Journal of Lie Theory
Volume **18** (2008) 391–411

The Klein Quadric and the Classification of Nilpotent Lie Algebras of Class Two

Markus Stroppel

Communicated by K. Strambach

Abstract. We collect information about the Klein quadric which is useful to determine the orbits of the group of all linear bijections of a four-dimensional vector space on the Grassmann manifolds of the exterior product. This information is used to classify nilpotent Lie algebras of small dimension, over arbitrary fields (including the characteristic 2 case). The invariants used are easy to read off from any set of structure constants.
Mathematics Subject Classification 2000: 17B30, 22E25, 15A63, 15A69, 15A72, 51A50, 51E24.
Key Words and Phrases: Klein quadric, Grassmann space, orbit, nilpotent Lie algebra, Heisenberg algebra, classification.

1. Introduction

The aim of the present paper is to give an easy, yet conceptual, approach to the classification of Lie algebras L with $L' := [L, L] \leq \mathfrak{z}(L)$, for small values of $\dim(L/\mathfrak{z}(L))$ and over arbitrary fields (including the characteristic two case). The need for a conceptual approach is felt sharply, we quote [13] p. 624: "Probably, all classifications that we know of 7-dimensional nilpotent Lie algebras contain some mistakes". As a consequence of our investigation, we spotted an error in [21] (see 7.9 below), and another error in [11] (see 7.10). Most of the existing attempts at classifications use special (and sometimes quite arbitrary) choices of structure constants. The invariants that we use are easy to read off from any given set of structure constants: one has to determine and compare quadratic forms on vector spaces of small dimension, up to similitudes. This facilitates the comparison with previous lists: our present approach allows to identify or distinguish isomorphism types in different lists easily, cf. 8.3, 8.4, 7.9, 7.10, or 7.8.

Classification lists of (nilpotent) Lie algebras appear to be an often used tool in the mathematical foundations of physics. The present investigation was motivated by such applications in [22] and in [21].

A thorough treatment of nilpotent Lie algebras over $\mathbb{C}$ is undertaken in the monograph [12]. While the classification of nilpotent Lie algebras becomes

ISSN 0949–5932 / \$2.50

impractical for higher dimensions, it is still possible (but tedious) for algebras of dimension at most 7. A first attempt is found in Umlauf's thesis [32] as early as 1891. Dixmier [10] showed that over any field, there are 16 types of nilpotent Lie algebras of dimension at most 5. Skjelbred and Sund [28] announced (in a more general setting) a result that includes a classification of all nilpotent Lie algebras of dimension 6 over the field $\mathbb{R}$. Unfortunately, the proofs were only published in preprint form, and are not available for the author. Gauger [11] introduces a conceptual approach, suitable for nilpotent algebras of class 2 over algebraically closed fields of odd characteristic. Lists of nilpotent Lie algebras of dimension at most 7 are found (among other places) in [1], [2], [20], and [23] (unfortunately, a list without any indications of proofs). A computer approach to the classification of finite nilpotent Lie algebras is presented in [27]: presently, this works for the fields of order 2, 3, 5, and 7.

Any attempt at such a classification will, in some way or other, use a classification of the orbits of $\mathrm{GL}(V)$ on the exterior product $V \wedge V$, cf. [13] Prop. 32, p. 636. Fortunately, there are quite beautiful geometric methods to treat this question in the case where $\dim V = 4$.

Our approach to the classification is similar to the one in [11]. However, we treat fields that are not algebraically closed. While Gauger [11] classifies Lie algebras with a given number of generators, we are interested in a classification that starts with the invariants $(\dim L, \dim \mathfrak{z}(L), \dim L')$, or, equivalently, with $(\dim(L/\mathfrak{z}(L)), \dim(\mathfrak{z}(L)/L'), \dim L')$. Note that $\dim(L/\mathfrak{z}(L)) + \dim(\mathfrak{z}(L)/L')$ gives the (minimal) number of generators for a nilpotent Lie algebra of class 2. In order to show that there are infinitely many isomorphism types of nilpotent Lie algebras with prescribed values for certain invariants, Gauger's methods are very useful, cf. 7.8 below. Gauger's duality (cf. also [26]) works well with his approach, but may be misleading in the present situation, cf. 7.2 below. Our present approach also makes it easy to find explicit structure constants, cf. [17], and to determine the group $\mathrm{Aut}(L)$ and its orbits on L, see [14] and [15].

For every vector space V, the group $\mathrm{GL}(V)$ acts linearly on the tensor product $V \otimes V$ and on the exterior product $V \wedge V$. Using the universal property of the tensor product offers an elegant way to prove this. However, we need an explicit version, later on. For this reason, we collect explicit definitions and the results we need in the first sections. Proofs are included, for the readers' convenience.

1.1 Notation. Let $\mathbb{K}$ be a field, and let $b_0, \dots, b_{n-1}$ denote the standard basis for $\mathbb{K}^n$. We will think of elements $v = \sum_{i<n} v_i b_i \in \mathbb{K}^n$ as columns, the transpose is written $v' := (v_0, \dots, v_{n-1})$. For $v, w \in \mathbb{K}^n$ we obtain the *decomposable tensor* $v \otimes w := vw' = (v_i w_j)_{i,j<n}$. The elements $b_i \otimes b_j$, with $i, j < n$, form the standard basis for the space $\mathbb{K}^{n\times n}$ of $n \times n$ – matrices with entries from $\mathbb{K}$.

