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The Dual of Kawazoe's Atomic Hardy Space $\mathbf{H}^1_{p,0}$

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Abstract. The purpose of this paper is to determine the dual of Kawazoe's atomic Hardy space for semisimple Lie groups. The conclusion is that it consists of functions, whose translates satisfy conditions which are similar to the conditions for Goldberg's Euclidean local bmo-space. We will also find the duals of the corresponding K-invariant and K-bi-invariant spaces.
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1. Introduction

To begin with, we recall Kawazoe's definition of the atomic Hardy space, $\mathbf{H^1_{p,0}(G)}$, for non-compact semisimple Lie groups, for more details about this space see [4]. Thus, let G be a connected non-compact semisimple Lie group with finite center and Cartan decomposition $G = K \exp \mathfrak{p}$. For $g = k \exp X$, we denote by $\sigma(g)$ the norm of X with respect to the Euclidean structure on $\mathfrak{p}$ induced by the Killing form. Let dg and dk be the Haar measures on G and K respectively, where the latter is assumed to be normalized to have total mass 1. By $B(r)$ we denote the ball centered at the origin of radius r, i.e. $B(r) = \{g \in G; \sigma(g) \leq r\}$. We observe that $KB(r)K = B(r)$ (That the ball is left K-invariant follows directly from the definition of the norm σ and that it is right K-invariant follows from the invariance of the Killing form). A well-known property for non-compact semisimple Lie groups is that they are of exponential growth, i.e. the volume of a ball with radius r grows exponentially as r tends to infinity.

For $1 < p \leq \infty$, a function a on G is called a $(1,p,0)$-atom if it satisfies the following conditions

(i) $\operatorname{supp} a \subset B(r)$ for some $r > 0$,

(ii) if $r \leq 1$, then $\|a\|_{\mathbf{L}^p} \leq |B(r)|^{-\frac{1}{p'}}$ and $\int_G a(g)\, dg = 0$,

(iii) if $r > 1$, then $\|a\|_{\mathbf{L}^p} \leq |B(r)|^{-1}$.

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Remark 1.1. In [4] it is assumed throughout that the group has real rank one, because the goal in that paper is to show that certain maximal operators are bounded. However, for the definition of the atomic Hardy space this restriction is not necessary.

Remark 1.2. We observe that if $p = \infty$ the conditions coincide with the conditions for Goldberg's local $\mathbf{h}^1$-atoms in the Euclidean setting, see [2] pages 36-37. The reason for the stronger bound (for local $\mathbf{h}^1$-atoms the bound would have been $|B(r)|^{-\frac{1}{p'}}$) in (iii) for general values of p is imposed to ensure boundedness of different maximal functions. For more details see [4] remarks 4.2 and 4.7.

Remark 1.3. In $\mathbf{R}^n$ the condition (ii) for different values of p give rise to the same Hardy space. However, for the Hardy spaces $\mathbf{H}^1_{p,0}$ defined below this is still an open question.

We introduce the notation $f_x(g) = f(xg),\ x, g \in G$. The atomic Hardy space on G is then defined as the space of linear combinations of translated atoms

$$\mathbf{H}^1_{p,0} := \Big\{ f = \sum_i \lambda_i (a_i)_{x_i};\, a_i \text{ is a } (1,p,0)\text{-atom },\, x_i \in G, \text{ and } \sum_i |\lambda_i| < \infty \Big\}.$$

The norm is $\|f\|_{\mathbf{H}^1_{p,0}} := \inf \sum_i |\lambda_i|$, where the infimum is taken over all representations $f = \sum_i \lambda_i (a_i)_{x_i}$.

Kawazoe also defines Hardy spaces of K-right-invariant and K-bi-invariant functions. Let

$$f^{\#}(g) = \int_K f(gk)\, dk, \quad f^{\flat}(g) = \int_K \int_K f(kgk')\, dk\, dk'.$$

then following Kawazoe we define $\mathbf{H}^{1,\#}_{p,0} := \{f^{\#}; f \in \mathbf{H}^1_{p,0}\}$, and $\mathbf{H}^{1,\flat}_{p,0} := \{f^{\flat}; f \in \mathbf{H}^1_{p,0}\}$. We will determine the dual of $\mathbf{H}^1_{p,0}$ in Theorem 2.3 and the dual of $\mathbf{H}^{1,\flat}_{p,0}$ and $\mathbf{H}^{1,\#}_{p,0}$ respectively in Corollary 3.1 and Corollary 3.2. There is also another way to define $\mathbf{H}^{1,\flat}_{p,0}$ from atoms without using translations and we will give a definition of its dual without the use of translations as well, see the end of Section 3.

