Linear Maps Preserving Fibers

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Abstract. Let $G \subset \text{GL}(V)$ be a complex reductive group where $\text{dim} \ V < \infty$, and let $\pi: V \to V/G$ be the categorical quotient. Let $N := \pi^{-1}(0)$ be the null cone of $V$, let $H_0$ be the subgroup of $\text{GL}(V)$ which preserves the ideal $I$ of $N$ and let $H$ be a Levi subgroup of $H_0$ containing $G$. We determine the identity component of $H$. In many cases we show that $H = H_0$. For adjoint representations we have $H = H_0$ and we determine $H$ completely. We also investigate the subgroup $G_F$ of $\text{GL}(V)$ preserving a fiber $F$ of $\pi$ when $V$ is an irreducible cofree $G$-module.

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1. Introduction

Our base field is $\mathbb{C}$, the field of complex numbers. Let $V$ be a finite dimensional $G$-module where $G \subset \text{GL}(V)$ is reductive. Let $R$ denote $\mathbb{C}[V]$. We have the categorical quotient $\pi: V \to V/G$ dual to the inclusion $R^G \subset R$. Let $N_G := \pi^{-1}(0)$ (or just $N$) denote the null cone. Let $G_0 = \{g \in \text{GL}(V) \mid f \circ g = f \text{ for all } f \in R^G\}$. Let $H_0$ denote the subgroup of $\text{GL}(V)$ which preserves $N_G$ schematically. Equivalently, $H_0$ is the group preserving the ideal $I = R^G_+R$ where $R^G_+$ is the ideal of invariants vanishing at 0. Let $G_1$ be a Levi factor of $G_0$ containing $G$ and let $H$ denote a Levi factor of $H_0$ containing $G_1$. We show that $H^0 \subset G_1 \text{GL}(V)^{G_1}$, hence that $H^0 \subset G_1 \text{GL}(V)^{G_1}$. In many cases $H_0$ and $G_0$ are reductive, for example, if $V$ is irreducible. In the case that $V = \mathfrak{g}$ is a semisimple Lie algebra and $G$ its adjoint group we show that $H = H_0 = (\mathbb{C}^*)^r \text{Aut}(\mathfrak{g})$ where $r$ is the number of simple ideals in $\mathfrak{g}$. We also obtain information about the subgroup of $\text{GL}(\mathfrak{g})$ preserving a fiber of $\pi$ (other than the zero fiber). We have similar results in the case that $V$ is a cofree $G$-module. Our results generalize those of Botta, Pierce and Watkins [1] and Watkins [12] for the case $\mathfrak{g} = \mathfrak{sl}_n$.

Finally, we show that if $G \subset G' \subset \text{GL}(V)$ where $G'$ is connected reductive such that $\pi$ and $\pi': V \to V/G'$ have a common fiber, then $R^G = R^{G'}$.

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2. Equal fibers

Let $G \subset G' \subset \text{GL}(V)$ be reductive where $G'$ is connected. We have quotient mappings $\pi: V \to V//G$ and $\pi': V \to V//G'$. Let $\rho: V//G \to V//G'$ denote the canonical map.

**Theorem 2.1.** Suppose that $\rho^{-1}(z')$ is a fiber in $X := V//G$. Since $\rho$ is surjective, the minimal dimension of any irreducible component of a fiber is the difference in the dimensions of $X$ and $X'$, so we have that $\dim X = \dim X'$. Then there is a nonempty open subset $U$ of $X'$ such that the fiber of $\rho$ over any point of $U$ is finite. But for $z' \in X'$, the fiber $\rho^{-1}(z')$ is connected since $G'$ is connected. Hence the fiber $\rho^{-1}(z') = \pi((\pi')^{-1}(z'))$ is connected. It follows that $\rho: \rho^{-1}(U) \to U$ is 1-1 and onto, hence birational. Thus $\rho$ is an isomorphism [3, II.3.4]

**Proof.** The hypothesis implies that there is a point $z' \in X' := V//G'$ such that $\rho^{-1}(z')$ is a point in $X := V//G$. Since $\rho$ is surjective, the minimal dimension of any irreducible component of a fiber is the difference in the dimensions of $X$ and $X'$, so we have that $\dim X = \dim X'$. Then there is a nonempty open subset $U$ of $X'$ such that the fiber of $\rho$ over any point of $U$ is finite. But for $z' \in X'$, the fiber $\rho^{-1}(z')$ is connected since $G'$ is connected. Hence the fiber $\rho^{-1}(z') = \pi((\pi')^{-1}(z'))$ is connected. It follows that $\rho: \rho^{-1}(U) \to U$ is 1-1 and onto, hence birational. Thus $\rho$ is an isomorphism [3, II.3.4]

**Remark 2.2.** Solomon [10, 11] has classified many of the pairs of groups $G \subset G' \subset \text{GL}(V)$ with the same invariants, including the case where $V$ is irreducible. Often, $R^G = R^{G'}$ forces $G = G'$. Suppose that $(V, G)$ is generic, i.e., it has trivial principal isotropy groups and the complement of the set of principal orbits has codimension two in $V$. Then $R^G = R^{G'}$ implies that $G = G'$ [9].

