\}$, $\mathcal{O}_{\mathbf{d}^{(k)}}$ is a nilpotent orbit of $\mathfrak{m}^{(k)}$, where $\mathfrak{m}^{(k)}$ is a classical simple Lie algebra of same type as $\mathfrak{g}_\varepsilon$ whose corresponding standard representation has dimension $N - 2i_0 - 2i_1 - \cdots - 2i_k$,

($\mathbf{c_2}$) for all $k \in \{1, \dots, p-1\}$, $\mathcal{O}_{\mathbf{d}^{(k-1)}} = \mathrm{Ind}_{\mathfrak{l}^{(k)}}^{\mathfrak{m}^{(k-1)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(k)}})$, where $\mathfrak{l}^{(k)}$ is a Levi subalgebra of $\mathfrak{m}^{(k-1)}$ of type $(i_k; N - 2i_0 - 2i_1 - \cdots - 2i_k)$.

Then, we define i_p, $\mathbf{d}^{(p)}$, $\mathfrak{l}^{(p)}$ and $\mathfrak{m}^{(p)}$ as in **Step 1**. More precisely, if $\mathbf{d}^{(p-1)} \in \mathcal{P}^*_\varepsilon(N - 2i_0 - 2i_1 \cdots - 2i_{p-1})$, we set $\mathbf{d}^{(p)} := \mathbf{d}^{(p-1)}$. Otherwise, there is $j \in \{1, \dots, N - 2i_0 - 2i_1 \cdots - 2i_{p-1}\}$ such that either $d_j^{(p-1)} \geq d_{j+1}^{(p-1)} + 2$, or $d_{j-1}^{(p-1)} > d_j^{(p-1)} = d_{j+1}^{(p-1)} > d_{j+2}^{(p-1)}$ with $d_j^{(p-1)}$ odd. Denote by i_p the smallest integer j such that one of those two situations happens.

(a) In the first situation, set:

$$\mathbf{d}^{(p)} := [d_1^{(p-1)} - 2, \dots, d_{i_p}^{(p-1)} - 2, d_{i_p+1}^{(p-1)}, \dots, d_{N-2i_0-2i_1\cdots-2i_{p-1}}^{(p-1)}].$$

(b) In the second situation, set:

$$\mathbf{d}^{(p)} := [d_1^{(p-1)} - 2, \dots, d_{i_p-1}^{(p-1)} - 2, d_{i_p}^{(p-1)} - 1, d_{i_p+1}^{(p-1)} - 1, d_{i_p+2}^{(p-1)}, \dots, d_{N-2i_0-2i_1\cdots-2i_{p-1}}^{(p-1)}].$$

As before, in both cases, $\mathbf{d}^{(p)}$ is an element of $\mathcal{P}_\varepsilon(N-2i_0-2i_1\cdots-2i_p)$ so that $\mathcal{O}_{\mathbf{d}^{(p)}}$ is a nilpotent orbit of $\mathfrak{m}^{(p)}$, where $\mathfrak{m}^{(p)}$ is a classical simple Lie algebra of the same type as $\mathfrak{g}_\varepsilon$ whose standard representation has dimension $N-2i_0-2i_1\cdots-2i_p$. And, by Proposition 3.7,

$$\mathcal{O}_{\mathbf{d}^{(p-1)}} = \mathrm{Ind}_{\mathfrak{l}^{(p)}}^{\mathfrak{m}^{(p-1)}}(\mathcal{O}_0\times\mathcal{O}_{\mathbf{d}^{(p)}}),$$

where $\mathfrak{l}^{(p)}$ is a Levi subalgebra of $\mathfrak{m}^{(p-1)}$ of type $(i_p; N-2i_0-2i_1\cdots-2i_p)$. Then $i_0,i_1,\ldots,i_p$, $\mathbf{d}^{(0)},\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(p)}$, $\mathfrak{m}^{(0)},\mathfrak{m}^{(1)},\ldots,\mathfrak{m}^{(p)}$ and $\mathfrak{l}^{(1)},\ldots,\mathfrak{l}^{(p)}$ satisfy Conditions $(\mathbf{c_1})$ and $(\mathbf{c_2})$. The process clearly ends after a finite number of steps, that is to say $\mathbf{d}^{(j)}=\mathbf{d}^{(j+1)}$, from some $j\in\mathbb{N}$.

Definition 3.8. We denote by $z(\mathbf{d})$ the smallest integer j such that $\mathbf{d}^{(j)}=\mathbf{d}^{(j+1)}$. If a partition $\mathbf{d}'$ is obtained from another partition $\mathbf{d}$ by a transformation of type **(a)** or **(b)** as described in **Step 1** or in **Step 2** (in particular $\mathbf{d}\neq\mathbf{d}'$), then we shall say that $\mathbf{d}'$ is *deduced from* $\mathbf{d}$ *by an elementary transformation.* In this case, it is clear that

$$z(\mathbf{d}') = z(\mathbf{d})-1. \tag{2}$$

What foregoes proves that $\mathcal{O}_{\mathbf{d}}$ is not rigid as soon as $z(\mathbf{d})\neq 0$. Using Proposition 3.7, we can easily prove that the converse holds, too. To summarize, we have obtained:

Corollary 3.9. *The nilpotent orbit corresponding to $\mathbf{d}\in\mathcal{P}_\varepsilon(N)$ is rigid if and only if $z(\mathbf{d})=0$.*

As $z(\mathbf{d})=0$ if and only if $\mathbf{d}\in\mathcal{P}^*_\varepsilon(N)$, we deduce from Corollary 3.9 the following result [4](Corollary 7.3.5):

Corollary 3.10. *The nilpotent orbit corresponding to $\mathbf{d}\in\mathcal{P}_\varepsilon(N)$ is rigid if and only if $\mathbf{d}\in\mathcal{P}_\varepsilon(N)^*$.*

