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# The Constants of Cowling and Haagerup

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**Abstract.** In this article we give a simpler proof of the main theorem of M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, Invent. Math. **96** (1989), 507–549, which reads as follows. Let $G$ be a connected real Lie group of real rank 1 with finite centre. If $G$ is locally isomorphic to $\mathrm{SO}_0(1,n)$ or $\mathrm{SU}(1,n)$, then $\Lambda_G = 1$. If $G$ is locally isomorphic to $\mathrm{Sp}(1,n)$, then $\Lambda_G = 2n-1$, while if $G$ is the exceptional rank one group $F_{4(-20)}$, then $\Lambda_G = 21$.
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## Introduction

Let $G$ be a locally compact group, with a left-invariant Haar measure. The space of compactly supported continuous functions is written $C_c(G)$, and the Lebesgue spaces $L^p(G)$ are defined as usual (where $1 \le p \le \infty$). We recall some basic results about the Fourier algebra of $G$.

For any continuous representation $(\pi, \mathcal{H}_\pi)$ of $G$ and any $\xi$ and $\eta$ in $\mathcal{H}_\pi$, the continuous function $\langle \pi(\cdot)\xi, \eta\rangle$ on $G$ is called a *matrix coefficient* of $\pi$. The Fourier–Stieltjes algebra $B(G)$ is defined to be the space of all matrix coefficients of all continuous unitary representations of $G$. Equipped with pointwise operations, and the norm

$$\|u\|_{B(G)} = \min\{\|\xi\|\,\|\eta\| : u = \langle \pi(\cdot)\xi, \eta\rangle\},$$

where the minimum is taken over all such representations of $u$, the space $B(G)$ is a Banach algebra. The von Neumann algebra associated to the left regular representation of $G$ is denoted by $VN(G)$. The predual of $VN(G)$ is identified with the closed ideal of $B(G)$ consisting of the matrix coefficients of the left regular representation of $G$. It is known as the Fourier algebra and is denoted by $A(G)$.

A function $\phi\colon G \to \mathbb{C}$ is called a *multiplier* of $A(G)$ if $\phi \cdot A(G) \subseteq A(G)$. If $\phi$ is a multiplier of $A(G)$, then the linear operator $m_\phi : u \mapsto \phi \cdot u$ is bounded. The space of all multipliers of $A(G)$ is denoted by $MA(G)$. Equipped with the operator norm, $MA(G)$ is a Banach algebra. A *completely bounded multiplier* is an

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element $\phi$ of $MA(G)$ such that $M_\phi$, the transpose of $m_\phi$, is completely bounded on $VN(G)$. We define $\|\phi\|_{M_0A(G)}$ to be the completely bounded operator norm of $M_\phi$; then the space of all completely bounded multipliers with this norm is a Banach algebra, denoted by $M_0A(G)$.

We say that $A(G)$ has an $L$-*completely bounded approximate identity* (for some positive real $L$, necessarily at least 1) if there exists a net $\{u_i\}_{i\in I}$ in $A(G)$ such that $\|u_i\|_{M_0A(G)} \leq L$ for all $i$ in $I$ and $\{u_i\}$ tends to 1 locally uniformly. Denote by $\Lambda_G$ the infimum of all $L$ for which there exists an $L$-completely bounded approximate identity $\{u_i\}$ in $A(G)$, with the convention that $\Lambda_G = \infty$ if $G$ does not admit a $L$-completely bounded approximate identity for any finite $L$. The group $G$ is said to be *weakly amenable* if $\Lambda_G < \infty$. We refer to Eymard [7], De Cannière and Haagerup [6] and Cowling and Haagerup [5] for more details.

Cowling and Haagerup [5] proved that any connected noncompact simple Lie group of real rank one with finite center is weakly amenable and calculated the constant $\Lambda_G$, as described in the abstract. We give a simpler proof of this theorem by using the multiplier theory of the Fourier algebra of a double coset hypergroup developed in Muruganandam [14, 15] and spherical Fourier analysis.

## 1. Notation and Preliminaries

Denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$. For a locally compact group $G$ with a compact subgroup $K$, denote by $G//K$ the space of all double cosets of $K$ in $G$. For any function space $\mathcal{F}$ on $G$, denote by $\mathcal{F}^\natural$ the set of all $K$-biinvariant functions in $\mathcal{F}$. Whenever possible, we identify $\mathcal{F}^\natural$ with the corresponding function space on $G//K$. For example $C_c(G)^\natural$ is identified with $C_c(G//K)$.

The rest of this section is devoted to preliminaries on connected noncompact semisimple Lie groups. We adhere to the notation and conventions of Cowling and Haagerup [5] as far as possible.

Throughout this article, unless specified otherwise, $G$ denotes a connected noncompact simple Lie group of real rank 1 with finite center, not locally isomorphic to $\mathrm{SO}_0(1, n)$. Let $K$ be a maximal compact subgroup of $G$ and $\theta$ be the corresponding Cartan involution of $G$; extend $\theta$ to the Lie algebra in the usual way. Let $KAN$ and $K\bar{A}_+K$ be the Iwasawa and Cartan decompositions of $G$. Denote by $S$ the maximal solvable subgroup $AN$ of $G$, by $M$ the centralizer of $A$ in $K$, and by $\bar{N}$ the subgroup $\theta(N)$.

Write $\mathfrak{g}$, $\mathfrak{a}$ and $\mathfrak{n}$ for the Lie algebras of $G$, $A$ and $N$, and $\bar{\mathfrak{n}}$ for $\theta(\mathfrak{n})$. Fix an order on $\mathfrak{a}$, and denote by $\alpha$ the indivisible positive root and by $2p$ and $q$ the dimensions of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$. Write $r$ for $p + q$. Fix $H_\alpha$ in $\mathfrak{a}$ such that $\alpha(H_\alpha) = 1$. Then $r = (1/2)\,\mathrm{tr}(\mathrm{ad}\, H_\alpha|_{\mathfrak{n}})$. For $t$ in $[0, \infty)$, denote by $a_t$ the element $\exp(\log(t)H_\alpha/2)$ of $A$. Then

$$a_s \exp(X + Y) a_s^{-1} = \exp(s^{1/2}X + sY), \qquad (1)$$

for all $s$ in $\mathbb{R}^+$, $X$ in $\mathfrak{g}_\alpha$ and $Y$ in $\mathfrak{g}_{2\alpha}$. We normalize the Haar measure on $K$ so that the total mass of $K$ is 1; then there is a function $\delta$ on $A$ such that

$$\int_G f(g)\, dg = \int_K \int_{A_+} \int_K f(ka_tk')\, \delta(t)\, dk\, dt\, dk',$$

where $dk$ and $dg$ are elements of Haar measure on $K$ and $G$. We may and shall assume that the Haar measure on $G$ is normalized so that $\lim_{t\to\infty} e^{-2rt}\delta(t) = 1$.

The Killing form on $\mathfrak{g}$ is written $B$; we recall that

$$B(H_\alpha, H_\alpha) = \operatorname{tr}(\operatorname{ad} H_\alpha)^2 = 2(2p+4q).$$

We equip $\mathfrak{g}$ with the inner product

$$\langle X, Y\rangle = \frac{-B(X,\theta(Y))}{B(H_\alpha, H_\alpha)} = \frac{-B(X,\theta(Y))}{2(2p+4q)}, \tag{2}$$

so that the length of the vector $H_\alpha$ is equal to $1$. For any unexplained terms in the text, we refer to Cowling and Haagerup [5] and Faraut [9].

**Proposition 1.1.** *Take $X$ in $\mathfrak{g}_{-\alpha}$ and $Y$ in $\mathfrak{g}_{-2\alpha}$. Then there exist $k'$ and $k''$ in $K$, and unique $k$ in $K$, $n$ in $N$, $s$ in $\mathbb{R}$ and $t$ in $[0,\infty)$, such that*

$$\exp(X+Y) = k\exp(sH_\alpha)n = k'\exp(tH_\alpha)k'',$$
$$2\cosh(2t) = 1 + |X|^2 + \big(1 + \frac{|X|^2}{2}\big)^2 + 2\,|Y|^2,$$
$$e^s = \Big(\big(1+\frac{|X|^2}{2}\big)^2 + 2\,|Y|^2\Big)^{1/2}.$$

*Therefore, the restriction of $u$ in $C_c^\infty(G//K)$ to $\bar{N}$ is of the form*

$$\exp(X+Y) \mapsto f(2\,|X|^2 + \frac{1}{4}\,|X|^4 + 2\,|Y|^2),$$

*for some function $f$ in $C_c^\infty(\mathbb{R})$.*

**Proof.** This is a restatement of Helgason [10, Theorem IX.3.8], with the modified inner product in (2). ∎

We briefly recall some results about the class one or spherical principal series of representations of $G$.