3. Groups preserving the ideal of $\mathcal{N}$

Let $V$ be a $G$-module. We assume that $G$ is a Levi subgroup of $G_0$. Let $H$ be a Levi subgroup of $H_0$ containing $G$. Our aim is to show that $H^0$ is generated by $\text{GL}(V)^G$ and $G^0$.

**Proposition 3.1.** Let $V$, $G$ and $H$ be as above. Then $G$ is normal in $H$.

**Proof.** Let $p_1, \ldots, p_r$ be a set of minimal homogeneous generators of $R^G$. Let $d_1 < d_2 < \cdots < d_s$ be the distinct degrees of the $p_i$. Then clearly $H$ preserves the span $W_1$ of the $p_i$ of degree $d_1$. Assuming that $s > 1$, let $W'_1$ be the span of the $p_i$ of degree $d_2$. Then $H$ stabilizes $W_0 := R_{d_2-d_1}W_1$ and $H$ stabilizes $W := W'_2 + W_0 = I \cap R_d$ where $R_d$ for $d \in \mathbb{N}$ denotes the elements of $R$ homogeneous of degree $d$. Note that $W'_2 \cap W_0 = W'_2 \cap R^G : W_1 = 0$. Since $H$ is reductive, there is an $H$-stable subspace $W'_2$ of $W$ complementary to $W_0$. Since $G$ acts trivially on $W'_2$, it acts trivially on $W/W_0$ and on $W_2$. Continuing in this way we obtain $H$-modules $W_1, \ldots, W_s$ consisting of $G$-invariant functions such that $W' := W_1 + \cdots + W_s$ generates $R^G$. Clearly $G$ is the kernel of the action of $H$ on $W'$.

**Corollary 3.2.** Suppose that $H_0$ is reductive. Then $G_0$ is reductive and normal in $H_0$.

Since $G^0$ is reductive, $H^0$ acts on $G^0$ by inner automorphisms. Hence $H^0 = H_1G^0$ where $H_1 := Z_H(G^0)^0$ is the connected centralizer of $G^0$ in $H$. 


Lemma 3.3. Let \( g \in G \). Then there is a homomorphism \( \theta : H_1 \rightarrow Z(G^0) \) such that \( ghg^{-1} = \theta(h)h \), \( h \in H_1 \).

Proof. Let \( h \in H_1 \). Since conjugation by \( h \) preserves the connected components of \( G \) there is an element \( \theta(h) \in G^0 \) such that \( h^{-1}g^{-1}h^{-1} = g^{-1}\theta(h) \). Let \( h_1 \in H_1 \). Then

\[
g^{-1}\theta(h_1h) = h_1hg^{-1}h^{-1}h_1^{-1} = h_1g^{-1}\theta(h)h_1^{-1} = h_1g^{-1}h_1^{-1}\theta(h) = g^{-1}\theta(h_1)\theta(h).
\]

Thus \( \theta \) is a homomorphism. From \( h^{-1}g^{-1}h^{-1} = g^{-1}\theta(h) \) it follows that \( ghg^{-1} = \theta(h)h \). Since \( h \) centralizes \( G^0 \), so does \( ghg^{-1} \), and we see that \( \theta(h) \) centralizes \( G^0 \). Thus \( \theta(h) \in Z(G^0) \).

Corollary 3.4. Suppose that \( G = G_0 \) and that \( G_0 \) is normal in \( H_0 \). Then \( H_0 \) is reductive.

Proof. As above, we have \( (H_0)_0 = H_2G^0 \) where \( H_2 \subset H_0 \) is connected and centralizes \( G^0 \), and \( H_0 \) is reductive if and only if \( H_2 \) is reductive. Let \( R \) be the unipotent radical of \( H_2 \). Corresponding to each \( g \in G \) there is a homomorphism \( \theta : H_2 \rightarrow Z(G^0) \), and since \( R \) is unipotent, \( \theta(R) = \{ e \} \). Thus \( R \subset GL(V)^G \) where \( GL(V)^G \) is obviously in \( H_2 \). Thus \( R \) is trivial and \( H_0 \) is reductive.

Write \( H^0 = H^0_sG^0_sT \) where \( H^0_s \) (resp. \( G^0_s \)) is the semisimple part of \( H_1 \) (resp. \( G^0 \)) and \( T := Z(H^0)_0 \subset H_1 \) is a torus. Set \( T_0 := Z(G^0)_0 \).

Corollary 3.5. The group \( H^0_s \) is contained in \( GL(V)^G \).

Theorem 3.6. Let \( V, G \) and \( H \) be as above. Then \( H^0 = GL(V)^G \).

