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Economizing Brackets to Define Filiform Lie Algebras

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Abstract. It is usual to define the law of a Lie algebra by giving explicitly the nonzero brackets between the elements of one of its bases. However, this paper shows that it is possible to reduce significatively the number of the brackets which are normally indicated when defining a filiform Lie algebra. Indeed, two particular families of brackets are considered and it is proved that the algebra can be defined by using only the elements of anyone of them.
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Introduction

Firstly, we would like to explain why we are dealing with complex *filiform* Lie algebras. The class of these algebras, within the nilpotent Lie algebras, was introduced by M. Vergne in the late 1960's (in her Ph. D. Thesis, later published in 1970 (see [12])), although before that, Blackburn [2] had already studied the analogous class of finite Lie p-groups and used the term *maximal class* to call them, which is now also used for Lie algebras. In fact, both terms (*filiform* and *maximal class*) are synonymous. Vergne showed that within the variety of nilpotent Lie multiplications on a fixed vector space, non-filiform ones can be relegated to small-dimensional components. Thus, from an intuitive point of view, it is possible to consider that *quite a lot* nilpotent Lie algebras are filiform. Moreover, filiform Lie algebras are the most structured subset of nilpotent Lie algebras, which allows us to study and classify them easier than the set of nilpotent Lie algebras.

Apart from that, it is well-known that the usual form to define explicitly a Lie algebra in general consists on giving the nonzero brackets between the elements constituting one of its bases. For instance, the Lie algebra h (belonging to $sl2$) given by the 2×2 real matrices with trace equal to zero (note that it is not filiform, but simple) is normally defined in an explicit way by the brackets

$$[e_1, e_2] = 2\,e_2, \quad [e_1, e_3] = -2\,e_3, \quad [e_2, e_3] = e_1,$$

where $\mathcal{B} = \{e_1, e_2, e_3\}$ is a basis of that algebra, with

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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So, it seems logical that all the nonzero brackets appearing in the definition of a Lie algebra are totally necessary to define it, in the sense that if some of them were missing, then the Lie algebra would not be well defined.

However, the main goal of this paper is precisely to show that this is not true, specially in the case of filiform Lie algebras. In fact, quite a lot of the nonzero brackets generally appearing in the definition of a filiform Lie algebra are unnecessary, since they can be obtained from the remaining ones.

Indeed, two examples shown in this paper prove that, for a filiform Lie algebra of dimension n, a reduction of the number of nonzero brackets from about $\frac{n^2}{2}$ to about n is possible. For instance, if $n = 8$, it can be seen in the examples that almost fifty per cent (even more in other dimensions) of the brackets appearing in the explicit definition of the filiform Lie algebra are unnecessary.

Finally, let us notice that it also happens in the case of any general Lie algebra, although in the non-filiform case, the number of nonzero brackets to reduce is very small. So, for instance, the middle bracket $[e_1, e_3]$ used to define the above Lie algebra h can be obtained from the other two, simply by using the Jacobi identity.

1. Definitions and notations

For a global overview of Lie algebras in general and nilpotent Lie algebras in particular, the reader can consult [11] and [9], respectively. Let us now recall some concepts on filiform Lie algebras.

A complex nilpotent Lie algebra $\mathbf{g}$ is said to be *filiform* if

$$dim\, \mathbf{g}^2 = n-2; \ \dots \ dim\, \mathbf{g}^k = n-k; \ \dots \ dim\, \mathbf{g}^n = 0,$$

where $dim\, \mathbf{g} = n$, and $\mathbf{g}^k = [\mathbf{g}, \mathbf{g}^{k-1}], \quad 2 \le k \le n$.

