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Algebraic Characterization of Differential Geometric Structures

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Abstract. We consider purely algebraic data generalizing the notion of a smooth differentiable manifold. It is given by a triple X, R, W where X is a set, R a commutative associative algebra over the ground field, W a Lie subalgebra and an R-submodule in the derivation algebra of R. Geometric structures studied in differential geometry can be defined on such triples. The main result answers the question about the existence and the uniqueness of an L-invariant unimodular, hamiltonian, contact, or pseudo-riemannian structure in terms of the isotropy subalgebras of points of X. The second major result generalizes a classical fact which says that the Lie algebra of infinitesimal automorphisms of a Riemann metric on a connected manifold is finite dimensional.
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Given a transitive Lie algebra L of vector fields on a differentiable manifold X of class C^∞, one may search for a geometric structure on X having all vector fields in L as infinitesimal automorphisms. The simplest conditions that can be imposed are formulated in terms of the representations of the isotropy subalgebras of L in the tangent spaces of X. The above problem is probably better known in the setup of formal transitive Lie algebras developed by Guillemin and Sternberg [8]. Here L is a Lie algebra with a subalgebra of finite codimension L^0 which contains no nonzero ideals of L. Denoting by ρ the representation of L^0 in L/L^0, it is possible to determine L to a large extent by knowing the image of L^0 in $\mathfrak{gl}(L/L^0)$.

For example, if $\rho(L^0)$ is the Lie algebra of linear transformations of L/L^0 with zero trace, then L is realized as a Lie algebra of formal vector fields annihilating the standard volume form $dt_1 \wedge \cdots \wedge dt_n$ where $t_1, \ldots, t_n$ are formal coordinates. If $\rho(L^0)$ is a symplectic Lie algebra, then L annihilates the standard hamiltonian form in a similar realization. These assertions are special cases of the embedding theorem for transitive Lie algebras due to Rim [18] and Hayashi [10]. In those papers an embedding of one transitive Lie algebra into another one is constructed by a sequence of approximations. At each step the embedding is determined modulo some terms of the canonical filtrations in the two Lie algebras.

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The possibility of making an adjustment to obtain an embedding modulo higher order terms depends on vanishing of some Spencer homology groups. In this approach the geometric structure associated with the given Lie algebra does not arise naturally. It is also desirable to have a global version of the result.

In the present paper we develop a purely algebraic framework in which the problem posed above can be generalized and solved. We work over the ground field $\mathbb{F}$, and the setup includes a triple X, R, W where X is just a set, R a commutative associative unital algebra, $W \subset \operatorname{Der} R$ a Lie subalgebra and an R-submodule in the derivation algebra of R. Furthermore, associated with each element of R there is an $\mathbb{F}$-valued function on X, although some nonzero elements of R may produce identically zero function. The precise conditions on X, R, W are listed in section 3, and the main result is presented in section 5.

Especially, our set of conditions is satisfied when X is a Hausdorff C^∞-manifold, $R = C^\infty(X)$ the ring of smooth functions, $W = \operatorname{Vect}(X)$ the Lie algebra of smooth vector fields on X. A Lie subalgebra $L \subset W$ is transitive if L contains vector fields in every direction. For each point $x \in X$ the isotropy subalgebra L_x^0 consists of all vector fields in L with zero value at x. The linear bijection $T_x(X) \cong L/L_x^0$ gives rise to a representation of L_x^0 in the tangent space $T_x(X)$. As an illustration we reformulate below one part of Theorem 5.1 for the triple X, R, W just defined:

Theorem. *Let L be a transitive Lie algebra of smooth vector fields on X such that for each $x \in X$ the image of L_x^0 in the Lie algebra of linear transformations of $T_x(X)$ is the symplectic Lie algebra $\mathfrak{sp}(\alpha_x)$ associated with a nondegenerate alternating bilinear form α_x on $T_x(X)$. Then there exists a smooth line bundle E over X, a flat connection on E, and an L-invariant E-valued hamiltonian form ω on X. This data is determined uniquely up to a naturally defined equivalence.*

There is a covering of X by open subsets U_i over which the bundle E and the connection on E trivialize. The restriction of ω to each U_i is represented by an ordinary hamiltonian form determined uniquely only up to a scalar multiple. Glueing of such local data necessitates introducing a more general class of hamiltonian forms. The bundle E is actually constructed as a factor bundle of $\bigwedge^2 T(X)$; the connection on E and the form ω arise naturally. A similar result is valid for different geometric structures. Besides hamiltonian forms we will consider volume forms, contact forms, and Riemann pseudometrics.

There are many other possible choices for X, R, W. For example, over a field of characteristic $p > 0$ the finite dimensional analogs of the infinite dimensional Lie algebras of Cartan type correspond to geometric structures on such triples where X contains just a single point and $\dim R < \infty$. This class of Lie algebras was introduced by Kostrikin and Shafarevich [15]. A thorough exposition of the characteristic p theory can be found in a treatise of Strade [22].

In section 6 we will generalize a classical fact which says that the Lie algebra of infinitesimal automorphisms of a Riemann metric on a connected manifold is finite dimensional. This assertion may be viewed as the infinitesimal part of a result, due to Myers and Steenrod, according to which the isometries of a Riemannian manifold form a Lie group. Our result is proved in the setup already described, but we need an additional assumption on the W-invariant ideals of R

which excludes, for example, nonconnected differentiable manifolds. In differential geometry a more general result on automorphisms of G-structures is known for a Lie group G with the Lie algebra of finite type (see [13, Ch. 1, Th. 5.1]). We do not attempt to obtain our result in this generality since this would involve a considerable amount of extra work.

One complication in our approach arise from the fact that we aim at algebraic versions of results which include fully the case of a differentiable manifold without the compactness assumption. To achieve this we have to work with a not so familiar class of R-modules studied in section 1 of the paper. Some readers might be content with simpler versions of results where one needs just the standard properties of finitely generated projective modules (see the last paragraph of section 1). Section 2 provides crucial arguments to verify the local freeness of certain modules which will be encountered later.

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1. Vector bundles interpreted in terms of modules

Given a smooth vector bundle E over a manifold X of class C^∞, the smooth global sections form a $C^\infty(X)$-module $\Gamma(X,E)$. When X is connected and second countable, E is a direct summand of a trivial vector bundle [7], which means that $\Gamma(X,E)$ is a direct summand of a finitely generated free $C^\infty(X)$-module, i.e. a projective module. In this case the category of smooth vector bundles is equivalent to the category of finitely generated projective $C^\infty(X)$-modules. The assignment

$$x \mapsto \mathfrak{m}_x = \{f \in C^\infty(X) \mid f(x) = 0\}$$

allows us to identify points of X with certain maximal ideals of $C^\infty(X)$. Unless X is compact, we obtain only a part of maximal ideals in this way (Grabowski [6] gives a characterization of the maximal ideals corresponding to points and discusses further subtleties). Note that the fibre of a vector bundle E above $x \in X$ can be reconstructed from the module $M = \Gamma(X,E)$ as

$$E(x) \cong M/\mathfrak{m}_x M.$$

Let R be a commutative ring, $X \subset \operatorname{Spec} R$ a set of prime ideals of R. The aim of this section is to introduce a class of R-modules which may be thought of as representing vector bundles on X. Our intention is to fully subsume the C^∞-example above. However, we won't be able to prove the projectivity of certain modules in the algebraic setup of section 3. For this reason we will have to be content with less restrictive conditions on modules.

For each $f \in R$ let R_f denote the ring of fractions of R with respect to the multiplicatively closed set of powers of f and $M_f = R_f \otimes_R M$ the corresponding localization of an R-module M. Denote by v_f the image in M_f of an element $v \in M$. For a homomorphism of R-modules $\varphi : M \to N$ let $\varphi_f : M_f \to N_f$ stand for its R_f-linear extension.

We say that M is *locally free near* $\mathfrak{p} \in \operatorname{Spec} R$ if there exists $f \in R \smallsetminus \mathfrak{p}$ such that M_f is a free R_f-module. For instance, a finitely generated R-module is projective if and only if it is locally free near all maximal ideals of R [1, Ch. II, §5, Th. 1]. We say that M is *locally free near* X if M is locally free near all $\mathfrak{p} \in X$. The *rank function* of such a module M assigns to each $\mathfrak{p} \in X$ the rank of the free module M_f with f chosen in $R \smallsetminus \mathfrak{p}$. This function is locally constant with respect to the Zariski topology.

A similar terminology will be used for several other concepts. For instance, M is *locally finitely generated near* X if for each $\mathfrak{p} \in X$ there exists $f \in R \smallsetminus \mathfrak{p}$ such that M_f is a finitely generated R_f-module. A submodule N of M is a *direct summand locally near* X if for each $\mathfrak{p} \in X$ there exists $f \in R \smallsetminus \mathfrak{p}$ such that N_f is a direct summand of the R_f-module M_f.

Denote

$$\mathcal{F} = \{\text{ideals } I \text{ of } R \mid I \not\subset \mathfrak{p} \text{ for any } \mathfrak{p} \in X\}.$$

An R-module M is *$\mathcal{F}$-torsion* if every element of M is annihilated by an ideal in $\mathcal{F}$. Given an exact sequence of R-modules $0 \to M' \to M \to M'' \to 0$, it is easy to see that M is $\mathcal{F}$-torsion if and only if so are both M' and M''. An arbitrary R-module M contains a largest $\mathcal{F}$-torsion submodule $t_{\mathcal{F}}(M)$. These properties mean that $\mathcal{F}$ is a *localizing filter* (also called an *idempotent topologizing filter* or a *Gabriel topology*) [4], [5], [21].

One says that M is *$\mathcal{F}$-torsionfree* if $t_{\mathcal{F}}(M) = 0$. Next, M is called *$\mathcal{F}$-closed* if M is $\mathcal{F}$-torsionfree and M is not a submodule of any larger $\mathcal{F}$-torsionfree module N such that the factor module N/M is $\mathcal{F}$-torsion. There are several equivalent characterizations of $\mathcal{F}$-closedness. In particular, M is $\mathcal{F}$-closed if and only if for each homomorphism of R-modules $P \to Q$ with $\mathcal{F}$-torsion kernel and cokernel the induced map $\operatorname{Hom}_R(Q, M) \to \operatorname{Hom}_R(P, M)$ is bijective. The localization functor $?_{\mathcal{F}}$ assigns to an arbitrary R-module M an $\mathcal{F}$-closed R-module

$$M_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} \operatorname{Hom}_R\big(I,\, M/t_{\mathcal{F}}(M)\big).$$

There is a canonical R-linear map $M \to M_{\mathcal{F}}$ with $\mathcal{F}$-torsion kernel and cokernel. In particular, for any $\mathcal{F}$-closed R-module N the R-linear maps $M \to N$ are in a canonical bijective correspondence with the R-linear maps $M_{\mathcal{F}} \to N$. In order that M be $\mathcal{F}$-torsionfree (resp. $\mathcal{F}$-closed), it is necessary and sufficient that $M \to M_{\mathcal{F}}$ be injective (resp. bijective). The general theory of localization in arbitrary abelian categories was created by Gabriel [5].

Lemma 1.1. *Let $\varphi : M \to N$ be a homomorphism of R-modules, and let $K = \operatorname{Ker} \varphi$.*

(i) *If M and N are both $\mathcal{F}$-closed, then so is K.*

(ii) *If K and N are $\mathcal{F}$-closed, and if $\varphi(M) = N$, then M is $\mathcal{F}$-closed.*

Proof. Both assertions are valid for an arbitrary localizing filter and are easy consequences of the fact that the functor $?_{\mathcal{F}}$ is left exact (see, e.g., [21, p. 199]). Thus this functor takes the exact sequence of R-modules $0 \to K \to M \to N$ to an exact sequence $0 \to K_{\mathcal{F}} \to M_{\mathcal{F}} \to N_{\mathcal{F}}$. Under the hypothesis of (i) both

$M \to M_{\mathcal{F}}$ and $N \to N_{\mathcal{F}}$ are bijective, whence so is $K \to K_{\mathcal{F}}$. In (ii) $K \to K_{\mathcal{F}}$ and $N \to N_{\mathcal{F}}$ are bijective, and standard diagram chasing shows that $M \to M_{\mathcal{F}}$ is bijective. ■

For $\mathfrak{p} \in X$ denote by $\kappa(\mathfrak{p})$ the field of fractions of the domain $R/\mathfrak{p}$. Put

$$M(\mathfrak{p}) = \kappa(\mathfrak{p}) \otimes_R M,$$

which is a vector space over $\kappa(\mathfrak{p})$. In particular, $\kappa(\mathfrak{m}) = R/\mathfrak{m}$ and $M(\mathfrak{m}) \cong M/\mathfrak{m}M$ when $\mathfrak{m}$ is a maximal ideal of R. For any $f \in R \smallsetminus \mathfrak{p}$ the canonical ring homomorphism $R \to \kappa(\mathfrak{p})$ factors through R_f, whence $M(\mathfrak{p}) \cong \kappa(\mathfrak{p}) \otimes_{R_f} M_f$. When M_f is a free R_f-module of rank r, we get $\dim M(\mathfrak{p}) = r$.

Lemma 1.2. *Let M be locally finitely generated near X. Then M is $\mathcal{F}$-torsion if and only if $M(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in X$, if and only if for each $\mathfrak{p} \in X$ there exists $f \in R \smallsetminus \mathfrak{p}$ such that $M_f = 0$.*

A homomorphism of R-modules $\varphi : N \to M$ has an $\mathcal{F}$-torsion cokernel if and only if the induced maps $N(\mathfrak{p}) \to M(\mathfrak{p})$ are surjective for all $\mathfrak{p} \in X$, if and only if for each $\mathfrak{p} \in X$ there exists $f \in R \smallsetminus \mathfrak{p}$ such that $\varphi_f : N_f \to M_f$ is surjective.

Proof. For $\mathfrak{p} \in X$ we have $M(\mathfrak{p}) \cong M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ where $M_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R M$ is a finitely generated module over the local ring $R_{\mathfrak{p}}$ of $\mathfrak{p}$. By Nakayama's Lemma $M(\mathfrak{p}) = 0$ if and only if $M_{\mathfrak{p}} = 0$. Since M is locally finitely generated near $\mathfrak{p}$, the latter equality is equivalent to the existence of $f \in R \smallsetminus \mathfrak{p}$ for which $M_f = 0$, and this is also equivalent to the condition that none of the annihilators of elements of M is contained in $\mathfrak{p}$.

The second assertion of the lemma follows from the first one applied to the locally finitely generated R-module $L = \operatorname{Coker} \varphi$. Indeed, $L(\mathfrak{p})$ is isomorphic to the cokernel of $N(\mathfrak{p}) \to M(\mathfrak{p})$ and L_f to the cokernel of $N_f \to M_f$. ■

Lemma 1.3. *Let $\varphi : N \to M$ be a homomorphism of R-modules with $\mathcal{F}$-torsion cokernel. Denote $K = \operatorname{Ker} \varphi$. If M is locally free of finite rank near X, then K is locally near X a direct summand of N and for each $\mathfrak{p} \in X$ the sequence of vector spaces $0 \to K(\mathfrak{p}) \to N(\mathfrak{p}) \to M(\mathfrak{p}) \to 0$ is exact.*

Proof. In view of Lemma 1.2 for each $\mathfrak{p} \in X$ there exists $f \in R \smallsetminus \mathfrak{p}$ such that φ_f is surjective. We may also assume that M_f is a free R_f-module, refining our choice of f. The exact sequence of R_f-modules $0 \to K_f \to N_f \to M_f \to 0$ then has to split. Hence K_f is a direct summand of N_f and the previous sequence remains exact after tensoring with $\kappa(\mathfrak{p})$. ■

Lemma 1.4. *Let $\varphi : M' \to M$ and $\psi : M \to M''$ be homomorphisms of R-modules such that $\psi \circ \varphi = 0$. Suppose that M' is locally finitely generated, M and M'' are locally free of finite rank near X, and the sequences*

$$0 \longrightarrow M'(\mathfrak{p}) \xrightarrow{\varphi(\mathfrak{p})} M(\mathfrak{p}) \xrightarrow{\psi(\mathfrak{p})} M''(\mathfrak{p}) \longrightarrow 0$$

are exact for all $\mathfrak{p} \in X$. *Then for each* $\mathfrak{p} \in X$ *there exists* $f \in R \setminus \mathfrak{p}$ *such that*

$$0 \longrightarrow M'_f \xrightarrow{\varphi_f} M_f \xrightarrow{\psi_f} M''_f \longrightarrow 0$$

is a split exact sequence of free R_f*-modules. Furthermore, if* M' *is* $\mathcal{F}$*-closed and* M *is* $\mathcal{F}$*-torsionfree, then* $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M''$ *is an exact sequence.*

Proof. Note that $\varphi(M') \subset K$ where $K = \operatorname{Ker}\psi$. By Lemma 1.2 $\operatorname{Coker}\psi$ is $\mathcal{F}$-torsion. Lemma 1.3 shows that for each $\mathfrak{p} \in X$ there exists $g \in R \setminus \mathfrak{p}$ such that M_g, M''_g are both free over R_g and $0 \to K_g \to M_g \to M''_g \to 0$ is a split exact sequence. In particular, K is locally free of finite rank. We see also from Lemma 1.3 that for each $\mathfrak{p}$ the sequence $0 \to K(\mathfrak{p}) \to M(\mathfrak{p}) \to M''(\mathfrak{p}) \to 0$ is exact; hence $M'(\mathfrak{p}) \to K(\mathfrak{p})$ is a bijection. By another invocation of Lemmas 1.2, 1.3 the R-linear map $\varphi : M' \to K$ has $\mathcal{F}$-torsion kernel and cokernel, and both are locally finitely generated. Hence for each $\mathfrak{p} \in X$ there exists $h \in R \setminus \mathfrak{p}$ such that $M'_h \to K_h$ is bijective; we may also assume K_h to be free over R_h. Taking $f = gh$, we obtain the required split exact sequence of free R_f-modules. If M is $\mathcal{F}$-torsionfree, then so is K; hence $M' \to K$ has to be bijective in the case of an $\mathcal{F}$-closed M'. ∎

Lemma 1.5. *Let* $\varphi : N \to M$ *be a homomorphism of* R*-modules. Suppose that* N *is* $\mathcal{F}$*-closed and locally finitely generated near* X, *while* M *is* $\mathcal{F}$*-torsionfree and locally free of finite rank near* X. *In order that* φ *be an isomorphism, it is necessary and sufficient that the maps* $N(\mathfrak{p}) \to M(\mathfrak{p})$ *induced by* φ *be bijective for all* $\mathfrak{p} \in X$.