The set of *alternating tensors* is the linear span $\mathbb{K}^n \wedge \mathbb{K}^n$ of the elements of the form $v \wedge w := v \otimes w - w \otimes v$. These are the skew-symmetric matrices with zero diagonal (the latter condition follows from the former unless $\mathrm{char}\,\mathbb{K} = 2$). The elements $S_k^j := b_j \wedge b_k$ with $j < k \le n$ form a basis for $\mathbb{K}^n \wedge \mathbb{K}^n$. With the bilinear map $\eta\colon \mathbb{K}^n \times \mathbb{K}^n \longrightarrow \mathbb{K}^n \wedge \mathbb{K}^n \colon (v, w) \mapsto v \wedge w$, we have $(\mathbb{K}^n \wedge \mathbb{K}^n, \eta)$ as an explicit model for the exterior product, satisfying the universal property: for every alternating bilinear map $\beta\colon V \times V \longrightarrow Z$, there is a unique linear map

$\hat{\beta}\colon \mathbb{K}^n \wedge \mathbb{K}^n \to Z$ such that $\beta = \hat{\beta} \circ \eta$.

The (linear) action of the group $\mathrm{GL}_n\mathbb{K}$ on $\mathbb{K}^n$ induces a linear action $\omega\colon \mathrm{GL}_n\mathbb{K} \times (\mathbb{K}^n \otimes \mathbb{K}^n) \to \mathbb{K}^n \otimes \mathbb{K}^n$, determined via extension of $(A, (v \otimes w)) \mapsto Av \otimes Aw = (Av)(Aw)' = Avw'A'$. Thus $\omega(A, M) = AMA'$ holds for all $A \in \mathrm{GL}_n\mathbb{K}$ and all $M \in \mathbb{K}^{n\times n}$. Obviously, the space of alternating tensors is invariant under this action. We will write $A.X := AXA'$.

1.2 Remarks. For fields of characteristic 2, one cannot distinguish between symmetric and skew-symmetric matrices, and skew-symmetric matrices are not necessarily alternating.

Using the linear map $d\colon \mathbb{K}^{n\times n} \to \mathbb{K}^{n\times n}\colon A \mapsto A - A'$ we see that $\mathbb{K}^n \wedge \mathbb{K}^n$ is the correct model for the space of alternating tensors: the kernel K of d is the subspace generated by all tensors of the form $v \otimes v$, whence the image (which consists of all alternating matrices) is isomorphic to $(\mathbb{K}^n \otimes \mathbb{K}^n)/K$, which is the abstract definition of the space of alternating tensors. The map d is $\mathrm{GL}_n\mathbb{K}$-equivariant, thus this model also gives the correct interpretation of the action.

If $\operatorname{char}\mathbb{K} \neq 2$ then the relation $d^2 = 2\,d$ gives that $\frac{1}{2}\,d$ is a projection. For $\operatorname{char}\mathbb{K} = 2$ we obtain that d is a nilpotent endomorphism of $\mathbb{K}^{n\times n}$.

2. Orbits on the Space of Alternating Tensors

The action $\omega\colon \mathrm{GL}_n\mathbb{K}\times(\mathbb{K}^n\wedge\mathbb{K}^n) \to \mathbb{K}^n\wedge\mathbb{K}^n$ is an action by linear transformations, and induces an action on the Grassmann manifold $\mathrm{Gr}^{\wedge}_{d,n}$ (consisting of all subspaces of dimension d in $\mathbb{K}^n \wedge \mathbb{K}^n$) for each d. The classification of alternating forms (cf. [3] Thm. 3.3 and Thm. 3.7 or [9] § 11 and § 6) yields:

2.1 Proposition. *The* $\mathrm{GL}_n\mathbb{K}$*-orbits on* $\mathbb{K}^n \wedge \mathbb{K}^n$ *are represented by the tensors* 0, S_1^0, $S_1^0 + S_3^2$, $\ldots$, $\sum_{m<\frac{n}{2}} S_{2m+1}^{2m}$, *and the* $\mathrm{GL}_n\mathbb{K}$*-orbits on* $\mathrm{Gr}^{\wedge}_{1,n}$ *are represented by the elements of the set* $\left\{ \left\langle \sum_{m<j} S_{2m+1}^{2m} \right\rangle_{\mathbb{K}} \,\middle|\, j \leq \frac{n}{2} \right\}$. ∎

For general n and $d > 1$, the orbit decomposition of $\mathrm{Gr}^{\wedge}_{d,n}$ will be quite complicated because $\mathrm{GL}_n\mathbb{K}$ induces a rather small[1] subgroup of $\mathrm{GL}(\mathbb{K}^n \wedge \mathbb{K}^n)$: in fact, the dimension of the first group is n^2, while the latter has dimension $(\frac{1}{2}n(n-1))^2 = \frac{1}{4}(n^4 - 2n^3 + n^2)$. The case $d = n-2$ amounts to the study of pairs of alternating forms; this is treated (for general n and over general fields) in [24].

One of the biggest cases for n where a complete determination of the orbits seems feasible is the case $n = 4$. Fortunately, we have an additional, strong tool at hand in this case: the orbit $\mathrm{Gr}^{\wedge}_{1,4}$ forms a quadric (known as the Klein quadric), and $\mathrm{GL}_4\mathbb{K}$ is a normal subgroup in the corresponding group of similitudes. On the projective space, the group $\mathrm{GL}_4\mathbb{K}$ induces a subgroup of index 2 in $\mathrm{PGO}_6\mathbb{K}$, see 3.11 below.

