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An Application of the Dieudonné Determinant: Detecting Non-tame Automorphisms

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Abstract. We show how to detect non-tame automorphisms by using a criterion which is based on the Dieudonné determinant and we construct some specific non-tame automorphisms of free metabelian Lie algebras and free Lie algebras of the form $F/\gamma_m(F)^{'}$.
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1. Introduction

Let F be the free Lie algebra generated by the set $X = \{x_1, \dots, x_n\}$ over the field K of characteristic 0. Denote by $F^{'}$ the derived subalgebra of F. We identify a free metabelian Lie algebra L of rank n, with F/F'' in the usual way , where F'' is the derived subalgebra of F'. If an automorphism φ of L can be lifted to an automorphism of F then we say that φ is a tame automorphism of L. The questions of lifting automorphism are naturally related to the problem of finding appropriate necessary and sufficent conditions for an endomorphism of F to be an automorphism.

Drensky and Gupta [5] have proved that free nilpotent Lie algebras have non-tame automorphisms. In the case of free metabelian Lie algebras Bahturin and Nabiyev [1] have established the existence of non-tame automorphisms.

Birman [3], Reutenauer [7],Umirbaev [10] and Yagzhev [11] have given a matrix characterization of automorphisms among arbitrary endomorphisms as follows: Define the Jacobian matrix $J_\varphi = (\frac{\partial \varphi(x_i)}{\partial x_j})_{1 \le i,j \le n}$,where $\frac{\partial}{\partial x_j}$ denotes partial Fox derivation with respect to x_j in the universal enveloping algebra $U(F)$ [6]. Then φ is an automorphism if and only if the matrix J_φ is invertible over $U(F)$. A generalization of this result has been proved by Shpilrain[8]. He has proved that any subset $\{y_1, \dots, y_n\}$ generates the free Lie algebra F modulo $R^{'}$ if and only if the matrix $(\sigma_R(\frac{\partial y_i}{\partial x_j})_{1 \le i,j \le n}$ is invertible over $U(F/R)$, where R is an ideal of F and $\sigma_R\colon\ U(F) \to U(F/R)$ is the natural homomorphism. This result has been used in obtaining a powerful necessary condition of tameness. An approach giving a

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necessary condition for a matrix over integral group ring $Z(G)$ of a free group G to be invertible is due to Shpilrain [9]. He has used a non-commutative determinant to obtain this necessary condition. This condition was earlier mentioned for free Lie algebras in [2] by Y. Bahturin and V. Shpilrain. Our purpose is to give some specific examples of non-tame automorphisms by using Bahturin's and Shpilrain's technique for free Lie algebras.

In this paper we use a necessary condition which is given in [2] for a matrix over $U(F)$ to be invertible. This condition yields a method for detecting non-tame automorphisms of the free metabelian Lie algebra $L = F/F''$. This method is explicitly based on a non-commutative determinant: We consider the image of the Jacobian matrix in an appropriate algebra $U(F)/\Delta^m$, where Δ is the augmentation ideal of $U(F)$, evaluate the Dieudonné determinant of this image and then observe that this determinant should be equal to an element of the field K. We also give some applications of this technique to detecting non-tame automorphisms. First we give two examples illustrating the difference between the usual determinant of the abelianized matrix and our non-commutative determinant. Then we present a non-tame automorphism of the free metabelian Lie algebra L. We denote the multiplication in F by the the commutator $[a, b]$.

2. Preliminaries

Let $U(F)$ be the universal enveloping algebra of the free Lie algebra F and Δ its augmentation ideal, that is, the kernel of the augmentation homomorphism $\varepsilon : U(F) \to K$ defined by $\varepsilon(x_i) = 0,\ i = 1, 2, \ldots, n$. If $R \neq F$ is an ideal of F then we denote by Δ_R the ideal of $U(F)$ generated by the ideal R. Note that Δ_R is the kernel of the natural homomorphism $\sigma_R : U(F) \to U(F/R)$.