Let $\mathbf{d}=[d_1,\ldots,d_N]$ be in $\mathcal{P}_\varepsilon(N)$. Put $r_i=\#\{j\mid d_j=i\}$ and $s_i=\#\{j\mid d_j\geq i\}$. Denote by $m(\mathbf{d})$ the half dimension of $\mathcal{O}_{\mathbf{d}}$. By standard results [4](Corollary 6.1.4), $m(\mathbf{d})$ is given by the following formulas:

$$m(\mathbf{d}) = \begin{cases} (2n^2+n-\frac{1}{2}\sum\limits_i s_i^2+\frac{1}{2}\sum\limits_{i\text{ odd}} r_i)/2, & \text{if } \mathfrak{g}=\mathfrak{so}_{2n+1}\\ (2n^2+n-\frac{1}{2}\sum\limits_i s_i^2-\frac{1}{2}\sum\limits_{i\text{ odd}} r_i)/2, & \text{if } \mathfrak{g}=\mathfrak{sp}_{2n}\\ (2n^2-n-\frac{1}{2}\sum\limits_i s_i^2+\frac{1}{2}\sum\limits_{i\text{ odd}} r_i)/2, & \text{if } \mathfrak{g}=\mathfrak{so}_{2n}. \end{cases} \tag{3}$$

The construction preceding Definition 3.8 gives a method to compute the number $z(\mathbf{d})$, for $\mathbf{d}\in\mathcal{P}_\varepsilon(N)$. Then, according to Corollary 3.9, we list in Table 2 the rigid nilpotent orbits, together with their dimensions computed following (3), for the types $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{C}_3$, $\mathbf{D}_4$, $\mathbf{D}_5$, $\mathbf{D}_6$ and $\mathbf{D}_7$. We list these cases since they all appear as simple factors of Levi subalgebras in the exceptional Lie algebras.

Proposition 3.11. *Let $\mathbf{d}=[d_1,\ldots,d_N]$ be in $\mathcal{P}_\varepsilon(N)$. Suppose that $\mathcal{O}_{\mathbf{d}}=\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}_\varepsilon}(\mathcal{O}_{\mathfrak{l}})$, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}_\varepsilon$ and $\mathcal{O}_{\mathfrak{l}}$ a rigid nilpotent orbit in $\mathfrak{l}$. Then* $\dim\mathfrak{z}_{\mathfrak{g}_\varepsilon}(\mathfrak{l})=z(\mathbf{d})$.

$\mathbf{B}_2$		$\mathbf{C}_3$		$\mathbf{B}_3$		$\mathbf{D}_4$		$\mathbf{D}_5$		$\mathbf{D}_6$		$\mathbf{D}_7$	
$[1^5]$	0	$[1^6]$	0	$[1^7]$	0	$[1^8]$	0	$[1^{10}]$	0	$[1^{12}]$	0	$[1^{14}]$	0
$[2^2,1]$	4	$[2,1^4]$	6	$[2^2,1^3]$	8	$[2^2,1^4]$	10	$[2^2,1^6]$	14	$[2^2,1^8]$	18	$[2^2,1^{10}]$	22
						$[3,2^2,1]$	16	$[3,2^2,1^3]$	24	$[2^4,1^4]$	28	$[2^4,1^6]$	36
										$[3,2^2,1^5]$	32	$[3,2^2,1^7]$	40
										$[3,2^4,1]$	36	$[3,2^4,1^3]$	48
												$[3^3,2^2,1]$	58

Table 2: Rigid nilpotent orbits with their dimensions in $\mathfrak{so}_5$, $\mathfrak{sp}_6$, $\mathfrak{so}_7$, $\mathfrak{so}_8$, $\mathfrak{so}_{10}$, $\mathfrak{so}_{12}$ and $\mathfrak{so}_{14}$

Remark 3.12. As noted on several occasions (see Remarks 2.15 and 2.16), a given nilpotent orbit can belong to different sheets. Nevertheless, Proposition 3.11 assures that, in the classical case, the dimension of all sheets containing a given nilpotent orbit have the same dimension. For example in $\mathfrak{so}_{12}$, we can readily check that the nilpotent orbit $\mathcal{O}_{[3^2,1^6]}$ is induced from two different ways: from the zero orbit in a Levi subalgebra of type $(2;8)$ and from the zero orbit in a Levi subalgebra of type $(1;10)$. The two corresponding sheets share the same dimension 35 (cf Table 5). For exceptional Lie algebras the statement is no longer true (see again the unique nilpotent orbit of dimension 66 in E_6 in Table 8 for a counterexample).

Proof. 1) If $\mathcal{O}_{\mathbf{d}}$ is rigid, then $\mathfrak{l} = \mathfrak{g}_\varepsilon$, and $\dim \mathfrak{z}_{\mathfrak{g}_\varepsilon}(\mathfrak{l}) = 0$. On the other hand, $z(\mathbf{d}) = 0$, according to Corollary 3.9.

2) We suppose that $\mathbf{d} \notin \mathcal{P}^*_\varepsilon(N)$, or equivalently that $\mathcal{O}_{\mathbf{d}}$ is not rigid. Then $\mathfrak{l}$ is strictly contained in $\mathfrak{g}_\varepsilon$. By Lemma 3.6, $\mathfrak{l}$ is G-conjugated to

$$\mathfrak{gl}_{i_S} \times \cdots \times \mathfrak{gl}_{i_1} \times \mathfrak{m},$$

for $2i_1 + \cdots 2i_S + R = N$, $i_1 \geq \cdots \geq i_S$, and $\mathfrak{m}$ a Lie algebra of same type as $\mathfrak{g}_\varepsilon$ whose standard representation has dimension R. In other words, $\mathfrak{l}$ has type $(\mathbf{i}; R)$, with $\mathbf{i} = [i_1, \ldots, i_S] \in \mathcal{P}(S)$. Notice that the center of $\mathfrak{l}$ has dimension S. By Theorem 3.1,

$$\mathcal{O}_{\mathfrak{l}} = \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{\mathbf{f}},$$

where $\mathbf{f}$ is an element of $\mathcal{P}^*_\varepsilon(R)$ and where the component of $\mathcal{O}_{\mathfrak{l}}$ on the $\mathfrak{gl}_{i_k}$ factor is zero, for $k = 1, \ldots, S$. Set $i_0 := 0$ and $\mathfrak{m}^{(0)} = \tilde{\mathfrak{l}}^{(0)} := \mathfrak{m}$. For $p = 1, \ldots, S$, denote by $\mathfrak{m}^{(p)}$ a Lie algebra of the same type as $\mathfrak{g}_\varepsilon$ whose standard representation has dimension

$$R_p := 2i_p + \cdots + 2i_1 + 2i_0 + R,$$

and by $\tilde{\mathfrak{l}}^{(p)}$ a Levi subalgebra of $\mathfrak{m}^{(p)}$ of type $(i_p; 2i_{p-1} + \cdots + 2i_1 + 2i_0 + R)$. Set $\mathfrak{l}^{(S)} := \mathfrak{g}_\varepsilon$ and, for $p = 0, \ldots, S-1$, denote by $\mathfrak{l}^{(p)}$ a Levi subalgebra of $\mathfrak{g}_\varepsilon$ of type $([i_{p+1}, \ldots, i_S]; 2i_p + \cdots + 2i_1 + 2i_0 + R)$ such that:

$$\mathfrak{l} = \mathfrak{l}^{(0)} \subseteq \cdots \subseteq \mathfrak{l}^{(S)} = \mathfrak{g}_\varepsilon.$$