From now on, $\mathbf{g}$ will denote a n-dimensional complex filiform Lie algebra, with $n \le 3$. It is already proved (see [6]) that there exists a (ordered) basis $\{e_1, \dots, e_n\}$ of $\mathbf{g}$, called an *adapted basis*, such that

$$\begin{array}{ll} [e_1, e_h] = e_{h-1}, & h = 3, \dots, n. \\ [e_2, e_h] = 0, & h = 1, \dots, n. \\ [e_3, e_h] = 0, & h = 4, \dots, n. \end{array} \tag{1}$$

These brackets will be called *filiformity brackets.*

It is easy to deduce that, with respect to that previous basis, it holds

$$\mathbf{g}^2 \equiv \{e_2, \dots, e_{n-1}\}, \ \mathbf{g}^3 \equiv \{e_2, \dots, e_{n-2}\}, \dots, \mathbf{g}^{n-1} \equiv \{e_2\}, \ \mathbf{g}^n \equiv \{0\}. \tag{2}$$

The filiform Lie algebra $\mathbf{g}$ is called a *model* one, if the only nonzero brackets between the elements of an adapted basis are $[e_1, e_h] = e_{h-1}$, $h = 3, \dots, n$. For a given dimension n $(n \ge 3)$, this model algebra is unique (up to isomorphism) and every filiform Lie algebra of this dimension is a deformation of the model (see [9]).

Let us now denote by $C_{\mathbf{g}} h$ the centralizer of a subalgebra h in $\mathbf{g}$. In [6] (although by using a different notation to denote it, which was later improved in [8]), the following integer was introduced

$$z_1 = z_1(\mathbf{g}) = max\{k \in \mathbb{N} \mid \ C_{\mathbf{g}}(\mathbf{g}^{n-k+2}) \supset \mathbf{g}^2 \}.$$

Note that this definition means that the ideal $\mathbf{g}^{n-z_1+2}$ is the greatest ideal whose centralizer contains $\mathbf{g}^2$, that is, the ideal whose centralizer is the ideal $\overline{\mathbf{g}}$, generated by $\{e_2, \ldots, e_{n-1}, e_n\}$.

Moreover, note as well that, according to that definition, $z_1(\mathbf{g})$ is an invariant of filiform Lie algebras. In terms of adapted basis, it is deduced in [6] that

$$z_1 = min\{k \in \mathbb{N} - \{1\} \mid [e_k, e_n] \neq 0\}.$$

However, let us observe that $z_1(\mathbf{g})$ may not exist. In such a case, it is also easy to see that $\mathbf{g}$ is a model algebra. This implies that the algebra with basis $\{e_2, \ldots, e_n\}$ is commutative, which allows us to give a new definition of model algebra, also independent of any adapted basis: a filiform Lie algebra $\mathbf{g}$ is said to be a model algebra if $C_{\mathbf{g}}(\mathbf{g}^{n-2})$ is commutative. Both definitions are equivalent, since $\mathbf{g}^{n-2}$ is the ideal with basis $\{e_2, e_3\}$ whose centralizer is the ideal with basis $\{e_2, \ldots, e_n\}$.

In [7] (also with a different notation) the integer $z_2(\mathbf{g})$ was introduced as follows

$$z_2 = z_2(\mathbf{g}) = max\{k \in \mathbb{N} \,|\, \mathbf{g}^{n-k+1} \text{ is commutative}\}.$$

Note that this definition means that the ideal $\mathbf{g}^{n-z_2+1} \equiv \{e_2, \ldots, e_{z_2}\}$ is the greatest commutative subalgebra in the nilpotency sequence.

In that paper the three following asserts were also proved: a) $z_2(\mathbf{g})$ is an invariant of complex filiform Lie algebras. b) there exists at least some bracket $[e_k, e_{k+1}] \neq 0$, for some $k < n$, in every non-model complex filiform Lie algebra of dimension n, and c) an equivalent definition of z_2 is

$$z_2(\mathbf{g}) = min\ \{k \in \mathbb{N} \mid [e_k, e_{k+1}] \neq 0\}.$$

Similarly to the case of $z_1(\mathbf{g})$, if the set of this definition is empty, then $\mathbf{g}$ is a model algebra. Otherwise, the smallest value of $z_2(\mathbf{g})$ is 4, because of being $[e_1, e_2] = [e_2, e_3] = [e_3, e_4] = 0$ with respect to any adapted basis. Finally, this relation between both invariants was also obtained

$$4 \leq z_1 \leq z_2 < n \leq 2\, z_2 - 2.$$

Now, in the following sections, two different families of brackets are considered. The first one will be formed by the brackets appearing in the definition of $z_1(\mathbf{g})$ and the second one for those appearing in the definition of $z_2(\mathbf{g})$. Both of them will allow us, separately, to reduce significatively the number of the brackets which normally appear when defining a filiform Lie algebra.