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2. Duality

As the definition of $(1,p,0)$-atoms is related to the definition of atoms for local Hardy spaces it is not surprising that the dual space should be related to bmo, the local BMO-space introduced by Goldberg.

Definition of $BMO^1_{p,0}$

In the following, let us denote the average of a function, f, over a measurable set, U, by f_U, i.e.

$$f_U := \frac{1}{|U|} \int_U f(g)\, dg.$$

The space $\mathrm{BMO}^1_{p,0}$ consists in the functions $b \in \mathbf{L}^1_{loc}(G)$ for which there exists a constant C such that

$$\sup_{x\in G}\sup_{r\leq 1}\left(\frac{1}{|B(r)|}\int_{B(r)}|b_x(g)-(b_x)_{B(r)}|^{p'}\,dg\right)^{\frac{1}{p'}} < C$$

and

$$\sup_{x\in G}\sup_{r>1}\frac{1}{|B(r)|}\left(\int_{B(r)}|b_x(g)|^{p'}\,dg\right)^{\frac{1}{p'}} < C,$$

with the smallest bound C as the norm.

If for $f \in \mathbf{L}^1_{loc}(G)$ we define

$$\begin{aligned}\tilde{f}(x) &:= \sup_{x\in B, r(B)\leq 1}\left(\frac{1}{|B|}\int_B |f-f_B|^{p'}\,dg\right)^{\frac{1}{p'}}\\ \tilde{\tilde{f}}(x) &:= \sup_{x\in B, r(B)>1}\frac{1}{|B|}\left(\int_B |f|^{p'}\,dg\right)^{\frac{1}{p'}},\end{aligned}$$

where the supremum is taken over all balls containing x with radius $r(B) \leq 1$ and > 1 respectively. Then because the measure dg is left invariant we obtain

$$\mathrm{BMO}^1_{p,0} = \{b \in \mathbf{L}^1_{loc}(G) : \tilde{b} \in \mathbf{L}^\infty(G), \tilde{\tilde{b}} \in \mathbf{L}^\infty\}$$

and the norm can be identified with

$$\|b\|_{\mathrm{BMO}^1_{p,0}} = \max(\|\tilde{b}\|_{\mathbf{L}^\infty}, \|\tilde{\tilde{b}}\|_{\mathbf{L}^\infty}).$$

Before proceeding we make the following observation, which is easy to prove using the triangle inequality.

Remark 2.1. If a function $b \in \mathbf{L}^1_{loc}(G)$ satisfies the condition

$$\left(\frac{1}{|B(r)|}\int_{B(r)}|b_x(g)-c|^{p'}\,dg\right)^{\frac{1}{p'}} \leq C$$

for a given constant c then this is true also with c replaced by the constant $(b_x)_{B(r)}$.

Remark 2.2. A related BMO-space has been introduced by Ionescu [3]. We will return to that space in Section 3.

Main theorem

We will denote by $(\mathbf{H}^1_{p,0})_{finite}$ the dense subspace of $\mathbf{H}^1_{p,0}$ consisting of finite linear combinations of atoms, or translates of atoms.

Theorem 2.3. $\mathrm{BMO}^1_{p,0} = (\mathbf{H}^1_{p,0})^*$. *This holds in the sense that* *i) if* $b \in \mathrm{BMO}^1_{p,0}$, *then we obtain a linear functional on* $(\mathbf{H}^1_{p,0})_{finite}$ *by setting*

$$L(f) = \int_G f(g)b(g)\,dg, \qquad f \in (\mathbf{H}^1_{p,0})_{finite}.$$

This linear functional has a unique bounded extension to $\mathbf{H}^1_{p,0}$ *which satisfies* $\|L\| \le c\|b\|_{\mathrm{BMO}^1_{p,0}}$.

ii) Conversely, for every bounded linear functional L *on* $\mathbf{H}^1_{p,0}$ *there exists an element* $b \in \mathrm{BMO}^1_{p,0}$ *such that*

$$L(f) = \int_G f(g)b(g)\,dg$$

for $f \in (\mathbf{H}^1_{p,0})_{finite}$ *and* $\|b\|_{\mathrm{BMO}^1_{p,0}} \le c'\|L\|$.