In [6] Fox gave a detailed account of the differential calculus in a free group ring . Since any associative algebra is naturally imbedded in a free group algebra, most of the technical results remain valid for free associative algebras.

We introduce here Fox derivations as the mappings $\frac{\partial}{\partial x_i} : U(F) \to U(F), i = 1, 2, \ldots, n$, satisfying the following conditions whenever $\alpha, \beta \in K,\ u, v \in U(F)$:

1. $\frac{\partial x_j}{\partial x_i} = \delta_{i,j}$ (Kronecker delta),
2. $\frac{\partial}{\partial xi}(\alpha u + \beta v) = \alpha \frac{\partial u}{\partial x_i} + \beta \frac{\partial v}{\partial x_i}$,
3. $\frac{\partial}{\partial xi}(uv) = \frac{\partial u}{\partial x_i}\varepsilon(v) + u \frac{\partial v}{\partial x_i}$.

It is an obvious consequence of the definitions that $\frac{\partial}{\partial x_i}(1) = 0$.

The ideal Δ is a free left $U(F)-$module with a free basis $\{x_1, \ldots, x_n\}$ and the mappings $\frac{\partial}{\partial x_i}$ are projections on the corresponding free cyclic direct summands. Thus any element $f \in \Delta$ can be uniquely written in the form $f = \sum\limits_i \frac{\partial f}{\partial x_i} x_i$.

Throught out this paper we will need the following technical lemmas. The first lemma is an immediate consequence of the definitions.

Lemma 2.1. *Let J be an arbitrary ideal of F and let $u \in \Delta$. Then $u \in J\Delta$ if and only if $\frac{\partial u}{\partial x_i} \in J$ for each $i, 1 \leq i \leq n$.*

Proof of the next lemma can be found in [11]

Lemma 2.2. *Let R be an ideal of F and let $u \in F$. Then $u \in \Delta_R\Delta$ if and only if $u \in R'$.*

3. The Dieudonné Determinant

In this section we consider an algebra $H_m = U(F)/\Delta^m$, $m \geq 2$, and we describe the construction of a non-commutative determinant corresponding to Dieudonné determinant.This construction is similar to that of [2] but more understandable and detailed than it. Any invertible square matrix over the universal enveloping algebra $U(F)$ (i.e. a matrix from the general lineer group $GL_k(U(F))$ for some $k \geq 1$) is also invertible over H_m. Such a matrix over H_m has at least one invertible element in every row and in every column (see the Remark 3 below). Therefore by using elementary transformations every invertible square matrix over H_m can be written as a product of elementary and diagonal matrices. By an elementary matrix we mean a matrix which differs from the identity matrix by a single entry outside the diagonal.

Now take the multiplicative group H_m^* of all invertible elements of H_m. It is clear that invertible elements of H_m are of the form $\alpha + v + \Delta^m$ with $v \in \Delta$, $0 \neq \alpha \in K$. Since $(\alpha + v)(\alpha^{-1} - \alpha^{-2}v + \alpha^{-3}v^2 + \ldots + (-1)^{m-1}\alpha^{-m}v^{m-1}) = 1(mod\Delta^m)$, we have modulo Δ^m:

$$(\alpha + v)^{-1} = \alpha^{-1} - \alpha^{-2}v + \alpha^{-3}v^2 + \ldots + (-1)^{m-1}\alpha^{-m}v^{m-1}.$$

Hence the commutator subgroup (H_m^*, H_m^*) of the group H_m^* is generated as a group modulo Δ^m by elements of the form

$$(1-v)(1-w)(1-v)^{-1}(1-w)^{-1} = (1-v)(1-w)(1+v+\ldots+v^{m-1})(1+w+\ldots+w^{m-1})$$

with $v, w \in \Delta$. Let S_m be the subsemigroup of $U(F)$ generated by all such elements .

