Journal of Lie Theory
Volume **18** (2008) 937–949

On the Decomposition of $L^2(\Gamma \setminus G)$ in the Cocompact Case

Goran Muić

Communicated by B. Ørsted

Abstract. Let G be a semisimple Lie group with a finite center and finitely many connected components. For example, G could be a group of $\mathbb{R}$–points of a semisimple Zariski connected algebraic group defined over $\mathbb{Q}$. Let Γ be a discrete cocompact subgroup of G. Using the spectral decomposition of compactly supported Poincaré series we discuss the existence of various types of irreducible unitary subrepresentations of $L^2(\Gamma \setminus G)$.
2000 Mathematics Subject Classification: 22Exx, 11F03.
Key Words and Phrases: Poincaré series, cocompact quotients.

1. Introduction

Let G be a semisimple Lie group with a finite center and finitely many connected components. For example, G could be a group of $\mathbb{R}$–points of a semisimple Zariski connected algebraic group defined over $\mathbb{Q}$. Let Γ be a discrete cocompact subgroup of G. By a well–known theorem of Gelfand–Graev–Piatetski Shapiro, the right–regular representation $L^2(\Gamma \setminus G)$ decomposes into a direct sum of irreducible unitary representations of G each appearing with a finite multiplicity. The spectral decomposition of $L^2(\Gamma \setminus G)$ was studied in the works such as [4], [5] [8], [16] usually assuming that Γ is torsion free and using the Selberg trace formula for compact quotients. In spite of those efforts, the decomposition of $L^2(\Gamma \setminus G)$ is still rather mysterious. In fact, except some partial results on representations in the discrete series, K–spherical representations, and cohomological representations, we do not know if $L^2(\Gamma \setminus G)$ contains a "significant" number of other types of irreducible unitary subrepresentations. The goal of this paper is to shed some light on those issues. The main result of this short note is the following theorem:

Theorem 1.1. *Let K be a maximal compact subgroup of G. Assume that Γ is a cocompact discrete subgroup of G but G is not compact. Then we have the following:*

(i) Every irreducible subrepresentation of $L^2(\Gamma \setminus G)$ contains a K–type from $L^2(K \cap \Gamma \setminus K)$ i.e., a K–type containing a non–zero vector invariant under $K \cap \Gamma$.

ISSN 0949–5932 / $2.50

(ii) Let δ be a K–type appearing in $L^2(K \cap \Gamma \setminus K)$. Then there exist infinitely many non–equivalent irreducible unitary subrepresentations of $L^2(\Gamma \setminus G)$ containing δ.

We remark that $K \cap \Gamma$ is usually trivial. For example, in the torsion free case. Theorem 1.1 is proved in Section 4. A major step towards the proof of (i) was done in Section 3 where we prove Theorem 3.1 which is of independent interest. More precisely, in Definition 2.1 we define the space of square–integrable automorphic forms $\mathcal{A}^{\in}(- \setminus \mathcal{G})$ for an arbitrary discrete subgroup $\Gamma \subset G$ by the analogy with the usual definition for an arithmetic group Γ [3] and, for the convenience of the reader, we include the standard theorem (see Theorem 2.3) which describes the relation between $\mathcal{A}^{\in}(- \setminus \mathcal{G})$ and $L^2(\Gamma \setminus G)$. Then Theorem 3.1 shows that for a non–zero $\varphi \in \mathcal{A}^{\in}(- \setminus \mathcal{G})$ the $(\mathfrak{g}, K)$–module generated by φ contains a K–type trivial on $K \cap \Gamma$. At this point the standard theorem (see Theorem 2.3 (i)) completes the proof of Theorem 1.1 (i). The proof of Theorem 1.1 (ii) occupies the major part of Section 4. It is self–contained and it is based on a new (and simple) method of the spectral decomposition of compactly supported Poincaré series. We develop this idea further in the case of the non–compact quotient in [12] but this is more arithmetic in its nature.

In Section 5 we collect some applications. Our intention is not to give an exhaustive list. First, in Proposition 5.1 we generalize the classical results about the existence of infinitely many $\mathcal{Z}(\mathfrak{g})$–eigenvalues on the space of automorphic forms for $\Gamma \setminus G/K$ when $\Gamma \setminus G$ is compact ([5], [13]). In Proposition 5.2 we show that given δ containing a non–zero vector invariant under $K \cap \Gamma$, only finitely many irreducible subrepresentations of $L^2(\Gamma \setminus G)$ containing δ are in the (limits) of the discrete series for G. This is interesting since when G poses discrete series (i.e., when its connected component has a compact Cartan subgroup [7]), then the trace formula ([4], [16]) or Poincaré series (see [11] and references there) can be used to show the existence of irreducible subrepresentations of $L^2(\Gamma \setminus G)$ which are in the discrete series for G. Next, Vogan's theory of representations attached to fine K–types ([15], Definition 4.3.9) generalizes the usual theory of K–spherical representations. In Theorem 5.4 we discuss the existence of the subrepresentations of $L^2(\Gamma \setminus G)$ of that form. We explain the case of $G = SL_2(\mathbb{R})$ in Example 5.5 which seems to be a rather new result.

We remark that it would be interesting to study the appearance of representations attached to fine K–types in the discrete spectrum of $L^2(\Gamma \setminus G)$ when $\Gamma \setminus G$ is not compact. We leave this for another occasion.

I would like to thank M. Tadić, G. Savin, J. Schwermer and D. Vogan for some useful discussions. The paper was written while I was a visitor of the Erwin Schrödinger Institute in Vienna. I would like to thank the Erwin Schrödinger Institute and J. Schwermer for their hospitality.