Proof. i) If a is an $(1,p,0)$-atom with $\operatorname{supp} a \subset B(r)$, $r \le 1$ then since the measure is left-invariant

$$\left|\int_{x^{-1}B(r)} a_x(g)b(g)dg\right| = \left|\int_{B(r)} a(g)b_{x^{-1}}(g)dg\right|.$$

By condition (ii) in the definition of a $(1,p,0)$-atom and Hölder's inequality, this is bounded by

$$\le \|a\|_{\mathbf{L}^p}\left(\int_{B(r)} |b_{x^{-1}}(g) - (b_{x^{-1}})_{B(r)}|^{p'}\,dg\right)^{\frac{1}{p'}}.$$

Again by condition (ii) we can estimate this by

$$\left(\frac{1}{|B(r)|}\int_{B(r)} |b_{x^{-1}}(g) - (b_{x^{-1}})_{B(r)}|^{p'}\,dg\right)^{\frac{1}{p'}} \le \|b\|_{\mathrm{BMO}^1_{p,0}}.$$

If a is a $(1,p,0)$-atom with $\operatorname{supp} a \subset B(r)$, $r > 1$ and $b \in \mathrm{BMO}^1_{p,0}$ then

$$\left|\int_{x^{-1}B(r)} a_x(g)b(g)\,dg\right| = \left|\int_{B(r)} a(g)b_{x^{-1}}(g)\,dg\right|$$

by the left-invariance of the measure. By Hölder's inequality and (iii) this is

$$\le \frac{1}{|B(r)|}\left(\int_{B(r)} |b_{x^{-1}}(g)|^{p'}\,dg\right)^{\frac{1}{p'}} \le \|b\|_{\mathrm{BMO}^1_{p,0}}.$$

Hence, if $f \in (\mathbf{H}^1_{p,0})_{finite}$, $f = \sum_{i=1}^N \lambda_i (a_i)_{x_i}$ and $b \in \mathrm{BMO}^1_{p,0}$ we have

$$\left|\int_G f(g)b(g)\,dg\right| \le \sum_{i=1}^N |\lambda_i|\|b\|_{\mathrm{BMO}^1_{p,0}}.$$

ii) The idea of this part of the proof is to show that an element of the dual is given locally by functions in $\mathbf{L}^{p'}$ that are compatible, and hence gives rise to a $\mathbf{L}^1_{loc}$ function on G. Finally we verify that this function belongs to $\mathrm{BMO}^1_{p,0}$. To see that the linear functionals on $\mathbf{H}^1_{p,0}$ are given locally by functions in $\mathbf{L}^{p'}$ we need some lemmas.

For a given subset U we will denote by $\mathbf{L}^p_{U,0} := \{f \in \mathbf{L}^p(U) : f_U = 0\}$. Let $L \in (\mathbf{H}^1_{p,0})^*$ with $\|L\| \le 1$.

Lemma 2.4. *If $r \leq 1$ then $\mathbf{L}^p_{xB(r),0} \subset \mathbf{H}^1_{p,0}$, for all $x \in G$.*

Proof. If $f \in \mathbf{L}^p_{xB(r),0}$ then $f = c \cdot a_{x^{-1}}$, where a is an $(1,p,0)$-atom and $c = |B(r)|^{\frac{1}{p'}} \|f\|_{\mathbf{L}^p}$. Hence, $\|f\|_{\mathbf{H}^1_{p,0}} \leq |B(r)|^{\frac{1}{p'}} \|f\|_{\mathbf{L}^p}$. ■

Hence, if $r \leq 1,$ L restricts to a linear functional on $\mathbf{L}^p_{xB(r),0}$ with $\|L\|_{(\mathbf{L}^p_{xB(r),0})^*} \leq |B(r)|^{\frac{1}{p'}}$. By the Hahn-Banach theorem L extends from $\mathbf{L}^p_{xB(r),0}$ to a linear functional $\tilde{L}$ on $\mathbf{L}^p_{xB(r)}$ with $\|\tilde{L}\|_{(\mathbf{L}^p_{xB(r)})^*} = \|L\|_{(\mathbf{L}^p_{xB(r),0})^*}$. By $\left(\mathbf{L}^p_{xB(r)}, \mathbf{L}^{p'}_{xB(r)}\right)$-duality, for each $x \in G$ there exists a function $\Phi_{x,r} \in \mathbf{L}^{p'}_{xB(r)}$ such that

$$\tilde{L}(f) = \int_{xB(r)} \Phi_{x,r}(g) f(g)\, dg \quad \text{ for } f \in \mathbf{L}^p_{xB(r)}$$

and $\|\Phi_{x,r}\|_{\mathbf{L}^{p'}} \leq |B(r)|^{\frac{1}{p'}}$. In particular

$$L(f) = \int_{xB(r)} \Phi_{x,r}(g) f(g)\, dg \quad \text{ for } f \in \mathbf{L}^p_{xB(r),0}.$$

Thus, for each $x \in G$ and $r \leq 1$, we have a function $\Phi_{x,r} \in \mathbf{L}^{p'}_{xB(r)}$ representing L on $\mathbf{L}^p_{xB(r),0}$. Furthermore, as the following remark shows, the different representants are compatible up to a constant on the intersection.