Proof. Write \( H^0 = H^0_sG^0_sT \) as above and set \( F := G/G^0 \). Then \( F \) normalizes \( T \) and by Lemma 3.3, \( F \) acts trivially on \( T/T_0 \). Thus \( T^F \) projects onto \( T/T_0 \). Choose a torus \( S \) in \( (T^F)^0 \) complementary to \( (T^F \cap T_0)^0 \). Then \( H^0 = H^0_sSG^0_0 \) where \( H^0_sS \) lies in \( GL(V)^G \).

Remark 3.7. Write \( V = \bigoplus_{i=1}^r m_iV_i \) where \( V_i \) are irreducible and pairwise non-isomorphic and \( m_iV_i \) denotes the direct sum of \( m_i \) copies of \( V_i \). Then the theorem shows that \( H^0 = G^0 \prod_{i=1}^r GL(m_i) \).

Example 3.8. Let \( \{ e \} \neq G \subset GL(V) \) be finite. Then \( N_G \), as a set, is just the origin, and it is preserved by \( GL(V) \). Thus it is essential in Theorem 3.6 that \( H \) preserve \( N_G \) schematically.

Corollary 3.9. Suppose that \( V = \bigoplus_{i=1}^r V_i \) where the \( V_i \) are irreducible, non-trivial and pairwise non-isomorphic. Let \( H' \subset GL(V) \) be semisimple. Then the following are equivalent:

1. \( H' \subset H_0 \).
2. \( H' \subset G_0 \).
Proposition 3.10. Suppose that $V$ is an irreducible $G$-module. Then $G_0$ and $H_0$ are reductive and $H^0 = C^*G^0$.

Proof. The fixed points of the unipotent radical $R$ of $G_0$ are a $G_0$-stable nonzero subspace of $V$. Thus $R$ acts trivially on $V$, i.e., $R = 0$. Hence $G_0$ is reductive. Similarly, $H_0$ is reductive. ■

Corollary 3.11. Suppose that $V = mW$ where $W$ is an irreducible $G$-module. Then $H_0$ is reductive.

Proof. The group $H$ contains $G \times \text{GL}(m)$ which acts irreducibly on $V \cong W \otimes \mathbb{C}^m$. Thus $H_0$ is reductive. ■

In the remainder of this section, we do not assume that $G$ is a Levi subgroup of $G_0$.

Corollary 3.12. Let $G \subset \text{GL}(W)$ and let $V = pW \oplus qW^*$ where $2 \leq p \leq q$ and the $G$-modules $W$ and $W^*$ are irreducible and non isomorphic. Then

1. $G_0$ and $H_0$ are reductive.
2. $G_0 \subset \text{GL}(W)$.
3. $H^0 = \text{GL}(p)\text{GL}(q)(G_0)^0$.

Proof. First we consider the case that $G = \text{GL}(W)$. Then Example 4.3 below shows that $G_0 = \text{GL}(W)$ and that $(H_0)^0 = \text{GL}(p)\text{GL}(q)\text{GL}(W)$. Now the invariants of $\text{GL}(W)$ are generated by those of degree 2 and the degree 2 invariants of $G$ and of $\text{GL}(W)$ are the same. Thus $G_0$ must be a subgroup of $\text{GL}(W)$ and $(H_0)^0$ must be a subgroup of $\text{GL}(p)\text{GL}(q)\text{GL}(W)$ containing $\text{GL}(p)\text{GL}(q)$. Hence $(H_0)^0 = \text{GL}(p)\text{GL}(q)H_1$ where $H_1 \subset \text{GL}(W)$. Note that $GH_1$ is a finite extension of $H_1$. Since $W$ is an irreducible $G$-module and $G_0$ and $GH_1$ contain $G$, both $G_0$ and $H_1$ (hence $(H_0)^0$) are reductive and we have (1) and (2). Theorem 3.6 gives (3). ■

Lemma 3.13. Suppose that $V^G = (0)$ and let $V = \bigoplus_{i=1}^r m_iV_i$ be the isotypic decomposition of $V$ where the $V_i$ are pairwise non-isomorphic $G$-modules. Suppose that $h_0(m_iV_i) \subset m_iV_i$ for all $i$. Then $H_0$ is reductive.

Proof. For any $i$, $G(H_0)^0$ is a finite extension of $(H_0)^0$ which contains the product $G \prod_i \text{GL}(m_i)$. The latter group acts irreducibly on $m_iV_i$, hence the image of $G(H_0)^0$ in $\text{GL}(m_iV_i)$ is reductive for all $i$. It follows that $(H_0)^0$ is reductive, hence that $H_0$ is reductive. ■