From now on, we will suppose that all the Lie algebras appearing in this paper are complex filiform ones and that all bases are adapted. We will denote by $J(a, b, c) = 0$ the Jacobi identity associated with vectors a, b and c.

Remark 1.1. In the last years, a new notation to denote the vectors belonging to an adapted basis of a filiform Lie algebra is being used:

Let $\mathbf{g}$ be a n-dimensional filiform Lie algebra. A (ordered) basis $\{e_1, \ldots, e_n\}$ of $\mathbf{g}$ is called an *adapted basis* if (compare with (1))

$$\begin{array}{ll} [e_1, e_h] = e_{h+1}, & h = 2, \ldots, n-1. \\ [e_h, e_n] = 0, & h = 1, \ldots, n. \\ [e_h, e_{n-1}] = 0, & h = 2, \ldots, n. \end{array}$$

In this case, the filiformity brackets would be those $[e_1, e_h] = e_{h+1}$, with $h = 2, \ldots, n-1$, and expressions (2) reduce to

$$\mathbf{g}^2 \equiv \{e_3, \ldots, e_n\},\ \mathbf{g}^3 \equiv \{e_4, \ldots, e_n\}, \ldots, \mathbf{g}^{n-1} \equiv \{e_n\},\ \mathbf{g}^n \equiv \{0\}.$$

Also, in a similar way, the definitions of integers $z_1(\mathbf{g})$ and $z_2(\mathbf{g})$ can be translated.

However, in this paper we will continue using the first notation (which, in any case, has not been completely abandoned), since all the classifications of filiform Lie algebras existing at the present time, which we will refer to in the last section of this paper, have been written by using it.

2. Reducing the number of brackets

Theorem 2.1. *Together with the filiformity brackets (1), the brackets belonging to the family*

$$\mathcal{F}_1 = \{[e_h, e_n],\ z_1 \le h < n\}$$

are enough to define explicitly a n-dimensional complex filiform Lie algebra, with respect to an adapted basis.

Proof. We firstly consider the Jacobi identity $J(e_1, e_{n-2}, e_n) = 0$

$$[[e_1, e_{n-2}], e_n] + [[e_{n-2}, e_n], e_1] + [[e_n, e_1], e_{n-2}] = 0.$$

Since the brackets $[[e_1, e_{n-2}], e_n] \equiv [e_{n-3}, e_n]$ and $[e_{n-2}, e_n]$ are known by hypothesis, the bracket $[e_{n-2}, e_{n-1}] \equiv [-e_{n-1}, e_{n-2}] \equiv [[e_n, e_1], e_{n-2}]$ is obtained.

Similarly, starting from the Jacobi identity $J(e_1, e_h, e_n) = 0$ and by using the previous result, the bracket $[e_h, e_{n-1}]$ is now obtained.

In this way, we obtain the brackets $[e_h, e_{n-2}]$ with $z_1 \le h < n-1$, starting from Jacobi identities and the brackets obtained in the previous steps. Since n is finite, all of the resting brackets can be consecutively deduced by applying repeatedly this procedure. □

Now, let us see how we can apply the above theorem with an example:

Example 2.2. The 8-dimensional complex filiform Lie algebra μ_8^4 (from Goze and Ancocheas's classification (see [1])) is usually defined, with respect to an adapted basis $B = \{e_1, \ldots e_8\}$, by the following nonzero brackets

$$\begin{array}{ll}
[e_1, e_h] = e_{h-1}\ (3 \le h \le 8), & \\
[e_4, e_7] = e_2, & [e_4, e_8] = e_3, \\
[e_5, e_6] = -e_2, & [e_5, e_8] = e_4, \\
[e_6, e_7] = e_2, & [e_6, e_8] = e_5 + e_3, \\
[e_7, e_8] = e_6 + e_4. &
\end{array}$$

Note that with the exception of the brackets corresponding to the filiformity of the algebra, that is, brackets $[e_1, e_h] = e_{h-1}$ $(3 \le h \le 8)$, other seven nonzero brackets are used to define explicitly this algebra.