3. The Klein Quadric

The contents of this section belong to classical projective geometry; for instance, see [16] Kap. I. Apparently, they are no longer known as well as they deserve to be.

[1] In fact, the number of orbits becomes infinite if $n \geq 6$ and $3 \leq d \leq \binom{n}{2} - 3$, see [11] 7.8. Cf. also [4].

For the reader's convenience, we give a formulation suitable to our present needs, and include proofs (for a slightly different approach, see [31] Ch. 12).

According to 2.1, we have exactly three orbits on $\mathbb{K}^4 \wedge \mathbb{K}^4$, represented by 0, S_1^0 and $S_1^0 + S_3^2$. Therefore, there are two orbits on $\mathrm{Gr}_{1,4}^{\wedge}$. The orbit of $\langle S_1^0 \rangle_{\mathbb{K}}$ consists of subspaces $\langle X \rangle_{\mathbb{K}}$ with $X \in \mathbb{K}^4 \wedge \mathbb{K}^4 \setminus \{0\}$ and $\det X = 0$. With respect to the basis $S_1^0, S_2^0, S_3^0, S_2^1, S_3^1, S_3^2$ we introduce homogeneous coordinates $[x_0, \ldots, x_5]$ for $\langle X \rangle_{\mathbb{K}}$, where

$$X = \begin{pmatrix} x_0 S_1^0 & + & x_1 S_2^0 & + & x_2 S_3^0 \\ & + & x_3 S_2^1 & + & x_4 S_3^1 \\ & & & + & x_5 S_3^2 \end{pmatrix} = \begin{pmatrix} 0 & x_0 & x_1 & x_2 \\ -x_0 & 0 & x_3 & x_4 \\ -x_1 & -x_3 & 0 & x_5 \\ -x_2 & -x_4 & -x_5 & 0 \end{pmatrix}.$$

The orbit of $\langle S_1^0 \rangle_{\mathbb{K}}$ may then be described as the set of zeros of the homogeneous polynomial

$$\begin{aligned} \det X = p(x_0, x_1, x_2, x_3, x_4, x_5) &= \quad x_0(-x_1x_4x_5 + x_2x_3x_5 + x_0x_5x_5) \\ &\quad -x_1(-x_1x_4x_4 + x_2x_3x_4 + x_0x_4x_5) \\ &\quad +x_2(\ \ x_0x_3x_5 - x_1x_3x_4 + x_2x_3x_3) \\ &= (x_0x_5 - x_1x_4 + x_2x_3)^2 . \end{aligned}$$

We have $p = q^2$ with $q(x_0, x_1, x_2, x_3, x_4, x_5) := x_0x_5 - x_1x_4 + x_2x_3$, and the polynomial q is a quadratic form of Witt index 3 on $\mathbb{K}^4 \wedge \mathbb{K}^4$. This gives:

3.1 Proposition. *The orbit of $\langle S_1^0 \rangle_{\mathbb{K}}$ in $\mathrm{Gr}_{1,4}^{\wedge}$ forms a hyperbolic quadric Q, known as the Klein quadric. The complement of that quadric is the second orbit in $\mathrm{Gr}_{1,4}^{\wedge}$.* ■

3.2 Notation. We re-arrange the basis, using $S_1^0, S_2^0, S_3^0, S_3^2, -S_3^1, S_2^1$. With respect to the new basis, the quadratic form itself may be described[2] as $q(v) = v' M_q v$, and the polar form f_q for the quadratic form q has the Gram matrix J, where M_q and J are defined as follows:

$$M_q := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad J := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The description with respect to this basis makes it easier to compute the group $\mathrm{GO}(q)$ of similitudes, its elements are described by the matrices in

$$\mathrm{GO}_6\mathbb{K} := \left\{ \begin{pmatrix} A & C \\ B & D \end{pmatrix} \;\middle|\; \begin{array}{c} A, B, C, D \in \mathbb{K}^{3\times 3} \\ \exists\, a \in \mathbb{K} \setminus \{0\} \colon A'D + B'C = a\,1 \\ A'B \text{ and } C'D \text{ are alternating} \end{array} \right\},$$

[2] We have to use a (somewhat arbitrary) non-symmetric matrix M_q since we include the characteristic two case.

where X' denotes the transpose of X. The orthogonal group $\mathrm{O}(q)$ is characterized by $a = 1$.

Note that the conditions on $A'B$ and $C'D$ reduce to $A'B + B'A = 0$ and $C'D + D'C = 0$ if $\operatorname{char} \mathbb{K} \neq 2$. In that case, the similitudes of q and those of the polar form f_q are the same.

3.3 Remarks. The group $\mathrm{GL}_4\mathbb{K}$ acts by similitudes with respect to q. This yields a homomorphism $\delta\colon \mathrm{GL}_4\mathbb{K} \to \mathrm{GO}(q)$, and a corresponding homomorphism $\tilde{\delta}\colon \mathrm{GL}_4\mathbb{K} \to \mathrm{GO}_6\mathbb{K}$. The kernel of these homomorphisms is $\{1, -1\}$.

We will be interested in the induced groups $\mathrm{PGL}_4\mathbb{K}$ and $\mathrm{PGO}_6\mathbb{K}$ on the projective spaces (or on the quadric). In fact, we shall see that $\tilde{\delta}$ induces an isomorphism π from $\mathrm{PGL}_4\mathbb{K}$ onto a subgroup of index 2 in $\mathrm{PGO}_6\mathbb{K}$, see 3.11 below.