Remark 3.1. Let $M = (m_{i,j}) \in GL_n(H_m)$. Consider the matrix $\varepsilon(M)$ of the augmentations of elements of M. It is clear that the matrix $\varepsilon(M)$ is invertible over K. This means that we have at least one element with non-zero augmentation in every row and in every column of the matrix M. Hence M has at least one invertible element in every row and in every column. This allows us to reduce the matrix M to a diagonal form by applying elementary transformations to its rows as the following:

Select an invertible element m_{ij} in the j-th column. Subtract the i-th row multiplied on the left by $m_{kj}m_{ij}^{-1}$ from the k-th row, where $1 \leq k \leq n$, $k \neq i$. Then all elements in the $j-th$ column will be zero except m_{ij}. Now apply this operation to all columns and then change the rows to obtain a diagonal matrix.

Now given a matrix $A \in GL_n(H_m)$ over H_m ,we define its Dieudonné determinant using the fact that every invertible matrix over H_m can be diagonalizable. For every arbitrary permutation $\sigma \in S_n$ we associate the permutation matrix $P(\sigma) = (\delta_{i,\sigma(j)})$, where δ denotes the Kronecker symbol. Then there exists a decomposition $A = TDP(\sigma)V$, where

$$T = \begin{bmatrix} 1 & * & * \\ & \ldots & * \\ 0 & & 1 \end{bmatrix}, \quad D = diag(d_1, \ldots, d_n), \quad V = \begin{bmatrix} 1 & . & 0 \\ * & .. & . \\ * & * & 1 \end{bmatrix},$$

σ a permutation, $P(\sigma)$ the permutation matrix corresponding to σ and D and σ are unique with these properties (see [4] Theorems 1 and 2 in chap.19).

Definition 3.2. Let $A \in GL_n(H_m)$ have the decomposition of the form $A = TDP(\sigma)V$, $D = diag(d_1, \dots, d_n)$. The Dieudonné determinant of A is $D_m(A) = \pi(sgn(\sigma)d_1 \cdots d_n)$, where π is the canonical mapping $H_m^* \to H_m^*/(H_m^*, H_m^*)$.

Now we can give the following proposition which is similar to Y. Bahturin and V. Shpilrain's result [2].

Proposition 3.3. *Let $A \in GL_n(U(F))$ and $\det_m(A)$ be an arbitrary preimage of $D_m(A)$ in $U(F)$. Then for any $m \geq 2$ we have*

$$det_m(A) = (\alpha + v)g_m + w_m,$$

where $\alpha \in K$, $\alpha \neq 0, v \in \Delta \backslash \Delta^m, g_m \in S_m, w_m \in \Delta^m$.

Proof. Let $A \in GL_n(U(F))$. Then it is invertible over H_m. Now consider the multiplicative group H_m^* of all invertible elements of H_m and come up with the determinant $D_m(A)$ using the fact that in the algebra H_m, every element with non-zero augmentation is invertible. Hence the result follows. ∎

Now we have

Corollary 3.4. ([2]) *Let φ be an automorphism of F and $\det_m(J_\varphi)$ be an arbitrary preimage of $D_m(J_\varphi)$ in $U(F)$. Then for any $m \geq 2$ we have*

$$det_m(J_\varphi) = \alpha g_m + w_m,$$

where $\alpha \in K, \alpha \neq 0, g_m \in S_m, w_m \in \Delta^m$.