Corollary 3.14. Suppose that $V_i$ is an irreducible nontrivial $G_i$-module where $G_i$ is reductive and $\mathbb{C}[V_i]^{G_i} \neq \mathbb{C}$, $i = 1, \ldots, r$. Let $V := \bigoplus_i m_iV_i$ with the canonical action of $G := G_1 \times \cdots \times G_r$ where $m_i \geq 1$ for all $i$. Then $H_0$ is reductive.
Proof. Suppose that $\mathfrak{h}_0$ is not contained in $\bigoplus_i \text{End}(m_i V_i)$. Since $\mathfrak{h}_0$ is $H$-stable, it must contain one of the irreducible $G_i \times \text{GL}(m_i) \times G_j \times \text{GL}(m_j)$-modules $\text{Hom}(m_i V_i, m_j V_j)$, $i \neq j$. Without loss of generality suppose that $\mathfrak{h}_0 \supset \text{Hom}(m_2 V_2, m_1 V_1)$. Let $f \in \mathcal{O}(m_1 V_1)^{G_i}$ be a nonconstant homogeneous invariant of minimal degree $d \geq 2$. Let $\varphi \in \text{Hom}(m_2 V_2, m_1 V_1)$. Then $\varphi$ sends $f$ to the function $h(v_1, v_2) := df(v_1)(\varphi(v_2))$ where $v_i \in m_i V_i$, $i = 1, 2$. Clearly there is a $\varphi$ such that $h \neq 0$. Thus $h$ is a nonzero element of bidegree $(d - 1, 1)$ in $\mathbb{C}[m_1 V_1 \oplus m_2 V_2]$. But by the minimality of $d$ and the fact that no nonzero invariant in $\mathbb{C}[m_2 V_2]$ has degree 1, there is no element of $\mathcal{I}$ of this bidegree. Hence $\text{Hom}(m_2 V_2, m_1 V_1)$ does not preserve $\mathcal{I}$, a contradiction. Thus $\mathfrak{h}_0$ is contained in $\bigoplus_i \text{End}(m_i V_i)$ and one can apply Lemma 3.13.

Corollary 3.15. Suppose that $G \subset \text{GL}(V)$ is a finite group generated by pseudoreflections. Then $H_0$ is reductive.

Proof. We have that $V = \bigoplus V_i$ and $G = \prod G_i$ where $G_i \subset \text{GL}(V_i)$ is an irreducible group generated by pseudoreflections. Now apply Corollary 3.14.

Proposition 3.16. Suppose that $V$ is an orthogonal representation of $G$ where $V^G = \{0\}$. Then $H_0$ is reductive.

Proof. We have an isotypic decomposition $V = \bigoplus_i m_i V_i \bigoplus n_j(W_j \oplus W_j^*)$ where the $V_i$ are irreducible nontrivial orthogonal representations of $G$ and the $W_j$ are irreducible nonorthogonal representations of $G$. Note that for each $i$ there is a quadratic invariant $p_i \in \mathbb{C}[m_i V_i]^G$ and for each $j$ a quadratic invariant (a contraction) $q_j \in \mathbb{C}[n_j(W_j \oplus W_j^*)]^G$. Suppose that $\mathfrak{h}_0$ is not contained in $\bigoplus_i \text{End}(m_i V_i) \bigoplus_j \text{End}(n_j(W_j \oplus W_j^*))$. For example, suppose that there is a nonzero element $\varphi$ of $\mathfrak{h}_0$ whose restriction to $m_2 V_2$ has nonzero projection to $m_1 V_1$. Then we have the function $h(v_1, v_2) := dp_1(v_1)(\varphi(v_2))$ for $v_1 \in m_1 V_1$ and $v_2 \in m_2 V_2$. As before, the actions of $G$ and the $\text{GL}(m_i)$ guarantee that we can assume that $h \neq 0$. Now the bidegree of $h$ is $(1, 1)$ and $h \in \mathcal{I}$. However, there are no nonconstant invariants of bidegree $(a, b)$ in $\mathbb{C}[m_1 V_1 \oplus m_2 V_2]$ for $a \leq 1$ and $b \leq 1$. Thus $h$ cannot lie in $\mathcal{I}$. One similarly gets contradictions for all the possible ways that $\mathfrak{h}_0 \not\subset \bigoplus_i \text{End}(m_i V_i) \bigoplus_j \text{End}(n_j(W_j \oplus W_j^*))$ can occur. Finally, note that the normalizer $N$ of the image of $G$ in $\text{GL}(n_j(W_j \oplus W_j^*))$ contains an element interchanging the copies of $W_j$ and $W_j^*$. Thus $N$ acts irreducibly and we can now apply the argument of Lemma 3.13.

Corollary 3.17. If $G$ is any one of the following groups, then $H_0$ is reductive for any representation $V$ of $G$ with $V^G = \{0\}$.

1. $\text{SO}(n)$, $n \geq 3$.
2. $G_2$, $F_4$, $E_8$.
3. $B_{4n+3}$ and $B_{4n+4}$, $n \geq 0$.
4. $D_{4n}$, $n \geq 1$. 
4. Some examples and a conjecture

We give examples where $G_0$ is not reductive and we give examples where $G_0$ is reductive but $H_0$ is not.

**Example 4.1.** Let $V$ and $W$ be $G$-modules such that $\mathcal{O}(V \oplus W)^G = \mathcal{O}(V)^G$. Then $\text{Hom}(V,W)$ is contained in the radical of $\mathfrak{g}_0$ so that $G_0$ and $H_0$ are not reductive. A concrete example is given by $G = \text{SL}_4$ and $V \oplus W = \wedge^2 \mathbb{C}^4 \oplus \mathbb{C}^3$ with the obvious $G$ action.