Proof. Necessity is obvious, while sufficiency follows from the special case of Lemma 1.4 when $M'' = 0$. ∎

Lemma 1.6. *Let* $H = \operatorname{Hom}_R(M, N)$ *where* M *is an arbitrary* R*-module. If* N *is an* $\mathcal{F}$*-torsionfree* (*resp.* $\mathcal{F}$*-closed*) R*-module, then so too is* H.

Proof. All maps $M \to N$ in the $\mathcal{F}$-torsion submodule of $\operatorname{Hom}_R(M, N)$ have images in $t_{\mathcal{F}}(N)$. Hence the first assertion. Suppose next that N is $\mathcal{F}$-closed. Given a homomorphism of R-modules $\varphi : P \to Q$ with $\mathcal{F}$-torsion kernel and cokernel, the induced map $\operatorname{Hom}_R(Q, N) \to \operatorname{Hom}_R(P, N)$ is bijective. Since

$$\operatorname{Hom}_R(?, H) \cong \operatorname{Hom}_R(? \otimes_R M, N) \cong \operatorname{Hom}_R\big(M, \operatorname{Hom}_R(?, N)\big)$$

naturally, the map $\operatorname{Hom}_R(Q, H) \to \operatorname{Hom}_R(P, H)$ induced by φ is also bijective. ∎

In Lemmas 1.7–1.16 we will need the following assumption about the ring R which will not be repeated in the statements:

(A) *there is an integer* $e > 0$ *with the property that* $f^e g = 0$ *for any pair of elements* $f, g \in R$ *such that* g *is annihilated by a power of* f.

For instance, if R is *reduced*, i.e. R has no nonzero nilpotent elements, then $e = 1$ will do. Note that for any $h \in R$ the ring R_h also satisfies (A) with the same e.

Lemma 1.7. *Suppose that M is $\mathcal{F}$-torsionfree and locally free near X. Then M_f is $\mathcal{F}$-torsionfree for any $f \in R$ and one has*

$$\operatorname{Ker}(M \to M_f) = \{v \in M \mid f^e v = 0\}.$$

Proof. The kernel K of the canonical map $M \to M_f$ consists of all elements of M annihilated by a power of f. We have to show that $f^e v = 0$ for any $v \in K$. If M is a free module with a basis $\{u_\alpha\}$, then $v = \sum g_\alpha u_\alpha$ where $g_\alpha \in R$, $g_\alpha \neq 0$ for finitely many α's, and each g_α is annihilated by a power of f. Therefore $f^e g_\alpha = 0$ for each α, which yields our claim. The general case reduces to the case of a free module as follows. For each $\mathfrak{p} \in X$ there exists $h \in R \setminus \mathfrak{p}$ such that M_h is a free R_h-module. Denote by v_h and f_h the images of v and f in M_f and R_f, respectively. Since v_h lies in the kernel of the map $M_h \to (M_h)_{f_h}$, we get $f_h^e v_h = 0$. Hence $f^e v$ lies in the kernel of the map $M \to M_h$, i.e. $f^e v$ is annihilated by a power of h. Denoting by I the annihilator of $f^e v$ in R, we deduce that $I \not\subset \mathfrak{p}$. Since this holds for any $\mathfrak{p} \in X$, it follows that $I \in \mathcal{F}$. Thus $f^e v \in t_{\mathcal{F}}(M) = 0$.

Each element of M_f can be written as w/f^n for some $w \in M$ and an integer $n \geq 0$. If $w/f^n \in t_{\mathcal{F}}(M_f)$, then $Jw \subset \operatorname{Ker}(M \to M_f)$ for some $J \in \mathcal{F}$; hence $f^e Jw = 0$, and therefore $f^e w \in t_{\mathcal{F}}(M) = 0$, yielding $w/f^n = 0$. This shows that $t_{\mathcal{F}}(M_f) = 0$. ∎

Lemma 1.8. *Let $\varphi : N \to M$ be a homomorphism of R-modules such that $\varphi_f = 0$ for some $f \in R$. If M is $\mathcal{F}$-torsionfree and locally free near X, then $f^e \varphi = 0$.*

Proof. Since $\varphi(N) \subset \operatorname{Ker}(M \to M_f)$, the conclusion follows from Lemma 1.7. ∎

Lemma 1.9. *Let M, N, P be locally free R-modules such that M is $\mathcal{F}$-closed, P is $\mathcal{F}$-torsionfree, M and N have rank r, while P has rank 1 near X. Let $\beta : M \times N \to P$ be an R-bilinear pairing, and $\varphi : M \to \operatorname{Hom}_R(N, P)$ the R-linear map defined by the rule $\varphi(u)(v) = \beta(u, v)$ for $u \in M$ and $v \in N$. If the induced pairings of vector spaces $M(\mathfrak{p}) \times N(\mathfrak{p}) \to P(\mathfrak{p})$ are nondegenerate for all $\mathfrak{p} \in X$, then φ is bijective.*

Proof. For each $\mathfrak{p}$ there exists $f \in R \setminus \mathfrak{p}$ such that the R_f-modules M_f and N_f are free of rank r, while $P_f \cong R_f$. Denote by d the determinant of the matrix of the R_f-bilinear pairing $\beta_f : M_f \times N_f \to P_f$ induced by β in any bases of the three R_f-modules. Since the image of d in $\kappa(\mathfrak{p})$ is the determinant of a nondegenerate pairing of vector spaces, it is nonzero. Hence $d \in R \setminus \mathfrak{p}$. Replacing f with fd, we find such an f for which β_f is perfect, so that the corresponding R_f-linear map φ_f is bijective. The commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & \operatorname{Hom}_R(N, P) \\ {\scriptstyle\text{can.}}\downarrow & & \downarrow{\scriptstyle\text{can.}} \\ M_f & \xrightarrow{\varphi_f} & \operatorname{Hom}_{R_f}(N_f, P_f) \end{array}$$

shows that $\operatorname{Ker}\varphi \subset \operatorname{Ker}(M \to M_f)$. Applying this observation to various $\mathfrak{p}$, we deduce that $\operatorname{Ker}\varphi$ is $\mathcal{F}$-torsion. Since M is $\mathcal{F}$-torsionfree, φ has to be injective. Given $\xi \in \operatorname{Hom}_R(N,P)$, there exists $u \in M$ and an integer $n \ge 0$ such that $\xi_f = \varphi_f(u/f^n)$. Putting $\eta = f^n\xi - \varphi(u)$, we get $\eta_f = 0$, i.e. $\eta(N) \subset \operatorname{Ker}(P \to P_f)$. It follows from Lemma 1.8 that $f^e\eta = 0$, and therefore $f^{n+e}\xi = \varphi(f^e u) \in \varphi(M)$. Hence $\operatorname{Coker}\varphi$ is $\mathcal{F}$-torsion. By Lemma 1.6 the R-module $\operatorname{Hom}_R(N,P)$ is $\mathcal{F}$-torsionfree, and it remains to use the $\mathcal{F}$-closedness of M. ∎

In order to obtain analogs of dual vector bundles we need yet another condition on modules. Assume an R-module M to be locally free of finite rank near X. Denote $M_R^* = \operatorname{Hom}_R(M,R)$. We say that M is *dualizable near X* if for each $\mathfrak{p} \in X$ there exists $f \in R \setminus \mathfrak{p}$ such that M_f is a free R_f-module and the canonical map

$$R_f \otimes_R M_R^* \to \operatorname{Hom}_{R_f}(M_f, R_f)$$

is bijective. Note that this map is automatically injective in view of (A), and therefore we only have to check that the R_f-module $\operatorname{Hom}_{R_f}(M_f,R_f)$ is generated by the image of M_R^*.

Lemma 1.10. *If M is finitely generated and locally free near X, then it is dualizable.*

Proof. Given $\mathfrak{p} \in X$, take any $f \in R \setminus \mathfrak{p}$ for which M_f is a free R_f-module of finite rank. Let $\psi : M_f \to R_f$ be any R_f-linear map. Denote by M' the image of M in M_f, by R' the image of R in R_f. Since M is finitely generated, there exists an integer $r \ge 0$ such that $f^r\psi(M') \subset R'$. By assumption (A) the kernel of the canonical surjection $R \to R'$ coincides with $\{g \in R \mid f^e g = 0\}$. Hence the multiplication by f^e in R induces an R-linear map $\lambda : R' \to R$. Denoting by φ the composite

$$M \xrightarrow{\text{can.}} M' \xrightarrow{f^r\psi} R' \xrightarrow{\lambda} R,$$

we get a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & R \\ {\scriptstyle\text{can.}}\downarrow & & \downarrow{\scriptstyle\text{can.}} \\ M' & \xrightarrow{f^{e+r}\psi} & R' \end{array}$$

which shows that $\varphi_f = f^{e+r}\psi$, i.e. $\psi = f^{-e-r}\varphi_f$ with $\varphi \in M_R^*$. ∎

If M is not finitely generated, but only locally finitely generated near X, then the conclusion of Lemma 1.10 remains valid when X is Zariski quasicompact and R is $\mathcal{F}$-closed. Indeed, under these assumptions there exists a finitely generated submodule $N \subset M$ such that M/N is $\mathcal{F}$-torsion. Since N is necessarily locally free, N is dualizable by Lemma 1.10. Since the map $M_R^* \to N_R^*$ is bijective, M is also dualizable.

Lemma 1.11. *Suppose that M is locally free of finite rank near X. If for each $\mathfrak{p} \in X$ the $\kappa(\mathfrak{p})$-vector space $M(\mathfrak{p})^* = \operatorname{Hom}_{\kappa(\mathfrak{p})}\big(M(\mathfrak{p}), \kappa(\mathfrak{p})\big)$ is linearly spanned by the image of M_R^*, then M is dualizable.*

Proof. Fixing some $\mathfrak{p} \in X$, we can find R-linear maps $\varphi_1, \ldots, \varphi_r : M \to R$ such that $\kappa(\mathfrak{p}) \otimes_R \varphi_1, \ldots, \kappa(\mathfrak{p}) \otimes_R \varphi_r$ are a basis for $M(\mathfrak{p})^*$ over $\kappa(\mathfrak{p})$. Take $f \in R \setminus \mathfrak{p}$ for which M_f is a free R_f-module. The rank of M_f is equal to $r = \dim M(\mathfrak{p})$. The dual R_f-module $\mathrm{Hom}_{R_f}(M_f, R_f)$ is also free of rank r. Picking any basis $\varepsilon_1, \ldots, \varepsilon_r$ for this module, we have $(\varphi_i)_f = \sum_{j=1}^r a_{ij}\varepsilon_j$ with $a_{ij} \in R_f$. Denote $d = \det(a_{ij})$. Since

$$M(\mathfrak{p})^* \cong \kappa(\mathfrak{p}) \otimes_{R_f} \mathrm{Hom}_{R_f}(M_f, R_f),$$

the images of a_{ij}'s in $\kappa(\mathfrak{p})$ are entries of the transition matrix between two bases for $M(\mathfrak{p})^*$. Since this matrix is invertible, we have $d \notin \mathfrak{p}R_f$. Adjusting f if necessary, we may assume therefore that d is invertible in R_f, which means that $(\varphi_1)_f, \ldots, (\varphi_r)_f$ are a basis for $\mathrm{Hom}_{R_f}(M_f, R_f)$ over R_f. ∎

Lemma 1.12. *Let M be locally free of finite rank and dualizable near X. For $\mathfrak{p} \in X$ there exists $f \in R \setminus \mathfrak{p}$ such that M_f is a free R_f-module and the canonical map*

$$R_f \otimes_R \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{R_f}(M_f, N_f) \qquad (*)$$

is surjective for any R-module N. If N is $\mathcal{F}$-torsionfree and locally free near X, then $(*)$ is bijective. In this case $H = \mathrm{Hom}_R(M, N)$ is locally free near X and we have $H(\mathfrak{p}) \cong \mathrm{Hom}_{\kappa(\mathfrak{p})}\big(M(\mathfrak{p}), N(\mathfrak{p})\big)$ for all $\mathfrak{p} \in X$.

Proof. There exist $f \in R \setminus \mathfrak{p}$ and R-linear maps $\varphi_1, \ldots, \varphi_r : M \to R$ such that M_f is a free R_f-module of rank r and $(\varphi_1)_f, \ldots, (\varphi_r)_f$ are a basis for $\mathrm{Hom}_{R_f}(M_f, R_f)$ over R_f. Each R_f-linear map $\psi : M_f \to N_f$ can be expressed as

$$\psi(u) = \sum (\varphi_i)_f(u)v_i, \qquad u \in M_f,$$

for some $v_1, \ldots, v_r \in N_f$. We have $v_i = w_i/f^n$ with $w_1, \ldots, w_r \in N$ and a sufficiently large integer $n > 0$. Now the assignment $u \mapsto \sum \varphi_i(u)w_i$ defines an R-linear map $\varphi : M \to N$ such that $f^n\psi = \varphi_f$. Hence $(*)$ is surjective.

Suppose that N is $\mathcal{F}$-torsionfree and locally free near X. If the homomorphism $\varphi \in \mathrm{Hom}_R(M, N)$ satisfies $\varphi_f = 0$, then $f^e\varphi = 0$ by Lemma 1.8, which means that $(*)$ is injective. We can find an element f with the additional property that N_f is also a free R_f-module. Then so too is $H_f \cong \mathrm{Hom}_{R_f}(M_f, N_f)$. Finally,

$$\begin{aligned}\mathrm{Hom}_{\kappa(\mathfrak{p})}\big(M(\mathfrak{p}), N(\mathfrak{p})\big) &\cong \mathrm{Hom}_{\kappa(\mathfrak{p})}\big(\kappa(\mathfrak{p}) \otimes_{R_f} M_f, \kappa(\mathfrak{p}) \otimes_{R_f} N_f\big) \\ &\cong \kappa(\mathfrak{p}) \otimes_{R_f} \mathrm{Hom}_{R_f}(M_f, N_f) \cong \kappa(\mathfrak{p}) \otimes_R H = H(\mathfrak{p}).\end{aligned}$$ ∎

Lemma 1.13. *Suppose that R is $\mathcal{F}$-torsionfree, M is locally free of finite rank and dualizable near X. Then:*

(i) $t_{\mathcal{F}}(M) = \bigcap_{\xi \in M_R^*} \mathrm{Ker}\,\xi$ *and $M/t_{\mathcal{F}}(M)$ is locally free of finite rank near X.*

(ii) *If R is $\mathcal{F}$-closed, then $M_{\mathcal{F}}$ is locally free of finite rank near X.*

Proof. Denote by T the intersection of kernels of all R-linear maps $M \to R$. Then

$$T_f = \bigcap_{\xi \in M_R^*} \operatorname{Ker} \xi_f$$

for any $f \in R$. Indeed, $v_f \in \operatorname{Ker} \xi_f$ for some $v \in M$ if and only if $\xi(v)$ has zero image in R_f, which is equivalent to the equality $f^e \xi(v) = 0$ by assumption (A); when this equality holds for every $\xi \in M_R^*$, we get $f^e v \in T$.

For each $\mathfrak{p} \in X$ there exist $f \in R \smallsetminus \mathfrak{p}$ and $\varphi_1, \ldots, \varphi_r \in M_R^*$ such that M_f is a free R_f-module of rank r and $(\varphi_1)_f, \ldots, (\varphi_r)_f$ are a basis for $\operatorname{Hom}_{R_f}(M_f, R_f)$ over R_f. Since $\bigcap_{i=1}^r \operatorname{Ker}(\varphi_i)_f = 0$, we get $T_f = 0$ by the previous description, which yields $(M/T)_f \cong M_f$. Letting $\mathfrak{p}$ vary, we deduce that $T \subset t_{\mathcal{F}}(M)$ and M/T is locally free near X. If $v \in t_{\mathcal{F}}(M)$, then $\xi(v) \in t_{\mathcal{F}}(R) = 0$ for all $\xi \in M_R^*$, whence $t_{\mathcal{F}}(M) \subset T$.