2. Preliminary results

Let G be a semisimple Lie group with a finite center and finitely many connected components. Let K be a maximal compact subgroup of G. We write $\mathfrak{g}$ for the (real) Lie algebra of G. The maximal compact subgroup K is a fixed point set of a Cartan involution Θ of G. The differential θ of Θ gives the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and -1 eigenspaces of θ. We have $\mathfrak{k} = Lie(K)$. Let $\mathfrak{a}$ be a maximal Abelian subalgebra of $\mathfrak{p}$. We choose some ordering of the roots $\Sigma(\mathfrak{a}, \mathfrak{g})$ so that we determine the positive roots $\Sigma^+(\mathfrak{a}, \mathfrak{g})$. Let N be the corresponding unipotent radical. This determines minimal parabolic subgroup $P = MAN$ of G, where $A = \exp(\mathfrak{a})$ and $M = Z_K(A)$.

We have the following diffeomorphism:

$$N \times A \times K \xrightarrow{(n,a,k)\mapsto n\cdot a\cdot k} G = NAK.$$

The Iwasawa decomposition implies that there exists unique C^∞–functions $a : G \to A$, $n : G \to N$, and $k : G \to K$ such that

$$g = n(g) \cdot a(g) \cdot k(g), \quad g \in G. \tag{1}$$

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of the complexified Lie algebra of $\mathfrak{g}$. Let $\mathcal{Z}(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$. We consider $\mathcal{U}(\mathfrak{g})$ as algebra of left–invariant differential operators on G:

$$X.f(g) = \frac{d}{dt} f(g \exp(tX))\Big|_{t=0}, \quad f \in C^\infty(G), \quad X \in \mathfrak{g}.$$

In this paper Γ denotes a discrete subgroup of G. We define a G–invariant measure on $\Gamma \setminus G$ as follows:

$$\int_{\Gamma\setminus G} P(f)(g)dg = \int_G f(g)dg, \tag{2}$$

for $f \in C_c(G)$ (the space of compactly supported complex continuous functions on G), where the *compactly supported Poincaré series* is defined as follows:

$$P(f)(g) \overset{\text{def}}{=} \sum_{\gamma\in\Gamma} f(\gamma \cdot g). \tag{3}$$

Obviously, for $f \in C_c^\infty(G)$ (the space of compactly supported smooth complex continuous functions on G), the function $P(f)$ belongs to the space $C_c^\infty(\Gamma \setminus G)$ (the subspace of $C^\infty(G)$ consisting of all left Γ–invariant functions compactly supported modulo Γ).

We use the measure on $\Gamma \setminus G$ defined by the right–hand side of (2) to define $L^2(\Gamma \setminus G)$ a right regular representation of G.

Let $\hat{K}$ be the set of equivalence of irreducible representations of K. Let $\delta \in \hat{K}$, then we write $d(\delta)$ and ξ_δ the degree and character of δ, respectively. We fix the normalized Haar measure dk on K. Let π be a Banach representation of G on the Banach space $\mathcal{B}$. Then, for $b \in \mathcal{B}$ and $\delta \in \hat{K}$, we let

$$E_\delta(b) = \int_K d(\delta)\overline{\xi_\delta(k)}\pi(k)b \; dk.$$

It belongs to δ–isotypic component $\mathcal{B}(\delta)$ of $\mathcal{B}$.

Definition 2.1. We let $\mathcal{A}^{\in}(-\backslash\mathcal{G})$ be the space of K–finite (square–integrable) automorphic forms on G. In the present set-up can be defined as in the arithmetic case (see ([3], 1.3)). Explicitly, $\mathcal{A}^{\in}(-\setminus\mathcal{G})$ consists of all functions $\varphi\in C^\infty(\Gamma\setminus G)$ satisfying the following:

a-1) φ is K–finite and $\mathcal{Z}(\mathfrak{g})$–finite [1]

a-2) $\varphi\in L^2(\Gamma\setminus G)$.

We will use the following result from the representation theory (see [6]):

Theorem 2.2. *Assume that Γ is cocompact in G. Then $L^2(\Gamma\setminus G)=\hat{\oplus}_j\mathcal{H}_j$, where $\mathcal{H}_j$ are closed irreducible G–invariant subspaces of $L^2(\Gamma\setminus G)$. Moreover, for each i, there exists finitely many j's such that $\mathcal{H}_j$ is equivalent with $\mathcal{H}_i$.*

Next, we observe the following standard theorem (due to Harish–Chandra):

Theorem 2.3. *Let Γ be an arbitrary discrete subgroup of G. Then we have the following:*

(i) Let $\mathcal{H}$ be an irreducible subspace of $L^2(\Gamma\setminus G)$. Then the $(\mathfrak{g},K)$–module on the space of K–finite vectors $\mathcal{H}_K$ of $\mathcal{H}$ is an irreducible submodule of $\mathcal{A}^{\in}(-\setminus\mathcal{G})$.

(ii) Let $\varphi\in\mathcal{A}^{\in}(-\setminus\mathcal{G})$ be a non–zero automorphic form. Then the (closed) subrepresentation U_φ generated by $\varphi\in L^2(\Gamma\setminus G)$ is a direct sum of finitely many irreducible subrepresentations.