Proof. Let φ be an automorphism of F. It is well known that φ is a composition of elementary automorphisms of F. It is routine to show that the Jacobian matrix of any elementary automorphism of F can be written as a product of elementary and diagonal matrices. This result and the equality $J_{\alpha\circ\beta} = \alpha(J_\beta)J_\alpha$ for the composition $\alpha \circ \beta$ of any two automorphisms α, β of F allows us to write the Jacobian matrix J_φ of φ in the form $J_\varphi = E \cdot D$, where E is a product of elementary matrices and D is a diagonal matrix with diagonal elements $d_1, d_2, \dots, d_n$. Since the only invertible elements of $U(F)$ are the elements of the field K , the diagonal elements of D must belong to K. Now consider the algebra $H_m = U(F)/\Delta^m$. The image of J_φ over H_m is also invertible. Let $\overline{J_\varphi} = \overline{E} \cdot \overline{D}$ be this image. Then the diagonal elements $d_i + \Delta^m$ of the matrix $\overline{D}$ and their product $\prod_{i=1}^{n}(d_i + \Delta^m) = d_1 \cdot d_2 \cdots d_n + \Delta^m$ cannot belong to the commutator subgroup (H_m^*, H_m^*). Hence the Dieudonné determinant of J_φ is of the form

$$D_m(J_\varphi) = \alpha + \Delta^m,$$

where $\alpha = d_1 \cdot d_2 \cdots d_n$. It is clear that an arbitrary preimage of $D_m(J_\varphi)$ in $U(F)$ is of the form $\alpha g_m + w_m$, where $g_m \in S_m, w_m \in \Delta^m$. ∎

Corollary 3.4 yields a necessary condition for an endomorphism of F to be an automorphism . The main point is that we have to check whether or not the condition of the Corollary 3.4 is contradicted.

The following examples show that the usual commutative determinant is only good for distinguishing automorphisms modulo F'' whereas a non-commutative determinant allows a more subtle analysis.

Example 3.5. Consider the endomorphism φ of F defined as

$$\begin{aligned}\varphi \ : \ & x_1 \to x_1 + \left[\left[x_1, \left[x_{j_1}, x_{j_2}\right]\right], x_{j_3}\right], \\ & x_i \to x_i, i \neq 1,\end{aligned}$$

where $j_\alpha \neq j_\beta$ for $1 \leq \alpha, \beta \leq 3$ and $j_\gamma \neq 1$ for $\gamma = 1, 2$. Then the image of the Jacobian matrix J_φ over $U(F/F')$ has zeroes below the diagonal and units on the diagonal. Thus it is invertible. This implies that φ induces an automorphism of L.

Let us consider the image

$$\begin{bmatrix} 1 + \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} & \dots & \frac{\partial g}{\partial x_n} \\ 0 & 1 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

of the Jacobian matrix J_φ over $H_4 = U(F)/\Delta^4$,where $g = [[x_1, [x_{j_1}, x_{j_2}]], x_{j_3}]$. Recall that the commutator subgroup $(H_4^*, H_4^*)$of the multiplicative group H_4^* is generated as a group by elements of the form

$$(1 - v)(1 - w)(1 + v + v^2 + v^3)(1 + w + w^2 + w^3) + \Delta^4$$

where $v, w \in \Delta$. Straightforward calculation shows that an element of this form may be written as

$$1 + vw - wv + v^2 w - vwv + wvw - w^2 v + \Delta^4.$$

Denote by $\langle \Delta, \Delta \rangle$ the subspace of $U(F)$ generated by all elements of the form $fg - gf$, $f, g \in \Delta$. Hence the elements of the commutator subgroup (H_4^*, H_4^*) of H_4^* are of the form $1 + z$, where $z \in \langle \Delta, \Delta \rangle$. Now we can compute $D_4(J_\varphi)$:

$$D_4(J_\varphi) = \pi(1 + [x_1, [x_{j_1}, x_{j_2}]] \frac{\partial x_{j_3}}{\partial x_1} - x_{j_3}[x_{j_1}, x_{j_2}]) = 1 - x_{j_3}[x_{j_1}, x_{j_2}] + \Delta^4.$$

Therefore an arbitrary preimage of $D_4(J_\varphi)$ in $U(F)$ must be

$$det_4(J_\varphi) = 1 + [x_1, [x_{j_1}, x_{j_2}]] \frac{\partial x_{j_3}}{\partial x_1} - x_{j_3}[x_{j_1}, x_{j_2}] (mod\ \Delta^4).$$