Suppose that R is $\mathcal{F}$-closed. Then each $\xi \in M_R^*$ extends uniquely to an R-linear map $\tilde{\xi} : M_{\mathcal{F}} \to R$. By (i) the submodule

$$\widetilde{T} = \bigcap_{\xi \in M_R^*} \operatorname{Ker} \tilde{\xi}$$

of $M_{\mathcal{F}}$ has zero intersection with the image of M in $M_{\mathcal{F}}$. Hence $\widetilde{T}$ embeds in the cokernel of $M \to M_{\mathcal{F}}$, and therefore $\widetilde{T}$ is $\mathcal{F}$-torsion. Since $M_{\mathcal{F}}$ is $\mathcal{F}$-torsionfree, we conclude that $\widetilde{T} = 0$. The maps $\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_r$ with $\varphi_1, \ldots, \varphi_r$ chosen as in the previous paragraph give a homomorphism of R-modules $\alpha : M_{\mathcal{F}} \to R^r$ where R^r is the direct sum of r copies of R. Since the composite of $\alpha_f : (M_{\mathcal{F}})_f \to R_f^r$ with $M_f \to (M_{\mathcal{F}})_f$ is an isomorphism of R_f-modules, α_f is surjective. Note that

$$\operatorname{Ker} \alpha_f = \bigcap_{i=1}^r \operatorname{Ker} (\widetilde{\varphi}_i)_f.$$

If $\xi \in M_R^*$, then $f^n \xi_f = \sum g_i (\varphi_i)_f$ for some integer $n \geq 0$ and $g_1, \ldots, g_r \in R$. This entails $f^e(f^n \xi - \sum g_i \varphi_i) = 0$, and $f^{n+e} \tilde{\xi} = \sum f^e g_i \widetilde{\varphi}_i$. Passing to localizations at f, we deduce that $\operatorname{Ker} \alpha_f \subset \operatorname{Ker} \tilde{\xi}_f$. Hence $\operatorname{Ker} \alpha_f \subset \widetilde{T}_f = 0$. This shows that α_f is an isomorphism of R_f-modules, and therefore $(M_{\mathcal{F}})_f$ is free of rank r over R_f. ∎

Lemma 1.14. *Suppose that R is $\mathcal{F}$-closed and M', M, M'' satisfy the hypothesis of Lemma 1.4. Then M is dualizable near X if and only if so are both M' and M''.*

Proof. When $\chi : P \to Q$ is a homomorphism of locally free R-modules of finite rank with $\mathcal{F}$-torsion kernel and cokernel, χ induces an isomorphism $P_R^* \cong Q_R^*$ and for each $\mathfrak{p} \in X$ there exists $f \in R \smallsetminus \mathfrak{p}$ such that χ_f is an isomorphism $P_f \cong Q_f$. It follows that P is dualizable if and only if so is Q. Denoting $K = \operatorname{Ker} \psi$, we deduce that M' is dualizable if and only if so is K, while M'' is dualizable if and only if so is M/K. Thus it suffices to consider the case where M' is a submodule of M and $M'' = M/M'$.

Suppose that M is dualizable. Given $\mathfrak{p} \in X$, pick f as in Lemma 1.4. By further refining our choice of f, we may assume that each R_f-linear map $\eta : M_f \to R_f$ can be written as $f^{-n} \xi_f$ with $n \geq 0$ and $\xi \in M_R^*$. Clearly

$\eta|_{M'_f} = f^{-n}(\xi|_{M'})_f$. As M'_f is a direct summand of M_f, each R_f-linear map $M'_f \to R_f$ extends to M_f, which shows that the R_f-module $\mathrm{Hom}_{R_f}(M'_f, R_f)$ is generated by the image of $(M')^*_R$, i.e. M' is dualizable. If $\eta = \eta'' \circ \psi_f$ where η'' is an R_f-linear map $M''_f \to R_f$, then η vanishes on M'_f. In this case $f^e\xi|_{M'} = 0$ by Lemma 1.8, whence $f^e\xi = \xi'' \circ \psi$ for some $\xi'' \in (M'')^*_R$, so that $\eta'' = f^{-n-e}\xi''_f$. Hence M'' is dualizable.

Suppose now that M' and M'' are dualizable. Put $T = t_{\mathcal{F}}(M'')$, and denote by N the preimage of T in M. Then $M' \subset N$ with $N/M' \cong T$ and $M/N \cong M''/T$. By Lemma 1.13 T and M''/T are locally free of finite rank near X. The same holds then for N. The observation at the beginning of the proof applied to the canonical homomorphisms $M' \to N$ and $M'' \to M''/T$ shows that N and M/N are both dualizable. Replacing M' with N and M'' with M/N, we may assume from the very beginning that M'' is $\mathcal{F}$-torsionfree.

Let $f \in R \setminus \mathfrak{p}$ be an element for which both the conclusion of Lemma 1.4 and the conclusion of Lemma 1.12 with M'' in place of M are true. The split epimorphism $\psi_f : M_f \to M''_f$ admits an R_f-linear retraction $M''_f \to M_f$. The latter can be written as $f^{-n}\sigma_f$ for some integer $n \ge 0$ and $\sigma \in \mathrm{Hom}_R(M'', M)$. Since $(f^n \mathrm{Id} - \psi \circ \sigma)_f = 0$, Lemma 1.8 yields $f^e(f^n \mathrm{Id} - \psi \circ \sigma) = 0$. Hence

$$\tau = f^{n+e}\,\mathrm{Id} - f^e\sigma \circ \psi : M \to M$$

satisfies $\psi \circ \tau = 0$, i.e. $\tau(M) \subset M'$. Given an R_f-linear map $\eta : M_f \to R_f$, we have

$$f^{n+e}\eta = f^e\eta \circ \sigma_f \circ \psi_f + \eta \circ \tau_f.$$

There exist an integer $m \ge 0$ and R-linear maps $\xi' : M' \to R$ and $\xi'' : M'' \to R$ such that

$$f^m\eta|_{M'_f} = \xi'_f \quad \text{and} \quad f^{m+e}\eta \circ \sigma_f = \xi''_f.$$

Hence $f^{m+n+e}\eta = \xi_f$ where $\xi = \xi'' \circ \psi + \xi' \circ \tau$. This shows that M is dualizable. ■

Lemma 1.15. *If two R-modules M, N are locally free of finite rank and dualizable near X, then so are the tensor product $M \otimes_R N$, arbitrary tensor powers $\bigotimes_R^k M$, exterior powers $\bigwedge_R^k M$, symmetric powers $S_R^k M$.*

Proof. Given any $\mathfrak{p} \in X$, we can find $f \in R \setminus \mathfrak{p}$ such that the R_f-modules M_f, N_f are both free of finite rank and their dual modules are generated by the images of M^*_R and N^*_R, respectively. Then $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$ is free of finite rank over R_f and its dual module is generated by $\{(\varphi \otimes \psi)_f \mid \varphi \in M^*_R,\ \psi \in N^*_R\}$ where $\varphi \otimes \psi : M \otimes_R N \to R$ is defined by the rule $u \otimes v \mapsto \varphi(u)\psi(v)$. This proves the conclusion for $M \otimes_R N$, and by induction for $\bigotimes_R^k M$.

The symmetric and exterior powers of finitely generated free modules are finitely generated free. Since the formation of symmetric and exterior powers commutes with the localization at f, the R-modules $S_R^k M$, $\bigwedge_R^k M$ are locally free of finite rank. Each $\varphi \in M^*_R$ extends to a homomorphism of R-algebras $\bigoplus_{k=0}^{\infty} S_R^k M \to R$. Let us denote by $\varphi^{(k)}$ the restriction of the latter to $S_R^k M$. When M is free of finite rank, $\mathrm{Hom}_R(S_R^k M, R)$ is the R-linear span of all $\varphi^{(k)}$'s. In general this observation shows that the R_f-module $\mathrm{Hom}_{R_f}\big((S_R^k M)_f, R_f\big)$ is generated by all $\varphi_f^{(k)}$ with $\varphi \in M_R^*$. Similarly, any $\varphi_1, \dots \varphi_k \in M_R^*$ give rise to an R-linear map $\bigwedge_R^k M \to R$ sending $u_1 \wedge \cdots \wedge u_k$ to $\det[\varphi_i(u_j)]$, and $\operatorname{Hom}_{R_f}\big((\bigwedge_R^k M)_f, R_f\big)$ is generated by extensions of such maps. Hence $S_R^k M$ and $\bigwedge_R^k M$ are dualizable. ∎

The next lemma shows that the $\mathcal{F}$-closedness of locally free modules is related to the glueing property for a collection of sections of a vector bundle over open subsets covering the base space.

Lemma 1.16. *Suppose that M is locally free near X. In order that M be $\mathcal{F}$-closed, it is necessary and sufficient that for any indexed set $\{f_\alpha\} \subset R$ contained in none of the ideals $\mathfrak{p} \in X$ and any collection of elements $v_\alpha \in M_{f_\alpha}$ such that v_α and v_β have the same image in $M_{f_\alpha f_\beta}$ for each pair of indices α, β there exists a unique $v \in M$ whose image in M_{f_α} coincides with v_α for each α.*

This lemma will not be used, and we will only indicate the main steps of the proof. The uniqueness in the statement of the lemma is equivalent to the condition that $t_{\mathcal{F}}(M) = 0$. Therefore M may be assumed to be $\mathcal{F}$-torsionfree. Then M embeds in

$$\check{H}^0(\{f_\alpha\}, M) = \{(v_\alpha) \in \prod M_{f_\alpha} \mid (v_\alpha)_{f_\alpha f_\beta} = (v_\beta)_{f_\alpha f_\beta} \text{ for all } \alpha, \beta\}$$

where the subscript $f_\alpha f_\beta$ indicates taking the image in $M_{f_\alpha f_\beta}$. The formula above defines a submodule of the $\mathcal{F}$-torsionfree R-module $\prod M_{f_\alpha}$. The crucial argument is that the factor module $\check{H}^0(\{f_\alpha\}, M)/M$ is $\mathcal{F}$-torsion; here assumption (A) is needed. The $\mathcal{F}$-closedness of M entails $\check{H}^0(\{f_\alpha\}, M) = M$. In the opposite direction, let N be any $\mathcal{F}$-torsionfree R-module containing M as a submodule with an $\mathcal{F}$-torsion factor module N/M. Given $w \in N$, there exists $I \in \mathcal{F}$ such that $Iw \subset M$. If $\{f_\alpha\}$ is any generating set for the ideal I, then w_{f_α}, the image of w in N_{f_α}, actually belongs to M_{f_α}, and $(w_{f_\alpha}) \in \check{H}^0(\{f_\alpha\}, M)$. The isomorphism $\check{H}^0(\{f_\alpha\}, M) \cong M$ yields $w \in M$. We omit further details of the proof.

Lemma 1.16 admits the following reformulation: if $\mathcal{M}$ denotes the quasicoherent sheaf associated with M on the affine scheme $\operatorname{Spec} R$, then M is $\mathcal{F}$-closed if and only if the canonical map $M \to \Gamma(U, \mathcal{M})$ is bijective for any open neighbourhood U of X with respect to the Zariski topology.

Thus the class of R-modules which are $\mathcal{F}$-closed, locally free of finite rank and dualizable near X provides all basic features of the category of vector bundles. When X contains all maximal ideals of R, all these concepts become trivial. Indeed, in this case the only $\mathcal{F}$-torsion module is 0, and therefore all modules are $\mathcal{F}$-closed. The term "locally finitely generated" is equivalent to "finitely generated", while "locally free of finite rank" means "finitely generated projective", and such modules are dualizable. Given a finitely generated projective module M, any its finitely generated submodule which is locally a direct summand of M is a direct summand by [1, Ch. II, §3, Cor. 1 to Prop. 12].

2. Detecting local freeness

Let R be a commutative associative algebra with 1 over a field $\mathbb{F}$, and let X be a set of prime ideals of R. The vector space $\operatorname{Der} R$ of all $\mathbb{F}$-linear derivations of

R is a Lie algebra over $\mathbb{F}$ with respect to the commutators of derivations. There is also an R-module structure on $\operatorname{Der} R$ such that

$$(fD)(g) = f \cdot D(g) \quad \text{for } f, g \in R \text{ and } D \in \operatorname{Der} R.$$

We will assume in this section that L is a Lie algebra over $\mathbb{F}$ which acts on R via derivations, so that a Lie algebra homomorphism $L \to \operatorname{Der} R$ is given. By a *Lie algebra action* of L on a vector space V we mean any representation given by a Lie algebra homomorphism $L \to \mathfrak{gl}(V)$. An (R, L)*-module* is an R-module M equipped with a Lie algebra action $L \times M \to M$ such that

$$D(fu) = (Df)u + f(Du) \quad \text{for all } D \in L,\ f \in R,\ u \in M.$$

Each derivation of R extends in a unique way to a derivation of R_h for any $h \in R$. In particular, L acts on R_h as a Lie algebra of derivations. Similarly, the induced action of L on M_h makes M_h into an (R_h, L)-module.

One can define several operations on (R, L)-modules. Given two (R, L)-modules M and N, there are actions of L on $M \otimes_R N$ and $\operatorname{Hom}_R(M, N)$ compatible with the R-module structures. Explicitly,

$$D(u \otimes v) = Du \otimes v + u \otimes Dv, \qquad (D\varphi)(u) = D\,\varphi(u) - \varphi(Du)$$

where $D \in L$, $u \in M$, $v \in N$ and $\varphi \in \operatorname{Hom}_R(M, N)$. In particular, the kth tensor power $\bigotimes_R^k M$ is an (R, L)-module for any $k > 0$. The symmetric and exterior powers of M carry the induced (R, L)-module structures.

Recall that $\kappa(\mathfrak{p})$ denotes the field of fractions of $R/\mathfrak{p}$ and $M(\mathfrak{p})$ denotes the vector space $\kappa(\mathfrak{p}) \otimes_R M$ over $\kappa(\mathfrak{p})$. The family of ideals $\mathcal{F}$ was introduced in section 1. For each ideal I of R denote by $(I : L)$ the largest L-invariant ideal of R contained in I. It consists of all elements $a \in I$ such that $D_1 \cdots D_k a \in I$ for any finite sequence of elements $D_1, \ldots, D_k \in L$.

Theorem 2.1. *Suppose that $\bigcap_{\mathfrak{p} \in X} (\mathfrak{p} : L) = 0$, i.e. R has no nonzero L-invariant ideals contained in all $\mathfrak{p} \in X$. Let M be an (R, L)-module locally finitely generated near X as an R-module. If $r \geq 0$ is an integer such that $\dim M(\mathfrak{p}) = r$ for all $\mathfrak{p} \in X$, then M is locally free of rank r near X.*

The proof of Theorem 2.1 is preceded by two lemmas.

Lemma 2.2. *Let $\mathfrak{p}$ be a prime ideal of R. If $\operatorname{char} \mathbb{F} = 0$, then the ideal $(\mathfrak{p} : L)$ is also prime. If $\operatorname{char} \mathbb{F} = p > 0$ and $f, g \in R$ satisfy $f^n g \in (\mathfrak{p} : L)$ for some $n > 0$, then either $f^p \in (\mathfrak{p} : L)$ or $g \in (\mathfrak{p} : L)$.*

Proof. The statement in the zero characteristic case is contained in [2, Lemma 3.3.2] (it is valid even for noncommutative algebras). Suppose that $\operatorname{char} \mathbb{F} = p > 0$. Then $D(f^p) = pf^{p-1} D(f) = 0$ for all $D \in \operatorname{Der} R$ and $f \in R$. Hence each ideal Rf^p is stable under all derivations. If $f \in \mathfrak{p}$, then this ideal is contained in $\mathfrak{p}$, and therefore also in $(\mathfrak{p} : L)$. Suppose that $f \notin \mathfrak{p}$. Then $\{h \in R \mid f^p h \in (\mathfrak{p} : L)\}$ is an L-invariant ideal of R contained in $\mathfrak{p}$, and therefore in $(\mathfrak{p} : L)$. This shows that the image of f^p in the factor ring $R/(\mathfrak{p} : L)$ is not a zero divisor, and the same holds then for any power of f. Hence $g \in (\mathfrak{p} : L)$. ∎

When M is a finitely generated R-module, valuable information can be derived from certain ideals of R called Fitting invariants of M (see, e.g., [3, Ch. 20]). Since we do not require M to be finitely generated, we are forced to consider Fitting invariants of finitely generated submodules. For each finite sequence $v_1, \dots, v_n \in M$ let

$$\mathrm{Rel}(v_1, \dots, v_n) = \{(a_1, \dots, a_n) \in R^n \mid a_1v_1 + \cdots + a_nv_n = 0\}$$

and denote by $I_k(v_1, \dots, v_n)$ the ideal of R generated by the $k \times k$ minors of all $k \times n$ matrices whose rows belong to $\mathrm{Rel}(v_1, \dots, v_n)$. Obviously, these ideals do not depend on the order of $v_1, \dots, v_n$. Our convention is that $I_0(v_1, \dots, v_n) = R$. In the next lemma we recall several well-known properties of Fitting invariants.