Remark 2.5. If $r_1 < 1$ and $r_2 < 1,$ $x_1,$ $x_2 \in G$, then

$$\int (\Phi_{x,r_2} - \Phi_{x,r_1}) f\, dx = 0 \quad \text{ for } f \in \mathbf{L}^p_{x_1 B(r_1) \cap x_2 B(r_2),0}.$$

Hence, $\Phi_{x_2,r_2} - \Phi_{x_1,r_1} = c_{x_1,x_2,r_1,r_2}$ on $x_1 B(r_1) \cap x_2 B(r_2)$ for some constant c_{x_1,x_2,r_1,r_2}. We will make a choice later to fix this constant.
Next we consider $r > 1$. In this case we have no moment conditions.

Lemma 2.6. *If $r > 1,$ then $\mathbf{L}^p_{xB(r)} \subset \mathbf{H}^1_{p,0}$ for all $x \in G$.*

Proof. If $f \in \mathbf{L}^p_{xB(r)}$ then $f = c a_{x^{-1}}$ where $c = |B(r)| \|f\|_{\mathbf{L}^p}$ and a is an $(1,p,0)$-atom. Hence, $\|f\|_{\mathbf{H}^1_{p,0}} \leq |B(r)| \|f\|_{\mathbf{L}^p}$. ■

Thus when $r > 1,$ L restricts to a linear functional on $\mathbf{L}^p_{xB(r)}$ with $\|L\|_{(\mathbf{L}^p_{xB(r)})^*} \leq |B(r)|$. The lemma and the $\left(\mathbf{L}^p_{xB(r)}, \mathbf{L}^{p'}_{xB(r)}\right)$- duality shows that there exists a function $\Phi_{x,r} \in \mathbf{L}^{p'}_{xB(r)}$ such that

$$L(f) = \int \Phi_{x,r}(g) f(g)\, dg \quad \text{ for } f \in \mathbf{L}^p_{xB(r)},$$

and $\|\Phi_{x,r}\|_{\mathbf{L}^{p'}_{xB(r)}} \leq |B(r)|$.

Remark 2.7. If $r_2 > 1$ and $r_1 > 1,$ $x_1,$ $x_2 \in G$ and $x_1 B(r_1) \subset x_2 B(r_2)$ then $\int (\Phi_{x_2,r_2} - \Phi_{x_1,r_1}) f\, dx = 0$ for $f \in \mathbf{L}^p_{x_1 B(r_1)}$ so $\Phi_{x_2,r_2} = \Phi_{x_1,r_1}$ on $x_1 B(r_1)$.

Remark 2.8. Assume now that $r_3 > 1$, $r_2 > 1$, $r_1 < 1$ and $x_1B(r_1) \subset x_3B(r_3) \cap x_2B(r_2)$, then we may choose r_4 large enough so that $x_3B(r_3) \cup x_2B(r_2) \subset x_1B(r_4)$. As $\mathbf{L}^p_{x_1B(r_1),0} \subset \mathbf{L}^p_{x_1B(r_4)}$ with preserved norms, we must have

$$\Phi_{x_1,r_4} - \Phi_{x_1,r_1} = c_{x_1,r_1,r_4}$$

on $x_1B(r_1)$. Hence, by the reasoning above and Remark 2.7 we have

$$\begin{aligned} \Phi_{x_3,r_3} - \Phi_{x_2,r_2} &= (\Phi_{x_3,r_3} - \Phi_{x_1,r_1}) - (\Phi_{x_2,r_2} - \Phi_{x_1,r_1}) \\ &= (\Phi_{x_1,r_4} - \Phi_{x_1,r_1}) - (\Phi_{x_1,r_4} - \Phi_{x_1,r_1}) = 0, \end{aligned}$$

on $x_1B(r_1)$. Thus there is a unique constant c_{x_1,r_1} such that $\Phi_{x_2,r_2} = \Phi_{x_1,r_1} + c_{x_1,r_1}$ on $x_1B(r_1)$ as soon as $r_1 < 1 < r_2$ and $x_1B(r_1) \subset x_2B(r_2)$.