(iii) Assume that Γ is cocompact in G. Then, using the notation of Theorem 2.2, we obtain the following decomposition into irreducible $(\mathfrak{g},K)$–modules:

$$\mathcal{A}^2(\Gamma\setminus G)=\oplus_j(\mathcal{H}_j)_K.$$

Proof. We include the standard proof for the reader's convenience. We prove (i). Since $\mathcal{H}$ is irreducible and unitary, it is admissible by a well–known theorem of Harisch–Chandra (see [15], Theorem 0.3.6). Hence, its $(\mathfrak{g},K)$–module $(\mathcal{H})_K$ is irreducible and admissible (see [15], Theorem 0.3.5). It is well–known that $(\mathcal{H})_K$ is $\mathcal{Z}(\mathfrak{g})$–finite (see [15], Proposition 0.3.19). This means that every $\psi\in\mathcal{H}_K$ is $\mathcal{Z}(\mathfrak{g})$–finite and K–finite in the sense of distributions. But then ψ is real analytic on G. In particular, $\psi\in C^\infty(\Gamma\setminus G)$. Now, (i) follows from Definition 2.1. The proof of (ii) the same as the proof of ([7], Lemma 77). We prove (iii). First, (i) implies $\oplus_j(\mathcal{H}_j)_K\subset\mathcal{A}^2(\Gamma\setminus G)$. Conversely, let $\varphi\in\mathcal{A}^{\in}(-\setminus\mathcal{G})$ be a non–zero automorphic form. We write $\varphi=\sum_j\varphi_j$ according to the decomposition $L^2(\Gamma\setminus G)=\hat{\oplus}_j\mathcal{H}_j$. It is clear that if φ_j is not trivial the projection $U_\varphi\to\mathcal{H}_j$ is a non–trivial G–equivariant bounded map. Hence, (ii) implies that there exists only finitely many such j's. This proves the converse inclusion. ■

[1] In what follows the adjective finite will be used with respect to the right action.

3. Frobenius Reciprocity for Automorphic Representations

The main result of this section is Theorem 3.1. It explains the restriction of an automorphic representation to K. If we think of $\mathcal{A}^{\in}(- \setminus \mathcal{G})$ and $L^2(\Gamma \setminus G)$ as induced representations from the trivial representation of Γ to G in an appropriate category, Theorem 3.1 (ii) is a sort of a Frobenius reciprocity for the restriction to K. Example 3.3 below shows that the result is the best possible.

Theorem 3.1. *Assume that Γ is a discrete subgroup of G. Let $\varphi \in \mathcal{A}^{\in}(- \setminus \mathcal{G})$ be a non–zero automorphic form. Then we have the following:*

(i) There exists $u \in \mathcal{U}(\mathfrak{g})$ and $k \in K$ such that such that $u.\varphi(k) \neq 0$.

(ii) The $(\mathfrak{g}, K)$–submodule of $\mathcal{A}^{\in}(- \setminus \mathcal{G})$ generated by φ contains a non–trivial isotypic component for some $\delta \in \hat{K}$ such that there is a non–zero $K \cap \Gamma$–invariant vector in the space of δ.

Proof. First, we prove (i). The proof rests on the following simple fact:

If Y is a finite–dimensional connected C^ω–manifold (this means real analytic) and if $f : Y \to \mathbb{C}$ is real analytic and trivial on a non–empty open subset of Y, then $f \equiv 0$ on Y.

It well–known that K meets all connected components of G. Also, being an automorphic form, φ is real–analytic on G. Hence, on a sufficiently small neighborhood of an $k \in K \subset G$, we have the following: $\varphi(k \cdot \exp X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n.\varphi(k)$. Therefore, if (i) is not true, then we obtain φ is identically equal to zero on every connected component of G. This is a contradiction. Now, we prove (ii). First, (i) implies that there exists an automorphic form ψ in the module generated by φ and $k_0 \in K$ such that $\psi(k_0) \neq 0$. Since ψ is K–finite, we can find $\delta_1, \ldots, \delta_r \in \hat{K}$ such that $\psi = \sum_{i=1}^{r} E_{\delta_i}(\psi)$. Note that the automorphic forms $E_{\delta_i}(\psi)$ are defined as follows: $E_{\delta_i}(\psi) = \int_K d(\delta_i)\overline{\xi_{\delta_i}(k)}\psi(gk)dk$. Since

$$0 \neq \psi(k_0) = \sum_{i=1}^{r} E_{\delta_i}(\psi)(k_0),$$

there exists i such that $E_{\delta_i}(\psi)(k_0) \neq 0$. Hence the claim follows from Lemma 3.2 below. ∎

Lemma 3.2. *Let $\delta \in \hat{K}$. Assume that $\varphi \in \mathcal{A}^{\in}(- \setminus \mathcal{G})$ belongs to the δ–isotypic component of $\mathcal{A}^{\in}(- \setminus \mathcal{G})$ and φ is not identically zero on K. Then there is a non–zero $K \cap \Gamma$–invariant vector in the space of δ.*

Proof. Since φ belongs to the δ–isotypic component of $\mathcal{A}(- \setminus \mathcal{G})$, $E_\delta(\varphi) = \varphi$. Explicitly,

$$\varphi(g) = \int_K d(\delta)\overline{\xi_\delta(k)}\varphi(gk)dk, \quad g \in G.$$

Since φ is not identically zero on K, $\varphi(k_0) \neq 0$ for some $k_0 \in K$. Now,

$$\varphi(k_0) = \int_K d(\delta)\overline{\xi_\delta(k)}\varphi(k_0 \cdot k)dk.$$