Lemma 2.3. *Let $v_1, \dots, v_n$ and $w_1, \dots, w_m$ be two finite sequences in M.*

(i) *If $w_j \in Rv_1 + \cdots + Rv_n$ for all j, then*

$$I_k(v_1, \dots, v_n) = I_{k+m}(v_1, \dots, v_n, w_1, \dots, w_m) \quad \text{for all } k \geq 0.$$

(ii) *$I_k\big((v_1)_f, \dots, (v_n)_f\big) = I_k(v_1, \dots, v_n)R_f$ for any $f \in R$.*

(iii) *If $I_1(v_1, \dots, v_n) = 0$, then $v_1, \dots, v_n$ are linearly independent over R.*

(iv) *If $Rv_1 + \cdots + Rv_n = M$ and $I_k(v_1, \dots, v_n) = R$, $I_{k+1}(v_1, \dots, v_n) = 0$ for some integer $k \geq 0$, then $k \leq n$ and M is a projective R-module of rank $n - k$.*

(v) *If M is an (R, L)-module and $Rv_1 + \cdots + Rv_n = M$, then $I_1(v_1, \dots, v_n)$ is an L-invariant ideal.*

Proof. (i) Since we may proceed by induction on m, it suffices to consider the case $m = 1$. Let $w_1 = c_1v_1 + \cdots + c_nv_n$ for some $c_1, \dots, c_n \in R$. Each $(n+1)$-tuple in $\mathrm{Rel}(v_1, \dots, v_n, w_1)$ can be written as $r \cdot (c_1, \dots, c_n, -1) + (a_1, \dots, a_n, 0)$ with $r \in R$ and $(a_1, \dots, a_n) \in \mathrm{Rel}(v_1, \dots, v_n)$. The conclusion now follows from elementary properties of determinants.

(ii) Note that $\mathrm{Rel}\big((v_1)_f, \dots, (v_n)_f\big)$ is the R_f-submodule of R_f^n generated by the image of $\mathrm{Rel}(v_1, \dots, v_n)$.

(iii) The equality $I_1(v_1, \dots, v_n) = 0$ is equivalent to $\mathrm{Rel}(v_1, \dots, v_n) = 0$.

(iv) We have to prove that for each prime ideal $\mathfrak{p}$ of R there exists $f \in R \smallsetminus \mathfrak{p}$ such that M_f is a free R_f-module of rank $n - k$. Let us fix $\mathfrak{p}$. The $\kappa(\mathfrak{p})$-vector space $M(\mathfrak{p})$ is spanned by the images $v_1', \dots, v_n'$ of $v_1, \dots, v_n$. Hence $M(\mathfrak{p})$ has dimension $r \leq n$. After renumbering we may assume that $v_1', \dots, v_r'$ are a basis for $M(\mathfrak{p})$. By Nakayama's Lemma there exists $f \in R \smallsetminus \mathfrak{p}$ such that the R_f-module M_f is generated by $(v_1)_f, \dots, (v_r)_f$. Part (ii) allows us to replace R with R_f and M with M_f. So we may assume that $v_1, \dots, v_r$ generate M. By (i)

$$I_j(v_1, \dots, v_r) = I_{j+n-r}(v_1, \dots, v_n)$$

for all j. Since these ideals form a descending chain when j grows, the hypothesis shows that $I_j(v_1, \dots, v_r)$ is equal to R when $j+n-r \leq k$ and is 0 otherwise. Note that $I_1(v_1, \dots, v_r) \subset \mathfrak{p}$ since $v_1', \dots, v_r'$ are linearly independent over $\kappa(\mathfrak{p})$. On the other hand, $I_0(v_1,\ldots,v_r)=R$. This entails $n-r=k$ and $I_1(v_1,\ldots,v_r)=0$. By (iii) $v_1,\ldots,v_r$ are a basis for M over R, which completes the proof.

(v) Let $D\in L$. Then for each i we have $Dv_i=\sum_{j=1}^n c_{ij}v_j$ for some $c_{ij}\in R$. If $(a_1,\ldots,a_n)\in \operatorname{Rel}(v_1,\ldots,v_n)$, then

$$0=D\bigl(\sum_{i=1}^n a_iv_i\bigr)=\sum_{j=1}^n\bigl(Da_j+\sum_{i=1}^n a_ic_{ij}\bigr)v_j$$

Hence $Da_j+\sum_{i=1}^n a_ic_{ij}$, and therefore Da_j, lie in $I_1(v_1,\ldots,v_n)$, for all j. ∎

Proof of Theorem 2.1. Fixing some $\mathfrak p\in X$, take any $v_1,\ldots,v_r\in M$ whose images in $M(\mathfrak p)$ form a basis for that vector space over $\kappa(\mathfrak p)$. There exists $h\in R\smallsetminus\mathfrak p$ such that M_h is a finitely generated R_h-module. Hence $M_{\mathfrak p}=R_{\mathfrak p}\otimes_R M\cong R_{\mathfrak p}\otimes_{R_h}M_h$ is a finitely generated module over the local ring $R_{\mathfrak p}$ of $\mathfrak p$. By Nakayama's Lemma the images of $v_1,\ldots,v_r$ in $M_{\mathfrak p}$ generate that module. It follows that there exists $f\in R\smallsetminus\mathfrak p$ such that $(v_1)_f,\ldots,(v_r)_f$ generate the R_f-module M_f.

Denote $I=I_1(v_1,\ldots,v_r)$. By (ii) and (v) of Lemma 2.3 IR_f is an L-invariant ideal of R_f. Hence the preimage I' of IR_f in R is an L-invariant ideal of R. In any relation $a_1v_1+\cdots+a_rv_r=0$ with coefficients in R we have $a_i\in\mathfrak p$ for all i by our choice of v_i's. Hence $I\subset\mathfrak p$. Since for each $g\in I'$ there exists $n>0$ such that $f^ng\in I$, it follows that $I'\subset\mathfrak p$ by the primeness of $\mathfrak p$, but then $I'\subset(\mathfrak p:L)$.

Consider an arbitrary $\mathfrak q\in X$. Let $w_1,\ldots,w_r\in M$ be any elements whose images in $M(\mathfrak q)$ form a basis over $\kappa(\mathfrak q)$. Put $J=I_{r+1}(v_1,\ldots,v_r,w_1,\ldots,w_r)$. By (i) and (ii) of Lemma 2.3 $JR_f=IR_f$. Then $J\subset I'$, and therefore $J\subset(\mathfrak p:L)$, as was shown in the previous paragraph. Interchanging $\mathfrak p$ and $\mathfrak q$, we deduce that $J\subset(\mathfrak q:L)$ by symmetry. Since for each $g\in I$ there exists $n>0$ such that $f^ng\in J$, Lemma 2.2 yields $f^eg\in(\mathfrak q:L)$ where $e=1$ if $\operatorname{char}\mathbb F=0$ and $e=\operatorname{char}\mathbb F$ otherwise.

We conclude that $f^eI\subset\bigcap_{\mathfrak q\in X}(\mathfrak q:L)=0$, and therefore $IR_f=0$. Parts (ii) and (iii) of Lemma 2.3 show that M_f is a free R_f-module. ∎

Theorem 2.4. *Suppose that none of the primes $\mathfrak p\in X$ contains any nonzero L-invariant locally finitely generated near X ideal of R. If N is an (R,L)-module, locally free of finite rank and dualizable near X as an R-module, then the rank function of N is constant on X, i.e. $\dim N(\mathfrak p)$ does not depend on $\mathfrak p\in X$. Moreover, given a homomorphism of (R,L)-modules $\varphi: M\to N$ where M, as an R-module, is locally finitely generated near X, there is an integer $r\ge 0$ such that*:

(i) *the $\kappa(\mathfrak p)$-linear map $M(\mathfrak p)\to N(\mathfrak p)$ induced by φ has rank r for each $\mathfrak p\in X$,*

(ii) *the R-module $\varphi(M)$ is locally near X a rank r direct summand of N.*

In the proof we will need another descending chain of ideals of R which can be defined for an arbitrary R-module N and its submodule M. Recall that N_R^* stands for $\operatorname{Hom}_R(N,R)$. Denote by $J_k(N_R^*,M)$ the ideal of R generated by the determinants of all $k\times k$ matrices of the form

$$\bigl[\xi_j(v_i)\bigr]_{1\le i,j\le k}\quad\text{with } \xi_1,\ldots,\xi_k\in N_R^* \text{ and } v_1,\ldots,v_k\in M;$$

when $k=0$ put $J_0(N_R^*,M)=R$. For each $f\in R$ the previous definition applies to the R_f-module N_f and its submodule M_f, giving an ideal of R_f.

Lemma 2.5. *Suppose that M is a submodule of an R-module N.*

(i) *If $(N_f)^*_{R_f}\cong R_f\otimes_R N_R^*$ for some $f\in R$ then $J_k\big((N_f)^*_{R_f},M_f\big)=J_k(N_R^*,M)R_f$.*

(ii) *If N_R^* and M are finitely generated R-modules then so is the ideal $J_k(N_R^*,M)$.*

(iii) *If M is generated by q elements then $J_k(N_R^*,M)=0$ for all $k>q$.*

(iv) *If $e_1,\dots,e_n$ are a basis for N and $\bar e_1,\dots,\bar e_n$ denote their images in N/M then $J_k(N_R^*,M)=I_k(\bar e_1,\dots,\bar e_n)$.*

(v) *If N is an (R,L)-module and M is L-stable then so is $J_k(N_R^*,M)$.*

Proof. If S and T are any generating sets for the R-modules M and N_R^*, respectively, then the ideal $J_k(N_R^*,M)$ is generated by the determinants of $k\times k$ matrices $\big[\xi_j(v_i)\big]$ with $\xi_1,\dots,\xi_k\in T$ and $v_1,\dots,v_k\in S$. When S and T are finite, we obtain a finite set of generators for $J_k(N_R^*,M)$. This proves (ii). If S has cardinality q, while $k>q$, then any sequence $v_1,\dots,v_k\in S$ contains at least two equal elements. Then the matrix $\big[\xi_j(v_i)\big]$ has two equal rows, and therefore its determinant is zero. Hence (iii). For (i) it remains to observe that the R_f-modules $(N_f)^*_{R_f}$ and M_f are generated by the images of N_R^* and M, respectively.

In (iv) N_R^* is a free R-module of rank n. Let $T=\{e_1^*,\dots,e_n^*\}$ be its basis dual to $e_1,\dots,e_n$. The ideal $J_k(N_R^*,M)$ is generated by the determinants of all $k\times k$ matrices $\big[\xi_j(v_i)\big]$ with $\xi_1,\dots,\xi_k\in T$ and $v_1,\dots,v_k\in M$. If $v=a_1e_1+\ldots+a_ne_n$ with $a_1,\dots,a_n\in R$, then $e_j^*(v)=a_j$; furthermore, the inclusion $v\in M$ is equivalent to the equality $a_1\bar e_1+\ldots+a_n\bar e_n=0$ in N/M. Hence $J_k(N_R^*,M)$ is generated by the $k\times k$ minors of the $k\times n$ matrices whose rows belong to $\mathrm{Rel}(\bar e_1,\dots,\bar e_n)$. This verifies the required equality of ideals.

Under the hypothesis of (v) N_R^* has a canonical (R,L)-module structure, and the map $N_R^*\otimes_R M\to R$ given by the assignment $\xi\otimes v\mapsto\xi(v)$ is a homomorphism of (R,L)-modules. It follows that for each permutation π of integers $1,\dots,k$ the map $\varphi_\pi:(\bigotimes_R^k N_R^*)\otimes_R(\bigotimes_R^k M)\to R$ such that

$$(\xi_1\otimes\cdots\otimes\xi_k)\otimes(v_1\otimes\cdots\otimes v_k)\mapsto\prod_{i=1}^k\xi_i(v_{\pi i})$$

is a homomorphism of (R,L)-modules, and so too is $\psi=\sum\mathrm{sgn}(\pi)\,\varphi_\pi$ where the sum is taken over all permutations π. Since

$$\psi\big((\xi_1\otimes\cdots\otimes\xi_k)\otimes(v_1\otimes\cdots\otimes v_k)\big)=\det\big[\xi_j(v_i)\big],$$

the ideal $J_k(N_R^*,M)$ coincides with the image of ψ, whence the conclusion. ∎

Proof of Theorem 2.4. Replacing M with $\varphi(M)$, we reduce the proof to the case where M is an (R,L)-submodule of N and φ is the inclusion. Since N is dualizable, for each $\mathfrak p\in X$ there exists $f\in R\setminus\mathfrak p$ such that N_f is free of finite rank over R_f and $(N_f)^*_{R_f}\cong R_f\otimes_R N_R^*$; we may also assume that M_f is finitely generated as an R_f-module. Then by (i) and (ii) of Lemma 2.5 $J_k(N_R^*, M)R_f$ is a finitely generated ideal of R_f for any k. Thus all ideals $J_k(N_R^*, M)$ are locally finitely generated near X. Furthermore, these ideals are L-invariant by Lemma 2.5(v). The hypothesis in Theorem 2.4 ensures that, whenever $J_k(N_R^*, M) \subset \mathfrak{p}$ for at least one $\mathfrak{p} \in X$, we must have $J_k(N_R^*, M) = 0$.

If f is chosen as before and the R_f-module M_f is generated, say, by q elements, then $J_{q+1}(N_R^*, M)R_f = 0$ by (i) and (iii) of Lemma 2.5. Therefore each element of $J_{q+1}(N_R^*, M)$ is annihilated by a power of f, which implies that $J_{q+1}(N_R^*, M) \subset \mathfrak{p}$. It follows that there exists an integer $r \geq 0$ such that $J_r(N_R^*, M) \not\subset \mathfrak{p}$ for all $\mathfrak{p} \in X$, while $J_{r+1}(N_R^*, M) = 0$.

Let us fix some $\mathfrak{p}$ again and refine our choice of f. Multiplying f by an arbitrary element in $J_r(N_R^*, M) \smallsetminus \mathfrak{p}$, we may assume that $f \in J_r(N_R^*, M)$. Lemma 2.5(i) now yields

$$J_r\big((N_f)^*_{R_f}, M_f\big) = R_f, \qquad J_{r+1}\big((N_f)^*_{R_f}, M_f\big) = 0.$$

If $\bar{e}_1, \ldots, \bar{e}_n$ denote the images in N_f/M_f of a basis for the free R_f-module N_f, then Lemma 2.5(iv) allows us to rewrite the previous equalities as

$$I_r(\bar{e}_1, \ldots, \bar{e}_n) = R_f, \qquad I_{r+1}(\bar{e}_1, \ldots, \bar{e}_n) = 0.$$

By Lemma 2.3(iv) N_f/M_f is a projective R_f-module of rank $n - r$, whence M_f is a direct summand of N_f. We may assume both M_f and N_f/M_f to be free over R_f, refining our choice of f once again. Then

$$\operatorname{rank} M_f = \operatorname{rank} N_f - \operatorname{rank} N_f/M_f = r,$$

so that $\dim M(\mathfrak{p}) = r$ as well, and this number does not depend on $\mathfrak{p}$. Tensoring with $\kappa(\mathfrak{p})$ over R_f, we deduce that $M(\mathfrak{p}) \to N(\mathfrak{p})$ is injective. In the special case where $M = N$ we obtain the conclusion about the rank function of N. ∎

Corollary 2.6. *Suppose that R has no L-invariant finitely generated ideals other than 0 and R itself. Let $\varphi : M \to N$ be a homomorphism of (R, L)-modules. If M is a finitely generated R-module, while N is finitely generated projective, then $\varphi(M)$ is an R-module direct summand of N and $\varphi(M)$ is projective of constant rank.*

This is a special case of Theorem 2.4 where we take X to be the set of all maximal ideals of R. The next proposition and its corollary present algebraic formulations of the fact that certain systems of linear differential equations have finite dimensional space of solutions. In differential algebra a result of this kind is known for differential vector spaces (cf. [23, (1.3)]). For an (R, L)-module M denote

$$M^L = \{v \in M \mid Dv = 0 \text{ for all } D \in L\}.$$

In particular, R^L is a subring of R. Clearly M^L is an R^L-submodule of M.