The Klein quadric provides a model for the space $\mathcal{L}$ of lines in the 3-dimensional projective space over $\mathbb{K}$, as follows:

3.4 Lemma. *The map $\lambda\colon \mathcal{L} \to Q\colon \langle u,\, v\rangle_{\mathbb{K}} \mapsto \langle u \wedge v\rangle_{\mathbb{K}}$ is well-defined and bijective.*

Proof. Replacing u, v by some other basis $au + bv, cu + dv$ for $\langle u,\, v\rangle_{\mathbb{K}}$ and then expanding $(au + bv) \wedge (cu + dv)$ bilinearly, we see that λ is well-defined. The rest is straightforward. ■

3.5 Lemma. *The group $\mathrm{GL}_4\mathbb{K}$ acts with three orbits on the set of pairs of lines, represented by (L_0, L_0), (L_0, K_0), and (L_0, K_1), where $L_0 := \langle b_0, b_1\rangle_{\mathbb{K}}$, and $K_j := \langle b_0 + jb_2, b_3\rangle_{\mathbb{K}}$.* ■

3.6 Lemma. *Two lines $K, L \in \mathcal{L}$ share a point if, and only if, their images $\lambda(K)$ and $\lambda(L)$ are orthogonal with respect to q.*

Proof. Using 3.5, we may assume $L = \langle b_0,\, b_1\rangle_{\mathbb{K}}$ and $K = \langle b_0 + kb_2,\, b_3\rangle_{\mathbb{K}}$, for some $k \in \{0, 1\}$. Then $\lambda(L) = \langle S_1^0\rangle_{\mathbb{K}}$ and $\lambda(K) = \langle S_3^0 + kS_3^2\rangle_{\mathbb{K}}$, and $\lambda(K) \perp \lambda(L)$ means

$$0 = q(S_3^0 + kS_3^2 + S_1^0) - q(S_3^0 + kS_3^2) - q(S_1^0) = k - 0 - 0 = k\,.$$

Now $K \cap L = \{0\} \iff k \neq 0$ yields the claimed equivalence. ■

3.7 Proposition. *The maximal totally singular subspaces with respect to q are just the images of maximal sets of pairwise confluent lines. There are two types of such sets: pencils $\mathcal{L}_p := \big\{L \in \mathcal{L} \mid\ p < L\big\}$ or, dually, line sets of planes $\mathcal{L}_P := \big\{L \in \mathcal{L} \mid\ L < P\big\}$.*

Proof. After 3.6, it suffices to show that a maximal set $\mathcal{K}$ of pairwise confluent lines is either contained in $\mathcal{L}_p$, or in $\mathcal{L}_P$, for some point p or some plane P. So assume that $\mathcal{K}$ contains three lines forming a triangle: the plane P we need is spanned by that triangle. ■

3.8 Corollary. *The action of* $\mathrm{GL}_4\mathbb{K}$ *on the set* $\mathcal{M}_3$ *of maximal totally singular subspaces has two orbits, represented by* $\lambda(\mathcal{L}_p) = \langle S_1^0, S_2^0, S_3^0\rangle_{\mathbb{K}}$ *and* $J(\lambda(\mathcal{L}_p)) = \lambda(\mathcal{L}_P) = \langle S_2^1, S_3^1, S_3^2\rangle_{\mathbb{K}}$, *where* $p = \langle b_0\rangle_{\mathbb{K}}$, *and* $P = \langle b_1, b_2, b_3\rangle_{\mathbb{K}}$. ∎

Since $\mathrm{GL}_4\mathbb{K}$ is transitive on pairs of different but confluent lines, we have:

3.9 Corollary. *The group* $\mathrm{GL}_4\mathbb{K}$ *acts transitively on the set* $\mathcal{M}_2$ *of* 2*-dimensional maximal totally singular subspaces.* ∎

3.10 Definition. Let $\mathrm{PGO}_6^+\mathbb{K}$ denote the subgroup of $\mathrm{PGO}_6\mathbb{K}$ that leaves the two orbits under $\mathrm{GL}_4\mathbb{K}$ in $\mathcal{M}_3$ invariant. Clearly, the homomorphism $\tilde{\delta}$ induces a homomorphism $\pi\colon \mathrm{PGL}_4\mathbb{K} \to \mathrm{PGO}_6^+\mathbb{K}$, and $\mathrm{PGO}_6^+\mathbb{K}$ is a (normal) subgroup of index 2 in $\mathrm{PGO}_6\mathbb{K}$.

3.11 Theorem. *The map* $\pi\colon \mathrm{PGL}_4\mathbb{K} \to \mathrm{PGO}_6^+\mathbb{K}$ *is an isomorphism. The subgroup generated by the image of the involution* J *in* $\mathrm{PGO}_6\mathbb{K}$ *forms a complement to* $\mathrm{PGO}_6^+\mathbb{K}$.

Proof. Every element of $\mathrm{PGO}_6^+\mathbb{K}$ acts on the two $\mathrm{GL}_4\mathbb{K}$-orbits in $\mathcal{M}_3$. Via the inverse of λ, the group $\mathrm{PGO}_6^+\mathbb{K}$ thus acts on the sets of points, lines, and planes of the projective space in such a way that incidences are preserved. This gives a homomorphism μ from $\mathrm{PGO}_6^+\mathbb{K}$ to the group $\mathrm{P\Gamma L}_4\mathbb{K}$ of all automorphisms of the projective space. It is easy to see that $\mu\circ\pi$ is the identity. Thus π is injective, it remains to show surjectivity.