Let $\phi_{x,r}(g) = \Phi_{x,r}(g) + c_{x,r}$ if $r \leq 1$ and $\phi_{x,r}(g) = \Phi_{x,r}(g)$ if $r > 1$. Set $b(g) = \phi_{x,r}(g)$ if $g \in xB(r)$.

Lemma 2.9. *The function b is well-defined.*

Proof. We need to show that the function $b(g)$ is independent of the choice of ball $xB(r)$ containing g. Thus, assuming that $g \in x_1B(r_1) \cap x_2B(r_2)$ then we want to show that $\phi_{x_1,r_1}(g) = \phi_{x_2,r_2}(g)$. There are three cases to consider

i) If $r_1 > 1$ and $r_2 > 1$ then there exists a ball $B(r_3)$ with $x_1B(r_1) \cup x_2B(r_2) \subset x_3B(r_3)$. Hence, by Remark 2.7, $\Phi_{x_3,r_3} = \Phi_{x_2,r_2}$ on $x_2B(r_2)$ and $\Phi_{x_3,r_3} = \Phi_{x_1,r_1}$ on $x_1B(r_1)$, which implies that $\Phi_{x_2,r_2} = \Phi_{x_1,r_1}$ on $x_1B(r_1) \cap x_2B(r_2)$. In particular, $\phi_{x_1,r_1}(g) = \phi_{x_2,r_2}(g)$.

ii) If $r_1 < 1$ and $r_2 > 1$ then if we choose r_3 large enough so that $x_1B(r_1) \cup x_2B(r_2) \subset x_1B(r_3)$ we obtain $\Phi_{x_1,r_3} = \Phi_{x_2,r_2}$ on $x_2B(r_2)$, by Remark 2.7 and $\Phi_{x_1,r_3} = \Phi_{x_1,r_1} + c_{x_1,r_1}$ on $x_1B(r_1)$, by Remark 2.8. Hence, $\Phi_{x_2,r_2} = \Phi_{x_1,r_1} + c_{x_1,r_1}$ on $x_1B(r_1) \cap x_2B(r_2)$. Thus, $\phi_{x_1,r_1}(g) = \phi_{x_2,r_2}(g)$.

iii) If $r_1 < 1$ and $r_2 < 1$ then there exists $r_3 > 1$ such that $x_1B(r_1) \cup x_2B(r_2) \subset x_3B(r_3)$. By remark 2.8 we have, on $x_1B(r_1)$, $\Phi_{x_3,r_3} = \Phi_{x_1,r_1} + c_{x_1,r_1}$, and on $x_2B(r_2)$, $\Phi_{x_3,r_3} = \Phi_{x_2,r_2} + c_{x_2,r_2}$. This implies that $\Phi_{x_1,r_1} + c_{x_1,r_1} = \Phi_{x_2,r_2} + c_{x_2,r_2}$ on $x_1B(r_1) \cap x_2B(r_2)$. Hence, $\phi_{x_1,r_1}(g) = \phi_{x_2,r_2}(g)$.

■

To summarize, given a linear functional L we have defined a function b such that $b \in \mathbf{L}^{p'}$ locally and

$$L(f) = \int_G f(g)b(g)\,dg,$$

for any function $f \in \mathbf{H}^1_{p,0}$. Next we have to show that this function belongs to our BMO-space.

Lemma 2.10. $b \in \mathrm{BMO}^1_{p,0}$.

Proof. For $xB(r)$ with $r \leq 1$ we have, according to the comment after Lemma 2.4,

$$\|b - c_{x,r}\|_{\mathbf{L}^{p'}_{xB(r)}} = \|\Phi_{x,r}\|_{\mathbf{L}^{p'}_{xB(r)}} \leq |B(r)|^{\frac{1}{p'}},$$

i.e.

$$\left(\frac{1}{|B(r)|}\int_{B(r)} |b_x(g) - c_{x,r}|^{p'}\, dg\right)^{\frac{1}{p'}} \leq C.$$

The first condition in our definition of $\mathrm{BMO}^1_{p,0}$ then follows from Remark 2.1.