If we fix the realization V_δ of δ. Let $(\ ,\ )$ be the K–invariant scalar product on V_δ, and let $(v_1, \dots, v_{d(\delta)})$ be an orthonormal basis of V_δ. Then

$$\xi_\delta(k) = \sum_{i=1}^{d(\delta)} (\delta(k)v_i, v_i).$$

The number of elements, say M, in $K \cap \Gamma$ is finite. We compute

$$\begin{aligned}
0 \neq M \cdot \varphi(k_0) &= \sum_{\gamma \in K \cap \Gamma} \varphi(\gamma \cdot k_0) \\
&= \sum_{\gamma \in K \cap \Gamma} \int_K d(\delta)\overline{\xi_\delta(k)}\varphi(\gamma k_0 \cdot k)dk \\
&= \sum_{\gamma \in K \cap \Gamma} \int_K d(\delta)\overline{\xi_\delta((\gamma k_0)^{-1}k)}\varphi(k)dk \\
&= \sum_{i=1}^{d(\delta)} \int_K d(\delta) \left(\sum_{\gamma \in K \cap \Gamma} (\delta(\gamma)\delta(k_0)v_i, \delta(k)v_i) \right) \varphi(k)dk \\
&= \sum_{i=1}^{d(\delta)} \int_K d(\delta) \left(\sum_{\gamma \in K \cap \Gamma} \delta(\gamma)\delta(k_0)v_i, \delta(k)v_i \right) \varphi(k)dk.
\end{aligned}$$

Hence, we see that

$$0 \neq \sum_{\gamma \in K \cap \Gamma} \delta(\gamma)\delta(k_0)v_i \in V_\delta^{K \cap \Gamma},$$

for some i. ■

Example 3.3. Let $G = SL_2$. Then $G = SL_2(\mathbb{R})$ and K can be identified with $U(1)$ as follows:

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \leftrightarrow \exp it = \cos t + i \sin t.$$

Then $K \cap SL_2(\mathbb{Z})$ is $\{\pm 1,\ \pm i\}$ in this identification. Let $m \in \mathbb{Z}_{\geq 2}$ and let $D_{\pm m}$ be the representation in the discrete series with the highest weight $-m$ or the lowest weight m, respectively. The K–types of $D_{\pm m}$ belong to $m + 2\mathbb{Z}$. Hence we see that if $D_{\pm m} \hookrightarrow \mathcal{A}^{\in}(\mathcal{SL}_{\in}(\mathbb{Z})\backslash\mathcal{SL}_{\in}(\mathbb{R}))$, then for some type $m+2k$ of $D_{\pm m}$, we must have $(-1)^{m+2k} = 1$ and $i^{m+2k} = 1$. This implies $m + 2k \equiv 0 (mod\ 4)$. Hence $D_{\pm m}$ with $m \equiv 1 (mod\ 2)$ do not appear. (This is well–known. It follows considering central characters.) But it is well–known that some of $D_{\pm m}$ with $m \equiv 0 (mod\ 2)$ do appear. We see that not all K–types of such $D_{\pm m}$ contain a vector invariant under $K \cap SL_2(\mathbb{Z})$.

4. The proof of Theorem 1.1

We begin this section by some general observations so we assume that Γ is an arbitrary discrete subgroup of G until we state differently.

The Hilbert space $L^2(K \cap \Gamma \setminus K)$ decomposes, under the right translations of K, as a Hilbert direct sum of irreducible representations $\delta \in \hat{K}$ containing a non–trivial vector invariant under $K \cap \Gamma$; such δ appears exactly $\dim V_\delta^{K\cap\Gamma}$ times, and δ–isotypic component consists of the following functions:

$$k \to (\delta(k)v, w), \quad v \in V_\delta, \ w \in V_\delta^{K\cap\Gamma}, \tag{4}$$

where V_δ is the space of δ and $(\ , \ )$ is K–invariant positive definite Hermitian form on V_δ. It is clear that the orthogonal projector E_δ fixes any such function.

We start this section proving the following non–vanishing result:

Lemma 4.1. *There exists a neighborhood U of 1 in N and a neighborhood V of 1 in A such that every function $\varphi \in C_c^\infty(G)$, supported in $U \cdot V \cdot K$ such that its restriction to K is non–trivial and it belongs to $L^2(K \cap \Gamma \setminus K)$, has a non–vanishing Poincaré series*

$$P(\varphi)(g) = \sum_{\gamma \in \Gamma} \varphi(\gamma \cdot g).$$

Proof. The main point is that there exists U and V small enough such that

$$\Gamma \cap (U \cdot V \cdot K) = \Gamma \cap K.$$

Indeed, let $U_1 \supset U_2 \supset \cdots$ and $V_1 \supset V_2 \supset \cdots$ be the bases of neighborhoods of identities in N and A consisting of compact sets, respectively. Then the compact sets in G defined by

$$W_n = U_n \cdot V_n \cdot K, \quad n \geq 1, \tag{5}$$

all have a finite intersection with Γ. Moreover, $\Gamma \cap W_1 \supset \Gamma \cap W_2 \supset \cdots$ implies that there exists n_0 such that $\Gamma \cap W_n = \Gamma \cap W_{n+1}$, for $n \geq n_0$. We show that we can take $U = U_{n_0}$ and $V = V_{n_0}$. Indeed, let $\gamma \in W_{n_0} = U_{n_0} \cdot V_{n_0} \cdot K$. Then, for $n \geq n_0$, we may write γ as follows $\gamma = u_n v_n k_n$, where $u_n \in U_n$, $v_n \in V_n$ and $k_n \in K$. Clearly, $u_n \to 1$ and $v_n \to 1$ as $n \to \infty$. Hence, $k_n \to \gamma$ as $n \to \infty$. Thus, $\gamma \in K$. This proves $\Gamma \cap U \cdot V \cdot K \subset \Gamma \cap K$. The converse inclusion is trivial since U and V are neighborhoods of identity.