As defined, $\mathfrak{l}^{(p)}$ is G-conjugated to

$$\mathfrak{gl}_{i_S} \times \cdots \times \mathfrak{gl}_{i_{p+1}} \times \mathfrak{m}^{(p)},$$

for any $p = 0, \ldots, S-1$. Set $\mathbf{d}^{(0)} := \mathbf{f}$ and define $\mathbf{d}^{(p)}$, for $p = 1, \ldots, S$, by induction as follows; $\mathbf{d}^{(p)}$ is the element of $\mathcal{P}_\varepsilon(R_p)$ such that:

$$\mathcal{O}_{\mathbf{d}^{(p)}} = \mathrm{Ind}_{\tilde{\mathfrak{l}}^{(p)}}^{\mathfrak{m}^{(p)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(p-1)}}),$$

where $\mathcal{O}_0$ is the zero orbit in the $\mathfrak{gl}_{i_p}$ factor. Check first that $\mathbf{d}^{(S)} = \mathbf{d}$. By Proposition 2.6, (ii), we can write:

$$\mathcal{O}_{\mathbf{d}} = \mathrm{Ind}_{\mathfrak{l}^{(S-1)}}^{\mathfrak{l}^{(S)}}(\ldots(\mathrm{Ind}_{\mathfrak{l}^{(0)}}^{\mathfrak{l}^{(1)}}(\underbrace{\mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{\mathbf{f}}}_{S \text{ factors}})))).$$

In addition, we easily see that

$$\begin{aligned}
\mathrm{Ind}_{\mathfrak{l}^{(p-1)}}^{\mathfrak{l}^{(p)}}(\underbrace{\mathcal{O}_0 \times \cdots \times \mathcal{O}_0}_{S-p+1 \text{ factors}} \times \mathcal{O}_{\mathbf{d}^{(p-1)}}) &\simeq \underbrace{\mathcal{O}_0 \times \cdots \times \mathcal{O}_0}_{S-p \text{ factors}} \times \mathrm{Ind}_{\tilde{\mathfrak{l}}^{(p)}}^{\mathfrak{m}^{(p)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(p-1)}}) \\
&= \underbrace{\mathcal{O}_0 \times \cdots \times \mathcal{O}_0}_{S-p \text{ factors}} \times \mathcal{O}_{\mathbf{d}^{(p)}},
\end{aligned}$$

for any $p = 1, \ldots, S$, since the $S-p$ first factors of $\mathfrak{l}^{(p-1)}$ and $\mathfrak{l}^{(p)}$ are the same. Then, by induction, we obtain:

$$\begin{aligned}
\mathcal{O}_{\mathbf{d}} &= \mathrm{Ind}_{\mathfrak{l}^{(S-1)}}^{\mathfrak{l}^{(S)}}(\ldots(\mathrm{Ind}_{\mathfrak{l}^{(0)}}^{\mathfrak{l}^{(1)}}(\underbrace{\mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{\mathbf{f}}}_{S \text{ factors}})) \\
&= \mathrm{Ind}_{\mathfrak{l}^{(S-1)}}^{\mathfrak{l}^{(S)}}(\ldots \mathrm{Ind}_{\mathfrak{l}^{(1)}}^{\mathfrak{l}^{(2)}}(\underbrace{\mathcal{O}_0 \times \cdots \times \mathcal{O}_0}_{S-1 \text{ factors}} \times \mathcal{O}_{\mathbf{d}^{(1)}}) \\
&\;\;\vdots \\
&= \mathrm{Ind}_{\mathfrak{l}^{(S-1)}}^{\mathfrak{l}^{(S)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(S-1)}}) \\
&= \mathrm{Ind}_{\tilde{\mathfrak{l}}^{(S)}}^{\mathfrak{m}^{(S)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(S-1)}}),
\end{aligned}$$

since $\mathfrak{l}^{(S)} \simeq \mathfrak{m}^{(S)} \simeq \mathfrak{g}_\varepsilon$ and $\tilde{\mathfrak{l}}^{(S)} \simeq \mathfrak{l}^{(S-1)}$. Now the definition of $\mathbf{d}^{(S)}$ forces $\mathbf{d} = \mathbf{d}^{(S)}$.

It remains to compute the number $z(\mathbf{d}^{(S)})$. By induction on $p \in \{0, \ldots, S\}$, let us prove that $z(\mathbf{d}^{(p)}) = p$ and that, for all $i < i_{p+1}$:

$$\begin{cases} \text{(i)}' \quad 0 \leq d_{i+1}^{(p)} \leq d_i^{(p)} \leq d_{i+1}^{(p)} + 1, \\ \\ \text{(ii)}' \quad \#\{j \mid d_j^{(p)} = i\} \neq 2, \text{ if } \varepsilon(-1)^i = -1. \end{cases}$$

Then we will deduce the expected result from the $p = S$ case, since the center of $\mathfrak{l}$ has dimension S.

$(p = 0)$: since $\mathcal{O}_{\mathbf{f}}$ is rigid in $\mathfrak{m}$, it follows from Corollary 3.9 that $z(\mathbf{f}) = z(\mathbf{d}^{(0)}) = 0$ and that the conditions (i)$'$ and (ii)$'$ hold.