For $B(r)$ with $r > 1$ we have

$$\|b\|_{\mathbf{L}^{p'}_{xB(r)}} = \|\Phi_{x,r}\|_{\mathbf{L}^{p}_{xB(r)}} \leq |B(r)|.$$

See the comment after Lemma 2.6. Hence,

$$\frac{1}{|B(r)|}\left(\int_{xB(r)} |b(g)|^{p'}\, dg\right)^{\frac{1}{p'}} \leq C,$$

which is equivalent to the second condition in our definition of $\mathrm{BMO}^1_{p,0}$. ■

This lemma concludes the proof of Theorem 2.3. ■

3. K-invariant cases

In this section we will determine the duals of $\mathbf{H}^{1,\#}_{p,0}$ and $\mathbf{H}^{1,\flat}_{p,0}$. For the latter space we will consider two different definitions, one with translations and one without.

The duals of $\mathbf{H}^{1,\#}_{p,0}$ and $\mathbf{H}^{1,\flat}_{p,0}$

Let $\mathrm{BMO}^{1,\#}_{p,0}$ denote the dual of $\mathbf{H}^{1,\#}_{p,0}$ and $\mathrm{BMO}^{1,\flat}_{p,0}$ the dual of $\mathbf{H}^{1,\flat}_{p,0}$, see the Introduction for definitions of $\mathbf{H}^{1,\#}_{p,0}$ and $\mathbf{H}^{1,\flat}_{p,0}$.

Corollary 3.1. *The space* $\mathrm{BMO}^{1,\#}_{p,0}$ *is easily determined from* $\mathrm{BMO}^1_{p,0}$, *namely*

$$\mathrm{BMO}^{1,\#}_{p,0} = \{b \in \mathrm{BMO}^1_{p,0};\, b \text{ is } K\text{-right-invariant}\}.$$

Proof. Since K is compact and the space $\mathbf{H}^1_{p,0}$ is normed, the first part follows directly from the fact that the dual of a space of K-invariant vectors is the corresponding space of K-invariant linear functionals, see [1, Theorem 1.1]. Hence,

$$(\mathbf{H}^{1,\#}_{p,0})^* = \{b \in \mathrm{BMO}^1_{p,0};\, b \text{ is } K\text{-right-invariant}\}.$$ ■

Corollary 3.2. *The space* $\mathrm{BMO}^{1,\flat}_{p,0}$ *can be characterized in a similar way. We have*

$$\mathrm{BMO}^{1,\flat}_{p,0} = \{b \in \mathrm{BMO}^1_{p,0};\, b \text{ is } K\text{-bi-invariant}\}.$$

Proof. The proof is a bit more complicated because $\mathbf{H}^{1,\flat}_{p,0}$ is not the subspace of K-left-invariant vectors in $\mathbf{H}^{1,\#}_{p,0}$. Instead we essentially repeat the proof of Theorem 2.3. First the fact that any K-bi-invariant element $b \in \mathrm{BMO}^1_{p,0}$ defines a linear functional on $\mathbf{H}^{1,\flat}_{p,0}$ follows directly from the proof of Theorem 2.3(i) once we note that if a is a $(1,p,0)$-atom with support in the set $B(r)$ then

$$\int_G (a_x)^\flat(g) b(g)\, dg = \int_G a_x(g) b(g)\, dg$$

because b is assumed to be K-bi-invariant and the measure is left- and right-invariant.

For the proof that any linear functional in $(\mathbf{H}^{1,\flat}_{p,0})^*$ can be represented by a K-bi-invariant element of $\mathrm{BMO}^1_{p,0}$ we follow the proof of Theorem 2.3(ii). The difference being that we have to replace the translated balls by sets of the form $KxB(r)$. Let $\mathbf{L}^{p,K}_{KxB(r)}$ be the space of K-bi-invariant $\mathbf{L}^p$-functions on the set $KxB(r)$ and $\mathbf{L}^{p,K}_{KxB(r),0}$ the subspace of such functions with integral zero. To obtain the analogues of Lemmas 2.4 and 2.6 we proceed as in the proof of Theorem 5.5 in [4] for showing that any element of $\mathbf{L}^{p,K}_{KxB(r),0}$ belongs to $\mathbf{H}^{1,\flat}_{p,0}$ for $r \le 1$ and that any element of $\mathbf{L}^{p,K}_{KxB(r)}$ belongs to $\mathbf{H}^{1,\flat}_{p,0}$ for $r > 1$.