Proposition 2.7. *Suppose that R is $\mathcal{F}$-closed and none of the primes $\mathfrak{p} \in X$ contains any nonzero L-invariant locally finitely generated near X ideal of R. Then R^L is a field. If an (R, L)-module M is $\mathcal{F}$-torsionfree, locally free of finite rank and dualizable near X as an R-module, then M^L is a finite dimensional vector space over R^L. Moreover, any basis for M^L over R^L is a basis for a free R-submodule of M which is locally near X a direct summand of M.*

Proof. If $v \in M^L$, then Rv is an (R, L)-submodule of M. By Theorem 2.4 there is an integer r such that the R-module Rv is locally free of rank r and is locally a direct summand of M. Since Rv is generated as an R-module by a single element, we must have $r \le 1$. If $r = 0$, then Rv is $\mathcal{F}$-torsion; in this case $v = 0$ since M is $\mathcal{F}$-torsionfree. Assuming that $v \ne 0$, we have therefore $r = 1$. Then for each $\mathfrak{p} \in X$ there exists $f \in R \setminus \mathfrak{p}$ such that v_f generates a cyclic free R_f-submodule of M_f; therefore the annihilator of v in R is contained in the kernel of $R \to R_f$. It follows that this annihilator is contained in the $\mathcal{F}$-torsion submodule of R, which is zero by the hypothesis. Thus v is a free generator of the R-module Rv.

In particular, for each $0 \ne a \in R^L$ the R-module Ra is free of rank 1. Since $Ra \not\subset \mathfrak{p}$ for any $\mathfrak{p} \in X$ by the hypothesis, we have $Ra \in \mathcal{F}$. Hence the R-module R/Ra is $\mathcal{F}$-torsion. Since R and $Ra \cong R$ are $\mathcal{F}$-closed, we get $Ra = R$, i.e. a is invertible. It follows that the map $R \to R$ given by the multiplication by a is an isomorphism of (R, L)-modules. Hence so too is the inverse map given by the multiplication by a^{-1}. This entails $a^{-1} \in R^L$, which verifies that R^L is a field.

We now prove that, whenever $v_1, \ldots, v_k \in M^L$ are linearly independent over R^L, then $v_1, \ldots, v_k$ are linearly independent over R and the R-module generated by these elements is locally a direct summand of M. The case $k = 1$ has been considered at the beginning of the proof. In particular, Rv_1 is locally a direct summand of M. Since M is locally free of finite rank, so too is $M' = M/Rv_1$. Any R-submodule T of M containing Rv_1 such that T/Rv_1 is $\mathcal{F}$-torsion must coincide with Rv_1 since $Rv_1 \cong R$ is $\mathcal{F}$-closed, while T is $\mathcal{F}$-torsionfree. In other words, M' is $\mathcal{F}$-torsionfree. Since M is dualizable, so is M' by Lemma 1.14.

Denote by v_i' the image of v_i in M'. Clearly $v_i' \in M'^L$. If

$$\lambda_2 v_2 + \cdots + \lambda_k v_k = f v_1$$

for some $\lambda_2, \ldots, \lambda_k \in R^L$ and $f \in R$, then $fv_1 \in M^L$; since the map $g \mapsto gv_1$ gives an isomorphism of (R, L)-modules $R \to Rv_1$, we deduce that $f \in R^L$, and the linear independence of $v_1, \ldots, v_k$ yields $\lambda_i = 0$ for all i. This shows that $v_2', \ldots, v_k'$ are linearly independent over R^L. Proceeding by induction on k, we may assume that $v_2', \ldots, v_k'$ are linearly independent over R and the R-module

$$M/(Rv_1 + \ldots + Rv_k) \cong M'/(Rv_2' + \ldots + Rv_k')$$

is locally free. Now our claim is immediate.

We conclude that the number k in the previous argument is bounded by the rank of M, and the proof is completed. ∎

It will later be convenient to have a reformulation of Proposition 2.7 in which the Lie algebra L is not mentioned explicitly. Let M be an R-module. An $\mathbb{F}$-linear transformation $\mathcal{D}$ of M will be called a *quasiderivation* if there exists $D \in \operatorname{Der} R$ such that

$$\mathcal{D}(fv) = D(f)v + f\mathcal{D}(v)$$

for all $f \in R$ and $v \in M$; we will also say that $\mathcal{D}$ is a *D-compatible quasiderivation.* The identity above can be rewritten as an equality of operators

$$[\mathcal{D}, f_M] = D(f)_M$$

where $g_M \in \operatorname{End}_{\mathbb{F}} M$ for $g \in R$ is afforded by the R-module structure on M. If M is a faithful R-module, so that $g_M \neq 0$ whenever $g \neq 0$, then each quasiderivation $\mathcal{D}$ is D-compatible for a uniquely determined D. In particular, this occurs when R is $\mathcal{F}$-torsionfree, M is locally free near X, and the rank function of M is nonzero at each $\mathfrak{p} \in X$. Indeed, under such assumptions any $g \in R$ with $g_M = 0$ has zero image in each localization R_f for which M_f is a nonzero free R_f-module, and therefore $g \in t_{\mathcal{F}}(R) = 0$.

Denote by $\operatorname{Qder} M$ the vector space of all quasiderivations of M. It is a Lie algebra with the Lie product given by commutators of operators. There is also an R-module structure on $\operatorname{Qder} M$ defined by the rule

$$f\mathcal{D} = f_M \circ \mathcal{D} \quad \text{for } f \in R \text{ and } \mathcal{D} \in \operatorname{Qder} M.$$

Corollary 2.8. *Let $\{D_\alpha\} \subset \operatorname{Der} R$ and $\{\mathcal{D}_\alpha\} \subset \operatorname{Qder} M$ be indexed sets such that $\mathcal{D}_\alpha$ is a D_α-compatible quasiderivation for each α. Suppose that R is $\mathcal{F}$-closed and none of the primes $\mathfrak{p} \in X$ contains any nonzero locally finitely generated near X ideal of R stable under all D_α. Put*

$$K = \{f \in R \mid D_\alpha f = 0 \textit{ for all } \alpha\}, \qquad V = \{v \in M \mid \mathcal{D}_\alpha v = 0 \textit{ for all } \alpha\}.$$

Then K is a field and V is a vector space over K. If M is $\mathcal{F}$-torsionfree, locally free of rank r and dualizable near X, then $\dim_K V \le r$.

Proof. Denote by L the set of all pairs $(D, \mathcal{D})$ where D is a derivation of R and $\mathcal{D}$ is a D-compatible quasiderivation of M. We have $(D_\alpha, \mathcal{D}_\alpha) \in L$ for each α. We will view L as a Lie algebra with respect to componentwise operations. In particular, the Lie product in L is given by componentwise commutators. The projection $L \to \operatorname{Der} R$ is a Lie algebra homomorphism. The Lie algebra L acts on M via second components, and this structure makes M into an (R, L)-module. We now meet all hypotheses of Proposition 2.7. It remains to note that $K = R^L$ and $V = M^L$. ■

Remark. The hypotheses of Theorems 2.1 and 2.4 include two different conditions on L-invariant ideals of R whose meaning is easy to understand in the case where X is a C^∞-manifold, $R = C^\infty(X)$ and $L = \operatorname{Vect}(X)$. The intersection $\bigcap_{x \in X} \mathfrak{m}_x$ is zero since it consists of functions vanishing at all points of X. In this case one has a much simpler version of Theorem 2.1 since any assumption about the Lie algebra may be omitted. Smooth functions flat at a chosen point x constitute an L-invariant ideal contained in $\mathfrak{m}_x$. However, this ideal is infinitely generated, even locally. Suppose that I is an L-invariant locally finitely generated ideal of R. Fixing x, pick finitely many functions $f_1, \ldots, f_k \in I$ whose images in R_h generate the ideal I_h of R_h for some $h \in R$ with $h(x) \neq 0$. Let $D_1, \ldots, D_n \in L$ be vector fields whose values at x give a basis for the tangent space $T_x(X)$. Since $D_i f_j \in I$, there are smooth functions g_{ijl} on the open neighbourhood $U = \{y \in X \mid h(y) \neq 0\}$ of x such that

$$D_i f_j = \sum_{l=1}^{k} g_{ijl} f_l.$$

Thus $f_1|_U, \ldots, f_k|_U$ are a solution of a system of differential equations. The initial values at x determine the solution uniquely in a neighbourhood of x. In particular, $f_1, \ldots, f_k$ are identically zero in a neighbourhood of x whenever $I \subset \mathfrak{m}_x$. This shows that the set of points where all functions in I attain zero value is not only closed, but also open. When X is connected, none of the ideals $\mathfrak{m}_x$ can contain any nonzero L-invariant locally finitely generated ideal. Thus Theorem 2.4 applies in this case. Its geometric meaning is as follows. Let $\varphi : E \to E'$ be a morphism of smooth vector bundles over X. Suppose that for each vector field D there are D-compatible quasiderivations on the modules of global sections of the two bundles and the map $\Gamma(X, E) \to \Gamma(X, E')$ induced by φ intertwines the two quasiderivations. Then $\varphi(E)$ is a subbundle of E'. A geometer would prove this statement by using local flows.

3. The setup

Let $\mathbb{F}$ be the ground field. We will consider a triple X, R, W where X is a set, R a commutative associative unital algebra, W a Lie subalgebra and an R-submodule of the derivation algebra $\operatorname{Der} R$. Furthermore, a homomorphism from R to the algebra of $\mathbb{F}$-valued functions on X is given, and the value of the function corresponding to an element $f \in R$ at a point $x \in X$ will be denoted by $f(x)$. With x we associate a maximal ideal

$$\mathfrak{m}_x = \{f \in R \mid f(x) = 0\}$$

of R for which $R = \mathbb{F} + \mathfrak{m}_x$, so that $R/\mathfrak{m}_x \cong \mathbb{F}$. This data is supposed to satisfy the following five assumptions:

(A1) *none of the nonzero W-invariant ideals of R is contained in* $\bigcap_{x \in X} \mathfrak{m}_x$,

(A2) *as an R-module, W is locally finitely generated near X,*

(A3) *there is an integer $n \geq 0$ such that* $\dim W/\mathfrak{m}_x W = n$ *for all $x \in X$,*

(A4) $\mathfrak{m}_x W$ *coincides with the stabilizer of* $\mathfrak{m}_x$ *in W,*

(A5) *R and W are $\mathcal{F}$-closed with respect to the localizing filter*

$$\mathcal{F} = \{\textit{ideals } I \textit{ of } R \mid I \not\subset \mathfrak{m}_x \textit{ for any } x \in X\}.$$

All local conditions near X should be understood as defined in section 1 with respect to the set of maximal ideals $\{\mathfrak{m}_x \mid x \in X\}$. In a sense the set X is considered with additional structure which serves as an algebraic replacement of the structure of an n-dimensional differentiable manifold. Here are three examples:

Example 1. Let $\mathbb{F}$ be the field of reals, X a Hausdorff C^∞-manifold, $R = C^\infty(X)$ the ring of smooth functions, $W = \operatorname{Vect}(X)$ the Lie algebra of smooth vector fields on X. In this case $\bigcap_{x \in X} \mathfrak{m}_x = 0$, so that (A1) is immediate. Applying the next lemma to the trivial onedimensional bundle and to the tangent bundle, we derive (A2), (A3) and (A5). Condition (A4) is also clear since $D \in W$ leaves $\mathfrak{m}_x$ stable if and only if $D(x) = 0$ in the tangent space $T_x(X) \cong W/\mathfrak{m}_x W$.

Lemma 3.1. *Let E be a smooth vector bundle over X. Then the R-module $M = \Gamma(X, E)$ is $\mathcal{F}$-closed and locally free of finite rank near X. The evaluation at any $x \in X$ gives a surjection $M \to E(x)$ whose kernel coincides with $\mathfrak{m}_x M$.*

Proof. Each $f \in R$ determines an open subset $X_f = \{y \in X \mid f(y) \neq 0\}$ of X, and one has a canonical map $M_f \to \Gamma(X_f, E)$. This map is injective since $fs = 0$ whenever $s \in M$ satisfies $s|_{X_f} = 0$.

Fixing some $x \in X$, let us choose an open neighbourhood U of x over which E trivializes and a function $f \in R$ with support in U such that $f(x) = 1$. Let $s_1, \dots, s_k$ be any basis for the free $C^\infty(U)$-module $\Gamma(U, E)$. Each local section fs_i extends to a global section $v_i \in M$ by assigning zero values outside the support of f. For any $w \in M$ we have $w|_U = \sum g_i s_i$ with $g_1, \dots, g_k \in C^\infty(U)$, and therefore

$$f^2 w = \sum h_i v_i$$

where $h_1, \dots, h_k \in R$ are extensions of $fg_1, \dots, fg_k$. Furthermore, if $p_1, \dots, p_k \in R$ satisfy $\sum p_i v_i = 0$, then $p_i|_{X_f} = 0$ for each i since $v_1(y), \dots, v_k(y)$ are a basis for the fibre $E(y)$ when $y \in X_f$. It follows that $fp_i = 0$ for all i. Hence $(v_1)_f, \dots, (v_k)_f$ are a basis for the R_f-module M_f, which verifies the local freeness, as defined in section 1.

Since $v_1(x), \dots, v_k(x)$ are a basis for $E(x)$, the map $M \to E(x)$ is surjective. The kernel of this map consists of those $w \in M$ for which $w(x) = 0$. Given such a w, we have $h_i(x) = 0$ for all i in the argument above, whence

$$w = (1 - f^2)w + \sum h_i v_i \in \mathfrak{m}_x M.$$

The $\mathcal{F}$-closedness of M can be verified by applying Lemma 1.16. Let $\{f_\alpha\} \subset R$ be an indexed subset such that for each $x \in X$ there exists α with $f_\alpha(x) \neq 0$, and let (v_α) be a collection of elements $v_\alpha \in M_{f_\alpha}$ such that v_α and v_β have the same image in $M_{f_\alpha f_\beta}$ for each pair of indices. Each v_α may be understood as a section of E over the open subset $X_{f_\alpha} \subset X$, and each pair of these local sections agree on the overlap of their domains. Hence there exists a unique $v \in M$ such that $v|_{X_{f_\alpha}} = v_\alpha$ for each α. The injectivity of the map $M_{f_\alpha} \to \Gamma(X_{f_\alpha}, E)$ implies that the image of v in M_{f_α} coincides with v_α. ∎

Example 2. Let $\mathbb{F}$ be any field of zero characteristic, $X = \{\mathrm{pt}\}$ a singleton, $R = \mathbb{F}[[t_1, \dots, t_n]]$ the formal power series algebra, $W = \operatorname{Der} R$ the Lie algebra of formal vector fields. We define $f(\mathrm{pt}) \in \mathbb{F}$ to be the constant term of $f \in R$. Here R has a single maximal ideal $\mathfrak{m} = \mathfrak{m}_{\mathrm{pt}}$, and all R-modules are $\mathcal{F}$-closed. Since W is a free R-module with a basis consisting of the partial derivatives in $t_1, \dots, t_n$, conditions (A2), (A3), (A4) are easily checked. Condition (A1) can be reformulated as follows: for each $0 \neq f \in R$ there exist $x \in X$ and a finite sequence $D_1, \dots, D_k \in W$ such that $(D_1 \cdots D_k f)(x) \neq 0$. In the case considered here the last inequality can always be achieved by using a suitable sequence of partial derivatives. This example explains why we want to allow more general algebras R than just the function algebras on X.

Example 3. Assume that $\mathbb{F}$ is algebraically closed of zero characteristic. Let $\widetilde{X}$ be a normal irreducible affine algebraic variety, X its nonsingular locus, R the ring of regular functions, $W = \operatorname{Der} R$ the Lie algebra of regular vector fields on $\widetilde{X}$. Denote by Ω_R the R-module of Kähler differentials of the $\mathbb{F}$-algebra R. Since R is a finitely generated algebra, this module is finitely presented. We have $W \cong \operatorname{Hom}_R(\Omega_R, R)$ and, by [1, Ch. II, §2, Prop. 19], for each maximal ideal $\mathfrak{m}$ of R

$$W_{\mathfrak{m}} = R_{\mathfrak{m}} \otimes_R W \cong \operatorname{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \otimes_R \Omega_R, R_{\mathfrak{m}})$$

where $R_{\mathfrak{m}}$ is the local ring of $\mathfrak{m}$. By [9, Ch. II, Prop. 8.2A] $R_{\mathfrak{m}} \otimes_R \Omega_R \cong \Omega_{R_{\mathfrak{m}}}$, the Kähler differentials of the $\mathbb{F}$-algebra $R_{\mathfrak{m}}$. If $\mathfrak{m} = \mathfrak{m}_x$ for $x \in X$, then [9, Ch. II, Th. 8.8] tells us that $\Omega_{R_{\mathfrak{m}}}$ is a free $R_{\mathfrak{m}}$-module of rank $n = \dim X$ since x is a nonsingular point of $\widetilde{X}$. It follows that the $R_{\mathfrak{m}}$-module $W_{\mathfrak{m}}$ is also free of rank n, and so too is the R_f-module W_f for a suitable $f \in R \smallsetminus \mathfrak{m}$. Thus W is locally free of rank n near X, which verifies (A2) and (A3). The freeness of $\Omega_{R_{\mathfrak{m}}}$ also yields

$$W/\mathfrak{m}W \cong W_{\mathfrak{m}}/\mathfrak{m}W_{\mathfrak{m}} \cong \operatorname{Hom}_{R_{\mathfrak{m}}}(\Omega_{R_{\mathfrak{m}}}, R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}) \cong \operatorname{Hom}_R(\Omega_R, R/\mathfrak{m}).$$

Furthermore, the R-linear maps $\Omega_R \to R/\mathfrak{m}$ are in a bijective correspondence with the $\mathbb{F}$-linear derivations $R \to R/\mathfrak{m}$. Let $D \in W$. Since $R = \mathbb{F} + \mathfrak{m}$, the containment $D(\mathfrak{m}) \subset \mathfrak{m}$ is equivalent to $D(R) \subset \mathfrak{m}$, which holds precisely when $D \in \mathfrak{m}W$ as the previous isomorphisms show. Hence (A4) is true.