$((p-1) \Rightarrow p)$: suppose that, for all $k \in \{0, \ldots, p-1\}$, $z(\mathbf{d}^{(k)}) = k$ and that conditions (i)$'$ and (ii)$'$ holds for $i < i_{k+1}$, for some $p \in \{1, \ldots, S\}$. We have to first prove that $z(\mathbf{d}^{(p)}) = p$. By Relation (2), it suffices to prove that $\mathbf{d}^{(p-1)}$ is deduced from $\mathbf{d}^{(p)}$ by an elementary transformation, since $z(\mathbf{d}^{(p-1)}) = p-1$ by the induction hypothesis. As $\tilde{\mathfrak{l}}^{(p)}$ is a maximal Levi subalgebra of $\mathfrak{m}^{(p)}$, we can apply Proposition 3.7 to

$$\mathcal{O}_{\mathbf{d}^{(p)}} = \mathrm{Ind}_{\tilde{\mathfrak{l}}^{(p)}}^{\mathfrak{m}^{(p)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(p-1)}}).$$

With the notations of Proposition 3.7, we have:

$$\mathbf{d}^{(p)} = (\widetilde{\mathbf{d}^{(p-1)}})_{\mathbf{T}}.$$

By the induction hypothesis, for all $i < i_p$, conditions (i)$'$ and (ii)$'$ hold for $\mathbf{d}^{(p-1)}$. Consequently the smallest integer l such that one of the situations **(a)** or **(b)** of **Step 1** happens in $\mathbf{d}^{(p)}$ is equal to i_p, because $\tilde{\mathfrak{l}}^{(p)}$ is of type $(i_p; R_p - 2i_p)$. We distinguish two cases: either $\widetilde{\mathbf{d}^{(p-1)}}_{\mathbf{T}}$ equals to $\widetilde{\mathbf{d}^{(p-1)}}$ or not. We easily check that in both situations, $\mathbf{d}^{(p-1)}$ is deduced from $(\widetilde{\mathbf{d}^{(p-1)}})_{\mathbf{T}}$ by an elementary transformation. Moreover, for all $i < i_{p+1}$, conditions (i)$'$ and (ii)$'$ hold for $\mathbf{d}^{(p)}$ because $i_{p+1} \leq i_p$. By induction, for all $p = 1, \ldots, S$, $z(\mathbf{d}^{(p)}) = p$ and conditions (i)$'$ and (ii)$'$ hold, for all $i < i_{p+1}$. In particular, with $p = S$, we have: $z(\mathbf{d}^{(S)}) = z(\mathbf{d}) = S = \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l})$. ■

We are now in a position to compute the dimension of the varieties $\mathfrak{g}_\varepsilon^{(m)}$. Recall that $m(\mathbf{d})$ is given by the formulas (3). We adopt the same conventions as in Theorem 3.3.

Theorem 3.13. *Let m be in $\mathbb{N}$. Then,* $\dim \mathfrak{g}_\varepsilon^{(m)} = \max\limits_{\substack{\mathbf{d}\in\mathcal{P}_\varepsilon(N) \\ m(\mathbf{d})=m}} (2m + z(\mathbf{d}))$.

Proof. Let m be in $\mathbf{N}_{\mathfrak{g}_\varepsilon}$ and let X be a sheet of $\mathfrak{g}_\varepsilon^{(m)}$ with data $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$. Let $\mathbf{d}$ be the partition associated to $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}_\varepsilon}(\mathcal{O}_{\mathfrak{l}})$. By Proposition 2.7, $m = m(\mathbf{d})$. In addition, by Proposition 3.11, $\dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}) = z(\mathbf{d})$. Then the statement results from Proposition 2.11. ■

In Tables 3, 4 and 5, we give for each $\mathbf{d} \in \mathcal{P}_\varepsilon(N)$, the integers $2m(\mathbf{d})$ and $z(\mathbf{d})$ and then the dimensions of the sets $\mathfrak{g}_\varepsilon^{(m(\mathbf{d}))}$, in the cases where $N = 3$ with $\varepsilon = \pm 1$ and $N = 6$ with $\varepsilon = 1$.

$2m \in \mathbf{N}_{\mathfrak{g}}$	$\mathbf{d} \in \mathcal{P}_\varepsilon(6)$, $m(\mathbf{d}) = m$	$z(\mathbf{d})$, $\mathbf{d} \in \mathcal{P}_\varepsilon(6)$	$\dim \mathfrak{g}_\varepsilon^{(m)}$
18	$[7]$	3	21
16	$[5, 1^2]$	2	18
14	$[3^2, 1]$	1	15
12	$[3, 2^2]$	1	13
10	$[3, 1^4]$	1	11
8	$[2^2, 1^3]$	0	8
0	$[1^7]$	0	0

Table 3: Dimensions of the sets $\mathfrak{g}_\varepsilon^{(m)}$ for $\mathfrak{g} = \mathfrak{so}_7$.

4. Computations in the exceptional cases

We suppose in this section that $\mathfrak{g}$ is a simple exceptional Lie algebra. We get now ready to compute the dimensions of all the sheets of $\mathfrak{g}$. Let us use the notations of Introduction, §1.3. For each pair (S, p), with $S \subset \Pi$ and $p \in \mathbf{N}_S$, set

$$\mathrm{d}_{S,p} := \dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g} - 2\#\Delta_+^S + p. \quad (4)$$

$2m \in \mathbf{N}_{\mathfrak{g}}$	$\mathbf{d} \in \mathcal{P}_{\varepsilon}(6)$, $m(\mathbf{d}) = m$	$z(\mathbf{d})$, $\mathbf{d} \in \mathcal{P}_{\varepsilon}(6)$	$\dim \mathfrak{g}_{\varepsilon}^{(m)}$
18	$[6]$	3	21
16	$[4, 2]$	2	18
14	$[3^2]$ / $[4, 1^2]$	1 / 1	15
12	$[2^3]$	1	13
10	$[2^2, 1^2]$	1	11
6	$[2, 1^4]$	0	6
0	$[1^6]$	0	0

Table 4: Dimensions of the sets $\mathfrak{g}_{\varepsilon}^{(m)}$ for $\mathfrak{g} = \mathfrak{sp}_6$.

By Proposition 2.6, (i), $\mathrm{d}_{S,p}$ is the dimension of any nilpotent orbit induced by a nilpotent orbit in $\mathfrak{l}_S$ of dimension p. Then, by Lemma 2.2, a G-Jordan class whose data $(\mathfrak{l}_S, \mathcal{O}_S)$ satisfies $\dim \mathcal{O}_S = p$, has dimension

$$\mathrm{d}_{S,p} + (\mathrm{rk}\,\mathfrak{g} - \#S), \tag{5}$$

since $\dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_S) = \mathrm{rk}\,\mathfrak{g} - \#S$. Moreover, by Proposition 2.10, the expression (5) corresponds to the dimension of a sheet of $\mathfrak{g}^{(m)}$, where $2m = \mathrm{d}_{S,p}$, if and only if $p \in \mathbf{N}_S^{\mathrm{rig}}$. When all the simple factors of $\mathfrak{l}_S$ are of classical type, the set $\mathbf{N}_S^{\mathrm{rig}}$ is given by Table 2. For the exceptional types, the rigid nilpotent orbits are listed in [20](Appendix of Chap. II), thanks to Elashvili's computations. Using all this, we present in Tables 6 – 12, the necessary data for each exceptional type to compute the dimension of the sheets. More precisely, for each subset $S \in \Pi$, we give per column:

I : the type of Δ^S, that is the type of the Levi subalgebra $\mathfrak{l}_S$,

II : the cardinality of S,

III : the cardinality of Δ_+^S,

IV : the set $\mathbf{N}_S^{\mathrm{rig}}$,

V : the set of the numbers $\mathrm{d}_{S,p}$, computed following (4), where p runs through the column **IV**,

VI : the set of the numbers $\mathrm{d}_{S,p} + (\mathrm{rk}\,\mathfrak{g} - \#S)$, where p runs through the column **IV**,

Like that, column **V** gives the set $\mathbf{N}_{\mathfrak{g}}$ of all the dimensions of nilpotent orbits while column **VI** gives the set of all the dimensions of the sheets of $\mathfrak{g}$, according to (5). Next, we list the dimension of the sets $\mathfrak{g}^{(m)}$ in Tables 13 – 17. This piece of work completes the exceptional case. We conclude by several remarks about the tables.

Remark 4.1. We might precise that our tables do not follow the usual notations of Bala and Carter. Recall that there is a natural 1-1 correspondence between nilpotent orbits of $\mathfrak{g}$ and G-conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ where $\mathfrak{l}$ is a

$2m \in \mathbf{N}_{\mathfrak{g}}$	$\mathbf{d} \in \mathcal{P}_{\varepsilon}(6)$, $m(\mathbf{d}) = m$	$z(\mathbf{d})$, $\mathbf{d} \in \mathcal{P}_{\varepsilon}(6)$	$\dim \mathfrak{g}_{\varepsilon}^{(m)}$
60	$[11, 1]$	6	66
58	$[9, 3]$	5	63
56	$[9, 1^3]$ / $[7, 5]$	4 / 4	60
54	$[7, 3, 1^2]$ / $[6^2]$	4 / 3	58
52	$[5^2, 1^2]$ / $[7, 2^2, 1]$	3 / 2	55
50	$[5, 3, 2^2]$	2	52
48	$[4^3]$ /$[4^2, 3, 1]$	2 / 2	50
46	$[4^2, 2^2]$ / $[5, 3, 1^4]$	2 / 2	48
44	$[4^2, 1^4]$ / $[5, 2^2, 1^3]$ / $[3^4]$	1 / 1 / 1	45
42	$[3^3, 1^3]$	1	43
40	$[3^2, 2^2, 1^2]$	1	41
36	$[5, 1^7]$ / $[3, 2^4, 1]$	2 / 0	38
34	$[3^2, 1^6]$	1	35
32	$[3, 2^2, 1^5]$	0	32
30	$[2^6]$	1	31
28	$[2^4, 1^4]$	0	28
18	$[2^2, 1^8]$	0	18
0	$[1^{12}]$	0	0

Table 5: Dimensions of the sets $\mathfrak{g}_{\varepsilon}^{(m)}$ for $\mathfrak{g} = \mathfrak{so}_{12}$.

Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_{\mathfrak{l}}$ a *distinguished* parabolic subalgebra of the semisimple part of $\mathfrak{l}$ [4](Theorem 8.2.12). The Bala-Carter notation for nilpotent orbits refer to the type of the corresponding Levi subalgebra via this correspondence. In this note, we have considered an other approach. In [20](Appendix of Chap. II), both approaches are presented in the tables. It is worth noting that isomorphic Levi subalgebras are not necessary G-conjugate. For instance there are two non-conjugate Levi subalgebras of type $\mathbf{A}_1$ (and of type $\mathbf{A}_2$) in $\mathbf{F}_4$. The one corresponding to a short root is usually denoted by $\widetilde{\mathbf{A}}_1$ in the Bala-Carter notations. Remark that $\mathrm{Ind}_{\mathbf{A}_1}^{\mathbf{F}_4}(\mathcal{O}_0) = \mathrm{Ind}_{\widetilde{\mathbf{A}}_1}^{\mathbf{F}_4}(\mathcal{O}_0)$ and $\mathrm{Ind}_{\mathbf{A}_2}^{\mathbf{F}_4}(\mathcal{O}_0) \neq \mathrm{Ind}_{\widetilde{\mathbf{A}}_2}^{\mathbf{F}_4}(\mathcal{O}_0)$, since there are two nilpotent orbits of dimension 42 and only one of dimension 46 (see Table 7). For completeness, we take this phenomenal in our tables into account. We refer to [20] or [4](§8.4) for more details about this. In fact, as we only need the dimension of Levi subalgebras of $\mathfrak{g}$ and of their center, we could disregard this fact.

I (Δ^S)	**II** ($\#S$)	**III** ($\#\Delta^S_+$)	**IV** ($\mathbf{N}_S^{\mathrm{rig}}$)	**V** : $\mathbf{N}_{\mathfrak{g}}$ ($\mathrm{d}_{S,p}$, $p \in \mathbf{N}_S^{\mathrm{rig}}$)	**VI** : Dimensions of the sheets ($\mathrm{d}_{S,p} + (\mathrm{rk}\,\mathfrak{g} - \#S)$, $p \in \mathbf{N}_S^{\mathrm{rig}}$)
$\varnothing$	0	0	0	12	14
$\mathbf{A}_1$, $\widetilde{\mathbf{A}}_1$	1	1	0 , 0	10 , 10	11 , 11
$\mathbf{G}_2$	2	6	8 / 6 / 0	8 / 6 / 0	8 / 6 / 0

Table 6: Dimension of the sheets for $\mathfrak{g}$ of type $\mathbf{G}_2$

I	**II**	**III**	**IV**	**V**	**VI**
$\varnothing$	0	0	0	48	52
$\mathbf{A}_1$, $\widetilde{\mathbf{A}}_1$	1	1	0 , 0	46 , 46	49 , 49
$\mathbf{A}_1 + \widetilde{\mathbf{A}}_1$	2	2	0	44	46
$\mathbf{A}_2$, $\widetilde{\mathbf{A}}_2$	2	3	0 , 0	42 , 42	44 , 44
$\mathbf{A}_1 + \widetilde{\mathbf{A}}_2$, $\widetilde{\mathbf{A}}_1 + \mathbf{A}_2$	3	4	0 , 0	40 , 40	41 , 41
$\mathbf{B}_2$	2	4	4 / 0	44 / 40	46 / 42
$\mathbf{B}_3$	3	9	8 / 0	38 / 30	39 / 31
$\mathbf{C}_3$	3	9	6 / 0	36 / 30	37 /31
$\mathbf{F}_4$	4	24	36 / 34 / 28 22 / 16 / 0	36 / 34 / 28 22 / 16 / 0	36 / 34 / 28 22 / 16 / 0