We will only give a proof in the first case since the second follows in the same way. Hence, given an element $f \in \mathbf{L}^{p,K}_{KxB(r),0}$, we define a to be

$$a(g) = \frac{f(x^{-1}g)}{I(x,r,(x^{-1}g)^{-1})} \chi_{B(r)}(g),$$

where $\chi_{B(r)}$ is the characteristic function for the ball $B(r)$ with radius r, and

$$I(x,r,y) = \int_K \chi_{B(r)}(xky^{-1})\, dk. \tag{1}$$

Since f and I are K-bi-invariant we obtain

$$\int_G a(g)\, dg = \int_G \frac{f(g)}{I(x,r,g^{-1})} \left(\int_K \chi_{B(r)}(xkg)\, dk \right) dg = \int_G f(g)\, dg = 0.$$

Next we want to know the $\mathbf{L}^p$-norm of a. First we observe that

$$\begin{aligned} |f(g)| &= \left| \int_K f(g)\, dk \right| = \left| \int_K f(g) \chi_{B(r)}(x^{-1}kg)\, dk \right| \\ &\le |f(g)| \left(\int_K \chi_{B(r)}(x^{-1}kg)\, dk \right)^{\frac{p-1}{p}} \end{aligned} \tag{2}$$

since the support of f is in $KxB(r)$ and f is K-bi-invariant. For the $\mathbf{L}^p$-norm we have the following estimate

$$\begin{aligned} \|a\|_p^p &= \int_G \left| \frac{f(x^{-1}g)}{I(x,r,(x^{-1}g)^{-1})} \right|^p \left(\int_K \chi_{B(r)}(xkg)\, dk \right) dg \\ &\le \int_G |f(g)|^p I(x,r,g^{-1})^{1-p}\, dg. \end{aligned}$$

By (2) we find that this is bounded by

$$\int_G |f(g)|^p \left(\int_K \chi_{B(r)}(x^{-1}kg)\,dk\right)^{p-1} I(x,r,g^{-1})^{1-p}\,dg = \int_G |f(g)|^p\,dg = \|f\|_{\mathbf{L}^p}^p.$$

Hence, a is a multiple of a $(1,p,0)$-atom and $\|a\|_{\mathbf{H}^1_{p,0}} \le |B(r)|^{\frac{1}{p'}}\|f\|_{\mathbf{L}^p}$. It is now easy to see that $f \in \mathbf{H}^{1,\flat}_{p,0}$ because

$$(a_x)^\flat(g) = \int_K\int_K a(xkgk')\,dk\,dk' = \int_K \frac{f(g)}{I(x,r,g^{-1})}\chi_{B(r)}(xkg)\,dk = f(g),$$

where we have used that f and I are K-bi-invariant and that $\chi_{B(r)}$ is K-right-invariant.

To define the function $\Phi_{x,r}$ we need $\left(\mathbf{L}^{p,K}_{KxB(r)}, \mathbf{L}^{p',K}_{KxB(r)}\right)$-duality, which is valid because of [1, Theorem 1.1]. Otherwise the proof is the same except for minor changes. ∎

Comparison with Ionescu's BMO

The space $\mathrm{BMO}^{1,\#}_{p,0}$ could also be considered as defined on the symmetric space G/K. In [3], Ionescu, defined a BMO-space on Riemannian symmetric spaces of rank one in the following way. Let for each $f \in \mathbf{L}^1_{loc}(G/K)$, in analogy with our earlier definition, a function $\tilde{f}$ be defined by

$$\tilde{f}(z) = \sup_{z\in B, r(B)\le 1} \frac{1}{|B|}\int_B |f(z') - f_B|\,dz'$$

where the supremum is taken over all balls with radius ≤ 1 containing z. The BMO-space defined by Ionescu is then

$$\mathrm{BMOI} := \left\{f \in \mathbf{L}^1_{loc}(G/K); \|\tilde{f}\|_{\mathbf{L}^\infty(G/K)} < C\right\}.$$

Comparing this with our definition of $\mathrm{BMO}^{1,\#}_{1,0}$ we find that the difference is that Ionescu does not assume any estimate for the balls with radius > 1. Hence, $\mathrm{BMO}^{1,\#}_{1,0} \subset \mathrm{BMOI}$. In particular this implies that the analytic interpolation theorem, [3, Proposition 2], will hold if we assume that the operator is bounded from $\mathbf{L}^\infty \to \mathrm{BMO}^{1,\#}_{1,0}$ instead of $\mathbf{L}^\infty \to \mathrm{BMOI}$. Essentially this says that, if T_τ is an analytic family of operators such that T_τ is bounded on $\mathbf{L}^2$ when $\mathrm{Re}(\tau) = 0$ and bounded from $\mathbf{L}^\infty$ to $\mathrm{BMO}^{1,\#}_{1,0}$ when $\mathrm{Re}(\tau) = 1$, it will also be bounded on $\mathbf{L}^p$ for any $p \in [2,\infty)$ when $\mathrm{Re}(\tau) = (p-2)/p$. As the referee has pointed out, this in itself is not so interesting because it is clearly more difficult to check whether a function belongs to $\mathrm{BMO}^{1,\#}_{1,0}$ than to check whether it belongs to BMOI. However, by duality, Corollary 3.1, this also implies that we also get analytic interpolation for values of p between 1 and 2 by replacing the $\mathbf{L}^\infty \to \mathrm{BMO}^{1,\#}_{1,0}$-estimate with an $\mathbf{H}^{1,\#}_{1,0} \to \mathbf{L}^1$ estimate.