However, Theorem 2.1 implies that it is enough to give only four of these seven brackets to define the algebra. Actually, these brackets are $[e_4, e_8]$, $[e_5, e_8]$,

$[e_6, e_8]$ and $[e_7, e_8]$, which belong to the family $\mathcal{F}_1$. In this way, *almost* half of the brackets usually given are unnecessary.

To check this fact, we just have to redo the proof. So, by using consecutively the Jacobi Identities given by the triples $(e_1, e_4, e_8), (e_1, e_6, e_7), (e_1, e_5, e_7)$, $(e_1, e_4, e_7), (e_1, e_5, e_6), (e_1, e_4, e_6)$ and (e_1, e_4, e_5), those non used brackets $[e_4, e_7] = e_2$, $[e_5, e_6] = -e_2$, $[e_4, e_6] = 0$ and $[e_4, e_5] = 0$ are obtained and the assert is proved. $\triangleleft$

Theorem 2.3. *Together with the filiformity brackets (1), the brackets belonging to the family:*

$$\mathcal{F}_2 = \{[e_k, e_{k+1}],\ z_2 \leq k < n\}$$

are enough to define explicitly a n-dimensional complex filiform Lie algebra, with respect to an adapted basis.

Proof. This proof is similar to that of Theorem 2.1. We firstly consider the Jacobi identity $J(e_1, e_{k-1}, e_k) = 0$, with $3 \leq k \leq n$, which implies $[e_{k-2}, e_k] = [e_1, [e_{k-1}, e_k]]$. This allows us to compute the bracket $[e_{k-2}, e_k]$, due to $[e_{k-1}, e_k]$ being known by hypothesis.

In a similar way, starting from the brackets

$$[e_{z_2}, e_{z_2+1}],\ \ [e_{z_2+1}, e_{z_2+2}], \ldots, [e_{n-1}, e_n] \qquad (3)$$

the brackets

$$[e_{z_2-1}, e_{z_2+1}],\ \ [e_{z_2}, e_{z_2+2}], \ldots, [e_{n-2}, e_n] \qquad (4)$$

are obtained.

Similarly, from $J(e_1, e_h, e_k) = 0$, with $3 \leq h < k$, we deduce that

$$[e_1, [e_h, e_k]] = [e_{h-1}, e_k] + [e_h, e_{k-1}] \qquad (5)$$

and thus, the bracket $[e_{n-3}, e_n]$ can be computed, since $[e_{n-2}, e_n]$ and $[e_{n-2}, e_{n-1}]$ are already obtained by (4) and (3), respectively. Similarly, $[e_{n-4}, e_{n-1}]$ can also be obtained from previous expressions.

Therefore, as n is finite, all of the resting brackets can be consecutively deduced by repeating this procedure. $\square$

Example 2.4. Let us consider again the same filiform Lie algebra μ_8^4 of Example 2.2. We already showed that, apart from the brackets due to the filiformity of the algebra, it was enough to consider only the four brackets $[e_4, e_8], [e_5, e_8], [e_6, e_8]$ and $[e_7, e_8]$ of the family $\mathcal{F}_1$ to define it, while the other three brackets were not needed to describe the algebra.

Now, Theorem 2.3 shows that it is enough to give only the three brackets $[e_5, e_6], [e_6, e_7]$ and $[e_7, e_8]$ of family $\mathcal{F}_2$ to define the algebra, since the other four brackets ($[e_4, e_7], [e_4, e_8]$, $[e_5, e_8]$ and $[e_6, e_8]$) can be obtained from them. Note that this gives us a saving of more than fifty per cent of the number of the brackets used in the usual definition of this algebra.

This fact can be verified as in Example 2.2, since the Jacobi identities formed by $(e_1, e_7, e_8), (e_1, e_6, e_7), (e_1, e_5, e_6)$, $(e_1, e_4, e_5), (e_1, e_6, e_8), (e_1, e_5, e_8)$ and (e_1, e_4, e_8) involve the following results: $[e_6, e_8] = e_3 + e_5$, $[e_5, e_7] = 0$, $[e_4, e_6] = 0$, $[e_5, e_8] = e_4$, $[e_4, e_8] = e_3$ and $[e_4, e_7] =_2 +e_5$. This finishes the example $\triangleleft$

3. Some applications

In this last section we show two applications of the previous results.