The subgroup $\mathrm{PGL}_4\mathbb{K}$ of $\mathrm{P\Gamma L}_4\mathbb{K}$ acts transitively on the set of ordered line sets of tetrahedra in the projective space. One of these is

$$\mathcal{S} := \left(\langle S_1^0\rangle_{\mathbb{K}}, \langle S_2^0\rangle_{\mathbb{K}}, \langle S_3^0\rangle_{\mathbb{K}}, \langle S_2^1\rangle_{\mathbb{K}}, \langle -S_3^2\rangle_{\mathbb{K}}, \langle S_3^1\rangle_{\mathbb{K}}\right) .$$

The stabilizer Σ of $\mathcal{S}$ in $\mathrm{PGO}_6^+\mathbb{K}$ consists of diagonal matrices. Using 3.2, we find that each element of Σ is induced by some diagonal matrix in $\mathrm{GL}_4\mathbb{K}$. Thus Σ is contained in $\pi(\mathrm{PGL}_4\mathbb{K})$. Frattini's lemma (see 4.1 below) now yields $\pi(\mathrm{PGL}_4\mathbb{K}) = \mathrm{PGO}_6^+\mathbb{K}$. ∎

4. Frattini's Lemma

The following very general but quite useful result is usually attributed to Frattini:

4.1 Lemma. *Let* G *be a group acting transitively on* X, *and consider a point* $a \in X$ *and a subgroup* H *of* G. *Then* H *is transitive on* X *if, and only if, there exists a subset* R *of the stabilizer* G_a *such that* $HR = G$.

Proof. Assume first that H is transitive. For each $g \in G$, there exists $h_g \in H$ such that $h_g.a = g.a$. Now $r_g := h_g^{-1}g$ belongs to G_a, and $R := \{r_g \mid g \in G\}$ satisfies our requirement.

Conversely, if there is $R \subseteq G_a$ such that $HR = G$, we compute the orbit $G.a = HR.a = H.a$, and H is transitive. ∎

We will apply 4.1 to transitive actions (i.e., orbits) of $\mathrm{PGO}_6\mathbb{K}$, in the following form:

4.2 Corollary. *Let H be a subgroup of G, of index 2. Then the orbit $G.a$ coincides with $H.a$ if, and only if, the set $G_a \setminus H$ is not empty. Explicitly, choose $J \in G \setminus H$: then the orbits coincide if, and only if, there exists $\varphi_a \in H$ such that $\varphi_a.a = J.a$.* ∎

5. Orbits in the Four-dimensional Case

The orbits on $\mathrm{Gr}^{\wedge}_{2,4}$, $\mathrm{Gr}^{\wedge}_{3,4}$, $\mathrm{Gr}^{\wedge}_{4,4}$ and $\mathrm{Gr}^{\wedge}_{5,4}$ (i.e., on the sets of lines, planes, three-spaces, and hyperplanes, respectively, in the projective space $\mathcal{P}$ coordinatized by $\mathbb{K}^4 \wedge \mathbb{K}^4$) may be described using the Klein quadric Q. We introduce some more notation.

5.1 Definitions. We consider the following lines in $\mathcal{P}$:

$$E := \langle S_1^0, S_2^0 \rangle_{\mathbb{K}}, \quad T := \langle S_1^0, S_3^0 + S_2^1 \rangle_{\mathbb{K}}, \quad \text{and} \quad S := \langle S_1^0, S_3^2 \rangle_{\mathbb{K}}.$$

The orthogonal spaces (with respect to q) are

$$\begin{aligned} E^\perp &= \langle S_1^0, S_2^0, S_3^0, S_2^1 \rangle_{\mathbb{K}}, \\ T^\perp &= \langle S_1^0, S_2^0, S_3^0 - S_2^1, S_3^1 \rangle_{\mathbb{K}} \\ \text{and} \quad S^\perp &= \langle S_2^0, S_3^0, S_2^1, S_3^1 \rangle_{\mathbb{K}}, \qquad \text{respectively.} \end{aligned}$$

We will also use the planes

$$\begin{aligned} F &:= \langle S_1^0, S_2^0, S_3^0 \rangle_{\mathbb{K}}, & E+T &:= \langle S_1^0, S_2^0, S_3^0 + S_2^1 \rangle_{\mathbb{K}}, \\ E+S &:= \langle S_1^0, S_2^0, S_3^2 \rangle_{\mathbb{K}}, & T+S &:= \langle S_1^0, S_3^0 + S_2^1, S_3^2 \rangle_{\mathbb{K}}. \end{aligned}$$

With respect to the given bases, the restriction of q to the subspace X may be described by an upper triangular matrix m_X, where

$$m_E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad m_T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad m_S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad m_F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$m_{E+T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad m_{E+S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_{T+S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$m_{E^\perp} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_{T^\perp} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_{S^\perp} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the Gram matrix for the polar form of the restriction is $m_X + (m_X)'$.

For some of these subspaces, geometric arguments are available to determine the orbits:

5.2 Examples. **1.** The line S joins two points of the quadric that correspond to a pair of skew lines in the projective space. As $\mathrm{GL}_4\mathbb{K}$ acts transitively on the pairs of skew lines, the orbits under $\mathrm{GL}_4\mathbb{K}$ and under $\mathrm{GO}_6\mathbb{K}$ coincide.

2. The plane $T+S$ meets Q in a non-degenerate conic, and is spanned by three points of Q that correspond to pairwise skew lines. Again, the orbits under $\mathrm{GL}_4\mathbb{K}$ and under $\mathrm{GO}_6\mathbb{K}$ coincide, because $\mathrm{GL}_4\mathbb{K}$ acts transitively on the triplets of pairwise skew lines.