**Example 4.2.** Let $W$ be an irreducible $G$-module where $W^G = \{0\}$ and $\mathcal{O}(W)^G \neq \mathbb{C}$. Let $V = W \oplus \mathbb{C}$ where $G$ acts trivially on $\mathbb{C}$. Then $\mathfrak{g}_0 \subset \mathfrak{gl}(W)$ while $\text{Hom}(\mathbb{C},W)$ is contained in the Lie algebra of the radical of $H_0$.

**Example 4.3.** Let $1 \leq p \leq q$ and consider the $G = \text{GL}(W)$ representation on $V = pW \oplus qW^*$ where $W = \mathbb{C}^n$, $n \geq 1$. (See Corollary 3.12.) By classical invariant theory, the $G$-invariants are just the contractions of elements of the copies of $W$ with elements of the copies of $W^*$. Let $U$ denote $W \oplus W^* \simeq \mathbb{C}^{2n}$.

Three cases arise:

Case 1: $p = q = 1$. Then our invariant is the bilinear form $(\ , \ )$ corresponding to the matrix $J := (0 \ I) \subset \text{GL}(2n)$, i.e., $(x, y) = x^t J y$, $x, y \in U$. Thus $G_0 = \text{O}(2n)$ and $H_0 = \mathbb{C}^* G_0$.

Case 2: $p = 1$, $q > 1$. Then $H_0$ contains a copy of $\text{GL}(q)$ and the action of $H_0$ on the invariants is a representation $H_0 \to \text{GL}(q)$ whose kernel is $G_0$. Thus $H_0 = \text{GL}(q) G_0$. A matrix computation shows that $G_0 = \text{GL}(W) \ltimes (\wedge^q (W^*) \otimes \mathbb{C})$. If $x \in W$ and $y_1, \ldots, y_q \in W^*$, then the unipotent radical of $G_0$ sends $(x, y_1, \ldots, y_q)$ to $(x, y_1 + B_1 x, \ldots, y_q + B_q x)$ where for each $j$, $B_j$ is a skew symmetric matrix, $B_j \in \wedge^2 (W^*) \subset \text{Hom}(W, W^*)$.

Case 3: $p \geq 2$. We show that $G_0 = \text{GL}(W)$, that $H_0 = H$ and that $H^0 = \text{GL}(p) \text{GL}(q) \text{GL}(W)$. We also determine $H$. First suppose that $p = q = 2$. Then $G_0$ preserves the inner products on $2U$, i.e., $G_0$ is a subgroup of $\text{O}(2n)$. Moreover, $G_0$ preserves the skew product on $2U$ sending $x$, $y$ to $x^t K y$ where $K = (0 \ I)$. Hence $G_0$ lies in the intersection of $\text{O}(2n)$ and $\text{Sp}(2n)$ which is the copy of $\text{GL}(W)$ acting on $U$ by the matrices $(A, 0, 0, A^{-1})$, $A \in \text{GL}(W)$. Clearly, as long as $2 \leq p \leq q$ we must have that $G_0 = G = \text{GL}(W)$. We have a representation $\varphi : H_0 \to \text{GL}(pq)$ given by the action of $H_0$ on the $pq$ generators of the invariants. The kernel of $\varphi$ is $G_0 = G$. Thus $H_0$ is reductive. By Theorem 3.6 we have $H^0 = \text{GL}(p) \text{GL}(q) \text{GL}(W)$. Let $h \in H$. If $h$ stabilizes $pW$ and $qW^*$, then $h$ induces an automorphism of $\text{GL}(W)$ which is trivial on $\mathbb{C}^* I$ and must be inner on $\text{SL}(W)$. Hence modulo an element of $\text{SL}(W)$, $h$ lies in the centralizer of $\text{GL}(W)$, which is $\text{GL}(p) \text{GL}(q)$. Hence $h \in H^0$. The only other possibility is that $h$ interchanges the copies of $pW$ and $qW^*$. This can only happen if $p = q$. Thus $H$ is connected if $p \neq q$ and $H/H^0$ has order two if $p = q$.

**Example 4.4.** Let $G = \mathbb{Z}/4\mathbb{Z} \subset \mathbb{C}^*$ and let $V = \mathbb{C}^2$ where $\xi(a, b) = (\xi^2 a, \xi b)$ for $(a, b) \in \mathbb{C}^2$, $\xi \in G$. Since $G$ is finite, $G_0 = G$. Let $x$ and $y$ be the usual coordinate functions on $V$. Then the invariants are generated by $x^2$, $xy^2$ and $y^4$. Consider the element $\varphi \in \text{End}(V)$ which sends $(a, b)$ to $(0, a)$ for $a, b \in \mathbb{C}$.
Then \( \varphi \) acts on \( \mathbb{C}[V] \) by the derivation \( x \partial / \partial y \). This derivation preserves \( \mathcal{I} \) and it follows that \( \varphi \) is a basis of the Lie algebra of the unipotent radical of \( H_0 \).