Now, since the restriction of φ to K is non–trivial, we can find $k \in K$ such that $\varphi(k) \neq 0$. We compute

$$P(\varphi)(k) = \sum_{\gamma \in \Gamma} \varphi(\gamma \cdot k) = \sum_{\gamma \in \Gamma \cap (U \cdot V \cdot K) \cdot k^{-1}} \varphi(\gamma \cdot k) = (\# K \cap \Gamma) \cdot \varphi(k) \neq 0.$$

■

Since G is not necessarily connected we need to be careful in defining the function φ. We prove the following lemma:

Lemma 4.2. *Let $\delta \in \hat{K}$ be a subrepresentation of $L^2(K \cap \Gamma \setminus K)$. Then there exists $\varphi \in C_c^\infty(G)$ such that the following hold:*

(i) $E_\delta(\varphi) = \varphi$.

(ii) $\psi = P(\varphi) \neq 0$.

(iii) *There exists a compact set $C \subset G$, right–invariant under K, such that $\Gamma \cdot C$ does not contain a connected component of G and $\mathrm{supp}(\psi) \subset \Gamma \cdot C$. The set $\Gamma \cdot C$ is closed in G.*

Proof. We use the notation introduced in Lemma 4.1. A function φ with the required properties (i) and (ii) can be constructed as follows. Let $\zeta \in C_c^\infty(U)$, $\zeta(1) \neq 0$, $\eta \in C_c^\infty(V)$, $\eta(1) \neq 0$, and ξ be a non–zero matrix coefficient of δ given by (4) with $v = w \neq 0$. Then the function $\varphi \in C_c^\infty(G_\infty)$ defined by (see (1))

$$\varphi(g) = \zeta(n(g))\eta(a(g))\xi(k(g)), \quad g \in G_\infty,$$

satisfies (i) and (ii).

Next, by the construction, the support of ψ is contained in the set of the form $\Gamma \cdot C$, where C is a compact set right–invariant under K. We show that $\Gamma \cdot C$ is closed in G. Indeed, if $\gamma_n \cdot c_n \to g$, then, by passing to a subsequence, we may assume that $c_n \to c \in C$, hence $\gamma_n \to g \cdot c^{-1}$. Since Γ is discrete, for n large enough, the sequence γ_n stabilizes. Hence $g \in \Gamma \cdot C$.

Finally, we show that we can shrink U and V in order to obtain that $\Gamma \cdot C$ does not contain a connected component of G. But since K meets all connected components of G and C is right–invariant under K, it is enough to show that we can shrink U and V in order to obtain that $\Gamma \cdot C \neq G$.

To accomplish this, we use the sequence of relatively compact neighborhoods W_n defined by (5) in the proof of Lemma 4.1. We prove that $\Gamma \cdot W_n \neq G$ for n large enough. Assume that this is not true. Then there is an increasing sequence $(n_l)_{l \geq 1}$ such that $\Gamma \cdot W_{n_l} = G$. Let us pick $g \in G$. Then we can write as follows

$$g = \gamma_l \cdot u_l \cdot v_l \cdot k_l$$

for some $\gamma_l \in \Gamma$, $u_l \in U_{n_l}$, $v_l \in V_{n_l}$, and $k_l \in K$. (See the proof of Lemma 4.1 for the notation.) Clearly, $u_l \to 1$, $v_l \to 1$, and, by passing to a subsequence, we may assume that $k_l \to k$. Hence $\gamma_l \to gk^{-1}$. Since Γ is discrete, the converging sequence must stabilize. We conclude that $g \in \Gamma \cdot K$. Hence $G = \Gamma \cdot K$. Now, since G is not compact, we have that N is not trivial. Let $g_l \to 1$ be an arbitrary sequence in G. We can write $g_l = \gamma_l k_l$, where the symbols have their obvious meaning, and, by passing to a subsequence, $k_l \to k$. Then $\gamma_l \to k^{-1}$. Hence, for large enough l, γ_l does not depend on l and it belongs to $K \cap \Gamma$. This implies that, for large enough l, $g_l \in K$. This is clearly impossible if we choose a sequence in N which satisfies $g_l \neq 1$ for l large enough. ■

Now, we begin the proof of Theorem 1.1. Hence, we assume that Γ is cocompact in G. First, Theorems 2.3 (i) and 3.1 prove Theorem 1.1 (i). Now, we prove Theorem 1.1 (ii). We take φ and $\psi = P(\varphi)$ as in Lemma 4.2. Then, Lemma 4.2 (i) implies $\psi = E_\delta(\psi)$. Next, we can write (see Theorem 2.2)

$$L^2(\Gamma \setminus G) = \hat{\oplus}_j \mathcal{H}_j,$$

where $\mathcal{H}_j$ are irreducible subspaces. We write according to that decomposition

$$\psi = \sum_j \psi_j. \tag{6}$$

Applying E_δ, we obtain

$$\psi = E_\delta(\psi) = \sum_j E_\delta(\psi_j).$$

The uniqueness of the expansion implies $E_\delta(\psi_j) = \psi_j$ for all j. Hence $\psi_j \in \mathcal{A}^{\in}(- \setminus \mathcal{G})$ applying Theorem 2.3 (i). Now, since ψ_j is real analytic for all j (see the proof of Theorem 2.3 (i)), the sum on the right–hand side of (6) cannot be finite since otherwise ψ would be real analytic. But this is not possible since it vanishes on a non–empty open set $G - \Gamma \cdot C$ which meets all connected components of G by Lemma 4.2 (iii). This is a contradiction. This proves Theorem 1.1 (ii).