Next, the $\mathfrak{m}$-adic completion $\hat{R}_{\mathfrak{m}}$ of $R_{\mathfrak{m}}$ is isomorphic to the algebra of formal power series in n indeterminates. The action of W on R extends to $\hat{R}_{\mathfrak{m}}$, and the image of W in $\operatorname{Der} \hat{R}_{\mathfrak{m}}$ generates the latter as an $\hat{R}_{\mathfrak{m}}$-module. It follows that for any W-invariant ideal I of R the ideal $I\hat{R}_{\mathfrak{m}}$ of $\hat{R}_{\mathfrak{m}}$ has to be stable under all derivations of $\hat{R}_{\mathfrak{m}}$. If $I \subset \mathfrak{m}$, then $I\hat{R}_{\mathfrak{m}}$ is contained in the maximal ideal of $\hat{R}_{\mathfrak{m}}$, whence $I\hat{R}_{\mathfrak{m}} = 0$, as we have seen in Example 2. Since R is a noetherian domain, R embeds in $\hat{R}_{\mathfrak{m}}$, and we deduce that $I = 0$. Thus none of the ideals $\mathfrak{m}_x$ with $x \in X$ contains any nonzero W-invariant ideal of R, which is stronger than (A1).

The normality of $\widetilde{X}$ is needed only to deduce (A5). This property implies that any rational function on $\widetilde{X}$ regular at all points of X is regular everywhere. Suppose that R embeds as a submodule in an $\mathcal{F}$-torsionfree R-module N with an $\mathcal{F}$-torsion factor module N/R. Then N embeds in the field of fractions of R, i.e. the field of rational functions on $\widetilde{X}$. If $g \in N$, then for any $x \in X$ there exists $s \in R \smallsetminus \mathfrak{m}_x$ such that $sg \in R$. Therefore g is regular on X, which yields $g \in R$. Thus $N = R$, and we conclude that R is $\mathcal{F}$-closed. By Lemma 1.6 W is $\mathcal{F}$-closed as well.

Remark. Our interpretation of X as a set of points makes the situation closer to particular cases of geometric origin. However, not X itself, but only the corresponding set of maximal ideals $\mathfrak{m}_x$ is relevant for further results. Therefore one might prefer to take X to be a set of maximal ideals of R such that the residue field $\kappa(\mathfrak{m})$ of each $\mathfrak{m} \in X$ coincides with $\mathbb{F}$. The assumption $\kappa(\mathfrak{m}) = \mathbb{F}$ is actually not needed, but without it one will have to redefine the isotropy algebra $L^0_{\mathfrak{m}}$ as a subalgebra of the Lie algebra $\kappa(\mathfrak{m}) \otimes L$ over $\kappa(\mathfrak{m})$ obtained from a given Lie algebra $L \subset W$ by field extension.

Up to the end of the paper we will keep our assumptions about X, R, W unchanged. In conformance with earlier notation put

$$M(x) = M/\mathfrak{m}_x M$$

for $x \in X$ and an R-module M. In particular, $W(x) = W/\mathfrak{m}_x W$. Recall that $(I : L)$ denotes the largest L-invariant ideal of R contained in an ideal I. For example, (A1) can be rewritten as $\bigcap_{x\in X}(\mathfrak{m}_x : W) = 0$. A Lie subalgebra $L \subset W$ will be called *transitive* if

$$W = L + \mathfrak{m}_x W \quad \text{for all } x \in X.$$

Lemma 3.2. *If $L \subset W$ is a transitive Lie subalgebra, then:*

(i) *The R-module W/RL is $\mathcal{F}$-torsion.*

(ii) *For each $x \in X$ we have $(\mathfrak{m}_x : L) = (\mathfrak{m}_x : W)$. Hence $\bigcap_{x\in X}(\mathfrak{m}_x : L) = 0$.*

Proof. Since $R/\mathfrak{m}_x \otimes_R W/RL = 0$ for all $x \in X$, part (i) follows from Lemma 1.2. Denote $I = (\mathfrak{m}_x : L)$ for some fixed x. Since I is an L-invariant ideal of R, so is

$$I' = I + \sum_{D\in W} D(I),$$

as one checks straightforwardly. By (i) for any $D \in W$ there exists $f \in R \smallsetminus \mathfrak{m}_x$ such that $fD \in RL$. Then $fD(I) \subset I \subset \mathfrak{m}_x$, whence $D(I) \subset \mathfrak{m}_x$ by the primeness of $\mathfrak{m}_x$. This shows that $I' \subset \mathfrak{m}_x$, but then we must have $I' = I$. Thus I is a W-invariant ideal contained in $\mathfrak{m}_x$, which yields $I \subset (\mathfrak{m}_x : W)$. The inverse inclusion is obvious since any W-invariant ideal is L-invariant. ■

Corollary 3.3. *Let $L \subset W$ be a transitive Lie subalgebra and M an (R, L)-module, locally finitely generated near X as an R-module. If $r \geq 0$ is an integer such that $\dim M(x) = r$ for all $x \in X$, then M is locally free of rank r near X. Any (R, L)-submodule N of M contained in $\bigcap_{x\in X} \mathfrak{m}_x M$ is $\mathcal{F}$-torsion, i.e. $N \subset t_{\mathcal{F}}(M)$.*

Proof. In view of Lemma 3.2 we meet the hypotheses of Theorem 2.1 which ensures the local freeness of M. Consider $M' = M/N$. Since $N(x) \to M(x)$ is a zero map, we get $M'(x) \cong M(x)$, and in particular $\dim M'(x) = r$, for each $x \in X$. Thus Theorem 2.1 shows that M' is also locally free. By Lemma 1.3 N is locally a direct summand of the R-module M, whence N is locally finitely generated. The same lemma shows that all maps $N(x) \to M(x)$ are injective, so that $N(x) = 0$ for each x. Now Lemma 1.2 completes the proof. ■

Lemma 3.4. *The ring R satisfies assumption* (A) *from section* 1 *with $e = 1$ when $\operatorname{char} \mathbb{F} = 0$ and $e = \operatorname{char} \mathbb{F}$ otherwise. If $\operatorname{char} \mathbb{F} = 0$, then R is reduced.*

Proof. Let $f, g \in R$ be two elements such that $f^m g = 0$ for some $m > 0$. By Lemma 2.2 we have $f^e g \in (\mathfrak{m}_x : W)$ for all $x \in X$, and it follows from (A1) that $f^e g = 0$. If $\operatorname{char} \mathbb{F} = 0$, then each factor ring $R/(\mathfrak{m}_x : W)$ is a domain, and therefore R cannot have any nonzero nilpotent elements. ■

Lemma 3.5. *The R-module W is locally free of finite constant rank and dualizable near X. Hence so are all its tensor, exterior and symmetric powers.*

Proof. We may regard W as an (R,W)-module with respect to the adjoint action of W. The local freeness of W follows from Corollary 3.3 applied to $L = W$, $M = W$, $r = n$. For $x \in X$ and $f \in R$ define a linear function $\xi_{x,f} : W(x) \to \mathbb{F}$ by the rule

$$\xi_{x,f}(D + \mathfrak{m}_x W) = (Df)(x), \qquad D \in W.$$

The set $V = \{\xi_{x,f} \mid f \in R\}$ is a subspace of the dual vector space $W(x)^*$. If $D \in W$ satisfies $\xi_{x,f}(D + \mathfrak{m}_x W) = 0$ for all $f \in R$, then $D(R) \subset \mathfrak{m}_x$, whence $D \in \mathfrak{m}_x W$ by (A4). Thus $\bigcap_{\xi \in V} \operatorname{Ker} \xi = 0$, which is only possible when $V = W(x)^*$. For any $f \in R$ the assignment $D \mapsto Df$ defines an R-linear map $\varphi : W \to R$ such that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\text{can.}} & W(x) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\xi_{x,f}} \\ R & \xrightarrow{\text{can.}} & R/\mathfrak{m}_x \cong \mathbb{F} \end{array}$$

commutes. Hence W is dualizable by Lemma 1.11. The last assertion follows from Lemma 1.15. ■

4. Lie algebroids, connections, differential forms

Here we recall several notions which have been used in purely algebraic context in many papers. Let R be a commutative algebra over a field $\mathbb{F}$. A *Lie algebroid*[2] over R is a pair (L, a) where L is a Lie $\mathbb{F}$-algebra and an R-module, $a : L \to \operatorname{Der} R$ is a homomorphism of Lie algebras and R-modules such that

$$[D, fD'] = f[D, D'] + a(D)(f)D' \quad \text{for all } D, D' \in L,\ f \in R.$$

One calls a the anchor map of the Lie algebroid (see [16]). We will often omit the explicit indication of a from the notation and write $Df = a(D)(f)$ for $D \in L$ and $f \in R$. For example, if L is a Lie subalgebra and an R-submodule of $\operatorname{Der} R$, then L is a Lie algebroid over R with $a : L \to \operatorname{Der} R$ being the inclusion.

Let L be a Lie algebroid over R, and let M be an R-module. An L-*connection* on M is an $\mathbb{F}$-linear map $\nabla : L \to \operatorname{End}_{\mathbb{F}} M$, $D \mapsto \nabla_D$, such that

$$\nabla_{fD} v = f\,\nabla_D v, \qquad \nabla_D(fv) = f\,\nabla_D v + (Df)v$$

for all $D \in L$, $f \in R$ and $v \in M$. With each connection ∇ one associates its curvature $\mathcal{R} : L \times L \to \operatorname{End}_R M$ defined by the rule

$$\mathcal{R}(D_1, D_2) = \nabla_{D_1}\nabla_{D_2} - \nabla_{D_2}\nabla_{D_1} - \nabla_{[D_1,D_2]}, \qquad D_1, D_2 \in L.$$

Note that $\mathcal{R}$ is an alternating R-bilinear map. A connection ∇ is called *flat* if $\mathcal{R} = 0$, i.e. if ∇ gives a Lie algebra action of L on M. When there is no need of

[2]Such structures appeared in the literature under different names (see [11], [12], [17], [19]). Mackenzie [16] reserves the term "Lie algebroid" only for structures defined on modules of sections of vector bundles.

distinction between different Lie algebra actions on M we will write $Dv = \nabla_D v$ with $D \in L$ and $v \in M$ in case of a flat connection. Thus an R-module M equipped with a flat L-connection is an (R,L)-module such that the Lie algebra action of L on M satisfies the additional condition that

$$(fD)v = f(Dv) \quad \text{for all } f \in R,\ D \in L,\ v \in M.$$

We will call any Lie algebra action of L on an R-module M which makes M into an (R,L)-module and satisfies the above property a *Lie algebroid action*. This is just an alternative name for a flat L-connection.

Since the anchor map a is R-linear, the action of L on R is a Lie algebroid action. Given Lie algebroid actions of L on R-modules M and N, the induced actions of L on $M \otimes_R N$, $\operatorname{Hom}_R(M,N)$, all tensor, exterior and symmetric powers of M are Lie algebroid actions as well.

An M-valued *differential form* of degree $k \ge 0$ on L is an element of the R-module

$$C_R^k(L,M) = \operatorname{Hom}_R(\textstyle\bigwedge_R^k L, M).$$

There are several classical operations on differential forms (see [16], [17]). We denote by $D \lrcorner\, \omega \in C_R^{k-1}(L,M)$ the contraction of $D \in L$ and $\omega \in C_R^k(L,M)$ defined as

$$(D \lrcorner\, \omega)(D_1 \wedge \cdots \wedge D_{k-1}) = \omega(D \wedge D_1 \wedge \cdots \wedge D_{k-1}), \qquad D_1, \ldots, D_{k-1} \in L.$$

When M is equipped with a flat L-connection, one has a Lie algebra action of L on each $C_R^k(L,M)$ and the exterior differential d which increases the degree of differential forms by 1. Explicitly,

$$(D\omega)(D_1 \wedge \cdots \wedge D_k) = D\,\omega(D_1 \wedge \cdots \wedge D_k) - \sum_{i=1}^{k} \omega(D_1 \wedge \cdots \wedge [D, D_i] \wedge \cdots \wedge D_k),$$

$$\begin{aligned}(d\omega)(D_1 \wedge \cdots \wedge D_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} D_i\, \omega(D_1 \wedge \cdots \hat{D}_i \cdots \wedge D_{k+1}) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([D_i, D_j] \wedge D_1 \wedge \cdots \hat{D}_i \cdots \hat{D}_j \cdots \wedge D_{k+1}).\end{aligned}$$

Further on X, R, W are assumed to satisfy all assumptions from section 3.

Lemma 4.1. *Let L be a Lie algebroid over R with the anchor map $a : L \to \operatorname{Der} R$. Then any (flat) L-connection ∇ on an R-module M extends to a (flat) L-connection ∇' on the $\mathcal{F}$-closure $M_{\mathcal{F}}$. If $a(L)$ is a transitive Lie subalgebra of W and*

$$\nabla_D(M) \subset t_{\mathcal{F}}(M) \quad \text{for all } D \in \operatorname{Ker} a,$$

then there is a (flat) W-connection ∇'' on $M_{\mathcal{F}}$ such that $\nabla'_D = \nabla''_{a(D)}$ for all $D \in L$. Moreover, given a homomorphism of R-modules $\varphi : M \to N$ which intertwines ∇ and an L-connection on N satisfying the same condition, the map $\varphi_{\mathcal{F}} : M_{\mathcal{F}} \to N_{\mathcal{F}}$ intertwines the induced W-connections on $M_{\mathcal{F}}$ and $N_{\mathcal{F}}$.

Proof. Denote $E_L(M) = \mathrm{Hom}_R(L, M) \oplus M$ with the R-module structure defined by the rule

$$f \cdot (\xi, v) = (f\xi + \alpha_{v,f},\ fv) \quad \text{for } f \in R,\ \xi \in \mathrm{Hom}_R(L, M),\ v \in M$$

where $\alpha_{v,f}(D) = (Df)v$ for $D \in L$. The projection $\pi_M : E_L(M) \to M$ onto the second summand is an epimorphism of R-modules with $\mathrm{Ker}\,\pi_M \cong \mathrm{Hom}_R(L, M)$. There is a bijective correspondence between L-connections on M and R-linear splittings of π_M. The splitting $\sigma : M \to E_D(M)$ corresponding to ∇ is defined by the assignment $v \mapsto (\beta_v, v)$ where $\beta_v(D) = \nabla_D v$ for $v \in M$ and $D \in L$.

Formation of $E_L(M)$ is functorial in M. So, replacing M with $M_{\mathcal{F}}$, we obtain an R-module $E_L(M_{\mathcal{F}})$ which is $\mathcal{F}$-closed since so are both $M_{\mathcal{F}}$ and $\mathrm{Hom}_R(L, M_{\mathcal{F}})$ in view of Lemma 1.6 and since the class of $\mathcal{F}$-closed modules is closed under extensions (see Lemma 1.1). It follows that there exists a unique R-linear map σ' rendering commutative the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\sigma} & E_L(M) & \xrightarrow{\pi_M} & M \\ \downarrow & & \downarrow & & \downarrow \\ M_{\mathcal{F}} & \xrightarrow{\sigma'} & E_L(M_{\mathcal{F}}) & \xrightarrow{\pi_{M_{\mathcal{F}}}} & M_{\mathcal{F}}. \end{array}$$

Since $\pi_M \circ \sigma = \mathrm{Id}$ and since the map $\mathrm{Hom}_R(M_{\mathcal{F}}, M_{\mathcal{F}}) \to \mathrm{Hom}_R(M, M_{\mathcal{F}})$ obtained by taking composites with $M \to M_{\mathcal{F}}$ is bijective by the $\mathcal{F}$-closedness of $M_{\mathcal{F}}$, we deduce that $\pi_{M_{\mathcal{F}}} \circ \sigma' = \mathrm{Id}$. Hence σ' corresponds to an L-connection ∇' on $M_{\mathcal{F}}$ such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\nabla_D} & M \\ \downarrow & & \downarrow \\ M_{\mathcal{F}} & \xrightarrow{\nabla'_D} & M_{\mathcal{F}}. \end{array}$$

commute for all $D \in L$. Let $\mathcal{R}$ and $\mathcal{R}'$ be the curvatures of ∇ and ∇', respectively. For $D_1, D_2 \in L$ the R-linear transformation $\mathcal{R}'(D_1, D_2)$ of $M_{\mathcal{F}}$ extends the R-linear transformation $\mathcal{R}(D_1, D_2)$ of M. If $\mathcal{R}(D_1, D_2) = 0$, we must have $\mathcal{R}'(D_1, D_2) = 0$.