3. The restriction of q to the plane $E+S$ is degenerate. This plane is spanned by a triplet of points on Q that corresponds to two skew lines and one line meeting both. Again, we have transitivity of $\mathrm{GL}_4\mathbb{K}$ on these configurations, and the orbits coincide.

It is well known (e.g., see [25] 1.3.5 p. 7 for the case $\operatorname{char}\mathbb{K} \neq 2$ and [25] p. 339 for $\operatorname{char}\mathbb{K} = 2$) that every quadratic form on a vector space of finite dimension can be described by an upper triangular matrix of quite restricted shape. We give an explicit formulation for the special cases that we need:

5.3 Lemma. *Let W be a vector space of finite dimension d over $\mathbb{K}$, and let $p\colon W \to \mathbb{K}$ be a quadratic form. In coordinates with respect to a suitable basis, we have $p(v) = v'Av$, where A may be chosen as follows:*

1. *For $d=2$, there are $a,c \in \mathbb{K}$ such that* $A \in \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} a & 1 \\ 0 & c \end{pmatrix} \right\}$.

2. *For $d=3$, there are $a,b,c \in \mathbb{K}$ such that* $A \in \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 1 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \right\}$.

3. *If $d=3$ and p has an isotropic vector in W then we may assume $b=0$.*

The non-diagonal matrices are only needed if $\operatorname{char}\mathbb{K} = 2$. ∎

5.4 Lemma. *Let $p\colon W \to \mathbb{K}$ be any quadratic form with $\dim W \leq 3$. Then there exists $W' \in \mathrm{Gr}^{\wedge}_{d,4}$ such that the restriction of q to W' is isometric to p.*

Proof. Without loss, we may assume $\dim W = 3$. In coordinates with respect to some suitable basis for W, the form p is given as $p(x,y,z) = ax^2 + by^2 + cz^2 + txz$, with $t \in \{0,1\}$. Now p is isometric to the restriction of q to the space $\langle S_1^0 + aS_3^2,\ S_2^0 - bS_3^1,\ S_3^0 + cS_2^1 + tS_3^2 \rangle_{\mathbb{K}}$. ∎

The subspaces introduced in the proof of 5.4 will be denoted by

$$\begin{aligned} P^t_{a,c} &:= \left\langle S_1^0 + aS_3^2,\ S_3^0 + cS_2^1 + tS_3^2 \right\rangle_{\mathbb{K}} \\ P^t_{a,b,c} &:= \left\langle S_1^0 + aS_3^2,\ S_2^0 - bS_3^1,\ S_3^0 + cS_2^1 + tS_3^2 \right\rangle_{\mathbb{K}} . \end{aligned}$$

Note that $T = P^0_{0,1}$ and $E + T = P^0_{0,0,1}$, and that $P^t_{a,c} \subseteq P^t_{a,0,c} \subseteq (S_2^0)^\perp$.

For $c \neq 0$ the space $P^0_{0,c}$ is clearly similar, but not necessarily isometric to $P^0_{0,1}$. A similitude is induced by a diagonal element of $\mathrm{GL}_4\mathbb{K}$, and we find:

5.5 Lemma. *The* $\mathrm{GL}_4\mathbb{K}$*-orbit of* $P^0_{0,1}$ *contains* $\left\{P^0_{0,c} \,\middle|\, c \in \mathbb{K}^\times\right\}$*, and the orbit of* $P^0_{0,0,1}$ *contains* $\left\{P^0_{0,0,c} \,\middle|\, c \in \mathbb{K}^\times\right\}$. ∎

5.6 Theorem. **1.** *The* $\mathrm{GL}_4\mathbb{K}$*-orbits in* $\mathrm{Gr}^\wedge_{2,4}$ *are represented by some set*

$$\{E, T, S\} \cup \mathcal{P}_1\,,$$

where $\mathcal{P}_1$ *denotes a (possibly empty) set of nonsingular lines.*

2. *The* $\mathrm{GL}_4\mathbb{K}$*-orbits in* $\mathrm{Gr}^\wedge_{3,4}$ *are represented by some set*

$$\{F, J(F), E+T, E+S, T+S\} \cup \mathcal{P}_2 \cup \mathcal{P}_3\,,$$

where $\mathcal{P}_2$ *denotes a (possibly empty) set of nonsingular planes, and* $\mathcal{P}_3$ *is a (possibly empty) set of planes of the form* $\langle S^0_2\rangle_{\mathbb{K}}+\ell$*, where* ℓ *is a nonsingular line contained in* $(S^0_2)^\perp$.

3. *The* $\mathrm{GL}_4\mathbb{K}$*-orbits in* $\mathrm{Gr}^\wedge_{4,4}$ *are represented by* $\mathcal{R}^\perp := \{R^\perp \mid R \in \mathcal{R}\}$*, where* $\mathcal{R}$ *is an arbitrary set of representatives in* $\mathrm{Gr}^\wedge_{2,4}$.

Proof. The orthogonality relation induced by q is invariant under the action of $\mathrm{GL}_4\mathbb{K}$. Thus the last assertion will follow from the first one.

Using 5.2, 5.5 and Witt's Theorem (see [25] § 9 or [9] § 11, p. 21 and § 16, p. 35, cf. [31] 7.4 or [3] Thm. 3.9 for $\operatorname{char}\mathbb{K} \neq 2$), we infer that the given sets contain representatives for the orbits under the full group $\mathrm{PGO}_6\mathbb{K}$ of similitudes. If the $\mathrm{PGO}^+_6\mathbb{K}$-orbit of some element of $\mathcal{P}_j$ should be smaller than the $\mathrm{PGO}_6\mathbb{K}$-orbit, we could simply enlarge the set $\mathcal{P}_j$. However, Lemma 5.7 below will show that this is never necessary.