For $\lambda$ in $\mathbb{C}$, define the character $\chi_\lambda$ on $MAN$ by

$$\chi_\lambda(m\exp(sH_\alpha)n) = e^{(\lambda+r)s} \qquad \forall m\in M \quad \forall s\in\mathbb{R} \quad \forall n\in N.$$

Denote by $\mathcal{H}^\lambda$ the completion of the space of all continuous functions $\xi$ on $G$ such that

$$\xi(xp) = \chi_\lambda(p^{-1})\,\xi(x) \qquad \forall x\in G \quad \forall p\in MAN,$$

where $\|\xi\|$ is defined by $(\int_K |\xi(k)|^2\,dk)^{1/2}$. The left translation representation of $G$ on $\mathcal{H}^\lambda$ is written $\pi_\lambda$.

The $K$-fixed unit vector $\xi_\lambda$ in $\mathcal{H}^\lambda$ is given by

$$\xi_\lambda(kp) = \chi_\lambda(p^{-1}) \qquad \forall k\in K \quad \forall p\in MAN.$$

If $X$ is in $\mathfrak{g}_{-\alpha}$ and $Y$ is in $\mathfrak{g}_{-2\alpha}$, then by Proposition 1.1,

$$\xi_\lambda(\exp(X+Y)) = \Big(\big(1+\frac{|X|^2}{2}\big)^2 + 2\,|Y|^2\Big)^{-(\lambda+r)/2}.$$

The spherical function $\phi_\lambda$ associated to $\pi_\lambda$ is given by

$$\phi_\lambda = \langle \pi_\lambda(\cdot)\xi_\lambda, \xi_{-\bar\lambda}\rangle.$$

The functions $\phi_\lambda$ are $K$-biinvariant, and $\phi_{-r} = \phi_r = 1$. Further, $\|\phi_\lambda\|_\infty = 1$ for every $\lambda$ in $\mathbb{C}$ such that $|\mathrm{Re}(\lambda)| \le r$. The functions $\phi_\lambda$ converge to 1 uniformly on compacta as $\lambda \to r^-$. The function $\phi_\lambda$ is positive definite if and only if $\lambda$ is in $\imath\mathbb{R} \cup [-\tau, \tau] \cup \{r\}$, where $\tau = \min\{p+1, r\}$.

We normalize the Haar measure on $\bar N$ by

$$\int_{\bar N} f(n)\, dn = \frac{1}{K_{p,q}} \int_{\mathfrak{g}_{-\alpha}} \int_{\mathfrak{g}_{-2\alpha}} f(v,z)\, dz\, dv, \tag{3}$$

where

$$K_{p,q} = \frac{\pi^{(2p+q+1)/2}}{\Gamma((2p+q+1)/2)\, 2^{(2p+3q-2)/2}}\,, \tag{4}$$

so that $\int_{\bar N} \xi_r(n)\, dn = 1$. (See Proposition 2.1 below for more on this). Then

$$\int_K \xi(k)\, \overline{\eta}(k)\, dk = \int_{\bar N} \xi(n)\, \overline{\eta}(n)\, dn \qquad \forall \xi \in \mathcal{H}^\lambda \quad \forall \eta \in \mathcal{H}^{-\bar\lambda},$$

for all $\lambda \in \mathbb{C}$, by Helgason [11, Theorem I.5.20]. Thus

$$\phi_\lambda(a\bar n) = \chi_\lambda(a) \int_{\bar N} \xi_\lambda|_{\bar N}(\bar n^{-1} a^{-1} \bar n' a)\, \xi_{-\lambda}|_{\bar N}(\bar n')\, d\bar n'.$$

It is notationally more convenient to work with $N$ rather than $\bar N$. Define $u_\lambda \colon N \to \mathbb{C}$ by

$$u_\lambda(\exp(X+Y)) = \Big(\big(1 + \frac{|X|^2}{2}\big)^2 + 2\,|Y|^2\Big)^{-(\lambda+r)/2} \tag{5}$$

for all $X$ in $\mathfrak{g}_\alpha$ and $Y$ in $\mathfrak{g}_{2\alpha}$.

The following theorem is a consequence of the previous proposition, and the interchange of $\alpha$ and $-\alpha$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, and $N$ and $\bar N$.

**Theorem 1.2.** *Normalize the Haar measure on $N$ as in (3). Then the spherical function $\phi_\lambda$, restricted to the group $S$, is given by*

$$\phi_\lambda(a_s n) = s^{-(\lambda+r)/2} \int_N u_\lambda(n^{-1} a_s^{-1} n' a_s)\, u_{-\lambda}(n')\, dn'. \tag{6}$$

*Further, for any function $u$ in $C_c^\infty(G//K)$, there exists $f$ in $C_c^\infty(\mathbb{R})$ such that*

$$u(\exp(X+Y)) = f(2\,|X|^2 + \frac{1}{4}\,|X|^4 + 2\,|Y|^2), \tag{7}$$

*for all $X$ in $\mathfrak{g}_\alpha$ and $Y$ in $\mathfrak{g}_{2\alpha}$.*

In order to compute the constant $\Lambda_G$, we need to estimate $\|\phi_\lambda\|_{M_0A(G)}$. By Cowling and Haagerup [5, Proposition 1.6],

$$\|\phi_\lambda\|_{M_0A(G)} = \|\phi_\lambda|_S\|_{B(S)}\,,$$

as $\phi_\lambda$ is $K$-biinvariant. To compute $\|\phi_\lambda|_S\|_{B(S)}$, we proceed as follows. Using the fact that $u_\lambda$ is $M$-biinvariant we apply the Plancherel–Parseval formula to the Gelfand pair of Korányi to calculate the right hand side of the equation (6). This involves finding an expression for the spherical Fourier transform of $u_\lambda$ in terms of Whittaker functions. Then we split $\phi_\lambda|_S$ into a sum of matrix coefficients $\{\phi_{\lambda,n}\}_{n=0}^\infty$ belonging to the representations of the double coset hypergroup $MS//M$. We complete the job by estimating the norms of $\phi_{\lambda,n}$.

## 2. The Gelfand pair of Korányi

For convenience, we denote $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ by $\mathfrak{v}$ and $\mathfrak{z}$ respectively, so that $\mathfrak{n} = \mathfrak{v} + \mathfrak{z}$, and an arbitrary element of $\mathfrak{n}$ is denoted by $(v, z)$, where $v$ is in $\mathfrak{v}$ and $z$ is in $\mathfrak{z}$. Moreover, for $(v, z)$ in $\mathfrak{n}$, the corresponding group element $\exp(v, z)$ is also denoted by $(v, z)$. In particular $u_\lambda$ in (5) is of the form

$$u_\lambda(v, z) = \Big( \big(1 + \frac{|v|^2}{2}\big)^2 + 2\,|z|^2 \Big)^{-(\lambda+r)/2}. \tag{8}$$

**Proposition 2.1.** *Suppose that* $\mathrm{Re}(\lambda) = \sigma$ *and that* $\sigma > 0$. *Then*

$$\int_{\mathfrak{v}} \int_{\mathfrak{z}} |u_\lambda(v, z)| \; dz\, dv = \frac{\pi^{(2p+q+1)/2}\, \Gamma(\sigma)}{\Gamma((\sigma + p + 1)/2)\, \Gamma((\sigma + r)/2)\, 2^{(2\sigma+q-2)/2}}\,.$$

*In particular,*

$$\int_{\mathfrak{v}} \int_{\mathfrak{z}} u_r(v, z)\, dz\, dv = \frac{\pi^{(2p+q+1)/2}}{\Gamma((2p + q + 1)/2)\, 2^{(2p+3q-2)/2}}\,.$$

**Proof.** The proof is elementary and follows as in [5, Lemma 3.1]. ∎

Observe that $MN$ forms a semidirect product with $M$ acting on $N$ by inner automorphisms, as the compact group $M$ normalizes $N$. Korányi [13] proved that $(MN, N)$ is a Gelfand pair. We list some of the useful properties of this pair.

When $q > 1$, the $M$-biinvariant functions on $MN$ may be identified with their restrictions to $N$ that are biradial, in the sense that, for example, $C_c(MN//M)$ is identified with the space of all functions $u$ in $C_c(N)$ for which there exists a function $f$ in $C_c(\mathbb{R}^2)$ such that $u(v, z) = f(|v|, |z|)$ for all $(v, z)$ in $N$. When $q = 1$, that is, when $G$ is $\mathrm{SU}(1, n)$, the class of $M$-biinvariant functions is larger. In fact, $u$ is $M$-biinvariant if and only if $u(v, t) = f(|v|, t)$. But this does not affect our considerations.