Table 7: Dimension of the sheets for $\mathfrak{g}$ of type $\mathbf{F}_4$

I	II	III	IV	V	VI
$\varnothing$	0	0	0	72	78
$\mathbf{A}_1$	1	1	0	70	75
$2\mathbf{A}_1$	2	2	0	68	72
$3\mathbf{A}_1$	3	3	0	66	69
$\mathbf{A}_2$	2	3	0	66	70
$\mathbf{A}_2 + \mathbf{A}_1$	3	4	0	64	67
$\mathbf{A}_2 + 2\mathbf{A}_1$	4	5	0	62	64
$2\mathbf{A}_2$	4	6	0	60	62
$2\mathbf{A}_2 + \mathbf{A}_1$	5	7	0	58	59
$\mathbf{A}_3$	3	6	0	60	63
$\mathbf{A}_3 + \mathbf{A}_1$	4	7	0	58	60
$\mathbf{D}_4$	4	12	16 / 10 / 0	64 / 58 / 48	66 / 60 / 50
$\mathbf{A}_4$	4	10	0	52	54
$\mathbf{A}_4 + \mathbf{A}_1$	5	11	0	50	51
$\mathbf{A}_5$	5	15	0	42	43
$\mathbf{D}_5$	5	20	24 / 14 / 0	56 / 46 / 32	57 / 47 / 33
$\mathbf{E}_6$	6	56	54 / 40 / 22 / 0	54 / 40 / 22 / 0	54 / 40 / 22 / 0

Table 8: Dimension of the sheets for $\mathfrak{g}$ of type $\mathbf{E}_6$

I	II	III	IV	V	VI
$\varnothing$	0	0	0	126	133
$\mathbf{A}_1$	1	1	0	124	130
$2\mathbf{A}_1$	2	2	0	122	127
$(3\mathbf{A}_1)'$, $(3\mathbf{A}_1)''$	3	3	0 , 0	120 , 120	124 , 124
$4\mathbf{A}_1$	4	4	0	118	121
$\mathbf{A}_2$	2	3	0	120	125
$\mathbf{A}_2 + \mathbf{A}_1$	3	4	0	118	122
$\mathbf{A}_2 + 2\mathbf{A}_1$	4	5	0	116	119
$\mathbf{A}_2 + 3\mathbf{A}_1$	5	6	0	114	116
$2\mathbf{A}_2$	4	6	0	114	117
$2\mathbf{A}_2 + \mathbf{A}_1$	5	7	0	112	114
$\mathbf{A}_3$	3	6	0	114	118
$(\mathbf{A}_3 + \mathbf{A}_1)'$, $(\mathbf{A}_3 + \mathbf{A}_1)''$	4	7	0 , 0	112 , 112	115 , 115
$\mathbf{A}_3 + 2\mathbf{A}_1$	5	8	0	110	112
$\mathbf{A}_3 + \mathbf{A}_2$	5	9	0	108	110

Table 9: Dimension of the sheets for $\mathfrak{g}$ of type $\mathbf{E}_7$, **1/2**

I	II	III	IV	V	VI
$\mathbf{A}_3+\mathbf{A}_2+\mathbf{A}_1$	6	10	0	106	107
$\mathbf{A}_4$	4	10	0	106	109
$\mathbf{A}_4+\mathbf{A}_1$	5	11	0	104	106
$\mathbf{A}_4+\mathbf{A}_2$	6	13	0	100	101
$\mathbf{D}_4$	4	12	16 / 10 / 0	118 / 112 / 102	121 / 115 / 105
$\mathbf{D}_4+\mathbf{A}_1$	5	13	16 / 10 / 0	116 / 110 / 100	118 / 112 / 102
$(\mathbf{A}_5)'$, $(\mathbf{A}_5)''$	5	15	0 , 0	96 , 96	98 , 98
$\mathbf{A}_5+\mathbf{A}_1$	6	16	0	94	95
$\mathbf{D}_5$	5	20	24 / 14 / 0	110 / 100 / 86	112 / 102 / 88
$\mathbf{D}_5+\mathbf{A}_1$	6	21	24 / 14 / 0	108 / 98 / 84	109 / 99 / 85
$\mathbf{A}_6$	6	21	0	84	85
$\mathbf{D}_6$	6	30	36 / 32 / 28 / 18 / 0	102 / 98 / 94 / 84 / 66	103 / 99 / 95 / 85 / 67
$\mathbf{E}_6$	6	36	54 / 40 / 22 / 0	108 / 94 / 76 / 54	109 / 95 / 77 / 55
$\mathbf{E}_7$	7	63	92 / 90 / 82 / 70 64 / 52 / 34 / 0	92 / 90 / 82 / 70 64 / 52 / 34 / 0	92 / 90 / 82 / 70 64 / 52 / 34 / 0

Table 10: Dimension of the sheets for $\mathfrak{g}$ of type $\mathbf{E}_7$, **2/2**

-

I	II	III	IV	V	VI
$\emptyset$	0	0	0	240	248
$\mathbf{A}_1$	1	1	0	238	245
$2\mathbf{A}_1$	2	2	0	236	242
$3\mathbf{A}_1$	3	3	0	234	239
$4\mathbf{A}_1$	4	4	0	232	236
$\mathbf{A}_2$	2	3	0	234	240
$\mathbf{A}_2 + \mathbf{A}_1$	3	4	0	232	237
$\mathbf{A}_2 + 2\mathbf{A}_1$	4	5	0	230	234
$\mathbf{A}_2 + 3\mathbf{A}_1$	5	6	0	228	231
$2\mathbf{A}_2$	4	6	0	228	232
$2\mathbf{A}_2 + \mathbf{A}_1$	5	7	0	226	229
$2\mathbf{A}_2 + 2\mathbf{A}_1$	6	8	0	224	226
$\mathbf{A}_3$	3	6	0	228	233
$\mathbf{A}_3 + \mathbf{A}_1$	4	7	0	226	230
$\mathbf{A}_3 + 2\mathbf{A}_1$	5	8	0	224	227
$\mathbf{A}_3 + \mathbf{A}_2$	5	9	0	222	225
$\mathbf{A}_3 + \mathbf{A}_2 + \mathbf{A}_1$	6	10	0	220	222
$2\mathbf{A}_3$	6	12	0	216	218
$\mathbf{A}_4$	4	10	0	220	224
$\mathbf{A}_4 + \mathbf{A}_1$	5	11	0	218	221
$\mathbf{A}_4 + 2\mathbf{A}_1$	6	12	0	216	218
$\mathbf{A}_4 + \mathbf{A}_2$	6	13	0	214	216