5. Some Applications of Theorem 1.1

We start by the following application which generalizes the classical results about the existence of infinitely many $\mathcal{Z}(\mathfrak{g})$–eigenvalues on the space of automorphic forms for $\Gamma \setminus G/K$ when $\Gamma \setminus G$ is compact ([5], [13]):

Proposition 5.1. *Assume that Γ is a cocompact discrete subgroup of G but G is not compact. Assume that $\delta \in \hat{K}$ is a K–type containing a non–zero vector invariant under $K \cap \Gamma$. Then there exist infinitely many infinitesimal characters $\chi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ such that χ–eigenspaces in $\mathcal{A}^{\in}(- \setminus \mathcal{G})(\delta)$ are non–trivial.*

Proof. First, we decompose $L^2(\Gamma \setminus G) = \hat{\oplus}_j \mathcal{H}_j$ as in Theorem 2.2, where $\mathcal{H}_j$ are closed irreducible G–invariant subspaces of $L^2(\Gamma \setminus G)$. Since an irreducible unitary representation on Hilbert space $\mathcal{H}$ is admissible (see [15], Theorem 0.3.5), we see $\mathcal{H}(\delta) = (\mathcal{H})_K(\delta)$. Hence, Theorem 2.3 (iii) implies that

$$\mathcal{A}^2(\Gamma \setminus G)(\delta) = \oplus_j (\mathcal{H}_j)_K(\delta) = \oplus_j \mathcal{H}_j(\delta).$$

Next, by Theorem 1.1 (ii), there exists infinitely many indices j such that $\mathcal{H}_j(\delta) \neq 0$. Since, for each i, there exists only finitely many j's such that $\mathcal{H}_j$ is equivalent with $\mathcal{H}_j$ and since there exists only finitely many non–equivalent irreducible $(\mathfrak{g}, K)$–modules with a fixed infinitesimal character (see [15], Corollary 5.4.17), the above decomposition of $\mathcal{A}^2(\Gamma \setminus G)(\delta)$ proves the claim. ■

We remark that the classical case corresponds to the case $\mathcal{A}^{\in}(- \setminus \mathcal{G})(\mathbf{1})$, where $\mathbf{1}$ is a trivial representation of K.

Now, we explain the representation–theoretic applications. The next proposition shows the existence of irreducible subspaces of $L^2(\Gamma \setminus G)$ which are not in the (limits) discrete series for G:

Proposition 5.2. *Assume that Γ is a cocompact discrete subgroup of G but G is not compact. Assume that $\delta \in \hat{K}$ is a K–type containing a non–zero vector invariant under $K \cap \Gamma$. Assume that G poses representations in the discrete series (i.e., its connected component has a compact Cartan subgroup [7].) Then there exists infinitely many irreducible unitary representations $(\pi, \mathcal{H})$ of G which are not in the limits of discrete series ([10], Section 1) for G, which contain δ, and the space of bounded G–equivariant maps $\mathrm{Hom}_G(\mathcal{H}, L^2(\Gamma \setminus G))$ is non–trivial.*

Proof. Following ([10], Section 1), we say that an irreducible unitary representation $(\pi, \mathcal{H})$ is in the discrete series (resp., in the limits of discrete series) if some irreducible (hence, all subrepresentations of $\pi|_{G^0}$) are in the discrete series (resp., in the limits of discrete series). Here G^0 is the connected component of G. Since G/G^0 is finite, we see that there are only finitely many irreducible subrepresentations of $\pi|_{G^0}$ (see ([10], Section 1) for a more precise description). This clearly reduces the proof to the case $G = G^0$. So, we assume that $G = G^0$. Now, the description of the K–type structure of the limits of discrete series for G^0 (see [9], Theorems 9.20 and 12.26) shows that there could be only finitely many of them containing any given K–type δ. Indeed, let $\mathfrak{t}$ be the Lie algebra of a compact Cartan subgroup $T \subset K \subset G$. Let $\mathfrak{t}' = \mathrm{Hom}_{\mathbb{R}}(\sqrt{-1}\mathfrak{t},\ \mathbb{R})$.

It is well–known that there exists a one–to–one correspondence between the sets of simple roots Δ of $\mathfrak{t}$ in $\mathfrak{g}_{\mathbb{C}}$ and Weyl chambers $C \subset \sqrt{-1}\mathfrak{t}$. It is given by $\Delta \leftrightarrow C = \{x \in \sqrt{-1}\mathfrak{t} :\ \alpha(x) > 0,\ \alpha \in \Delta\}$. Let ρ be the half–sum of all positive roots of $\mathfrak{t}$ in $\mathfrak{g}_{\mathbb{C}}$ determined by Δ. Also, the choice of the Weyl chamber C determines the positive roots for $\mathfrak{t}$ in $\mathfrak{k}_{\mathbb{C}}$; we write the half–sum of the positive roots as ρ_c and the set of simple roots by Δ_c.

A limit of discrete series π of G is parametrized by a pair (C, λ), $\pi = \pi(C, \lambda)$, consisting of a Weyl chamber $C \subset \sqrt{-1}\mathfrak{t}$ and a C–dominant weight λ which is not orthogonal to any compact C–simple root. Also, every K–type of π has its highest weight of the form $\lambda + \rho - 2\rho_c + \sum_{\alpha \in \Delta} n_\alpha \alpha$ ($\lambda \in \mathfrak{t}'$), where $n_\alpha \in \mathbb{Z}_{\geq 0}$.