The bounded spherical functions of this Gelfand pair are of two types:

(a) $\varphi_\mu$, where $\mu \geq 0$, is defined by

$$\varphi_\mu(v, z) = j^{2p}(2^{-1/2}\mu v),$$

where $j^{2p}$ is the "Bessel-like function" $v \mapsto \int_{S_{\mathfrak{v}}} e^{i\sigma\cdot v}\, d\sigma$. Here $S_{\mathfrak{v}}$ is the unit sphere in $\mathfrak{v}$, and $d\sigma$ denotes the normalized surface measure.

(b) $\psi_{\nu,n}$, where $\nu > 0$ and $n \in \mathbb{N}$, is defined by

$$\psi_{\nu,n}(v, z) = c_{\nu,n}\, j^q(2^{-1/2}\nu z) \exp(-\frac{\nu}{4}\,|v|^2)\, L_n^{p-1}(\frac{\nu}{2}\,|v|^2), \tag{9}$$

where $L_n^{p-1}$ denotes the $n$th Laguerre polynomial of order $p - 1$, and $c_{\nu,n}$ is the normalizing constant such that $\psi_{\nu,n}(e) = 1$. In fact,

$$c_{\nu,n} = \frac{1}{L_{p-1}^n(0)} = \frac{\Gamma(p)\, n!}{\Gamma(n + p)}\,.$$

See Cowling [4, appendix] for a quick tutorial on these special functions.

The above are eigenfunctions of the left invariant differential operators $\Delta_1$ and $\Delta_2$ on $N$ which are explicitly given in Faraut [9] (see also Cowling [3]). We define $\Delta_2$ only:

$$\Delta_2 = \sum_{j=1}^{q} \frac{\partial^2}{\partial z_j^2}\,. \tag{10}$$

Then

$$\Delta_2 \psi_{\nu,n} = -\frac{\nu^2}{2}\, \psi_{\nu,n}\,, \tag{11}$$

and, if $u(v,z) = f(|v|^2, |z|^2)$ for some function $f$ in $C_c^\infty(\mathbb{R}^2)$, then

$$\Delta_2 u(v,z) = 4t\, \frac{\partial^2 f(|v|^2, t)}{\partial t^2} + 2q\, \frac{\partial f(|v|^2, t)}{\partial t}\,, \tag{12}$$

where $t = |z|^2$. See [9, Section I, paragraph 4 and Section II, paragraph 1] for further details. In particular, for $u$ and $f$ as in (7), and for all $k > 0$,

$$\Delta_2^k u(v,0) = \frac{2^{3k}\,\Gamma(\frac{q}{2}+k)}{\Gamma(\frac{q}{2})}\, f^{(k)}(2\,|v|^2 + \frac{1}{4}\,|v|^4). \tag{13}$$

We now write down the Plancherel–Godement formula for this Gelfand pair explicitly. For the general theory of Gelfand pairs we refer to [8]. The set of spherical functions of type (a) is a set of Plancherel–Godement measure zero.

In the next theorem, $\Omega_\ell$ denotes the surface measure of the unit sphere $S^{\ell-1}$ in $\mathbb{R}^\ell$, and $K_{p,q}$ is as in (4). For $f \in C_c(MN//M)$, the spherical Fourier transform is defined by

$$\widehat{f}(\nu,n) = \int_N f(n)\, \psi_{\nu,n}(n)\, dn. \tag{14}$$

**Theorem 2.2.** *For every $f$ in $C_c(MN//M)$,*

$$\int_N |f(n)|^2\, dn = \frac{\Omega_{2p}\,\Omega_q\,K_{p,q}}{\pi^{2p+q}\,2^{(2p+3q+2)/2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+p)}{n!} \int_0^\infty \left|\widehat{f}(\nu,n)\right|^2 \nu^{r-1}\, d\nu$$
$$= \frac{2^{2-2r-q}\,\sqrt{\pi}}{\Gamma(p)\,\Gamma(\frac{q}{2})\,\Gamma\big(\frac{r+p+1}{2}\big)} \sum_{n=0}^{\infty} \frac{\Gamma(n+p)}{n!} \int_0^\infty \left|\widehat{f}(\nu,n)\right|^2 \nu^{r-1}\, d\nu.$$

**Proof.** Korányi [13] sketches the proof. It may also be found in Cowling [4, Theorem 5.2.1], with different normalizations. ∎

Denote by $d\pi(\nu, n)$ the element of the Plancherel–Godement measure appearing above. The corresponding unitary operator is called the Plancherel–Godement transformation.

**Theorem 2.3.** *If $u_\lambda$ is as in (8) and $\mathrm{Re}(\lambda) > -(p+1)$, then*

$$\widehat{u}_\lambda(\nu,n) = \frac{2^r\,\Gamma\big(\frac{2p+q+1}{2}\big)}{\Gamma(\frac{\lambda+r}{2})\,\Gamma(\beta)} \int_0^\infty \frac{x^{n+\beta-1}\,e^{-2x-\nu/2}}{(x+\nu/2)^{n+p-\beta+1}}\, dx,$$

*where $\beta = (\lambda+p+1)/2$.*

**Proof.** We first show that the Bessel transform of $u_\lambda$ is equal to

$$\begin{aligned}&\int_{\mathfrak{z}} u_\lambda(v,z)\, j^q(2^{-1/2}\nu z)\, dz \\ &= \frac{\pi^{(q+1)/2}\, 2^{(2-q)/2}}{\Gamma(\beta)\,\Gamma(\frac{\lambda+r}{2})} \int_0^\infty \big(x(x+\nu/2)\big)^{\beta-1} \exp\!\big(-\big(1+\frac{|v|^2}{2}\big)(2x+\nu/2)\big)\, dx.\end{aligned} \tag{15}$$

Let $\omega$ be a unit vector in $\mathfrak{z}$. Then

$$\begin{aligned}&\int_{\omega^\perp} u_\lambda(v, t\omega+z)\, dz \\ &= \int_{\omega^\perp} \Big(\big(1+\frac{|v|^2}{2}\big)^2 + 2(t^2+|z|^2)\Big)^{-(\lambda+r)/2} dz \\ &= \Big(\big(1+\frac{|v|^2}{2}\big)^2 + 2t^2\Big)^{(q-1-\lambda-r)/2} \int_{\omega^\perp} (1+|z'|^2)^{-(\lambda+r)/2}\, dz' \\ &= \frac{\pi^{(q-1)/2}\,\Gamma(\beta)}{\Gamma(\frac{\lambda+r}{2})\, 2^{(q-1)/2}} \Big(\big(1+\frac{|v|^2}{2}\big)^2 + 2t^2\Big)^{-\beta},\end{aligned}$$

where, again, $\beta = (\lambda+p+1)/2$. Since $u_\lambda$ is biradial,

$$\begin{aligned}&\int_{\mathfrak{z}} u_\lambda(v,z)\, j^q(2^{-1/2}\nu z)\, dz \\ &= \int_{\mathfrak{z}} u_\lambda(v,z) \exp\big(-i2^{-1/2}\nu\langle z,\omega\rangle\big)\, dz \\ &= \frac{\pi^{(q-1)/2}\,\Gamma(\beta)}{\Gamma(\frac{\lambda+r}{2})\, 2^{(q-1)/2}} \int_{\mathbb{R}} \Big(\big(1+\frac{|v|^2}{2}\big)^2 + 2t^2\Big)^{-\beta} \exp\big(-i2^{-1/2}\nu t\big)\, dt.\end{aligned} \tag{16}$$

We now show that

$$\begin{aligned}&\int_{\mathbb{R}} \Big(\big(1+\frac{|v|^2}{2}\big)^2 + 2t^2\Big)^{-\beta} \exp\big(-i2^{-1/2}\nu t\big)\, dt \\ &= \frac{\sqrt{2}\pi}{\Gamma(\beta)^2} \int_0^\infty \big(x(x+\nu/2)\big)^{\beta-1} \exp\big(-(2x+\nu/2)\big) \exp\big(-\frac{|v|^2}{2}(2x+\nu/2)\big)\, dx.\end{aligned}$$

As both expressions are analytic in $\{\lambda\in\mathbb{C} : \mathrm{Re}(\lambda) > -(p+1)\}$, it is sufficient to prove the equality when $\lambda$ is real. We therefore assume that $\lambda$ is real.