The orbits of F, $J(F)$, and E have been discussed in 3.8 and 3.9, and those of S, $T+S$ and $E+S$ were determined in 5.2.

According to 4.2, it remains to find for each subspace $X \in \{T, E+T\}$ an element of $\mathrm{PGO}_6\mathbb{K} \setminus \mathrm{PGO}^+_6\mathbb{K}$ that leaves X invariant. In other words, we search for an element $\varphi_X \in \mathrm{GL}_4\mathbb{K}$ moving $J(X)$ back to X: we use

$$\varphi_T := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \varphi_{E+T} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$ ∎

5.7 Lemma. *For each element of* $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$*, the orbits under* $\mathrm{GO}_6\mathbb{K}$ *and under* $\mathrm{GL}_4\mathbb{K}$ *coincide.*

Proof. According to 5.4, the orbits of elements of $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ under $\mathrm{GO}_6\mathbb{K}$ contain representatives of the form $P^t_{a,c}$ or $P^t_{a,b,c}$, where $a, b, c \in \mathbb{K}$ and $t \in \{0, 1\}$. Our assumptions about anisotropy secure $a \neq 0 \neq c$. Now the matrix

$$\psi_c := \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -c & 0 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_4\mathbb{K}$$

maps $P^t_{a,b,c}$ to $\langle cS_3^2 + acS_1^0,\, cS_3^1 - bcS_2^0,\, cS_2^1 + c^2S_3^0 + tcS_1^0\rangle_{\mathbb{K}} = J(P^t_{a,b,c})$ and $P^t_{a,c}$ to $J(P^t_{a,c})$, respectively. According to 4.2, the orbits of $P^t_{a,b,c}$ or $P^t_{a,c}$ under $\mathrm{GO}_6\mathbb{K}$ and under $\mathrm{GL}_4\mathbb{K}$ coincide: we use $\varphi_{P^t_{a,b,c}} := \psi_c =: \varphi_{P^t_{a,c}}$. ■

5.8 Remarks. **1.** If $\mathbb{K}$ is a euclidian field (i.e., an ordered field such that every positive element is a square, e.g. $\mathbb{K} = \mathbb{R}$), each of the sets $\mathcal{P}_j$ contains exactly one element. For instance, we may choose the subspaces $P^0_{1,1}$, $P^0_{1,1,1}$, and $P^0_{1,0,1}$ from 5.4: the diagonal matrix $\mathrm{diag}(1,-1,-1,-1) \in \mathrm{GL}_4\mathbb{K}$ interchanges the positive (semi-)definite spaces $P^0_{1,1}$, $P^0_{1,1,1}$, and $P^0_{1,0,1}$ with the negative (semi-)definite ones $P^0_{-1,-1}$, $P^0_{-1,-1,-1}$, and $P^0_{-1,0,-1}$.

2. Let X be a line that contains no point of the Klein quadric: this means that the restriction of q to X is anisotropic. We know that there are $a, c \in \mathbb{K}$ and $t \in \{0,1\}$ such that X is isometric to (and thus in the $\mathrm{GL}_4\mathbb{K}$-orbit of) $P^t_{a,c}$. Using the diagonal matrix $\mathrm{diag}(1,a,1,1) \in \mathrm{GL}_4\mathbb{K}$ we see that X is in the orbit of $P^t_{1,d}$, where $d = ac$. Since the restricted form is anisotropic, the polynomial $X^2 + tX + d$ is irreducible over $\mathbb{K}$.

Adjoining a root of this polynomial, we obtain a quadratic field extension $\mathbb{L}$ over $\mathbb{K}$, giving rise to an interesting Heisenberg algebra corresponding to the orthogonal spaces $\left(P^0_{1,d}\right)^{\perp} = \langle S_2^0, S_3^1, S_1^0 - S_3^2, S_3^0 - dS_2^1\rangle_{\mathbb{K}}$ and $\left(P^1_{1,d}\right)^{\perp} = \langle S_2^0, S_3^1, S_1^0 - S_3^2 - S_2^1, S_3^0 - dS_2^1\rangle_{\mathbb{K}}$, cf. 8.3 below.

3. There is a connection between nonsingular subspaces of dimension 3 and Heisenberg algebras defined using a quaternion field over $\mathbb{K}$, see 8.4 below.

4. If $\mathbb{K}$ is quadratically closed (e.g., if $\mathbb{K} = \mathbb{C}$) then there are no nonsingular lines, and each of the sets $\mathcal{P}_j$ is empty.

5. If $\mathbb{K}$ is a finite field then there are nonsingular lines (leading to elements in $\mathcal{P}_1$ and in $\mathcal{P}_3$), but no nonsingular planes, see [31] 11.2, cf. [3] III.6 or [18] 2.41 (where only the case $\mathrm{char}\,\mathbb{K} \neq 2$ is treated), and $\mathcal{P}_2$ is empty. From Remark 2 above we know that any two non-singular lines belong to the same orbit under $\mathrm{GL}_4\mathbb{K}$, represented by the subspace $P^t_{1,d}$ where X^2+tX+d is irreducible over $\mathbb{K}$ and $t = 0$ or $t = 1$ if $\mathrm{char}\,\mathbb{K} \neq 2$ or $\mathrm{char}\,\mathbb{K} = 2$, respectively: with respect to the basis $S_1^0+S_3^2$, $S_3^0+dS_2^1+tS_3^2$, the restriction of the form is then given as $\tilde{q}(x,y) = x^2 + dy^2 + txy$. The space $P^t_{1,0,d}$ may be used as the (unique) element of $\mathcal{P}_3$.