Now, assume that π contains given K–type δ. Let μ be the highest weight of δ. Then

$$\mu = \lambda + \rho - 2\rho_c + \sum_{\alpha \in \Delta} n_\alpha \alpha, \tag{7}$$

for some $n_\alpha \in \mathbb{Z}_{\geq 0}$. Since λ is C–dominant, we have the following:

$$((\mu - \rho + 2\rho_c), \beta) \geq \sum_{\alpha \in \Delta} n_\alpha (\alpha, \beta), \quad \beta \in \Delta, \tag{8}$$

where $(\ ,\ )$ is a (suitable) scalar product on $\mathfrak{t}'$. Multiplying (8) with n_β and summing over $\beta \in \Delta$, we obtain the following:

$$0 \geq \sum_{\alpha, \beta \in \Delta} (\alpha, \beta)\, n_\alpha n_\beta - \sum_{\beta \in \Delta} A_\beta n_\beta, \tag{9}$$

where we write $A_\beta = ((\mu - \rho + 2\rho_c), \beta)$, for $\beta \in \Delta$. Since G is semisimple, Δ is a basis of $\mathfrak{t}'$. It is obvious that the matrix $A = ((\alpha, \beta))_{\alpha, \beta \in \Delta}$ is symmetric and positive definite. Now, by the change of coordinates we can diagonalize the matrix A and by "completing the squares" in new coordinates we can see that the set of all $(x_\alpha)_{\alpha \in \Delta} \in \mathbb{R}^\Delta$ given by

$$0 \geq \sum_{\alpha, \beta \in \Delta} (\alpha, \beta)\, x_\alpha x_\beta - \sum_{\beta \in \Delta} A_\beta n_\beta$$

is compact. Hence, there exists only finitely many integral solutions $(n_\alpha)_{\alpha \in \Delta} \in \mathbb{Z}^\Delta$ to the inequality (9). Hence, given μ and C, there are only finitely many λ's as above such that (7) holds. Finally, since there are finitely many Weyl chambers, we obtain the claim. ■

Let $P = MAN$ be the minimal parabolic subgroup of G given by its Langlands decomposition described in Section 2. We let $\mathfrak{a}$ be the real Lie algebra of A and $\mathfrak{a}^*$ its complex dual (see Section 2). Then we have the following result:

Proposition 5.3. *Assume that $K \cap \Gamma$ is trivial. Let $\delta \in \hat{K}$. Then there exists an irreducible representation ϵ contained in the restriction of K to M such that $L^2(\Gamma \backslash G)$ contains an irreducible subquotient of $\mathrm{Ind}_{MAN}^G(\epsilon \otimes \exp \nu(\ ))$ for infinitely many values $\nu \in \mathfrak{a}^*$.*

Proof. Applying Harish–Chandra's subquotient theorem ([15], Theorem 4.1.9), every irreducible representation (or rather a $(\mathfrak{g}, K)$–module) of G containing δ must belong to one of the principal series $\mathrm{Ind}_{MAN}^G(\epsilon' \otimes \exp \nu(\ ))$, where ϵ' is contained δ upon the restriction to M. Now, we apply Theorem 1.1 (ii). ∎

This result is more transparent when G is quasi–split in view of Vogan's theory of minimal K–types ([14], [15]). Assume that $\epsilon \in \hat{M}$ is fine ([15], Definition 4.3.8) i.e., ϵ, upon a restriction to the identity component of $M \cap G'$, is trivial. Here G' is the commutator group of G. We remark that when G is split (and semisimple) then all representations of M are fine. (See the comment after ([15], Definition 4.3.8). In this case M is a finite abelian group.)

Let $\epsilon \in \hat{M}$ be fine. Following ([15], Definition 4.3.15), we let $A(\epsilon)$ is the set of K–types δ such that δ is fine ([15], Definition 4.3.9) and ϵ occurs in $\delta|_M$. Applying ([15], Theorem 4.3.16), we obtain that $A(\epsilon)$ is not empty and for $\delta \in A(\epsilon)$, we have the following:

$$\delta|_M = \oplus_{\epsilon' \in \{w(\epsilon);\ w \in W\}} \epsilon', \tag{10}$$

where $W = N_K(A)/M$ is the Weyl group of A in G. Since the restriction map implies $\mathrm{Ind}_{MAN}^G(\epsilon \otimes \exp \nu(\ )) \simeq \mathrm{Ind}_M^K(\epsilon)$ as K–representations, Frobenius reciprocity and (10) imply that for every $\nu \in \mathfrak{a}^*$ there exists a unique irreducible subquotient $J_{\epsilon \otimes \nu}(\delta)$ of $\mathrm{Ind}_{MAN}^G(\epsilon \otimes \exp \nu(\ ))$ containing K–type δ.

One important example is the case $\epsilon = \mathbf{1}_M$. Then $\mu = \mathbf{1}_K \in A(\mathbf{1}_M)$, and $J_{\epsilon \otimes \nu}(\delta)$ is the unique K–spherical irreducible subquotient of $\mathrm{Ind}_{MAN}^G(\epsilon \otimes \exp \nu(\ ))$.