If $F$ and $G$ are the Laplace transforms of (suitable) functions $f$ and $g$, then

$$\int_{\mathbb{R}} F(a+ib)\, \overline{G}(a+ib)\, e^{-i\nu b}\, db = 2\pi \int_0^\infty f(x)\, \overline{g}(x+\nu)\, e^{-a(2x+\nu)}\, dx.$$

See for instance, [5, equation (3.8)]. We apply this result where

$$f(x) = g(x) = \begin{cases} \dfrac{x^{\beta-1}\, e^{-x}}{\Gamma(\beta)} & \text{if } x>0 \\ 0 & \text{if } x \le 0. \end{cases}$$

Observe that $F(a+ib) = G(a+ib) = (1+a+ib)^{-\beta}$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}} \Big( \big(1 + \frac{|v|^2}{2}\big)^2 + 2t^2 \Big)^{-\beta} \exp\big(-i2^{-1/2}\nu t\big)\, dt \\
&= \int_{\mathbb{R}} F\Big(\frac{|v|^2}{2} + i\sqrt{2}t\Big)\, \overline{G}\Big(\frac{|v|^2}{2} + i\sqrt{2}t\Big) \exp\big(-i2^{-1/2}\nu t\big)\, dt \\
&= \frac{\sqrt{2}\pi}{\Gamma(\beta)^2} \int_0^\infty \big(x(x+\nu/2)\big)^{\beta-1} \exp\big(-(2x+\nu/2)\big) \exp\Big(-\frac{|v|^2}{2}(2x+\nu/2)\Big)\, dx.
\end{aligned}$$

We combine this equation and (16) to prove (15).

Finally, from the definition of the Laguerre polynomials,

$$\begin{aligned}
&\int_{\mathfrak{v}} \exp\Big(-\frac{|v|^2}{2}(2x+\nu/2)\Big) \exp\Big(-\frac{\nu}{4}|v|^2\Big) L_n^{p-1}\Big(\frac{\nu}{2}|v|^2\Big)\, dv \\
&= \Omega_{2p} \int_0^\infty \exp\Big(\frac{-s^2}{2}(2x+\nu)\Big) L_n^{p-1}\Big(\frac{\nu}{2}s^2\Big) s^{2p-1}\, ds \\
&= \frac{\Omega_{2p}\, 2^{p-1}}{(2x+\nu)^p} \sum_{m=0}^{n} \frac{(-1)^m\, \Gamma(n+p)}{\Gamma(m+p)(n-m)!m!} \int_0^\infty \Big(\frac{\nu s}{2x+\nu}\Big)^m e^{-s} s^{p-1}\, ds \\
&= \frac{\Omega_{2p}\, \Gamma(n+p)\, x^n}{2\, n!\, (x+\nu/2)^{n+p}}.
\end{aligned} \tag{17}$$

Therefore, by (14) and the definition of $\psi_{\nu,n}$ in (9),

$$\begin{aligned}
\widehat{u}_\lambda(\nu, n) &= \frac{c_{\nu,n}}{K_{p,q}} \int_{\mathfrak{v}} \int_{\mathfrak{z}} u_\lambda(v,z)\, j^q(2^{-1/2}\nu z) \exp\big(-\frac{\nu}{4}|v|^2\big) L_n^{p-1}\big(\frac{\nu}{2}|v|^2\big)\, dz\, dv \\
&= \frac{2^r\, \Gamma\Big(\frac{2p+q+1}{2}\Big)}{\Gamma(\frac{\lambda+r}{2})\, \Gamma(\beta)} \int_0^\infty \frac{x^{n+\beta-1}\, e^{-2x-\nu/2}}{(x+\nu/2)^{n+p-\beta+1}}\, dx,
\end{aligned}$$

where $\beta = (\lambda+p+1)/2$, by (15), (17) and (4). ∎

The following expression of the spherical Fourier transform of $u_\lambda$ in terms of the Whittaker function will be useful later. See Section 16.12 of Whittaker and Watson [17], for the definition and more details of Whittaker functions.

**Corollary 2.4.** *For all $\lambda$ in $\mathbb{C}$,*

$$\widehat{u}_\lambda(\nu, n) = \frac{2^{r-\lambda}\, \Gamma(\frac{2p+q+1}{2})\, \Gamma(n+\beta)}{\Gamma(\frac{\lambda+r}{2})\, \Gamma(\beta)}\, \nu^{(\lambda-1)/2}\, W_{-n-p/2,\,\lambda/2}(\nu),$$

*where $W_{-n-p/2,\,\lambda/2}$ denotes the Whittaker function.*

**Proof.** When $\mathrm{Re}(\lambda) > -(p+1)$, the corollary holds by the definition of the Whittaker functions and the theorem. Since the function $\lambda \mapsto \widehat{u}_\lambda$ extends analytically to $\mathbb{C}$, the above identity holds for other values of $\lambda$ also. ∎

**Corollary 2.5.** *For all $\nu$ in $\mathbb{R}^+$, all $n$ in $\mathbb{N}$, and all $\lambda$ in $\mathbb{C}$,*

$$\nu^{\lambda/2}\,\widehat{u}_{-\lambda}(\nu,n) = \frac{2^{2\lambda}\,\Gamma(\frac{\lambda+r}{2})\,\Gamma(\frac{2n-\lambda+p+1}{2})\,\Gamma(\frac{\lambda+p+1}{2})}{\Gamma(\frac{-\lambda+r}{2})\,\Gamma(\frac{2n+\lambda+p+1}{2})\,\Gamma(\frac{-\lambda+p+1}{2})}\,\nu^{-\lambda/2}\,\widehat{u}_{\lambda}(\nu,n). \tag{18}$$

**Proof.** For all $k$, $m$, and $x$,

$$W_{k,\,m}(x) = W_{k,\,-m}(x),$$

by Whittaker and Watson [17, Section 16.4, Equation (C)]. The result follows immediately. ■

**Theorem 2.6.** *Suppose that $\lambda = \sigma + \imath\gamma$, where $0 \le \sigma < r$. Then there exists a constant $C(p,q)$, depending only on $p$ and $q$, such that for all $n$ in $\mathbb{N}$,*

$$\int_0^\infty \left|\nu^{-\lambda/2}\,\widehat{u}_\lambda(\nu,n)\right|^2\,\nu^{r-1}\,d\nu \le C(p,q)\,e^{2|\gamma|}\,(r-\sigma)^{-1},$$

*Further,*

$$\limsup_{\sigma\to r^-}\,(r-\sigma)\int_0^\infty \left|\nu^{-\sigma/2}\,\widehat{u}_\sigma(\nu,n)\right|^2\,\nu^{r-1}\,d\nu \le 1.$$

**Proof.** We first observe that

$$\begin{aligned}\left|\int_0^\infty \frac{x^{(2n+\lambda+p-1)/2}\,e^{-2x-\nu/2}}{(x+\nu/2)^{(2n+p-\lambda+1)/2}}\,dx\right| &\le e^{-\nu/2}\int_0^\infty e^{-2x}\,x^{(\sigma-1)/2}(x+\nu/2)^{(\sigma-1)/2}\,dx\\ &= 2^{-\sigma}\,\nu^{(\sigma-1)/2}\,\Gamma\Big(\frac{\sigma+1}{2}\Big)\,W_{0,\,\sigma/2}(\nu).\end{aligned}$$

Therefore,

$$\left|\nu^{-\lambda/2}\,\widehat{u}_\lambda(\nu,n)\right| \le |Q(\lambda)|\;\nu^{-1/2}\,W_{0,\,\sigma/2}(\nu), \tag{19}$$

where

$$Q(\lambda) = \frac{2^{r-\lambda}\,\Gamma(\frac{2p+q+1}{2})\,\Gamma(\frac{\lambda+1}{2})}{\Gamma(\frac{\lambda+2p+q+1}{2})\,\Gamma(\frac{\lambda+p+1}{2})}.$$

From the asymptotic behaviour of the $\Gamma$-function,

$$|Q(\lambda)| \le C_1(p,q)\,e^{2|\gamma|};$$

see for example, Titchmarsh [16, Section 4.4.2].

For all $\nu > 1$, the Whittaker function satisfies

$$W_{0,\,\sigma/2}(\nu) = e^{-\nu/2}\bigg(1 + \frac{(\sigma/2)^2-(1/2)^2}{\nu} + O\Big(\int_0^\infty x^{(\sigma+3)/2}\,(1+x)^{|\sigma-1|}\,\nu^{-2}e^{-x}\,dx\Big)\bigg)$$

(see Whittaker and Watson [17, Section 16.3]). From the last two equations and (19),

$$\int_1^\infty \left|\nu^{-\lambda/2}\,\widehat{u}_\lambda(\nu,n)\right|^2\,\nu^{r-1}\,d\nu \le C_2(p,q)\,e^{2|\gamma|}. \tag{20}$$

By Proposition 2.1,

$$\int_0^1 \left|\nu^{-\lambda/2}\, \widehat{u}_\lambda(\nu,n)\right|^2 \,\nu^{r-1}\, d\nu \le \|u_\lambda\|_1^2 \int_0^1 \left|\nu^{-\lambda/2}\right|^2 \,\nu^{r-1}\, d\nu = \frac{\|u_\sigma\|_1^2}{(r-\sigma)}\,. \tag{21}$$