**Example 4.5.** Let \( G = \mathbb{C}^* \) and let \( V \) be the \( p + q + r \) dimensional representation with weights \(-1\) of multiplicity \( p \), \( 1 \) of multiplicity \( q \) and \( 2 \) of multiplicity \( r \) where \( p, q, r \in \mathbb{N} \) and \( pqr \neq 0 \). If \( x_i, y_j \) and \( z_k \) are corresponding coordinate functions, then the invariants are generated by the monomials \( x_i y_j \) and \( x_i x_j z_k \). We have \( G_0 = G \) while the radical of \( H_0 \) has Lie algebra spanned by the polynomial ring and \( \pi \).

**Example 4.6.** Let \( V \oplus W = S^2(\mathbb{C}^n) \oplus \mathbb{C}^n \) with the obvious action of \( G = \text{SL}_n \), \( n \geq 2 \). Then using classical invariant theory [6] one computes that the invariants have homogeneous generators \( p \) and \( q \) of bidegrees \((n,0)\) and \((n-1,2)\), respectively. Now \( \text{Hom}(V,W) \) contains a copy of \( W^* \) where \( \xi \in W^* \) sends \( v \in V \) to \( i_\xi(v) \in W \) (contraction). Then this copy of \( W^* \) acts on \( \mathbb{C}[V \oplus W] \) sending a polynomial \( f(v,w) \) into \( df(v,w)(0,i_\xi(v)) \), \( v \in V, w \in W \). This action annihilates \( p \) and sends \( q \) to a subspace of \( \mathcal{O}(V \oplus W) \) of bidegree \((n,1)\) transforming under \( G \) as \( W^* \). But the only way to get a copy of \( W^* \) in this bidegree is to multiply \( p \) times the copy of \( W^* \) in degree \( 1 \) in \( \mathcal{O}(V \oplus W) \). Thus \( \mathcal{I} \) is preserved. It is now easy to establish that the unipotent radical of \( H_0 \) has Lie algebra the copy of \( W^* \) in \( \text{Hom}(V,W) \).

**Conjecture 4.7.** If \( G \) is semisimple and \( V \) is generic (see 2.2) with \( V^G = (0) \), then \( H_0 \) is reductive.

5. Cofree Representations

Recall that \( V \) is cofree if \( R \) is a free module over \( R^G \). Equivalently, \( R^G \) is a polynomial ring and \( \pi: V \rightarrow V/\!/G \) is equidimensional [8, 17, 29]. If \( p_1, \ldots, p_d \) are minimal homogeneous generators of \( R^G \), then we can identify \( \pi \) with the polynomial map \( p = (p_1, \ldots, p_d): V \rightarrow \mathbb{C}^d \). Cofreeness is equivalent to the fact that the \( p_i \) form a regular sequence in \( \mathbb{C}[V] \). See [7] for the classification of cofree representations of the simple algebraic groups and [4] for the classification of irreducible cofree representations of semisimple algebraic groups.

We say that \( G' \subset \text{GL}(V) \) stabilizes a fiber \( F \) of \( \pi \) if \( G' \) preserves \( F \) schematically, i.e., preserves the ideal \( I_F \) of \( F \).

**Proposition 5.1.** Suppose that \( G \) is reductive and \( V \) is a cofree \( G \)-module. If \( G' \subset \text{GL}(V) \) stabilizes a fiber of \( \pi: V \rightarrow V/\!/G \), then \( G' \) stabilizes \( \mathcal{N}_G \).

**Proof.** Let \( F \) be a fiber of \( \pi \). Then there are constants \( c_i, i = 1, \ldots, d \), such that \( I_F \) is the ideal generated by \( p_i - c_i, i = 1, \ldots, d \). Let \( 0 \neq f \in I_F \) and let \( \text{gr} f \) denote the nonzero homogeneous part of \( f \) of largest degree. Then the elements \( \text{gr} f \) for \( 0 \neq f \in I_F \) generate a homogeneous ideal \( I \) which obviously contains \( \mathcal{I} \). We show that \( I \subset \mathcal{I} \) so that \( I = \mathcal{I} \). If \( G' \) preserves \( I_F \), it preserves \( I = \mathcal{I} \), and we have the proposition.

Let \( d_i \) be the degree of \( p_i, i = 1, \ldots, d \). Let \( 0 \neq f \in I_F \) where \( \text{gr} f \) is homogeneous of degree \( r \). We have \( f = \sum a_i (p_i - c_i) \) where \( a_1, \ldots, a_n \in R \). Let
\[ s = \max_i \{ \deg a_i + d_i \} \]. Let \( a'_i \) denote the homogeneous part of \( a_i \) of degree \( s - d_i \).

If \( s > r \), then we must have that \( \sum_i a'_i p_i = 0 \). Since the \( p_i \) are a regular sequence, this relation is generated by the Koszul relations \( p_j p_i - p_i p_j = 0 \), \( 1 \leq i < j \leq d \).