3.1. Relations between Graph Theory and Lie algebras.

A novel recent research tries to progress in Lie Theory by using Graph Theory as a tool. It is true that, at the present, there are just a few works relating both theories, Lie and Graphs, one to each other, but it is a fact that some research papers are appearing in this sense (for instance, see [4], [5] and [10]). The idea lies in the representation of each Lie algebra by a certain graph. In this way, the properties of these graphs can be studied by considering Graph Theory and then be translated to Lie algebras.

For example, other colleagues and myself considered in [4] the family $\mathcal{L}$ of n-dimensional Lie algebras over the field $Z/2Z$, with a basis $\{e_1, \ldots, e_n\}$, in such a way that if $r, s < n$, then $[e_r, e_s] = 0$ and $[e_r, e_n]$ is a linear combination of $e_1, \ldots, e_{n-1}$ (note, however, that these algebras are not filiform). We represented each of them by a square matrix $(n-1) \times (n-1)$, where the element i, j, which can only 0 or 1, is the coefficient of e_j in the bracket $[e_i, e_n]$.

Next, we defined in a natural way a map between $\mathcal{L}$ and the set of simple directed pseudo-graphs (i.e., directed pseudo-graphs with at most one loop in each vertex and without double edges) in such a way that each Lie algebra of $\mathcal{L}$ corresponds with the simple directed pseudo-graph whose adjacency matrix coincides with the matrix of the algebra.

In this way, by using the properties of such pseudo-graphs and an appropriate equivalence relation, we concluded that there exist, up to isomorphism, $4, 6, 14$ and 34 Lie algebras of this family of dimensions $2, 3, 4$ and 5, respectively, over $Z/2Z$.

Moreover, a similar study has been done in [10] by the author jointly with other colleagues for the family of n-dimensional Lie algebras over the field $Z/3Z$, having a basis $\{u_1, \ldots, u_n\}$, such that if $r, s < n$, then $[u_r, u_s] = 0$ and $[u_r, u_n]$ is a linear combination of the basic elements $u_1, \ldots, u_{n-1}$ (note that u_n does not appear in this combination). Now, a Lie algebra of such a type can be represented by a $(n-1) \times (n-1)$ square matrix, where the element i, j is the coefficient of u_j in the bracket $[u_i, u_n]$, which can only be 0, 1 or 2. By using now the set of directed pseudo-graphs in which, at most, two double edges are allowed, we obtained that there exist, at most, 41 Lie algebras belonging to that family.

Then, it is obvious that the less possible number of brackets appearing in the definition of the algebra implies that the corresponding graphs will be easier to deal with. Therefore, the reductions can constitute a step forward to tackle the open problem of the classifications of general Lie algebras.

3.2. Classification of complex filiform Lie algebras of dimension less than or equal to 8.

This subsection is devoted to show the explicit classification (up to isomorphisms) of complex filiform Lie algebras (from now on, CFLAs) of dimension $n \leq 8$, by showing only the brackets belonging to the family $\mathcal{F}_1$ (although in a same way, the family $\mathcal{F}_2$ could also have been used). The brackets $[e_1, e_h] = e_{h-1}$ $(3 \leq h \leq n)$ corresponding to the filiformity are omitted in each algebra for reasons of length, although they must be supposed.

It can be noted that these classifications are very simple up to dimension 7. However, getting the classification for larger dimensions constitutes a quite hard and complicated process, which normally requires the help of a computer. Note also that there exist 34, 104 and 149 CFLAs of dimensions 9, 10 and 11, respectively. The largest dimension for which the explicit classification of these algebras is known is 12. Indeed, there are 496 algebras of this dimension (see [3]).