Table 11: Dimension of the sheets for $\mathfrak{g}$ of type $\mathbf{E}_8$, **1/2**

I	II	III	IV	V	VI
$\mathbf{A}_4+\mathbf{A}_2+\mathbf{A}_1$	7	14	0	212	213
$\mathbf{A}_4+\mathbf{A}_3$	7	16	0	208	209
$\mathbf{D}_4$	4	12	16 / 10 / 0	232 / 226 / 216	236 / 230 / 220
$\mathbf{D}_4+\mathbf{A}_1$	5	13	16 / 10 / 0	230 / 224 / 214	233 / 227 / 217
$\mathbf{D}_4+\mathbf{A}_2$	6	15	16 / 10 / 0	226 / 220 / 210	228 / 222 / 212
$\mathbf{A}_5$	5	15	0	210	213
$\mathbf{A}_5+\mathbf{A}_1$	6	16	0	208	210
$\mathbf{D}_5$	5	20	24 / 14 / 0	224 / 214 / 200 /	227 / 217 / 203
$\mathbf{D}_5+\mathbf{A}_1$	6	21	24 / 14 / 0	222 / 212 / 198	224 / 214 / 200
$\mathbf{D}_5+\mathbf{A}_2$	7	23	24 / 14 / 0	218 / 208 / 194	219 / 209 / 195
$\mathbf{A}_6$	6	21	0	198	200
$\mathbf{A}_6+\mathbf{A}_1$	7	22	0	196	197
$\mathbf{D}_6$	6	30	36 / 32 / 28 18 / 0	216 / 212 / 208 198 / 180	218 / 214 / 210 200 / 182
$\mathbf{E}_6$	6	36	54 / 40 / 22 / 0	222 / 208 / 190 / 168	224 / 210 / 192 / 170
$\mathbf{E}_6+\mathbf{A}_1$	7	37	54 / 40 / 22 / 0	220 / 206 / 188 / 166	221 / 207 / 189 / 167
$\mathbf{A}_7$	7	28	0	184	185
$\mathbf{D}_7$	7	42	58 / 48 / 40 36 / 22 / 0	214 / 204 / 196 192 / 178 / 156	215 / 205 / 197 193 / 179 / 157
$\mathbf{E}_7$	7	63	92 / 90 / 82 / 70 64 / 52 / 34 / 0	206 / 204 / 196 / 184 178 / 166 / 148 / 114	207 / 205 / 197 / 185 179 / 167 / 149 / 115
$\mathbf{E}_8$	8	120	202 / 200 / 188 / 182 176 / 172 / 168 / 164 162 / 154 / 146 / 136 128 / 112 / 92 / 58 / 0	202 / 200 / 188 / 182 176 / 172 / 168 / 64 162 / 154 / 146 / 136 128 / 112 / 92 / 58 / 0	202 / 200 / 188 / 182 176 / 172 / 168 / 164 162 / 154 / 146 / 136 128 / 112 / 92 / 58 / 0

Table 12: Dimension of the sheets for $\mathfrak{g}$ of type $\mathbf{E}_8$, **2/2**

$2m \in \mathbf{N}_{\mathfrak{g}}$	12	10	8	6	0
$\dim \mathfrak{g}^{(m)}$	14	11	8	6	0

Table 13: Dimensions of the subsets $\mathfrak{g}^{(m)}$ for $\mathfrak{g}$ of type $\mathbf{G}_2$

$2m \in \mathbf{N}_{\mathfrak{g}}$	48	46	44	42	40	38	36	34	30	28	22	16	0
$\dim \mathfrak{g}^{(m)}$	52	49	46	44	42	39	37	34	31	28	22	16	0

Table 14: Dimensions of the subsets $\mathfrak{g}^{(m)}$ for $\mathfrak{g}$ of type $\mathbf{F}_4$

$2m \in \mathbf{N}_{\mathfrak{g}}$	72	70	68	66	64	62	60	58	56	54	52	50	48	46	42	40	32	22	0
$\dim \mathfrak{g}^{(m)}$	78	75	72	70	67	64	63	60	57	54	54	51	50	47	43	40	33	22	0

Table 15: Dimensions of the subsets $\mathfrak{g}^{(m)}$ for $\mathfrak{g}$ of type $\mathbf{E}_6$

$2m \in \mathbf{N}_{\mathfrak{g}}$	126	124	122	120	118	116	114	112	110	108	106	104	102	100	98
$\dim \mathfrak{g}^{(m)}$	133	130	127	125	122	119	118	115	112	110	109	106	105	102	99
$2m \in \mathbf{N}_{\mathfrak{g}}$	96	94	92	90	86	84	82	76	70	66	64	54	52	34	0
$\dim \mathfrak{g}^{(m)}$	98	95	92	90	88	85	82	77	70	67	64	55	52	34	0

Table 16: Dimensions of the subsets $\mathfrak{g}^{(m)}$ for $\mathfrak{g}$ of type $\mathbf{E}_7$

$2m \in \mathbf{N}_{\mathfrak{g}}$	240	238	236	234	232	230	228	226	224	222	220	218	216	214	212	210	208
$\dim \mathfrak{g}^{(m)}$	248	245	242	240	237	234	233	230	227	225	224	221	220	217	214	213	210
$2m \in \mathbf{N}_{\mathfrak{g}}$	206	204	202	200	198	196	194	192	190	188	184	182	180	178	176	172	168
$\dim \mathfrak{g}^{(m)}$	207	205	202	203	200	197	195	193	192	189	185	182	182	179	176	172	170
$2m \in \mathbf{N}_{\mathfrak{g}}$	166	164	162	156	154	148	146	136	128	114	112	92	58	0			
$\dim \mathfrak{g}^{(m)}$	167	164	162	157	154	149	146	136	128	115	112	92	58	0			

Table 17: Dimensions of the subsets $\mathfrak{g}^{(m)}$ for $\mathfrak{g}$ of type $\mathbf{E}_8$

Remark 4.2. Since we consider in our tables the subtle circumstances about G-conjugacy classes of Levi subalgebras which we discussed above (see Remark 4.1), our tables give also the exact number of sheets contained in each $\mathfrak{g}^{(m)}$ for any m. For example, there are three sheets in $\mathfrak{g}^{(20)}$ for $\mathfrak{g}$ of type $\mathbf{F}_4$.