Hence there are \( b_{ij} \in R \), \( b_{ij} = -b_{ji} \), such that
\[
\sum_i a'_i(p_i - c_i) = \sum_{i \neq j} b_{ij}(p_j(p_i - c_i) - p_i(p_j - c_j)) = \sum_{i \neq j} b_{ij}(c_j(p_i - c_i) - c_i(p_j - c_j))
\]
where for fixed \( i \), \( \deg \sum_{j \neq i} b_{ij} c_j < s - d_i \). Thus we may replace each \( a_i \) by a polynomial of degree less than \( s - d_i \) without changing \( f \). Continuing inductively we reduce to the situation that \( \deg a_i \leq s - d_i \) for all \( i \). Let \( a'_i \) denote the homogeneous degree \( r - d_i \) term in \( a_i \), \( i = 1, \ldots, d \). Then \( \text{gr } f = \sum_i a'_i p_i \in \mathcal{I} \).

**Example 5.2.** Let \( G = \mathbb{C}^* \) and \( V = \mathbb{C}^3 \) with coordinate functions \( x, y \) and \( z \) corresponding to weights \(-1, 1 \) and \( 2 \). The fiber defined by \( xy = 1 \) and \( x^2 z = 0 \) is the fiber defined by \( xy = 1 \) and \( z = 0 \), and it has a symmetry which interchanges \( x \) and \( y \). However, this is not a symmetry of the ideal generated by the invariants. Thus Proposition 5.1 does not hold in case the representation is not cofree.

**Remark 5.3.** Let \( F \) be a principal fiber of \( \pi \) where \( V \) is cofree. Then \( d\pi \) has rank \( d = \dim V/\mathcal{G} \) on \( F \) so that \( F \) is smooth. It follows that \( G' \) preserves \( I_F \) if and only if \( G'' \) preserves the set \( F \).

**Corollary 5.4.** Let \( V = \bigoplus_{i=1}^r V_i \) where the \( V_i \) are pairwise non-isomorphic nontrivial \( G \)-modules and \( V \) is cofree. Suppose that \( G \subseteq G' \subseteq \text{GL}(V) \) where \( G' \) is connected semisimple. Then the following are equivalent.

1. \( R^G = R^{G'} \).
2. \( G' \) preserves a fiber of \( \pi : V \to V/\mathcal{G} \).
3. \( G' \) preserves \( \mathcal{N}_G \).

**Proof.** Use Corollary 3.9 and Proposition 5.1.

**Corollary 5.5.** Let \( V \) be an irreducible nontrivial cofree \( G \)-module with \( R^G \neq \mathbb{C} \). Let \( F \neq \mathcal{N} \) be a fiber of \( \pi : V \to V/\mathcal{G} \) and let \( G_F \) be the subgroup of \( \text{GL}(V) \) stabilizing \( F \). Then

1. \( G_0 \subset G_F \subset H_0 \) are reductive.
2. \( H^0 = \mathbb{C}^*(G_0)^0 \).
3. \( G_F/G_0 \) is finite.

**Proof.** Parts (1) and (2) are clear. Since \( F \neq \mathcal{N} \), it is only stabilized by a finite subgroup of \( \mathbb{C}^* \), hence we have (3).

It would be nice to find an example of an irreducible module \( V \) of a semisimple group \( G \) with \( G = (G_0)^0 \) such that the subgroup of \( \text{GL}(V) \) fixing a fiber \( F \) of \( \pi : V \to V/\mathcal{G} \), \( F \neq \mathcal{N}_G \), has dimension bigger than \( \dim G \).
Remark 5.6. Let $V$ be an irreducible nontrivial cofree representation of a simple algebraic group $G$ such that $R^G \neq \mathbb{C}$. The cases for which $G \neq (G_0)^0$ are as follows (we use the numbering and notation of [6]).

1. $(\varphi_3, B_3)$.
2. $(\varphi_4, B_4)$.
3. $(\varphi_5, B_5)$.
4. $(\varphi_1, G_2)$.

6. The adjoint case

Let $\mathfrak{g}$ be a simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a base $\Pi$ of the root system. Choose $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$, $\alpha \in \Pi$, such that $(x_\alpha, y_\alpha, [x_\alpha, y_\alpha])$ is a standard $\mathfrak{sl}_2$ triple. Then there is a unique order 2 automorphism $\psi$ of $\mathfrak{g}$ which is $-1$ on $\mathfrak{t}$ and sends $x_\alpha$ to $-y_\alpha$, $\alpha \in \Pi$.

Now let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where the $\mathfrak{g}_i$ are simple ideals of the Lie algebra $\mathfrak{g}$. Let $\psi_i \in \text{Aut}(\mathfrak{g}_i)$ be as above. Let $G$ denote the adjoint group of $\mathfrak{g}$ and let $G_0$, $H_0$ and $H$ be as in the introduction.