If φ were an automorphism, then we would have

$$1 + [x_1, [x_{j_1}, x_{j_2}]] \frac{\partial x_{j_3}}{\partial x_1} - x_{j_3}[x_{j_1}, x_{j_2}] = \alpha g_4 (mod\ \Delta^4).$$

for some $0 \neq \alpha \in K$, $g_4 \in S_4$. This yields $\alpha = 1$ and $g_4 - 1 \in \langle \Delta, \Delta \rangle$ and

$$x_{j_3}[x_{j_1}, x_{j_2}] = 0(mod\ \langle \Delta, \Delta \rangle + \Delta^4).$$

But this is impossible since

$$x_{j_3}[x_{j_1}, x_{j_2}] = x_{j_3} x_{j_1} x_{j_2} - x_{j_3} x_{j_2} x_{j_1}.$$

cannot belong to $\langle \Delta, \Delta \rangle + \Delta^4$. This contradiction proves that φ is not an automorphism.

Example 3.6. Let ψ be the endomorphism of F defined as

$$\begin{aligned}\psi \ : \ & x_1 \to x_1 + [[x_1,[x_2,x_3]],x_4], \\ & x_i \to x_i, i \neq 1.\end{aligned}$$

Consider the Jacobian matrix J_ψ:

$$J_\psi = \begin{bmatrix} 1 + x_4\,[x_2,x_3] & x_4x_1x_3 & -x_4x_1x_2 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & .. & 0 \\ . \quad . & . \quad . & . & .. & . \\ 0 & 0 & 0 & .. & 1 \end{bmatrix}$$

The image of elements of J_ψ in $U(F/F')$ determines the image

$$\begin{bmatrix} 1 & x_4x_1x_3 & -x_4x_1x_2 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & .. & 0 \\ . \quad . & . \quad . & . & .. & . \\ 0 & 0 & 0 & .. & 1 \end{bmatrix}$$

of J_ψ over $U(F/F')$. Hence above matrix is invertible over $U(F/F')$. This implies that ψ is an automorphism of the free metabelian Lie algebra L.

Now, let us consider the image of J_ψ over $H_4 = U(F)/\Delta^4$ and compute $D_4(J_\psi)$:

$$D_4(J_\psi) = 1 + x_4[x_2,x_3] + \Delta^4.$$

If ψ were an automorphism of F we would have

$$det_4(J_\psi) = 1 + x_4[x_2,x_3] = \alpha g_4(mod\ \Delta^4)$$

for some $0 \neq \alpha \in K$, $g_4 \in S_4$ by Corollary 3.4. Then it follows that $\alpha = 1$ and $1 + x_4[x_2,x_3] \in S_4(mod\ \Delta^4)$. This yields $x_4[x_2,x_3] = 0(mod\ \langle\Delta,\Delta\rangle + \Delta^4)$. Which is impossible. Thus ψ can not be an automorphism .

These examples illustrate the difference between the usual commutative determinant and the non-commutative determinant.

Remark 3.7. By Proposition 3.3 we obtain a condition for detecting non-invertibility of a square matrix M over $U(F)$. First we compute $det_m(M)$ starting from $m = 1$ and carry on the computation until we have the condition of the Proposition 3.3 contradicted.

4. Applications of Non-Commutative Determinants: Non-tame Automorphisms

In this section we are going to give some applications of the Dieudonné determinant. Let L be the free metabelian Lie algebra F/F''. We denote by $\gamma_n(F)$ the n-th term of the lower central series of F.