Finally, from equations (20) and (21),

$$\begin{aligned}\int_0^\infty \left|\nu^{-\lambda/2}\, \widehat{u}_\lambda(\nu,n)\right|^2 \,\nu^{r-1}\, d\nu &\le \frac{\|u_\sigma\|_1^2}{(r-\sigma)} + C_2(p,q)\, e^{2|\gamma|}\\ &\le C_3(p,q)\,(r-\sigma)^{-1}\, e^{2|\gamma|}.\end{aligned}$$

To conclude, observe that equations (20) and (21) also imply that

$$\limsup_{\sigma\to r^-} (r-\sigma) \int_0^\infty \left|\nu^{-\sigma/2}\, \widehat{u}_\sigma(\nu,n)\right|^2 \,\nu^{r-1}\, d\nu \le \limsup_{\sigma\to r^-} \|u_\sigma\|_1^2 = \|u_r\|_1^2 = 1,$$

as required. ■

## 3. The double coset hypergroup $MS//M$

Recall that if $a_{s_1}(v_1,z_1)$ and $a_{s_2}(v_2,z_2)$ are in the group $S$ (that is, $AN$), then by (1),

$$\begin{aligned}a_{s_1}(v_1,z_1)\cdot a_{s_2}(v_2,z_2) &= a_{s_1}a_{s_2}\big((a_{s_2})^{-1}(v_1,z_1)a_{s_2}\big)\cdot(v_2,z_2)\\ &= a_{s_1}a_{s_2}(s_2^{-1/2}v_1, s_2^{-1}z_1)\cdot(v_2,z_2).\end{aligned}$$

Since $M$ commutes with $A$ and acts on $N$ by automorphisms, the action of $M$ extends to $S$ and forms another semidirect product $MS$. Recall that the space of double cosets $MS//M$ forms a hypergroup, called a double coset hypergroup. Notice that an arbitrary element $(a_s(v,z))^\diamond$ of $MS//M$ is of the form $a_s(v,z)^\diamond$, which is uniquely determined by $s$, $|v|$, and $|z|$. For the definition and basic properties of hypergroups, see Bloom and Heyer [1] and Jewett [12].

Now we construct a series of representations of the hypergroup $MS//M$ that are weakly contained in the left regular representation of $MS//M$.

Suppose that $G//K$ is a double coset hypergroup, where $K$ is a compact subgroup of a locally compact group $G$. Let $(\rho,\mathcal{H})$ be a unitary representation of $G$, and write $\mathcal{K}$ for $[\rho(L^1(G//K))(\mathcal{H})]^-$. If $\mathcal{K}\ne\{0\}$, then by Muruganandam [15, Remark 3.2]

$$f\mapsto \rho(f)|_{\mathcal{K}} \qquad L^1(G//K)\to BL(\mathcal{K})$$

defines a nondegenerate representation of $L^1(G//K)$ and so a representation of $G//K$. We shall denote this representation by $\tilde\rho$.

In particular, the representation $\tilde\lambda$ arising from the left regular representation $\lambda$ of $G$ is the left regular representation of $G//K$ on $L^2(G//K)$. More precisely, denote the coset $KxK$ in $G//K$ by $x^\diamond$, and define the generalized left translate $f(x^\diamond * y^\diamond)$ of $f$ by $x^\diamond$ to be $\int_K f(xky)\,dk$. Then

$$\tilde\lambda(x^\diamond)f(y^\diamond) = f((x^{-1})^\diamond * y^\diamond) = \int_K f(x^{-1}ky)\,dk \qquad \forall f\in L^2(G//K). \tag{22}$$

If $\rho$ and $\tilde{\rho}$ are as above and $\rho$ is weakly contained in $\lambda$, then $\tilde{\rho}$ is weakly contained in $\tilde{\lambda}$. For $\|\lambda(f)\| = \|\tilde{\lambda}(f)\|$ when $f$ is in $L^1(G//K)$, by Muruganandam [15, Theorem 3.15(1) and Remark 3.14]. Therefore

$$\|\tilde{\rho}(f)\| \leq \|\rho(f)\| \leq \|\lambda(f)\| = \|\tilde{\lambda}(f)\|. \tag{23}$$

Hence $\tilde{\rho}$ is weakly contained in $\tilde{\lambda}$. See Muruganandam [14] for more about representations of hypergroups and weak containment.

**Lemma 3.1.** *For $a_s$ in $A$, define the map $\delta_s$ on $L^2(N)$ by*

$$\delta_s f(v,z) = s^{-r/2} f(a_s^{-1}(v,z)a_s) = s^{-r/2} f(s^{-1/2}v, s^{-1}z).$$

*Then $\delta_s$ is a unitary operator on $L^2(N)$, and $\delta_s^\star = \delta_{s^{-1}}$. Further, $\delta_s$ leaves invariant the space $L^2(MN//N)$ of $M$-biinvariant functions in $L^2(N)$. Moreover, for all $f$ in $L^1 \cap L^2(MN//N)$, all $s$ and $\nu$ in $\mathbb{R}_+$, and all $n$ in $\mathbb{N}$,*

$$(\delta_s f)\widehat{\ }(\nu,n) = s^{r/2} \widehat{f}(s\nu,n).$$

**Proof.** This is elementary. ■

Denote by $\pi$ the representation of $S$ unitarily induced from the trivial character on $A$. That is, $\pi$ is realized on $L^2(N)$ by

$$\pi(a_s n)f(n') = s^{-r/2} f(n^{-1}a_s^{-1}n'a_s) = \delta_s\lambda(n)f(n') \qquad \forall n' \in N. \tag{24}$$

Denote a typical element of the Hilbert space $L^2(\mathbb{R}_+ \times \mathbb{N};\, d\pi(\nu,n))$ by $\widehat{f}$.

**Theorem 3.2.** *The map $\rho\colon MS//M \to BL(L^2(\mathbb{R}_+ \times \mathbb{N};\, d\pi(\nu,n)))$, defined by*

$$\rho((a_s(v,z))^\diamond)\widehat{f}(\nu,n) = s^{r/2}\,\psi_{s\nu,n}(v,z)\,\widehat{f}(s\nu,n)$$

*is a representation of the hypergroup $MS//M$ that is weakly contained in the left regular representation $\tilde{\lambda}$ of $MS//M$.*

*The restriction $\rho_n$ of $\rho$ to each copy of $L^2(\mathbb{R}_+, \nu^{r-1}\,d\nu)$ is a representation of $MS//M$ and is also weakly contained in $\tilde{\lambda}$.*

**Proof.** We extend the representation $\pi$ of $S$ in (24) to the semidirect product $MS$ by taking $\pi(m)$ to be the identity for all $m$ in $M$. We denote by $\pi$ this extended representation, and by $\tilde{\pi}$ the corresponding representation of $MS//M$ on the Hilbert space $[\pi(L^1(MS//M))(L^2(N))]^-$, which is equal to $L^2(N)$.

By Lemma 3.1, we see that $\tilde{\pi}$ leaves $L^2(MN//N)$ invariant. Denote this representation of $MS//M$ on $L^2(MN//N)$ by $\pi_1$. By (22),

$$\pi_1((a_s(v,z))^\diamond)f = \delta_s\big(\tilde{\lambda}((v,z)^\diamond)f\big). \tag{25}$$

Since the group $MS$ is amenable, $\pi$ is weakly contained in $\lambda$ and so by (23) (and the discussion thereof), $\pi_1$ is weakly contained in $\tilde{\lambda}$.

If $\mathcal{F}$ denotes the Plancherel–Godement transformation on the Gelfand pair $(MN, M)$ in Theorem 2.2, then $L^2(MN//N)$ and $L^2(\mathbb{R}_+ \times \mathbb{N};\, d\pi(\nu,n))$ are unitarily equivalent via $\mathcal{F}$. Using (25), it can be easily verified that

$$\rho(a_s(v,z)^\diamond) = \mathcal{F} \circ \pi_1(a_s(v,z)^\diamond) \circ \mathcal{F}^{-1} \qquad \forall\ a_s(v,z)^\diamond \in MS//M, \tag{26}$$

since $(\tilde{\lambda}((v,z)^\diamond)f)\widehat{\ }(\nu,n) = \psi_{\nu,n}(v,z)\,\widehat{f}(\nu,n)$. Thus $\rho$ defines a representation of $MS//M$, which is weakly contained in $\tilde{\lambda}$. The theorem follows. ■

For all $\lambda$ in $\mathbb{C}$ and $n$ in $\mathbb{N}$, define

$$\widehat{v}_{\lambda,n}(\nu) = \nu^{-\lambda/2}\,\widehat{u}_\lambda(\nu, n).$$

By Theorem 2.6, $\widehat{v}_{\lambda,n}$ is in $L^2(\mathbb{R}_+, \nu^{r-1}\,d\nu)$. By (18), $\widehat{v}_{-\lambda,n}$ also belongs to $L^2(\mathbb{R}_+, \nu^{r-1}\,d\nu)$. Thus we may define the matrix coefficient $\phi_{\lambda,n}$:

$$\phi_{\lambda,n} = \langle \rho_n(\cdot)\widehat{v}_{\lambda,n}, \widehat{v}_{-\lambda,n}\rangle. \tag{27}$$