- CFLAs of dimension 2: λ_2^1: Model algebra.
- CFLAs of dimension 3: λ_3^1: Model algebra.
- CFLAs of dimension 4: λ_4^1: Model algebra.
- CFLAs of dimension 5: λ_5^1: Model algebra. λ_5^2: $[e_4, e_5] = e_2$.
- CFLAs of dimension 6:

$$
\begin{array}{lll}
\lambda_6^1: & \text{Model algebra.} & \\
\lambda_6^2: & [e_5, e_6] = e_2. & \\
\lambda_6^3: & [e_4, e_6] = e_2, & [e_5, e_6] = e_3. \\
\lambda_6^4: & [e_4, e_6] = e_3, & [e_5, e_6] = e_4. \\
\lambda_6^5: & [e_4, e_6] = e_2 + e_3, & [e_5, e_6] = e_3 + e_4.
\end{array}
$$

- CFLAs of dimension 7 (α is a complex parameter):

$$
\begin{array}{llll}
\lambda_7^1: & \text{Model algebra.} & & \\
\lambda_7^2: & [e_6, e_7] = e_2. & & \\
\lambda_7^3: & [e_5, e_7] = e_2 & [e_6, e_7] = e_3. & \\
\lambda_7^4: & [e_5, e_7] = e_2, & [e_6, e_7] = e_3 + e_2. & \\
\lambda_7^5: & [e_4, e_7] = e_2, & [e_5, e_7] = e_3, & [e_6, e_7] = e_4. \\
\lambda_7^6: & [e_4, e_7] = e_2, & [e_5, e_7] = e_3, & [e_6, e_7] = e_4 + e_2. \\
\lambda_7^7: & [e_4, e_7] = -e_2, & [e_6, e_7] = e_3. & \\
\lambda_7^8: & [e_4, e_7] = \alpha\, e_2, & [e_5, e_7] = (\alpha + 1)\, e_3, & [e_6, e_7] = (\alpha + 1)\, e_4.
\end{array}
$$

- CFLAs of dimension 8

A previous remark is convenient at this point. By applying Theorem 2.1 to the general law of these algebras (with respect to an adapted basis), we obtain

$$[e_4, e_8] = a_{4,7}e_3 + a_{4,8}e_2,$$

$$[e_5, e_8] = a_{4,7}e_4 + (a_{4,8} + a_{5,7})e_3 + a_{5,8}e_2,$$

$$[e_6, e_8] = a_{4,7}e_5 + (a_{4,8} + 2a_{5,7})e_4 + (a_{5,8} + a_{6,7})e_3 + a_{6,8}e_2,$$

$$[e_7, e_8] = a_{4,7}e_6 + (a_{4,8} + 2a_{5,7})e_5 + (a_{5,8} + a_{6,7})e_4 + a_{6,8}e_3 + a_{7,8}\, e_2.$$

where $a_{4,7}, \ldots, a_{7,8}$ are complex parameters satisfying a restriction due to Jacobi identities: $a_{4,7}\,(2a_{4,8} + 5a_{5,7}) = 0$. This implies that some parameter families of algebras can appear. Therefore, by using them, the following classification of CFLAs of dimension 8 is obtained, where previous parameters have been replaced by Greek characters.