Theorem 4.1. *The endomorphism φ defined as*

$$\begin{aligned} \varphi \ : \ & x_1 \to x_1 + [[x_1,[x_{j_1},x_{j_2}]],x_{j_3}], j_\alpha \neq j_\beta, 1 \leq \alpha,\beta \leq 3, j_\gamma \neq 1, \gamma = 1,2 \\ & x_i \to x_i, i \neq 1 \end{aligned}$$

is a non-tame automorphism of L.

Proof. Suppose that for some $u_j \in F'', j = 1,2,\ldots,n$ we have an automorphism ψ of F induced by

$$\begin{aligned} \psi \ : \ & x_1 \to x_1 + [[x_1,[x_{j_1},x_{j_2}]],x_{j_3}] + u_1, j_\alpha \neq j_\beta, 1 \leq \alpha,\beta \leq 3, j_\gamma \neq 1, \gamma = 1,2 \\ & x_i \to x_i + u_i, i \neq 1 \end{aligned}$$

Since ψ is an automorphism , the Jacobian matrix

$$J_\psi = \begin{bmatrix} 1 + \frac{\partial g}{\partial x_1} + \frac{\partial u_1}{\partial x_1} & \frac{\partial g}{\partial x_2} + \frac{\partial u_1}{\partial x_2} & \ldots & \frac{\partial g}{\partial x_n} + \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & \ldots & \frac{\partial u_2}{\partial x_n} \\ . & . & \ldots & . \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \ldots & 1 + \frac{\partial u_n}{\partial x_n} \end{bmatrix}$$

is invertible over $U(F)$,where $g = [[x_1,[x_{j_1},x_{j_2}]],x_{j_3}], j_\alpha \neq j_\beta, 1 \leq \alpha,\beta \leq 3, j_\gamma \neq 1, \gamma = 1,2$. Denote $R = F'$. Then the diagonal elements of J_ψ have the form $1+v, v \in \Delta_R$ and they are all invertible modulo Δ_R^2 . All the other elements of J_ψ except those in the first row belong to Δ_R by Lemma 2.1 and Lemma 2.2. Note that $\Delta_R^2 \subset \Delta^4$. Let us consider the image of J_ψ under the canonical mapping $\eta : U(F) \to U(F)/\Delta^4$. We can reduce the matrix $\eta(J_\psi)$ to a diagonal form by applying elementary transformations to its rows. Let $w_1 = \frac{\partial g}{\partial x_1} + \frac{\partial u_1}{\partial x_1}$. Then the inverse of the diagonal element $1 + w_1$ of $\eta(J_\psi)$ is $1 - w_1$.

Now we are going to apply the following elementary transformations:

Subtract the first row multiplied on the left by $\frac{\partial u_i}{\partial x_1}(1 - w_1)$ from the i-th row , $i = 2,\ldots,n$. We get

$$\begin{bmatrix} 1 + w_1 & \frac{\partial g}{\partial x_2} + \frac{\partial u_1}{\partial x_2} & \ldots & \frac{\partial g}{\partial x_n} + \frac{\partial u_1}{\partial x_n} \\ 0 & 1 + \frac{\partial u_2}{\partial x_2} & \ldots & \frac{\partial u_2}{\partial x_n} \\ . \quad . & . \quad . & \ldots & . \\ 0 & \frac{\partial u_n}{\partial x_2} & \ldots & 1 + \frac{\partial u_n}{\partial x_n} \end{bmatrix}$$

over $U(F)/\Delta^4$. Using the diagonal elements $a_{22},\ldots,a_{nn}$ we apply similar elementary transformations to clear all the off diagonal elements of the matrix . After applying all of these transformations we obtain the following diagonal matrix over $U(F)/\Delta^4$.

$$\begin{bmatrix} 1 + w_1 & 0 & \ldots & 0 \\ 0 & 1 + \frac{\partial u_2}{\partial x_2} & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & 1 + \frac{\partial u_n}{\partial x_n} \end{bmatrix}.$$