Theorem 6.1. We have that $H = (\mathbb{C}^*)^r \text{Aut}(\mathfrak{g})$ and that $G_0 \simeq (\mathbb{Z}/2\mathbb{Z})^r G$ where the $i$th copy of $\mathbb{Z}/2\mathbb{Z}$ is generated by $-\psi_i$.

Proof. By Dixmier [2], $(G_0)^0 = G$, and using Corollary 3.14 we obtain that $H_0 = H$ where $H_0 = (\mathbb{C}^*)^r G$. Hence if $\varphi \in H$, we obtain an automorphism $\sigma$ of $\mathfrak{g} \simeq \text{ad} \mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{gl}(\mathfrak{g})$ where $\text{ad}(\sigma(X)) = \varphi \circ \text{ad} X \circ \varphi^{-1}$, $X \in \mathfrak{g}$. Clearly $\text{Aut}(\mathfrak{g}) \subset H$, so replacing $\varphi$ by $\varphi \circ \sigma^{-1}$ we can arrange that $\varphi \circ \text{ad} X \circ \varphi^{-1} = \text{ad} X$ for all $X \in \mathfrak{g}$. Then by Schur’s lemma, $\varphi \in (\mathbb{C}^*)^r \subset H^0$ so that $H = (\mathbb{C}^*)^r \text{Aut}(\mathfrak{g})$.

If we start with $\varphi \in G_0$, then since $\varphi$ induces the identity on $\mathbb{C}[\mathfrak{g}]^G$, so does $\sigma$, and it follows from Schur’s lemma that $\varphi$ is a product $\prod \lambda_i \sigma_i$, where for all $i$, $\sigma_i : \mathfrak{g}_i \to \mathfrak{g}_i$ is an automorphism and $\lambda_i \in \mathbb{C}^*$ acts via multiplication on $\mathfrak{g}_i$. But $\varphi$ has to preserve the invariants of degree 2 of each $\mathfrak{g}_i$, hence $\lambda_i = \pm 1$ for all $i$.

Now [9, Theorem 2.5] shows that, for each $i$, $\lambda_i \sigma_i \in G_i$ or $\lambda_i \sigma_i \in (-\psi_i)G_i \neq G_i$, where $G_i$ is the adjoint group of $\mathfrak{g}_i$. Hence $G_0 \simeq (\mathbb{Z}/2\mathbb{Z})^r G$. $lacksquare$

Corollary 6.2. (See [1]). Let $\mathfrak{g} = \mathfrak{sl}_n$. Then $H$ is generated by $G$, $\mathbb{C}^*$ and transposition.

Proof. In the case of $\mathfrak{sl}_n$ with the usual choice of $\mathfrak{t}$ and $\Pi$, the automorphism $\psi$ is $X \mapsto -X^t$, $X \in \mathfrak{sl}_n$. Then $\psi$ generates the group of outer automorphisms of $\mathfrak{sl}_n$ (which is the trivial group for $n = 2$). Hence $H$ is generated by $G$, $\mathbb{C}^*$ and transposition. $lacksquare$
Corollary 6.3. (See[12]). Let $G_F$ be the subgroup of $\text{GL}(\mathfrak{gl}_n)$ which preserves the $G := \text{PGL}(n)$-orbit $F$ of an element $x_0$ of $\mathfrak{gl}_n$ which has nonzero trace and distinct eigenvalues. Then $G_F$ is generated by $G$ and transposition.

Proof. The condition on $x_0$ implies that $F$ is a smooth fiber of the quotient mapping (see Remark 5.3). Write $x_0 = \mu I + y_0$ where $\mu \in \mathbb{C}^*$, $y_0 \in \mathfrak{sl}_n$ and $I$ is the $n \times n$ identity matrix. Then $F$ is just $\mu I + F_1$ where $F_1 = G \cdot y_0$. We may write an element of $G_F$ as $(\begin{array}{cc} 1 & 0 \\ c & \lambda g \end{array})$ where $c \in \mathfrak{sl}_n$, $g \in \text{GL}(<\mathfrak{sl}_n>$) is a linear mapping preserving the schematic null cone of $\mathfrak{sl}_n$ and $\lambda \in \mathbb{C}^*$ (use 3.10 and 5.1). Then $g$ is in $G$ or $g$ is an element of $G$ composed with transposition. Applying the inverse of $g$ we obtain an element $h$ of the form $y \mapsto \lambda y + c$, $y \in F_1$. We need to show that $c = 0$. Suppose not. Let $g \in G$ such that $gc \neq c$. Then $h^{-1}gh^{-1}(y) = y + c'$, $0 \neq c' \in \mathfrak{sl}_n$, $y \in F_1$. Thus $F_1 = F_1 + c'$. It follows that for any invariant polynomial $p$ on $\mathfrak{sl}_n$, $p(y + nc') = p(y)$ for all $y \in F_1$ and $n \in \mathbb{Z}$. Thus $dp(y)(c') = 0$ for any $y \in F_1$. But the covectors $dp(y)$ for $y \in F_1$ span $(\mathfrak{sl}_n)^*$. Thus $c' = 0$, a contradiction. □

References


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