$$\begin{array}{lll}
\lambda_8^1: & \text{Model algebra.} & \\
\lambda_8^2: & [e_7,e_8]=e_2. & \\
\lambda_8^3: & [e_6,e_8]=e_2, & [e_7,e_8]=e_3. \\
\lambda_8^4: & [e_6,e_8]=e_2, & [e_7,e_8]=e_3+e_2. \\
\lambda_8^5: & [e_5,e_8]=e_2, & [e_6,e_8]=e_3, \quad [e_7,e_8]=e_4. \\
\lambda_8^6: & [e_5,e_8]=e_2, & [e_6,e_8]=e_3+e_2, \quad [e_7,e_8]=e_4+e_3. \\
\lambda_8^7: & [e_5,e_8]=\beta\, e_2, & [e_6,e_8]=(\beta+1)\, e_3, \quad [e_7,e_8]=(\beta+1)e_4. \\
\lambda_8^8: & [e_5,e_8]=\beta\, e_2, & [e_6,e_8]=(\beta+1)\, e_3+e_2, \\
 & [e_7,e_8]=(\beta+1)e_4+e_3. & \\
\lambda_8^9: & [e_4,e_8]=e_2, & [e_5,e_8]=e_3, \\
 & [e_6,e_8]=e_4, & [e_7,e_8]=e_5. \\
\lambda_8^{10}: & [e_4,e_8]=e_2, & [e_5,e_8]=e_3, \\
 & [e_6,e_8]=e_4, & [e_7,e_8]=e_5+e_2. \\
\lambda_8^{11}: & [e_4,e_8]=e_2, & [e_5,e_8]=e_3, \\
 & [e_6,e_8]=e_4+e_2, & [e_7,e_8]=e_5+e_3+\gamma\, e_2. \\
\lambda_8^{12}: & [e_4,e_8]=e_2, & [e_5,e_8]=e_3, \\
 & [e_6,e_8]=e_4+a_{6,7}\, e_3+\delta\, e_2, & [e_7,e_8]=e_5+a_{6,7}\, e_4+\delta\, e_3. \\
\lambda_8^{13}: & [e_4,e_8]=-\, e_2, & [e_6,e_8]=e_4+e_3, \quad [e_7,e_8]=e_5+e_4. \\
\lambda_8^{14}: & [e_4,e_8]=\alpha\, e_2, & [e_5,e_8]=(\alpha+1)\, e_3, \\
 & [e_6,e_8]=(\alpha+2)\, e_4, & [e_7,e_8]=(\alpha+2)\, e_5. \\
\lambda_8^{15}: & [e_4,e_8]=\alpha\, e_2, & [e_5,e_8]=(\alpha+1)\, e_3+e_2, \\
 & [e_6,e_8]=(\alpha+2)\, e_4+e_3, & [e_7,e_8]=(\alpha+2)\, e_5+e_4.
\end{array}$$

$$\begin{array}{llll}
\lambda_8^{16}: & [e_4,e_8]=e_3, & [e_5,e_8]=e_4, & [e_6,e_8]=e_5, \\
 & [e_7,e_8]=e_6. & & \\
\lambda_8^{17}: & [e_4,e_8]=e_3, & [e_5,e_8]=e_4, & [e_6,e_8]=e_5+e_2, \\
 & [e_7,e_8]=e_6+e_3. & & \\
\lambda_8^{18}: & [e_4,e_8]=e_3, & [e_5,e_8]=e_4+e_2, & [e_6,e_8]=e_5+e_3+e_2, \\
 & [e_7,e_8]=e_6+e_4. & & \\
\lambda_8^{19}: & [e_4,e_8]=e_3, & [e_5,e_8]=e_4+e_2, & [e_6,e_8]=e_5+e_3+e_2, \\
 & [e_7,e_8]=e_6+e_4+e_3. & & \\
\lambda_8^{20}: & [e_4,e_8]=e_3-\frac{5}{2}\, e_2, & & [e_5,e_8]=e_4-\frac{3}{2}\, e_3+\frac{1}{2}\, e_2, \\
 & [e_6,e_8]=e_5-\frac{1}{2}\, e_4+\frac{1}{2}\, e_3-\frac{27}{8}\, e_2, & & [e_7,e_8]=e_6-\frac{1}{2}\, e_5+\frac{1}{2}\, e_4-\frac{27}{8}\, e_3. \\
\lambda_8^{21}: & [e_4,e_8]=e_3-\frac{5}{2}\, e_2, & & [e_5,e_8]=e_4-\frac{3}{2}\, e_3+\frac{1}{2}\, e_2, \\
 & [e_6,e_8]=e_5-\frac{1}{2}\, e_4+\frac{1}{2}\, e_3, & & [e_7,e_8]=e_6-\frac{1}{2}\, e_5+\frac{1}{2}\, e_4. \\
\lambda_8^{22}: & [e_4,e_8]=e_3-\frac{5}{2}\, e_2, & & [e_5,e_8]=e_4-\frac{3}{2}\, e_3, \\
 & [e_6,e_8]=e_5-\frac{1}{2}\, e_4, & & [e_7,e_8]=e_6-\frac{1}{2}\, e_5.
\end{array}$$

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