The dual of $\mathbf{H}^{1,\flat}_{p,0}$ without translations

There is also a different way of defining $\mathbf{H}^{1,\flat}_{p,0}$ without translating the atoms. Throughout this section we will assume that the rank of G is 1. For $g \in G$ and $r > 0$ let

$$R(x,r) = \{g \in G; \sigma(x) - r \le \sigma(g) \le \sigma(x) + r\}.$$

and

$$\|f\|_{x,r,p} = \left(\int_G |f(g)|^p I(x, r_0, g^{-1})^{1-p}\, dg \right)^{\frac{1}{p}},$$

where I is defined in (1), $r_0 = 2r$ if $r \le 1$ and $r_0 = r+1$ if $r > 1$. Following Kawazoe we now define a $(1, p, 0, \natural)$-atom to be a function, a, satisfying

i) a is K-bi-invariant and $\operatorname{supp} a \subset R(x,r)$ for some $x \in G$ and $r > 0$.

ii) For $r \le 1$,

$$\|a\|_{x,r,p} \le |B(r)|^{-\frac{1}{p}} \quad \text{and} \quad \int_G a(g)\, dg = 0.$$

iii) For $r > 1$,

$$\|a\|_{x,r,p} \le |B(r)|^{-1}.$$

Then the Hardy space $\mathbf{H}^{1,\natural}_{p,0}$ is defined to be

$$H^{1,\natural}_{p,0}(G) = \left\{ f = \sum \lambda_i a_i;\, a_i \text{ is a } (1,p,0,\natural)-\text{atom on } G \text{ and } \sum |\lambda_i| < \infty \right\},$$

with norm $\|f\|_{\mathbf{H}^{1,\natural}_{p,0}} = \inf \sum |\lambda_i|$.

In [4] it is shown, see [4, Theorem 5.5], that $\mathbf{H}^{1,\natural}_{p,0} = \mathbf{H}^{1,\flat}_{p,0}$ so $(\mathbf{H}^{1,\natural}_{p,0})^* = \mathrm{BMO}^{1,\flat}_{p,0}$. However, we would like to define the dual without translations. Let r_0 be as above and set $\mathrm{BMO}^{1,\natural}_{p,0}$ to be the space of K-bi-invariant functions $b \in \mathbf{L}^1_{loc}(G)$ for which there is a constant C such that

$$\sup_{x\in G} \sup_{r \le 1} \left(\frac{1}{|B(r)|} \int_{R(x,r)} I(x, r_0, g^{-1}) |b(g) - b_{R(x,r)}|^{p'}\, dg \right)^{\frac{1}{p'}} \le C$$

and

$$\sup_{x\in G} \sup_{r > 1} \frac{1}{|B(r)|} \left(\int_{R(x,r)} I(x, r_0, g^{-1}) |b(g)|^{p'}\, dg \right)^{\frac{1}{p'}} < C\}$$

Then it is possible to show that $(\mathbf{H}^{1,\natural}_{p,0})^* = \mathrm{BMO}^{1,\natural}_{p,0}$. The idea of the proof is to show that $\mathrm{BMO}^{1,\natural}_{p,0} \cong \mathrm{BMO}^{1,\flat}_{p,0}$. In fact, this follows, for $r \le 1$, from

$$\left(\int_{B(r)} |(b)_x(g) - c|^{p'}\, dg \right)^{\frac{1}{p'}} = \left(\int_G I(x, r, g^{-1}) |b(g) - c|^{p'}\, dg \right)^{\frac{1}{p'}}$$

and, for $r > 1$, from the identity

$$\left(\int_{B(r)} |(b)_x(g)|^{p'}\, dg \right)^{\frac{1}{p'}} = \left(\int_G I(x, r, g^{-1}) |b(g)|^{p'}\, dg \right)^{\frac{1}{p'}},$$

which both are easily obtained using invariance and the definition of the function I, (1).

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