Theorem 5.4. *Assume that $K \cap \Gamma$ is trivial, and G is quasi–split. Let $\epsilon \in \hat{M}$ be fine. Then, for every $\delta \in A(\epsilon)$, there exists infinitely many $\nu \in \mathfrak{a}^*$ such that $J_{\epsilon \otimes \nu}(\delta)$ is an irreducible subrepresentation of $L^2(\Gamma \backslash G)$.*

Proof. We remark that the principal series

$$\mathrm{Ind}_{MAN}^G(\epsilon \otimes \exp \nu(\ )) \text{ and } \mathrm{Ind}_{MAN}^G(w(\epsilon) \otimes \exp w(\nu)(\ ))$$

have the equivalent composition series ([15], Theorem 4.1.4). On the other hand, there are infinitely many irreducible unitary representations of G which contains δ and appear in $L^2(\Gamma \backslash G)$ by Theorem 1.1 (ii). Now, (10) and Harish–Chandra's subquotient theorem ([15], Theorem 4.1.9) imply that all of them must be subquotients of $\mathrm{Ind}_{MAN}^G(\epsilon \otimes \exp \nu(\ ))$ for various $\nu \in \mathfrak{a}^*$. ∎

We finish the paper with an example:

Example 5.5. Let $G = SL_2(\mathbb{R})$. Then K can be identified with the the group $U(1)$ (see Example 3.3). Then $\hat{K} = \mathbb{Z}$ as explained in Example 3.3. Let Γ be a cocompact discrete subgroup of G such that $K \cap \Gamma$ is trivial. The examples of such groups can be constructed out of quaternion algebras over $\mathbb{Q}$ (see [2], page

46, 3.20.4 and references there). Here $M = \{\pm 1\}$ (which is also the center of G). Let sgn be the non–trivial character of M. It is well–known that a non–spherical principal series $\mathrm{Ind}_{MAN}^{G}(sgn \otimes \exp s(\ ))$, $s \in \mathbb{C}$, reduces if and only if $s = 0$. We have the following:

$$J_{sgn\otimes s}(\delta) \simeq \mathrm{Ind}_{MAN}^{G}(sgn \otimes \exp s(\ )), \quad s \in \mathbb{C} - \{0\},$$

for all $\delta \in \mathbb{Z}$, $(-1)^{\delta} = -1$ (i.e., δ is odd). Next, Theorem 5.4 implies that $L^2(\Gamma \setminus G)$ contains $J_{sgn\otimes s}(\delta)$ for infinitely many $s \in \mathbb{C} - \{0\}$. Since all of them must be unitary, we conclude that $s \in \sqrt{-1}\mathbb{R}$ applying a well–known classification of unitary representations of $SL_2(\mathbb{R})$. This means that there exists infinitely many irreducible non–spherical unitary principal series which appear in $L^2(\Gamma \setminus G)$.

References

[1] Borel, A., *Reduction theory for arithmetic groups*, (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math. **IX**, Amer. Math. Soc., Providence, R.I. (1966), 19–24.

[2] — “Automorphic forms on $SL_2(\mathbb{R})$,” Cambridge Tracts in Mathematics Cambridge University Press, 1997.

[3] Borel, A., and H. Jacquet, *Automorphic forms and automorphic representations*, (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math. **XXXIII**, Amer. Math. Soc., Providence, R.I. (1979), 189–202.

[4] DeGeorge, D. L., and N. Wallach, *Limit formulas for multiplicities in* $L^2(\Gamma \setminus G)$, Annals of Math. **107** (1978), 133–150.

[5] Duistermaat, J. J., J. A. C. Kolk, and V. S. Varadarajan, *Spectra of compact locally symmetric manifolds of negative curvature*, Inventiones math. **52** (1979), 27–93.

[6] Godement, R., *The spectral decomposition of cusp forms*, (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 3, Proc. Sympos. Pure Math. **IX**, Amer. Math. Soc., Providence, R.I. (1966), 225–234.

[7] Harish–Chandra, *Discrete series for semisimple Lie groups II,* Acta Math. **116** (1966), 1-111.

[8] Hotta, R., and R. Parthasarathy, *Multiplicity formulae for discrete series,* Invent. Math. **26** (1974), 133–178.

[9] Knapp, A. W., “Representation theory of semisimple groups. An overview based on examples,” Princeton University Press **36**, 1986, Princeton, NJ.

[10] Knapp, A. W., and G. J. Zuckerman *Classification of irreducible tempered representations of semisimple groups,* Ann. of Math. **116** (1982), 389–455.

[11] Muić, G., *On a Construction of Certain Classes of Cuspidal Automorphic Forms via Poincaré Series,* Math. Ann. **343** (2009), 207-227.

[12] —, *Spectral Decomposition of Compactly Supported Poincaré Series and Existence of Cusp Forms,* Preprint: `http://web.math.hr/gmuic/vita/bern_poin_adelic.pdf`.

[13] Müller, W., *Weyl's law in the theory of automorphic forms, Survey,* in: Groups and Analysis, The Legacy of Hermann Weyl, Cambridge Univ. Press, to appear.

[14] Vogan, D., *The algebraic structure of the representation of semisimple groups I,* Annals of Math. **109** (1979), 1–60.

[15] —, "Representations of real reductive Lie groups," Progress in Mathematics, **15**, Birkhäuser, Boston, 1981, vii+754pp.

[16] Wallach, N., *On the Selberg trace formula in the case of compact quotient,* Bull. Amer. Math. Soc. **82** (1976), 171–195.

Goran Muić
Department of Mathematics
University of Zagreb
Bijenička 30
10000 Zagreb
Croatia
gmuic@math.hr

Received October 13, 2008
and in final form January 15, 2009