**Theorem 3.3.** *Suppose that $\lambda = \sigma + \imath\gamma$, where $|\sigma| < r$, and that $n \in \mathbb{N}$. Then the matrix coefficient $\phi_{\lambda,n}$ belongs to $B(S)$. Moreover, when $\sigma \in [-R, R]$ and $p < R < r$, and $n \in \mathbb{N}$,*

$$\|\phi_{\lambda,n}\|_{B(S)} \le C(p, q)\,(1 + n)^{-R}\,(r - R)^{-1}\,e^{6|\gamma|}.$$

**Proof.** By the preceding theorem and Theorem 4.3 of Muruganandam [15], we deduce that $\phi_{\lambda,n}$ belongs to $B(S)$ and

$$\|\phi_{\lambda,n}\|_{B(S)} \le \|\widehat{v}_{\lambda,n}\|_2\,\|\widehat{v}_{-\lambda,n}\|_2\,.$$

Now we estimate $\|\widehat{v}_{\lambda,n}\|_2\,\|\widehat{v}_{-\lambda,n}\|_2$. By Corollary 2.5,

$$\begin{aligned}
&\|\widehat{v}_{\lambda,n}\|_2\,\|\widehat{v}_{-\lambda,n}\|_2\\
&= \left|\frac{2^{2\lambda}\,\Gamma(\frac{\lambda+r}{2})\,\Gamma(\frac{2n-\lambda+p+1}{2})\,\Gamma(\frac{\lambda+p+1}{2})}{\Gamma(\frac{-\lambda+r}{2})\,\Gamma(\frac{2n+\lambda+p+1}{2})\,\Gamma(\frac{-\lambda+p+1}{2})}\right| \int_0^\infty \left|\nu^{-\lambda/2}\,\widehat{u}_\lambda(\nu, n)\right|^2\,\nu^{r-1}\,d\nu\\
&\le 2^{2\sigma}\left|\frac{\Gamma(\frac{\lambda+r}{2})}{\Gamma(\frac{-\lambda+r}{2})}\right| \prod_{k=1}^{n}\left|\frac{2k-\lambda+p-1}{2k+\lambda+p-1}\right| C(p, q)\,(r-\sigma)^{-1}\,e^{4|\gamma|}.
\end{aligned}$$

To complete the proof, we follow Cowling and Haagerup [5]. If $\lambda = \sigma + \imath\gamma$, where $\sigma$ is in $[p, r]$, and $n$ is in $\mathbb{N}$, then

$$\prod_{k=1}^{n}\left|\frac{2k-\lambda+p-1}{2k+\lambda+p-1}\right| \le C(p, q)\,(1 + |\gamma|)^{\sigma}\,n^{-\sigma},$$

by [5, equation (4.10)]. Therefore,

$$\|\phi_{\lambda,n}\|_{B(S)} \le C_2(p, q)\,(1 + n)^{-\sigma}\,e^{6|\gamma|}\,(r - \sigma)^{-1}.$$

Similarly, when $-r \le \sigma \le -p$,

$$\|\phi_{\lambda,n}\|_{B(S)} \le C_2(p, q)\,(1 + n)^{-\sigma}\,e^{6|\gamma|}\,(r + \sigma)^{-1}.$$

Define the strip $\mathcal{E}$ to be

$$\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \in (-r, r)\}.$$

Consider the analytic $B(S)$-valued function $\psi\colon \lambda \mapsto \cos(\lambda/r)^{-6\gamma}\,\phi_{\lambda,n}$ on the strip $\mathcal{E}$. This function is bounded in $B(S)$-norm on each closed substrip $\{\lambda \in \mathbb{C} : |\mathrm{Re}(\lambda)| \le R\}$ when $p < R < r$, and satisfies the condition

$$\|\psi_\lambda\|_{B(S)} \le C_3(p, q)\,(1 + n)^{-R}\,(r - R)^{-1}$$

when $\mathrm{Re}(\lambda) = \pm R$. By the three lines theorem for Banach spaces, this estimate also holds inside the strip. Therefore, when $p \le R < r$ and $\mathrm{Re}(\lambda) \in [-R, R]$,

$$\|\phi_{\lambda,n}\|_{B(S)} \le C(p, q)\,(1 + n)^{-R}\,(r - R)^{-1}\,e^{6|\gamma|},$$

as required. ∎

**Theorem 3.4.** *Suppose that $G$ is $\mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$ or $F_{4(-20)}$. If $\mathrm{Re}(\lambda)=0$, then $\phi_\lambda|_S$ belongs to $B(S)$ and $\|\phi_\lambda|_S\|_{B(S)}=1$. The family $\phi_\lambda|_S$ of $B(S)$-valued functions continues analytically into the strip $\mathcal{E}$ and, for $\lambda$ in $\mathcal{E}$,*

$$\|\phi_\lambda|_S\|_{B(S)} \leq C(p,q)\,(r-\sigma)^{-1}\,e^{6|\gamma|}$$

*and*

$$\limsup_{\sigma\to r^-} \|\phi_\sigma|_S\|_{B(S)} \leq \frac{\sqrt{\pi}\,\Gamma(\frac{r}{2})}{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{q}{2})}.$$

**Proof.** When $\mathrm{Re}(\lambda)=0$, the function $\phi_\lambda$ is positive definite, and we see that $\phi_\lambda \in B(G)$ and so $\phi_\lambda|_S \in B(S)$; further, $\|\phi_\lambda|_S\|_{B(S)}=1$.

Define the function $\widehat{v}_\lambda$ on $\mathbb{R}_+\times\mathbb{N}$ by

$$\widehat{v}_\lambda(\nu,n)=\nu^{-\lambda/2}\,\widehat{u}_\lambda(\nu,n).$$

When $\mathrm{Re}(\lambda)=0$, both $\widehat{v}_\lambda$ and $\widehat{v}_{-\lambda}$ are in $L^2(\mathbb{R}_+\times\mathbb{N};\,d\pi(\nu,n))$ as $\|\widehat{v}_\lambda\|_2=\|\widehat{u}_\lambda\|_2$.

Since $\phi_\lambda|_S$ in (6) is biradial, it is $M$-biinvariant. Therefore,

$$\phi_\lambda(a_s(v,z))=\phi_\lambda(a_s(v,z)^\diamond).$$

By (6), (24), (25) and (26),

$$\begin{aligned}\phi_\lambda(a_s(v,z)) &= s^{-(\lambda+r)/2}\int_N u_\lambda(n^{-1}a_s^{-1}n'a_s)u_{-\lambda}(n')\,dn'\\ &= s^{-\lambda/2}\langle \pi_1(a_s(v,z)^\diamond)u_\lambda,u_{-\lambda}\rangle_{L^2(MN//N)}\\ &= s^{-\lambda/2}\langle \rho(a_s(v,z)^\diamond)\widehat{u}_\lambda,\widehat{u}_{-\lambda}\rangle_{L^2(\mathbb{R}_+\times\mathbb{N};\,d\pi(\nu,n))}\\ &= \langle \rho(a_s(v,z)^\diamond)\widehat{v}_\lambda,\widehat{v}_{-\lambda}\rangle_{L^2(\mathbb{R}_+\times\mathbb{N};\,d\pi(\nu,n))}.\end{aligned}$$

By Theorem 2.2 and the definition of $\phi_{\lambda,n}$ in (27),

$$\phi_\lambda=\frac{2^{2-2r-q}\sqrt{\pi}}{\Gamma(p)\,\Gamma(\frac{q}{2})\,\Gamma(\frac{r+p+1}{2})}\sum_{n=0}^{\infty}\frac{\Gamma(n+p)}{n!}\,\phi_{\lambda,n} \tag{28}$$

when $\mathrm{Re}(\lambda)=0$. Both sides of this expression extend analytically when $\lambda$ varies in $\mathcal{E}$. Moreover,

$$\|\phi_\lambda|_S\|_{B(S)}\leq\frac{2^{2-2r-q}\sqrt{\pi}}{\Gamma(p)\,\Gamma(\frac{q}{2})\,\Gamma(\frac{r+p+1}{2})}\sum_{n=0}^{\infty}\frac{\Gamma(n+p)}{n!}\,\|\widehat{v}_{\lambda,n}\|_2\,\|\widehat{v}_{-\lambda,n}\|_2. \tag{29}$$

Since $\Gamma(n+p)\leq n!\,(p+n-1)^{p-1}$,

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+p)}{n!}\,(1+n)^{-R}<\infty,$$

when $p<R<r$. Therefore, by Theorem 3.3 and (29),

$$\|\phi_\lambda|_S\|_{B(S)}\leq C(p,q)\,(r-R)^{-1}\,e^{6|\gamma|},$$

whenever $-R\leq\mathrm{Re}(\lambda)\leq R$, and $R$ is in $(p,r)$. Therefore (28) and (29) hold for all $\lambda$ in the strip $\mathcal{E}$.