Now consider the algebra $H_4 = U(F)/\Delta^4$. Recall that the invertible elements of the multiplicative subgroup H_4^* are of the form

$$\alpha + w + \Delta^4,$$

where $0 \neq \alpha \in K, w \in \Delta$. Hence elements from the commutator subgroup of the group H_4^* have the form

$$1 + v + \Delta^4, v \in \langle \Delta, \Delta \rangle$$

(see the computation in the example 3.6). Applying Corollary 3.4 we obtain

$$1 + [x_1, [x_{j_1}, x_{j_2}]] \frac{\partial x_{j_3}}{\partial x_1} - x_{j_3}[x_{j_1}, x_{j_2}] + \sum \frac{\partial u_i}{\partial x_i} = \alpha g_4 (mod\ \Delta^4),$$

for some $0 \neq \alpha \in K, g_4 \in S_4$. This implies $\alpha = 1$ and $g_4 - 1 \in \langle \Delta, \Delta \rangle$. Since $[x_1, [x_{j_1}, x_{j_2}]] \frac{\partial x_{j_3}}{\partial x_1} \in \langle \Delta, \Delta \rangle$, we have $x_{j_3}[x_{j_1}, x_{j_2}] + \sum \frac{\partial u_i}{\partial x_i} = 0 (mod\ \langle \Delta, \Delta \rangle + \Delta^4)$. We know that the element $h = x_{j_3}[x_{j_1}, x_{j_2}]$ does not belong to $\langle \Delta, \Delta \rangle + \Delta^4$. Hence we have to compensate it by $\sum \frac{\partial u_i}{\partial x_i}$. Since $h \in \Delta^3$ we will examine monomials of weight 4 from F'' . Since h involves only the generators $x_{j_1}, x_{j_2}, x_{j_3}$, it is sufficient to consider the monomials of the following form:

$$\begin{aligned} u_{j_1} &= [[x_{j_2}, x_{j_1}], [x_{j_3}, x_{j_1}]], \\ u_{j_2} &= [[x_{j_1}, x_{j_2}], [x_{j_3}, x_{j_2}]], \\ u_{j_3} &= [[x_{j_1}, x_{j_3}], [x_{j_2}, x_{j_3}]]. \end{aligned}$$

Consider the expansions of $\frac{\partial u_{j_k}}{\partial x_{j_k}}$ modulo $\langle \Delta, \Delta \rangle + \Delta^4, k = 1, 2, 3$.

$$\begin{aligned} \frac{\partial u_{j_1}}{\partial x_{j_1}} &= [x_{j_2}, x_{j_1}]x_{j_3} - [x_{j_3}, x_{j_1}]x_{j_2} (mod\ \langle \Delta, \Delta \rangle + \Delta^4), \\ \frac{\partial u_{j_2}}{\partial x_{j_2}} &= [x_{j_1}, x_{j_2}]x_{j_3} - [x_{j_3}, x_{j_2}]x_{j_1} (mod\ \langle \Delta, \Delta \rangle + \Delta^4), \\ \frac{\partial u_{j_3}}{\partial x_{j_3}} &= [x_{j_1}, x_{j_3}]x_{j_2} - [x_{j_2}, x_{j_3}]x_{j_1} (mod\ \langle \Delta, \Delta \rangle + \Delta^4). \end{aligned}$$

Since

$$\sum_{k=1}^{3} \frac{\partial u_{j_k}}{\partial x_{j_k}} = 2[x_{j_1}, x_{j_3}]x_{j_2} (mod\ \langle \Delta, \Delta \rangle + \Delta^4),$$

$\sum\limits_{k=1}^{3} \frac{\partial u_{j_k}}{\partial x_{j_k}}$ can not compensate h. This completes the proof. ∎

Theorem 4.2. *The endomorphism*

$$\begin{aligned} \varphi \ :\ & x_1 \to x_1 + [x_1, v], \\ & x_i \to x_i, i \neq 1 \end{aligned}$$

induces a non-tame automorphism of $F/\gamma_m(F)'$, *where*

$$v = [[\ldots [x_{j_1}, x_{j_2}], \ldots], x_{j_m}], \quad j_k \neq 1, k = 1, .., m, \quad m \geq 3.$$