In the proof of the second assertion let $K = \mathrm{Ker}\, a$, and let M' denote the image of M in $M_{\mathcal{F}}$. Since $M_{\mathcal{F}}$ is $\mathcal{F}$-torsionfree, the condition on ∇ shows that $\nabla'_D(M') = 0$ for all $D \in K$. Note that ∇'_D is an R-linear transformation of $M_{\mathcal{F}}$ for each $D \in K$. Since $M_{\mathcal{F}}/M'$ is an $\mathcal{F}$-torsion R-module, so too is the image of ∇'_D, i.e. $\nabla'_D = 0$. Hence ∇' induces a (flat) $a(L)$-connection on $M_{\mathcal{F}}$. Replacing L with $a(L)$, we may assume from the very beginning that L is a transitive Lie subalgebra and an R-submodule of W. Since W/L is $\mathcal{F}$-torsion by Lemma 3.2, we have $\mathrm{Hom}_R(W, M_{\mathcal{F}}) \cong \mathrm{Hom}_R(L, M_{\mathcal{F}})$, and therefore $E_W(M_{\mathcal{F}}) \cong E_L(M_{\mathcal{F}})$. This yields a bijective correspondence between L-connections and W-connections on $M_{\mathcal{F}}$. Denote by $\mathcal{R}''$ the curvature of ∇''. If $\mathcal{R}' = 0$, we get $\mathcal{R}'' = 0$ on $L \times L$. Then $\mathcal{R}''(W, W)$ is contained in the $\mathcal{F}$-torsion submodule of $M_{\mathcal{F}}$, whence $\mathcal{R}'' = 0$ everywhere.

The intertwining of L-connections by φ and $\varphi_{\mathcal{F}}$ can be expressed by means of commutative diagrams

$$\begin{array}{ccc} M & \longrightarrow & E_L(M) \\ \downarrow & & \downarrow \\ N & \longrightarrow & E_L(N) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M_{\mathcal{F}} & \longrightarrow & E_L(M_{\mathcal{F}}) \\ \downarrow & & \downarrow \\ N_{\mathcal{F}} & \longrightarrow & E_L(N_{\mathcal{F}}), \end{array}$$

respectively, where horizontal arrows represent the splittings corresponding to the L-connections involved. By composing the two composite maps $M_{\mathcal F} \to E_L(N_{\mathcal F})$ in the second diagram with the canonical map $M \to M_{\mathcal F}$, the verification of commutativity reduces to that for the first diagram. Hence $\varphi_{\mathcal F}$ intertwines the induced L-connections on $M_{\mathcal F}$ and $N_{\mathcal F}$. Passing further to W-connections, we may assume that L is a Lie subalgebroid of W and use the fact that the functors E_L and E_W take the same values on $\mathcal F$-closed modules. ■

5. Recognizing a geometric structure by a Lie algebra

We will now define volume forms, hamiltonian forms, contact forms, and Riemann pseudometrics. The first three notions were considered earlier in a purely algebraic framework which didn't include any set X as a part of the data [20]. We will work in the setup of section 3.

Let P be an $\mathcal F$-closed R-module, locally free of rank 1 near X. In particular, the vector space $P(x) = P/\mathfrak m_x P$ has dimension 1 for each $x \in X$. We also assume that P is equipped with a flat W-connection in all cases except when contact forms are considered. Given $\omega \in C^k_R(W,P)$, denote by

$$\omega_x : \textstyle\bigwedge^k W(x) \to P(x)$$

the linear map obtained from ω by reduction modulo $\mathfrak m_x$. Lemmas 1.6, 1.12 and 3.5 show that the R-module $C^k_R(W,P)$ is $\mathcal F$-closed and locally free of rank $\binom{n}{k}$ near X. In particular, $C^k_R(W,P) = 0$ for $k > n$ and $C^n_R(W,P)$ is locally free of rank 1.

A differential form $\omega \in C^n_R(W,P)$ is a *volume form* on X if $\omega_x \ne 0$ for each $x \in X$. Lemma 1.5 applied to the R-linear map $R \to C^n_R(W,P)$, $f \mapsto f\omega$, shows that ω is a volume form if and only if $C^n_R(W,P)$ is a cyclic free R-module generated by ω. Another equivalent condition is the bijectivity of the R-linear map

$$i_\omega : W \to C^{n-1}_R(W,P), \qquad D \mapsto D \lrcorner\, \omega.$$

A differential form $\omega \in C^2_R(W,P)$ is a *hamiltonian form* on X if ω is closed, i.e. $d\omega = 0$, and ω_x gives a nondegenerate alternating bilinear form on $W(x)$ for each $x \in X$. By Lemma 1.9 the nondegeneracy condition implies that

$$i_\omega : W \to C^1_R(W,P), \qquad D \mapsto D \lrcorner\, \omega,$$

is an isomorphism of R-modules. We say that $D \in W$ is an *infinitesimal automorphism* of ω if $D\omega = 0$, and the same definition will be used for volume forms. Since $d\omega = 0$, the classical formula

$$D\omega = d(D \lrcorner\, \omega) + D \lrcorner\, d\omega$$

shows that $D\omega = 0$ if and only if $i_\omega(D)$ is a closed 1-form (a closed $(n-1)$-form in the case of a volume form).

Given $\omega \in C^1_R(W,P)$, denote $Q = \operatorname{Ker}\omega$ and define an alternating R-bilinear map $\beta : Q \times Q \to P$ by the rule

$$\beta(D,D') = \omega([D,D']) \quad \text{for } D, D' \in Q.$$

We call ω a *contact form* on X if the linear maps ω_x are surjective and the alternating bilinear forms

$$\beta_x : Q(x) \times Q(x) \to P(x)$$

induced by β are nondegenerate for all $x \in X$. In this case Q is a *contact distribution* in W. Suppose that ω is a contact form. By Lemma 1.4 Q is locally free of rank $n-1$ near X. Moreover, Q is $\mathcal{F}$-closed since so are W and P (see Lemma 1.1). By Lemma 1.9 the R-linear map $Q \to \operatorname{Hom}_R(Q, P)$ induced by β is bijective. This map extends to an $\mathbb{F}$-linear map

$$\varphi : W \to \operatorname{Hom}_R(Q, P)$$

defined by the rule $\varphi(D)(D') = \omega([D, D'])$ for $D \in W$ and $D' \in Q$. The bijectivity of $\varphi|_Q$ entails $W = Q \oplus \operatorname{Ker} \varphi$. Note that $\varphi(D) = 0$ if and only if $[D, Q] \subset Q$. Hence $W = Q \oplus N_W(Q)$ where

$$N_W(Q) = \{D \in W \mid [D, Q] \subset Q\}.$$

We say that $D \in W$ is an *infinitesimal automorphism* of the contact structure on X if $D \in N_W(Q)$.

Assume that $\operatorname{char} \mathbb{F} \neq 2$. A *Riemann pseudometric* on X is an R-bilinear symmetric map $\omega : W \times W \to P$ such that the induced bilinear forms

$$\omega_x : W(x) \times W(x) \to P(x)$$

are nondegenerate for all $x \in X$. The R-bilinear symmetric maps $W \times W \to P$ may be identified with elements of the (R, W)-module $\operatorname{Hom}_R(S_R^2 W, P)$. Thus we have a Lie algebra action of W on such bilinear maps. We say that $D \in W$ is an infinitesimal automorphism of ω if $D\omega = 0$, i.e. if

$$D\,\omega(D_1, D_2) = \omega([D, D_1], D_2) + \omega(D_1, [D, D_2]) \quad \text{for all } D_1, D_2 \in W.$$

Given another differential form of the same type or a pseudometric ω' with values in an $\mathcal{F}$-closed rank 1 locally free R-module P' equipped with a flat W-connection (when needed), we say that ω is *equivalent* to ω' if there exists an isomorphism of (R, W)-modules (an isomorphism of R-modules in the contact case) $\xi : P \to P'$ such that $\omega' = \xi \circ \omega$.

A *unimodular, hamiltonian, contact or pseudo-Riemannian structure* on X is just a volume, a hamiltonian, a contact form or a Riemann pseudometric, respectively. Such a structure is L*-invariant* where $L \subset W$ is a Lie subalgebra if all elements of L are infinitesimal automorphisms of this structure.

Let L be a transitive Lie subalgebra of W. For each $x \in X$ denote by L_x^0 the stabilizer of $\mathfrak{m}_x$ in L, so that $L_x^0 = L \cap \mathfrak{m}_x W$ according to (A4). Put

$$\mathfrak{g}_x^{-1} = L/L_x^0.$$

By transitivity the inclusion $L \to W$ induces a linear isomorphism $\mathfrak{g}_x^{-1} \cong W(x)$. The adjoint action induces a representation of L_x^0 in $\mathfrak{g}_x^{-1}$. We denote by $\mathfrak{g}_x^0$ the image of L_x^0 in the Lie algebra of linear transformation of $\mathfrak{g}_x^{-1}$. In fact $\mathfrak{g}_x^{-1}$ and $\mathfrak{g}_x^0$ are two homogeneous components of the graded Lie algebra $\bigoplus_{i=-1}^{\infty} \mathfrak{g}_x^i$ associated with a certain filtration of L. However, we will not need the components of higher degree. Consider the following conditions:

(a) $\mathfrak{g}_x^0 \subset \mathfrak{sl}(\mathfrak{g}_x^{-1})$, i.e. $\mathfrak{g}_x^0$ consists of linear transformations with trace 0;

(b) $\mathfrak{g}_x^0 = \mathfrak{sp}(\alpha_x)$, the symplectic Lie algebra associated with a nondegenerate alternating bilinear form α_x on $\mathfrak{g}_x^{-1}$;

(c) the subspace $V_x = [\mathfrak{g}_x^0, \mathfrak{g}_x^0] \cdot \mathfrak{g}_x^{-1}$ has codimension 1 in $\mathfrak{g}_x^{-1}$ and the alternating bilinear form $\alpha_x : V_x \times V_x \to \mathfrak{g}_x^{-1}/V_x$ induced by the Lie product in L is nondegenerate;

(d) $\mathfrak{g}_x^0 = \mathfrak{o}(\sigma_x)$, the orthogonal Lie algebra associated with a nondegenerate symmetric bilinear form σ_x on $\mathfrak{g}_x^{-1}$.

Theorem 5.1. *Suppose that $L \subset W$ is a transitive Lie subalgebra such that one of conditions* (a)–(d) *holds for all $x \in X$. Then in the respective cases there exists an L-invariant unimodular, hamiltonian, contact or pseudo-Riemannian structure on X. It is unique up to equivalence.*

Proof. Consider $R \otimes L$ (tensor product over $\mathbb{F}$) with the R-module structure given by multiplications on the first tensorand and the Lie bracket defined by the rule

$$[f \otimes D,\, g \otimes E] = fg \otimes [D,E] + fD(g) \otimes E - gE(f) \otimes D$$

where $f, g \in R$ and $D, E \in L$. Then $R \otimes L$ is a Lie algebroid with respect to the anchor map $a : R \otimes L \to W$ sending $f \otimes D$ to fD. The image of a coincides with RL. Denote $K = \operatorname{Ker} a$. Since W/RL is $\mathcal{F}$-torsion by Lemma 3.2, it follows from Lemma 1.3 that the reduction of a modulo a maximal ideal $\mathfrak{m}_x$ with $x \in X$ yields an exact sequence of vector spaces

$$0 \longrightarrow K(x) \longrightarrow R/\mathfrak{m}_x \otimes_R (R \otimes L) \cong L \xrightarrow{a_x} W(x) \longrightarrow 0$$

where a_x is just the projection. Thus $\operatorname{Ker} a_x = L_x^0$.

There is a Lie algebroid action of $R \otimes L$ on W defined by the rule

$$(f \otimes D) \cdot E = f[D,E] \quad \text{for } f \in R,\ D \in L,\ E \in W.$$

It induces a Lie algebroid action on exterior powers $\bigwedge_R^k W$ and symmetric powers $S_R^k W$. In general, given a Lie algebroid action of $R \otimes L$ on an R-module N, the restriction $K \times N \to N$ of this action is R-bilinear. The latter gives rise to a Lie algebra action of $K(x) \cong L_x^0$ on $N(x)$. When $N = W$, we obtain the action of L_x^0 on $W(x)$ induced by the adjoint action of L on W. Furthermore, $W(x) \cong \mathfrak{g}_x^{-1}$, and L_x^0 acts on this vector space via the projection onto $\mathfrak{g}_x^0$. We now consider separately the four cases of Theorem 5.1.

(a) The R-module $M = \bigwedge_R^n W$ is locally free of rank 1. By the assumption about $\mathfrak{g}_x^0$ the Lie algebra L_x^0 annihilates $M(x) \cong \bigwedge^n W(x)$. This means that $K \cdot M \subset \mathfrak{m}_x M$ for each $x \in X$. Note that $K \cdot M$ is an (R,L)-submodule of M since K is an ideal of $R \otimes L$. By Corollary 3.3 $K \cdot M \subset t_{\mathcal{F}}(M)$. Now put $P = M_{\mathcal{F}}$, so that P is $\mathcal{F}$-closed and locally free of rank 1 by Lemma 1.13. The Lie algebroid action of $R \otimes L$ on M gives rise to a Lie algebroid action of W on P by Lemma 4.1.

Let $\omega : M \to P$ be the canonical map. All maps $\omega_x : M(x) \to P(x)$ are bijective by Lemmas 1.2, 1.3 and ω is L-equivariant by construction. Thus ω is an L-invariant volume form. Let $\omega' : M \to P'$ be another L-invariant volume form. Since P' is an $\mathcal{F}$-closed R-module, there is a bijective correspondence between the R-linear maps $M \to P'$ and the R-linear maps $P \to P'$, so that $\omega' = \xi \circ \omega$ for a homomorphism of R-modules $\xi : P \to P'$. We have $\omega'_x \ne 0$ for each $x \in X$. Since both vector spaces $M(x)$ and $P'(x)$ have dimension 1, ω'_x is bijective, whence so too is the linear map $P(x) \to P'(x)$ induced by ξ. By Lemma 1.5 ξ is bijective. With the identification $P' \cong P'_{\mathcal{F}}$ the localization functor $?_{\mathcal{F}}$ takes ω' to ξ. Since ω' intertwines the Lie algebra action of L, and therefore also the Lie algebroid action of $R \otimes L$, in M and P', the last assertion of Lemma 4.1 shows that ξ intertwines the action of W in P and P'. In other words, ξ is an isomorphism of (R, W)-modules, and so ω' is equivalent to ω.

(b) For each $x \in X$ the vector space of all $\mathfrak{g}_x^0$-invariant alternating bilinear forms on $\mathfrak{g}_x^{-1}$ is one-dimensional. In other words, $U_x = L_x^0 \cdot \bigwedge^2 W(x)$ is a subspace of codimension 1 in $\bigwedge^2 W(x)$. Denoting

$$M = \textstyle\bigwedge_R^2 W \,/\, (K \cdot \bigwedge_R^2 W),$$

we get $M(x) \cong \bigwedge^2 W(x) \,/\, U_x$, and therefore $\dim M(x) = 1$, for each x. Clearly the Lie algebroid action of $R \otimes L$ on $\bigwedge_R^2 W$ passes to M. In particular, we may regard M as an (R, L)-module. Hence M is locally free of rank 1 near X by Corollary 3.3. Now Lemma 1.14 ensures that M is dualizable near X since so is $\bigwedge_R^2 W$. By Lemma 1.13 the $\mathcal{F}$-closed R-module $P = M_{\mathcal{F}}$ is locally free of rank 1 near X. Since K annihilates M, Lemma 4.1 gives a flat W-connection on P. Now ω is taken to be the composite $\bigwedge_R^2 W \to M \to P$ of two canonical maps. Applying Lemma 1.3, we deduce that $M(x) \cong P(x)$ and

$$\operatorname{Ker} \omega_x = \operatorname{Ker}\bigl(\textstyle\bigwedge^2 W(x) \to M(x)\bigr) = U_x.$$

This means that the bilinear form on $W(x)$ given by ω_x is nonzero and $\mathfrak{g}_x^0$-invariant. Hence ω_x is a nonzero scalar multiple of α_x; in particular, ω_x is nondegenerate.

We have to check that $d\omega = 0$. Since the exterior differential d commutes with the Lie algebra action of W, the 3-form $\theta = d\omega$ is L-invariant. Hence the reduction $\theta_x : \bigwedge^3 W(x) \to P(x)$ is L_x^0-invariant with the action of L_x^0 on $P(x)$ being trivial. Now note that $\mathfrak{g}_x^0$ has no nonzero invariant elements in the dual of $\bigwedge^3 \mathfrak{g}_x^{-1}$. In fact, the weights of any maximal diagonalizable subalgebra of $\mathfrak{g}_x^0$ on $\mathfrak{g}_x^{-1}$ are of the form $\pm\varepsilon_1, \dots, \pm\varepsilon_n$ with linearly independent $\varepsilon_1, \dots, \varepsilon_n$, and it follows that all weights on $\bigwedge^3 \mathfrak{g}_x^{-1}$ are nonzero. Hence $\theta_x = 0$ for all x. This means that the image of θ is an (R, L)-submodule of P contained in $\bigcap_{x \in X} \mathfrak{m}_x P$. By Corollary 3.3 $\theta = 0$, as claimed.