Fix $\sigma < r$ satisfying $r-\sigma<1$. Take $\epsilon$ such that $r-\sigma<\epsilon<1$. Then

$$\begin{aligned}&\|\widehat{v}_{\sigma,n}\|_2\,\|\widehat{v}_{-\sigma,n}\|_2\\ &= \left|\frac{2^{2\sigma}\,\Gamma(\frac{\sigma+r}{2})\,\Gamma(\frac{2n-\sigma+p+1}{2})\,\Gamma(\frac{\sigma+p+1}{2})}{\Gamma(\frac{-\sigma+r}{2})\,\Gamma(\frac{2n+\sigma+p+1}{2})\,\Gamma(\frac{-\sigma+p+1}{2})}\right|\int_0^\infty \left|\nu^{-\sigma/2}\,\widehat{u}_\sigma(\nu,n)\right|^2\,\nu^{r-1}\,d\nu,\end{aligned}$$

by Corollary 2.5. By Theorem 2.6,

$$\begin{aligned}\frac{\Gamma(n+p)}{n!}\,\|\widehat{v}_{\sigma,n}\|_2\,\|\widehat{v}_{-\sigma,n}\|_2 &\le C_5(p,q)\,\frac{2^{2\sigma}\,\Gamma(\frac{\sigma+r}{2})\,\Gamma(p)}{\Gamma(\frac{-\sigma+r}{2})(r-\sigma)}\cdot\frac{[p]_n\,\left|[\frac{-\sigma+p+1}{2}]_n\right|}{n!\,[\frac{\sigma+p+1}{2}]_n}\\ &\le C_6(p,q,\sigma)\,\frac{[p]_n\,\left|[\frac{-\sigma+p+1}{2}]_n\right|}{n!\,[\frac{\sigma+p+1}{2}]_n},\end{aligned}$$

where $C_6(p,q,\sigma)$ is a constant depending on $p,q$ and $\sigma$. Here $[a]_n$ denotes the "shifted factorial" (or Pochhammer symbol) $a(a+1)\cdots(a+n-1)$.

Since $|-\sigma+p+1|\le q-1+\epsilon$,

$$\frac{\Gamma(n+p)}{n!}\,\|\widehat{v}_{\sigma,n}\|_2\,\|\widehat{v}_{-\sigma,n}\|_2 \le C_6(p,q,\sigma)\,\frac{[p]_n\,[\frac{q-1+\epsilon}{2}]_n}{n!\,[\frac{r+p+1-\epsilon}{2}]_n}\qquad \forall n\in\mathbb{N}.$$

Now $2p+q-1+\epsilon<r+p+1-\epsilon$ as $\epsilon<1$, and so by [17, equation (14.11)], the sum over $n$ of the fractions on the right hand side is finite and is equal to ${}_2F_1(p,\frac{q-1+\epsilon}{2},\frac{r+p+1-\epsilon}{2},1)$, where ${}_2F_1$ is the Gaussian hypergeometric function.

Thus we see that for every such fixed $\sigma$, the right hand side of (29) is finite. Therefore we can apply the Lebesgue dominated convergence theorem for the variable $\sigma$ in the sum of the equation (29).

$$\begin{aligned}&\limsup_{\sigma\to r^-}\|\phi_\sigma|_S\|_{B(S)}\\ &\le \frac{2^{2-2r-q}\,\sqrt{\pi}}{\Gamma(p)\,\Gamma(\frac{q}{2})\,\Gamma(\frac{r+p+1}{2})}\sum_{n=0}^\infty\frac{\Gamma(n+p)}{n!}\lim_{\sigma\to r^-}\|\widehat{v}_{\sigma,n}\|_2\,\|\widehat{v}_{-\sigma,n}\|_2\,.\end{aligned}\tag{30}$$

Now by the second part of Theorem 2.6,

$$\begin{aligned}\lim_{\sigma\to r^-}\|\widehat{v}_{\sigma,n}\|_2\,\|\widehat{v}_{-\sigma,n}\|_2 &\le \lim_{\sigma\to r^-}\left|\frac{2^{2\sigma}\,\Gamma(\frac{\sigma+r}{2})\,\Gamma(\frac{2n-\sigma+p+1}{2})\,\Gamma(\frac{\sigma+p+1}{2})}{\Gamma(\frac{-\sigma+r}{2})\,\Gamma(\frac{2n+\sigma+p+1}{2})\,\Gamma(\frac{-\sigma+p+1}{2})\,(r-\sigma)}\right|\\ &= \lim_{\sigma\to r^-}\left|\frac{2^{2\sigma}\,\Gamma(\frac{\sigma+r}{2})}{\Gamma(\frac{-\sigma+r}{2})\,(r-\sigma)}\right|\prod_{k=1}^n\left|\frac{2k-\sigma+p-1}{2k+\sigma+p-1}\right|\\ &= 2^{2r-1}\,\Gamma(r)\prod_{k=1}^n\left|\frac{2k-q-1}{2k+r+p-1}\right|\\ &= \frac{2^{2r-1}\,\Gamma(r)\,\Gamma(\frac{r+p+1}{2})\,\Gamma(\frac{q+1}{2})}{\Gamma(\frac{r+p+1+2n}{2})\,\Gamma(\frac{q+1-2n}{2})}\,.\end{aligned}$$

Therefore, by (30)

$$\limsup_{\lambda\to r^-} \|\phi_\sigma|_S\|_{B(S)} \le \frac{\Gamma(r)\,2^{1-q}\sqrt{\pi}}{\Gamma(\frac{q}{2})\,\Gamma(\frac{r+p+1}{2})} \sum_{n=0}^{\infty} \frac{(-1)^n\,[p]_n\,[\frac{1-q}{2}]_n}{n!\,[\frac{r+p+1}{2}]_n}\,. \tag{31}$$

But then

$$\sum_{n=0}^{\infty} \frac{(-1)^n\,[p]_n\,[\frac{1-q}{2}]_n}{n!\,[\frac{r+p+1}{2}]_n} = {}_2F_1(p, \frac{1-q}{2}, \frac{r+p+1}{2}, -1) = \frac{2^{-p}\,\Gamma(\frac{r+p+1}{2})\sqrt{\pi}}{\Gamma(\frac{p+q+1}{2})\,\Gamma(\frac{p+1}{2})}$$

by [2, Section 2.8 (47)]. Hence by (31) and the Legendre Duplication Formula, (see for instance, [17, Section 12.15, Corollary]),

$$\limsup_{\sigma\to r^-} \|\phi_\sigma|_S\|_{B(S)} \le \frac{2^{1-r}\,\Gamma(r)\,\pi}{\Gamma(\frac{r+1}{2})\,\Gamma(\frac{p+1}{2})\,\Gamma(\frac{q}{2})} = \frac{\sqrt{\pi}\,\Gamma(\frac{r}{2})}{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{q}{2})},$$

as claimed. ■

**Theorem 3.5.** *Suppose that $G$ is $\mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$ or $F_{4(-20)}$. Then $G$ is weakly amenable and*

$$\Lambda_G \le \frac{\sqrt{\pi}\,\Gamma(\frac{r}{2})}{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{q}{2})}.$$

**Proof.** This follows as in De Cannière and Haagerup [6, Theorem 3.7]. ■

## 4. The lower bound of $\Lambda_G$

We prove the reverse inequality to the inequality of Theorem 3.5 to conclude. As $\Lambda_G$ is always at least 1, there is nothing to do for the groups $\mathrm{SU}(1,n)$. Therefore, we assume that $G$ is $\mathrm{Sp}(1,n)$ or $F_{4(-20)}$.

Recall the definition (10) of $\Delta_2$. For positive $R$, define the radial measure $\mu_R$ in the spherical measure algebra $M(MN//M)$ by

$$\langle f, \mu_R\rangle = \int_{\mathfrak{v}} f(v,0)\exp\Big(-\frac{R}{4}\,|v|^2\Big)\,dv \qquad \forall f \in C_c^\infty(N).$$

**Proposition 4.1.** *Suppose that $u$ is in $A(N)\cap C_c^\infty(MN//M)$. Then*

$$\Big|\langle \Delta_2^{p/2} u, \mu_R\rangle\Big| \le (2^{3/2}\,\pi)^p\,\|u\|_{B(N)}\,.$$

**Proof.** First, $A(MN)|_N = A(N)$, so $u$ is in $A(MN)^\natural$. By [15, Theorem 3.10], $u$ is in the Fourier algebra $A(MN//M)$ of the double coset hypergroup $MN//M$, and $\|u\|_{B(N)} = \|u\|_{B(MN//N)}$.