Proof. Suppose by way of contradiction that for some $u_j \in \gamma_m(F)'$ we have an automorphism ψ of F induced by

$$\begin{aligned} \psi \ :\ & x_1 \to x_1 + [x_1, v] + u_1, \\ & x_i \to x_i + u_i, i \neq 1. \end{aligned}$$

Now the proof goes along the same lines as in Theorem 4.1. But we consider the image of the Jacobian matrix J_ψ over the algebra $H_{2m-1} = U(F)/\Delta^{2m-1}$. As we have seen in the proof of Theorem 4.1, we obtain

$$det_{2m-1}(J_\psi) = 1 - v \ (mod\ \Delta^{2m-1}).$$

Applying Corollary 3.4 we get

$$1 - v = 1 - [[\ldots[x_{j_1}, x_{j_2}], \ldots], x_{j_m}] \in S_{2m-1}(mod\ \Delta^{2m-1}).$$

Which is not the case; indeed the element v has the form $uw - wu$, where $u = [[\ldots[x_{j_1}, x_{j_2}], \ldots], x_{j_{m-1}}], w = x_{j_m}$. Straightforward calculations show that

$$1 - v = 1 - [u, w] = (1+u)(1-w)(1+u)^{-1}(1-w)^{-1} + wuw - w^2u(mod\ \Delta^{2m-1}).$$

If the form of the element $1 - v$ were

$$(1-a)(1-b)(1-a)^{-1}(1-b)^{-1}(mod\ \Delta^{2m-1})$$

then we would have $wuw - w^2u = 0(mod\ \Delta^{2m-1})$. This contradiction completes the proof. ∎

References

[1] Bahturin, Yu., and S. Nabiyev, *Automorphisms and Derivations of Abelian Extensions of some Lie Algebras*, Abh. Math. Sem. Uni. Hamburg **62** (1992), 43–54.

[2] Bahturin, Yu., and V. Shpilrain, *On Lie algebras with wild automorphisms*, Results Math. **28** (1995), 209–213.

[3] Birman, J. S., *An Inverse Function Theorem for Free Groups*, Proc. Amer. Math. Soc. **41** (1973), 634–638.

[4] Draxl, P. K., "Skew Fields," London Mathematical Society Lecture Notes Series 83, Cambridge Univ. Press, 1983.

[5] Drensky, V., and C. K. Gupta, *Automorphisms of Free Nilpotent Lie Algebras*, Can J. Math **11** (1990), 259–279.

[6] Fox, R. H., *Free Differantial Calculus, I.Derivation in the Free Group Ring*, Ann. of Math. **2** (1953), 547–560.

[7] Reutenauer, C., *Applications of a Noncommutative Jacobian Matrix*, J. Pure Appl. Alg. **77** (1992), 169–181.

[8] Shpilrain, V., *On Generators of L/R^2 Lie Algebras*, Proc. Amer. Math. Soc. **119** (1993), 1039–1043.

[9] —, *Non-commutative Determinants and Automorphisms of Groups*, Comm. Algebra **25** (1997), 559–574.

[10] Umirbaev, U. U., *On Schreier Varieties of Algebras*, Comm. Algebra **33** (1994), 317–340; English transl.: Algebra and Logic **33** (1994), 180–193.

[11] Yagzhev, A. V., *Endomorphism of Free Algebras*, Sibirsk. Math. Zh. **21** (1980), 181–192; English transl.: Siberian Math. J. **21** (1980), 133–141.

[12] Yunus, I. A., *On the Fox Problem for Lie Algebras*, Uspekhi Math.Nauk **39** (1984), 251–252 ; English transl.: Russian Math. Surveys **39** (1984), 221–222.

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