Thus ω is a hamiltonian form. It is L-invariant by construction. Suppose now that $\omega' : \bigwedge_R^2 W \to P'$ is another L-invariant hamiltonian form. Letting $R \otimes L$ operate on P' via the action of W, we may regard ω' as a homomorphism of $R \otimes L$-modules. Since $K \cdot P' = 0$, we must have $K \cdot \bigwedge_R^2 W \subset \operatorname{Ker} \omega'$. Hence ω' factors through M and moreover through P by the $\mathcal{F}$-closedness of P'. Thus $\omega' = \xi \circ \omega$ where $\xi : P \to P'$ is an R-linear map. Since ω_x (resp. ω'_x) is a linear surjection of $\bigwedge^2 W(x)$ onto the onedimensional vector space $P(x)$ (resp. $P'(x)$), the map $P(x) \to P'(x)$ induced by ξ is bijective, for each $x \in X$. The same argument as in (a) shows that ξ is an isomorphism of (R, W)-modules.

(c) Denote $M = W/([K,K] \cdot W)$ and $P = M_{\mathcal{F}}$. Since K is a Lie ideal of $R \otimes L$, the Lie algebroid action of $R \otimes L$ on W passes to M and P. In particular, M and P are (R, L)-modules. However, K does not annihilate P in this case, so that there is no canonical way to obtain an action of W on P. All vector spaces

$$M(x) \cong W(x)/\big([\mathfrak{g}_x^0, \mathfrak{g}_x^0] \cdot W(x)\big) \cong \mathfrak{g}_x^{-1}/V_x, \qquad x \in X,$$

have dimension 1 by condition (3). Hence M is locally free of rank 1 near X by Corollary 3.3, and so too is the $\mathcal{F}$-closed R-module P by Lemmas 1.13, 1.14. Take ω to be the composite $W \to M \to P$. Then ω_x is surjective for each x. Since ω is R-linear and L-equivariant by construction, its kernel $Q = \operatorname{Ker} \omega$ is an (R, L)-submodule of W. By Lemma 1.3 ω induces an exact sequence of $\mathfrak{g}_x^0$-modules

$$0 \to Q(x) \to W(x) \to P(x) \to 0$$

for each x. Since $P(x) \cong M(x)$, we have $Q(x) \cong V_x$. Consider the R-bilinear map $\beta : Q \times Q \to W/P$ associated with ω and its reduction β_x at some fixed x. Given $D_1, D_2 \in Q$, there exist $D_1', D_2' \in L$ such that $D_i - D_i' \in \mathfrak{m}_x W$. Since $\mathfrak{m}_x W$ is a Lie subalgebra of W, we get

$$[D_1 - D_1',\, D_2 - D_2'] \in \mathfrak{m}_x W,$$

and the inclusion $[L, Q] \subset Q$ yields $[D_1, D_2] + [D_1', D_2'] \in Q + \mathfrak{m}_x W$. Hence

$$\omega([D_1, D_2]) + \omega([D_1', D_2']) \in \mathfrak{m}_x P.$$

Since $D_i' + \mathfrak{m}_x W \in Q(x)$, we have $D_i' + L_x^0 \in V_x$ for $i = 1, 2$, and making use of the linear bijection $\mathfrak{g}_x^{-1}/V_x \cong P(x)$ induced by ω_x, we get

$$\beta_x(D_1 + \mathfrak{m}_x Q,\, D_2 + \mathfrak{m}_x Q) = -\alpha_x(D_1' + L_x^0,\, D_2' + L_x^0).$$

Now the nondegeneracy of β_x follows from the nondegeneracy of α_x. Thus ω is indeed a contact form. Since $[L, Q] \subset Q$, all elements of L are infinitesimal automorphisms of the contact structure on X given by ω.

Consider another contact form $\omega' : W \to P'$ on X with the contact distribution $Q' = \operatorname{Ker} \omega'$ satisfying $[L, Q'] \subset Q'$. Then Q' is stable under the action of $R \otimes L$ on W. The R-module W/Q' embeds in P'. Since P' is $\mathcal{F}$-closed, while $\operatorname{Coker} \omega'$ is $\mathcal{F}$-torsion by Lemma 1.2, we may identify P' with the $\mathcal{F}$-closure $(W/Q')_{\mathcal{F}}$. By Lemma 4.1 the Lie algebroid action of $R \otimes L$ on W/Q' extends to one on P'. All elements of K act on P' as R-linear transformations. Hence $[K, K]$ annihilates each onedimensional vector space $P'(x)$, i.e.

$$[K, K] \cdot P' \subset \bigcap_{x \in X} \mathfrak{m}_x P'.$$

Since $[K, K] \cdot P'$ is an (R, L)-submodule of P', it must be equal to 0 according to Corollary 3.3. This means that $[K, K] \cdot W \subset Q'$. But then ω' factors through M and P. We get $\omega' = \xi \circ \omega$ for some R-linear map $\xi : P \to P'$. Since both ω_x and

ω'_x are surjective, the map $\xi_x : P(x) \to P'(x)$ has to be bijective for each $x \in X$. It follows that ξ is an isomorphism of R-modules.

(d) Here $P = M_{\mathcal{F}}$ where $M = S^2_R W/(K \cdot S^2_R W)$, and $\omega : W \times W \to P$ is taken to be the symmetric R-bilinear map corresponding to the composite of the canonical R-linear maps $S^2_R W \to M \to P$. The nondegeneracy of ω and the uniqueness are checked as in (b). ∎

6. Infinitesimal automorphisms of Riemann pseudometrics

This section generalizes one classical fact to the algebraic setup of section 3. Let $\omega : W \times W \to P$ be a Riemann pseudometric on X. There is a unique W-connection on W, the *Levi-Civita connection*, such that the two identities

$$\begin{gathered} D\omega(D', D'') = \omega(\nabla_D D', D'') + \omega(D', \nabla_D D''), \\ \nabla_D D' - \nabla_{D'} D = [D, D'] \end{gathered}$$

hold for all $D, D', D'' \in W$. This is established exactly as in the classical differential geometry, e.g. [14, Ch. IV]: the value $\nabla_D D'$ is determined from the identity

$$\begin{aligned} 2\omega(\nabla_D D', D'') \;=\; & D\omega(D', D'') + D'\omega(D, D'') - D''\omega(D, D') \\ & + \omega([D, D'], D'') - \omega(D', [D, D'']) - \omega(D, [D', D'']), \end{aligned} \tag{$*$}$$

which is possible since, by Lemma 1.9, the contraction with ω gives an R-linear bijection

$$i_\omega : W \to \mathrm{Hom}_R(W, P).$$

Theorem 6.1. *Suppose that none of the ideals $\mathfrak{m}_x$ with $x \in X$ contains any nonzero W-invariant locally finitely generated near X ideal of R. Then the Lie algebra L of all infinitesimal automorphisms of a Riemann pseudometric ω has finite dimension not exceeding $(n^2 + n)/2$.*

The proof will be preceded by a lemma which characterizes elements of L. For each $E \in W$ denote by ad_E the adjoint linear transformation of W given by the assignment $D \mapsto [E, D]$ for $D \in W$.

Lemma 6.2. *If $A \in L$, then $[\mathrm{ad}_A, \nabla_D] = \nabla_{[A,D]}$ for all $D \in W$.*

Proof. Denote by $\varphi(D, D', D'')$ the right hand side of $(*)$. Since ω is L-invariant, so too is the multilinear map $\varphi : W \times W \times W \to P$ (in other words, the corresponding linear map $W \otimes_{\mathbb{F}} W \otimes_{\mathbb{F}} W \to P$ is a homomorphism of L-modules). Define a bilinear map $\psi : W \times W \to W$ by the rule $\psi(D, D') = \nabla_D D'$. Equality $(*)$ shows that $i_\omega \circ \psi$ is an L-invariant bilinear map $W \times W \to \mathrm{Hom}_R(W, P)$. Since the bijection i_ω is also L-invariant, we conclude that ψ is L-invariant. This means that

$$[A, \nabla_D D'] = \nabla_{[A,D]} D' + \nabla_D \big([A, D']\big)$$

for all $A \in L$ and $D, D' \in W$, which can be rewritten as the desired equality. ∎

Proof of Theorem 6.1. We may exclude from consideration the trivial case $W = 0$. Then W is a faithful R-module since W is locally free of nonzero rank near X. If M is any faithful R-module, then we may regard $\operatorname{Qder} M$ as a Lie algebroid with the anchor map $a : \operatorname{Qder} M \to \operatorname{Der} R$ that assigns to a quasiderivation $\mathcal{D}$ of M the unique derivation D of R such that $\mathcal{D}$ is D-compatible (see section 2). This will be used for $M = W$. Put

$$J_1W = \{\mathcal{D} \in \operatorname{Qder} W \mid a(\mathcal{D}) \in W\},$$

which is a Lie subalgebroid of $\operatorname{Qder} W$ (we interpret J_1W as the first order jet prolongation of W). For each $D \in W$ both ad_D and ∇_D are D-compatible quasiderivations of W. Hence $a(J_1W) = W$. Clearly $\operatorname{Ker} a = \operatorname{End}_R W$, the R-linear endomorphisms. As W is $\mathcal{F}$-torsionfree, so is $\operatorname{End}_R W$ by Lemma 1.6, and it follows from the exact sequence

$$0 \to \operatorname{End}_R W \to J_1W \to W \to 0$$

that J_1W is also an $\mathcal{F}$-torsionfree R-module. For each $E \in W$ define an $\mathbb{F}$-linear transformation φ_E of J_1W by the formula

$$\varphi_E(\mathcal{D}) = [\nabla_E, \mathcal{D}] + \nabla_{\mathcal{D}(E)}, \qquad \mathcal{D} \in J_1W.$$

Note that the map $\mathcal{D} \mapsto \nabla_{\mathcal{D}(E)}$ is R-linear. On the other hand,

$$[\nabla_E, f_W \circ \mathcal{D}] = [\nabla_E, f_W] \circ \mathcal{D} + f_W \circ [\nabla_E, \mathcal{D}] = (Ef)\mathcal{D} + f[\nabla_E, \mathcal{D}]$$

for $f \in R$. Therefore φ_E is an E-compatible quasiderivation of the R-module J_1W. Now Corollary 2.8 applied to the set of derivations W and the set of quasiderivations $\{\varphi_E \mid E \in W\}$ shows that the subring of W-invariant elements R^W is a field and

$$V = \{\mathcal{D} \in J_1W \mid \varphi_E(\mathcal{D}) = 0 \text{ for all } E \in W\}$$

is a finite dimensional vector space over R^W. Since $\mathbb{F} \subset R^W$, while R^W embeds in $R/\mathfrak{m}_x \cong \mathbb{F}$ for any $x \in X$, we deduce that $R^W = \mathbb{F}$. If $A \in L$, then

$$\varphi_E(\mathrm{ad}_A) = [\nabla_E, \mathrm{ad}_A] + \nabla_{[A,E]} = 0$$

by Lemma 6.1. Thus the assignment $A \mapsto \mathrm{ad}_A$ gives an $\mathbb{F}$-linear map $L \to V$. Since $a(\mathrm{ad}_A) = A$, this map is injective, and so $\dim L \le \dim V$. As the rank of the locally free R-module J_1W equals $n^2 + n$, this does not yet give the required bound.

Denote by B the R-module of all R-bilinear symmetric maps $W \times W \to P$. There is a Lie algebroid action of J_1W on B defined by the rule

$$(\mathcal{D}\beta)(D_1, D_2) = a(\mathcal{D})\beta(D_1, D_2) - \beta\big(\mathcal{D}(D_1), D_2\big) - \beta\big(D_1, \mathcal{D}(D_2)\big)$$

for $\mathcal{D} \in J_1W$, $\beta \in B$ and $D_1, D_2 \in W$. The Lie subalgebroid

$$\operatorname{Stab}(\omega) = \{\mathcal{D} \in J_1W \mid \mathcal{D}\omega = 0\} \subset J_1W$$

coincides with the kernel of the R-linear map $\psi : J_1W \to B$, $\mathcal{D} \mapsto \mathcal{D}\omega$. For each β define $\mathcal{D}_\beta \in \operatorname{End}_R W$ by the rule $\mathcal{D}_\beta = i_\omega^{-1} \circ i_\beta$ where $i_\beta : W \to \operatorname{Hom}_R(W, P)$ is the R-linear map corresponding to β. Then

$$\beta(D_1, D_2) = \omega\big(\mathcal{D}_\beta(D_1), D_2\big) \quad \text{for all } D_1, D_2 \in W.$$

Since ω and β are symmetric, we also have $\beta(D_1, D_2) = \omega\big(D_1, \mathcal{D}_\beta(D_2)\big)$. Noting that $a(\mathcal{D}_\beta) = 0$, we get $\beta = -\frac{1}{2}\psi(\mathcal{D}_\beta)$. This shows that ψ is surjective. Since B is locally free of rank $(n^2+n)/2$ near X, the R-module $\operatorname{Stab}(\omega)$ is also locally free of rank

$$\operatorname{rank} \operatorname{Stab}(\omega) = \operatorname{rank} J_1W - \operatorname{rank} B = (n^2+n)/2.$$

The first identity defining the Levi-Civita connection ensures that $\nabla_E \in \operatorname{Stab}(\omega)$ for all $E \in W$. Therefore $\operatorname{Stab}(\omega)$ is stable under all quasiderivations φ_E, and we may apply Corollary 2.8 as earlier, but replacing J_1W with $\operatorname{Stab}(\omega)$. Finally, since $\operatorname{ad}_A \in \operatorname{Stab}(\omega)$ for all $A \in L$, we conclude that $\dim L \le \operatorname{rank} \operatorname{Stab}(\omega)$. ∎

Remark. The upper bound $\dim L = (n^2+n)/2$ is attained under the assumptions of case (d) in Theorem 5.1 (actually $\dim L$ can be larger if no connectedness assumptions are imposed). Furthermore, the curvature $\mathcal{R}$ of the Levi-Civita connection in that case is explicitly determined as

$$\mathcal{R}(D, D')D'' = k\omega(D', D'')D - k\omega(D, D'')D' \quad \text{for } D, D', D'' \in W$$

where $k \in R^W$ is some fixed W-invariant element of R. In differential geometry this identity with a scalar $k \in \mathbb{R}$ characterizes spaces of constant curvature. To prove the above identity one may use the following lemma:

Lemma 6.3. *Let $L \subset W$ be a transitive Lie subalgebra, M an (R, L)-module, and $u \in M$ an L-invariant element. Suppose that M is $\mathcal{F}$-torsionfree and locally free of constant rank near X. Suppose also that for each $x \in X$ the subspace of $\mathfrak{g}_0^x$-invariant elements in $M(x)$ is onedimensional and is spanned by the coset of u. Then M^L is a cyclic free R^W-module generated by u.*

When $n \ne 1, 4$ take $M = \operatorname{Hom}_R\big(\bigwedge_R^2 W, \mathfrak{o}(\omega)\big)$ where

$$\mathfrak{o}(\omega) = \operatorname{Stab}(\omega) \cap \operatorname{End}_R W$$

is the orthogonal Lie algebra associated with ω. Define $u, v \in M^L$ by the formulas

$$\begin{aligned} u(D_1 \wedge D_2)D_3 &= \omega(D_2, D_3)D_1 - \omega(D_1, D_3)D_2, \\ v(D_1 \wedge D_2)D_3 &= \mathcal{R}(D_1, D_2)D_3 \end{aligned}$$

for $D_1, D_2, D_3 \in W$. We have $R/\mathfrak{m}_x \otimes_R \mathfrak{o}(\omega) \cong \mathfrak{o}(\omega_x) \cong \mathfrak{g}_x^0$ for each $x \in X$. Note that $\mathfrak{g}_x^0$ is a central simple Lie algebra when $n > 4$ or when $n = 3$. If $n = 2$, then $\dim \mathfrak{g}_x^0 = 1$. Furthermore, the $\mathfrak{g}_x^0$-module $\bigwedge^2 W(x)$ is isomorphic to the adjoint module. The absolute irreducibility of these modules entails $\dim M(x)^{\mathfrak{g}_x^0} = 1$. Thus Lemma 6.3 applies, yielding $v \in R^W u$. In the case $n = 4$ take M to be the module of R-linear maps $\xi : \bigwedge_R^2 W \to \mathfrak{o}(\omega)$ satisfying the additional condition

$$\xi(D_1 \wedge D_2)D_3 + \xi(D_2 \wedge D_3)D_1 + \xi(D_3 \wedge D_1)D_2 = 0.$$

Then the equalities $\dim M(x)^{\mathfrak{g}_x^0} = 1$ can be verified and $u, v \in M^L$. Finally, $\mathcal{R} = 0$ in the case $n = 1$.

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