By [14, Proposition 4.2] and Jewett [12, Theorem 12.2(C)], we observe that $u$ is the inverse Fourier transform of $\widehat{u}$ and $\|u\|_{B(MN//N)} = \|\widehat{u}\|_1$. Therefore,

$$\|u\|_{B(N)} = \|\widehat{u}\|_1\,. \tag{32}$$

The spherical Fourier–Stieltjes transform of $\mu_R$ can easily be calculated. In fact,

$$\widehat{\mu}_R(\nu,n) = \Big(\frac{4\pi}{\nu+R}\Big)^p \Big(\frac{R-\nu}{R+\nu}\Big)^n,$$

and so

$$|\widehat{\mu}_R(\nu,n)| \le \Big(\frac{4\pi}{\nu+R}\Big)^p. \tag{33}$$

Since $\Delta_2$ is $M$-biinvariant, we see from (11) that

$$(\Delta_2 u)\widehat{\ }(\nu,n) = \int_{\mathfrak{v}}\int_{\mathfrak{z}} u(v,z)\,\Delta_2\psi_{\nu,n}(v,z)\,dz\,dv = -\frac{\nu^2}{2}\,\widehat{u}(\nu,n).$$

Therefore,

$$\begin{aligned}\langle \Delta_2^{p/2}u,\mu_R\rangle &= \int (\Delta_2^{p/2}u)\widehat{\ }(\nu,n)\,\widehat{\mu}_R(\nu,n)\,d\pi(\nu,n)\\ &= (-1)^{p/2}\,2^{-p/2}\int_{\mathbb{R}_+\times\mathbb{N}} \nu^p\,\widehat{u}(\nu,n)\,\widehat{\mu}_R(\nu,n)\,d\pi(\nu,n),\end{aligned}$$

and so, by (32) and (33),

$$\begin{aligned}\Big|\langle \Delta_2^{p/2}u,\mu_R\rangle\Big| &\le (\pi 2\sqrt{2})^p \int_{\mathbb{R}_+\times\mathbb{N}} \Big(\frac{\nu}{\nu+R}\Big)^p\,|\widehat{u}(\nu,n)|\;d\pi(\nu,n)\\ &\le (\pi 2\sqrt{2})^p\,\|u\|_{B(N)}\,,\end{aligned}$$

as required. ■

**Lemma 4.2.** *Suppose that $\{f_i\}_{i\in I}$ is a net in $C_c^\infty(\mathbb{R})$, that $\|f_i\|_\infty \le L$ for each $i$ in $I$, and that $\lim_i f_i = 1$ uniformly on compacta. Then*

$$\lim_i \int_0^\infty f_i^{(p/2)}(2s^2+\frac{1}{4}s^4)\,s^{2p-1}\,ds = (-1)^{p/2}\,2^{p-2}\,\Gamma\Big(\frac{p}{2}\Big).$$

**Proof.** This follows Cowling and Haagerup [5, Proposition 5.3]. ■

**Theorem 4.3.** *Suppose that $G$ is isomorphic to $\mathrm{Sp}(1,n)$, where $n\ge 2$, or to $F_{4(-20)}$. Let $\{u_i\}_{i\in I}$ be a net in $A_c(G)$ such that*

*(i) there exists $L>0$ such that $\|u_i\|_{M_0A(G)} \le L$ for all $i$ in $I$*

*(ii) $\{u_i\}$ tends to $1$, uniformly on compacta.*

*Then*

$$L \ge \frac{\sqrt{\pi}\,\Gamma(\frac{r}{2})}{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{q}{2})}\,.$$

**Proof.** By [5, Propositions 1.1 and 1.6], we may assume that each $u_i$ is in $A(G)\cap C_c^\infty(G//K)$. Thus, we may suppose that $u_i|_N \in A(N)\cap C_c^\infty(MN//M)$ and $\|u_i\|_{B(N)} \le L$ for every $i$, and $\{u_i\}$ tends to 1 uniformly on compacta.

By the proposition above,

$$(\pi 2\sqrt{2})^p\,\|u_i\|_{B(N)} \ge \Big|\langle \Delta_2^{p/2}u_i,\mu_R\rangle\Big| \qquad \forall i\in I.$$

Allowing $R$ to tend to 0 and applying (13), we deduce that

$$\begin{aligned}(\pi 2\sqrt{2})^p \left\|u_i\right\|_{B(N)} &\geq \left|\int_{\mathfrak{v}} \Delta_2^{p/2} u_i(v,0)\, dv\right| \\ &\geq \frac{(2\sqrt{2})^p\, \Gamma(\frac{p+q}{2})}{\Gamma(\frac{q}{2})} \left|\int_{\mathfrak{v}} f_i^{(p/2)}(2\left|v\right|^2 + \frac{1}{4}\left|v\right|^4)\, dv\right| \\ &= \frac{(2\sqrt{2})^p\, \Gamma(\frac{r}{2})}{\Gamma(\frac{q}{2})}\, \Omega_{2p} \left|\int_0^\infty f_i^{(p/2)}(2s^2 + \frac{1}{4}s^4)\, s^{2p-1}\, ds\right|.\end{aligned}$$

The functions $f_i$ converge to 1 locally uniformly on compacta on $[0,\infty)$. Therefore, by Lemma 4.2 above and the Legendre Duplication formula,

$$\limsup_i \left\|u_i\right\|_{B(N)} \geq \frac{2^{p-1}\, \Gamma(\frac{r}{2})\, \Gamma(\frac{p}{2})}{\Gamma(\frac{q}{2})\, \Gamma(p)} = \frac{\sqrt{\pi}\, \Gamma(\frac{r}{2})}{\Gamma(\frac{p+1}{2})\, \Gamma(\frac{q}{2})},$$

as claimed. ■

In conclusion, we have proved the following theorem.

**Theorem 4.4.** *Suppose that $G$ is a connected real Lie group with finite center. If $G$ is locally isomorphic to* $\mathrm{SU}(1,n)$*, then* $\Lambda_G = 1$*; if $G$ is locally isomorphic to* $\mathrm{Sp}(1,n)$*, then* $\Lambda_G = 2n-1$*; and if $G$ is the exceptional rank one group* $F_{4(-20)}$*, then* $\Lambda_G = 21$*.*


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## References


[1] Bloom, W. R., and H. Heyer, "Harmonic Analysis of Probability Measures on Hypergroups," de Gruyter Studies in Mathematics, vol. **20**, Walter de Gruyter, 1995.

[2] Erdélyi, A., W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Higher Transcendental Functions, Vol. I," Bateman Manuscript Project, McGraw-Hill, New York–Toronto–London, 1953.

[3] Cowling, M., *Unitary and uniformly bounded representations of some simple Lie groups*, in: "Harmonic Analysis and Group Representations," C.I.M.E. II ciclo 1980, Liguori, Napoli, 1982, 49–128.

[4] —, *The radial Haagerup property*, in: "Locally Compact Groups with the Haagerup Property," Progress in Mathematics, vol. **197**, Birkhäuser, Basel, 2001, 507–549.

[5] Cowling, M., and U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*, Invent. Math. **96** (1989), 507–549.

[6] De Cannière, J., and U. Haagerup, *Multipliers of the Fourier algebra of some simple Lie groups and their discrete subgroups*, Amer. J. Math. **107** (1984), 455–500.

[7] Eymard, P., *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France. **92** (1964), 181–236.

[8] Faraut, J., *Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques*, in: "Analyse harmonique," J. L. Clerc et al., ed., C.I.M.P.A., Nice, 1982, 315–446.

[9] —, *Un théorème de Paley-Wiener pour la transformation de Fourier sur un espace riemannien symétrique de rang un*, J. Funct. Anal. **49** (1982), 230–268.

[10] Helgason, S., "Differential Geometry, Lie Groups and Symmetric Spaces," Academic Press, New York, 1978.

[11] —, "Groups and Geometric Analysis," Academic Press, New York, 1984.

[12] Jewett, R. I., *Spaces with an abstract convolution of measures*, Advances in Math. **18** (1975), 1–101.

[13] Korányi, A., *Some applications of Gel'fand pairs in classical analysis*, in: Harmonic Analysis and Group Representations, C.I.M.E. II ciclo 1980, Liguori, Napoli, 1982, 333–348.

[14] Muruganandam, V., *Fourier algebra of a hypergroup. I*, J. Aust. Math. Soc. **82** (2007), 59–83.

[15] —, *Fourier algebra of a hypergroup – II, spherical hypergroups*, Math. Nachr. (2008), to appear.

[16] Titchmarsh, E. C., "The Theory of Functions," Oxford University Press, Oxford, 1978.

[17] Whittaker, E. T., and G. N. Watson, "A Course in Modern Analysis," Cambridge University Press, Cambridge, 1963.

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