Remark 4.3. The computations made by Elashvili ([20](Appendix of Chap. II) or [8]) give much more information than we actually need for our purpose. In fact, if we are only interested in the dimensions of the sets $\mathfrak{g}^{(m)}$, we can even avoid to refer to those computations. To get exactly those of all the sheets, we have to be a bit careful. Let us explain more precisely. We observe that the present method "almost" allows to recover in a simpler way the dimensions of the rigid nilpotent orbits in the exceptional types. Indeed, if for some $m \in \mathbf{N}_{\mathfrak{g}}$, there is no pair (S, p), with $S \subset \Pi$, $S \neq \Pi$, and $p \in \mathbf{N}_S^{\mathrm{rig}}$ such that $\mathrm{d}_{S,p} = m$, then m is in $\mathbf{N}_{\mathfrak{g}}^{\mathrm{rig}}$. Unfortunately, the converse is not true in general, since a non-rigid nilpotent orbit can be induced in different ways. Nevertheless, when there is only one nilpotent orbit of a given dimension m, we can decide if m belongs to $\mathbf{N}_{\mathfrak{g}}^{\mathrm{rig}}$ or not. Whatever the case, if we intend to compute the dimension $\mathfrak{g}^{(m)}$, this approach is sufficient because computing the dimension of a G-Jordan class possibly not dense in a sheet does not affect the final result. In that case, the corresponding dimension is not the dimension of a sheet and will not appear as a dimension of some $\mathfrak{g}^{(m)}$. For example, in $\mathbf{G}_2$ we do not need the precise Elashvili's computations since there is only one nilpotent orbit of a given dimension. In $\mathbf{F}_4$, there are two critical cases concerning the dimensions 30 and 42. Dimension 30 appear twice as dimension of nontrivial induced nilpotent orbit (see Table 7), but we cannot deduce that these two occurrences correspond to two different nilpotent orbits.

References

[1] Bourbaki, N., "Lie groups and Lie algebras. Chapters 4–6," Translated from the 1968 French original by Andrew Pressley, Springer-Verlag, Berlin, 2002.

[2] Borho, W., *Über Schichten halbeinfacher Lie-Algebren*, Inventiones Mathematicae **65** (1981/82), 283–317.

[3] Borho, W., and H. Kraft, *Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen*, Commentarii Mathematici Helvetici **54** (1979), 61–104.

[4] Collingwood, D., and W. M. McGovern, "Nilpotent orbits in semisimple Lie algebras," Van Nostrand Reinhold Co., New York, 1993.

[5] Dixmier, J., *Polarisations dans les algèbres de Lie semi-simples complexes*, Bulletin des Sciences Mathématiques, 2e Série, **99** (1975), 45–63.

[6] Dergachev V., and A. A. Kirillov, *Index of Lie algebras of seaweed type*, J. of Lie Theory **10** (2000), 331–343.

[7] Dvorsky, A., *Index of parabolic and seaweed subalgebras of* $\mathfrak{so}_n$, Lin. Alg. Appl. **374** (2003), 127–142.

[8] Elashvili, A. G., *Sheets of the exceptional Lie algebras*, in: "Issledovaniya po algebre," Tbilisi, 1985, (Russian), pp. 171–194.

[9] Gerstenhaber, M., *Dominance over the classical groups*, Ann. of Math. **74** (1961), 532–569.

[10] Im Hof, A., "The Sheets of Classical Lie Algebra," Thesis, Preprint available at `http://aleph.unibas.ch`, 2005.

[11] Joseph, A., *On semi-invariants and index for biparabolic (seaweed) algebras. I*, J. of Algebra **305** (2006), 485–515.

[12] Katsylo, P. I., *Sections of sheets in a reductive algebraic Lie algebra*, (Russian) Izv. Akad. Nauk SSSR, Ser. Mat. **46** (1982), 477–486.

[13] Kempken, G., *Induced conjugacy classes in classical Lie-algebras*, Abh. Math. Sem. Univ. Hamburg **53** (1983), 53–83.

[14] Kirillov, A. A., *Représentations unitaires des groupes de Lie nilpotents*, Uspehi Mat. Nauk. **17** (1962), 57–110.

[15] —, *Two more variations on the triangular theme*, in: "The orbit method in geometry and physics," Marseille, 2000, Birkhäuser Boston Inc., Boston, MA, 2003, pp. 243–258.

[16] Moreau, A., *Indice du normalisateur du centralisateur d'un élément nilpotent dans une algèbre de Lie semi-simple*, Bulletin de la Société Mathématique de France **134** (2006), 83–117.

[17] —, *Indice et décomposition de Cartan d'une algèbre de Lie semi-simple réelle*, J. of Algebra **303** (2006), 382–406.

[18] Panyushev, D. I., *Inductive formulas for the index of seaweed Lie algebras*, Moscow Math. J. 1 **1** (2001), 221–241, 303.

[19] —, *The index of a Lie algebra, the centralizer of a nilpotent element, and the normalizer of the centralizer*, Math. Proc. Camb. Phil. Soc. **134** (2003), 41–59.

[20] Spaltenstein, N., *Classes unipotentes et sous-groupes de Borel*, Springer-Verlag, Berlin, 1982.

[21] Tauvel, P., and R. W. T. Yu, "Lie algebras and algebraic groups," Monographs in Mathematics, Springer Berlin etc., 2005.

[22] —, *Sur l'indice de certaines algèbres de Lie*, Annales de l'Institut Fourier **54** (2005), 1793–1810.

[23] Veldkamp, F. D., *The center of the universal enveloping algebra of a Lie algebra in characteristic p*, Annales Scientifiques de L'École Normale Supérieure **5** (1972), 217–240.

[24] Yakimova, O., *The index of centralizers of elements in classical Lie algebras*, Funktsional. Anal. i Prilozhen. **40** (2006), 52–64, 962.

Anne Moreau
ETH Zürich
Departement Mathematik
Rämistrasse 101
HG G66.4
8092 Zürich, Switzerland
anne.moreau@math.ethz.ch

Received April 24, 2008
and in final form September 8, 2008