6. Heisenberg Algebras

6.1 Definitions. Let V and Z be vector spaces, and let $\beta\colon V \times V \longrightarrow Z$ be an alternating bilinear map (i.e., such that $\beta(v,v) = 0$ holds for each $v \in V$). On the vector space $V \times Z$, define the binary operation $[(v,x),(w,y)]_\beta := (0, \beta(v,w))$. Then $\mathfrak{gh}(V,Z,\beta) := (V \times Z, [\cdot,\cdot]_\beta)$ is called the *generalized Heisenberg algebra* corresponding to β.

If the image $\beta(V \times V)$ generates Z, and $\{v \in V \mid \forall\, w \in V\colon \beta(v,w) = 0\} = \{0\}$, we call $\mathfrak{gh}(V,Z,\beta)$ a *reduced Heisenberg algebra.*

6.2 Theorem. *Every nilpotent Lie algebra of class 2 is isomorphic to the direct sum of a reduced Heisenberg algebra and an abelian Lie algebra.*

Proof. Let L be a nilpotent Lie algebra of class 2, this means $L' \le \mathfrak{z}(L)$. Choose a vector space complement V for $\mathfrak{z}(L)$ in L, and a complement A for L' in $\mathfrak{z}(L)$. Now $\beta(v,w) := [v,w]$ defines a bilinear map $\beta\colon V \times V \longrightarrow L'$, and $((v,x),a) \mapsto v+x+a$ defines an isomorphism from $\mathfrak{gh}(V,L',\beta) \times A$ onto L. ■

6.3 Remark. The bilinear map β constructed in the proof of 6.2 does not depend on the choice of the complements. Thus the resulting reduced Heisenberg algebra is determined, up to isomorphism, by the isomorphism type of L.

In order to classify the nilpotent Lie algebras of class 2, it remains to classify the reduced Heisenberg algebras.

Using the universal property of the tensor product, we obtain:

6.4 Lemma. *For every reduced Heisenberg algebra $\mathfrak{gh}(V,Z,\beta)$, there is a unique linear surjection $\hat\beta\colon V \wedge V \to Z$ such that $\hat\beta(v \wedge w) = \beta(v,w)$, for all $v,w \in V$. Moreover, the kernel of $\hat\beta$ satisfies*

$$\forall\, v \in V \setminus \{0\}\colon \eta(\{v\} \times V) \not\subseteq \ker\hat\beta\,. \qquad (*)$$

Conversely, every linear surjection $\gamma\colon V \wedge V \to Z$ satisfying condition $(*)$ yields a reduced Heisenberg algebra $\mathfrak{gh}(V,Z,\gamma\circ\eta))$. ■

6.5 Lemma. *Let $\varphi\colon \mathfrak{gh}(V,X,\beta) \to \mathfrak{gh}(W,Y,\gamma)$ be an isomorphism between reduced Heisenberg algebras. Then there are linear bijections $\sigma\colon V \to W$ and $\sigma'\colon X \to Y$ and a linear map $\tau\colon V \to Y$ such that $\varphi(v,x) = (\sigma(v), \sigma'(x)+\tau(v))$, and*

$$\forall\, u,v \in V\colon \sigma'(\beta(u,v)) = \gamma(\sigma(u),\sigma(v))\,. \qquad (**)$$

Writing $\sigma\wedge\sigma := \widehat{\psi}$, where $\psi := \eta_W \circ (\sigma\times\sigma)\colon (u,v) \mapsto \sigma(u)\wedge\sigma(v)$, we can translate the latter condition into $\sigma'\circ\hat\beta = \hat\gamma\circ(\sigma\wedge\sigma)$. In particular, the linear bijection $\sigma\wedge\sigma$ maps the kernel of $\hat\beta$ onto the kernel of $\hat\gamma$.

$$\begin{tikzcd}
V \arrow[ddd, "\sigma"'] & V\times V \arrow[rr, "\beta"] \arrow[dr, "\eta_V"] \arrow[ddr, "\psi"'] \arrow[ddd, "\sigma\times\sigma"'] & & X \arrow[ddd, "\sigma'"] \\
& & V\wedge V \arrow[ur, "\hat\beta"'] \arrow[d, dotted, "\sigma\wedge\sigma"] & \\
& & W\wedge W \arrow[dr, "\hat\gamma"] & \\
W & W\times W \arrow[ur, "\eta_W"'] \arrow[rr, "\gamma"'] & & Y
\end{tikzcd}$$

Proof. We use the fact that φ maps the commutator algebra $\{0\}\times X$ of $\mathfrak{gh}(V,X,\beta)$ onto the commutator algebra $\{0\}\times Y$ of $\mathfrak{gh}(W,Y,\gamma)$ to obtain σ, σ', and τ. The condition $\varphi([(u,x),(v,z)]_\beta) = [(\sigma(u),\sigma'(x)+\tau(u),(\sigma(v),\sigma'(z)+\tau(v))]_\gamma$ then means condition $(**)$. ■

7. Classification of Nilpotent Lie Algebras

In order to determine the reduced Heisenberg algebras $\mathfrak{gh}(V,Z,\beta)$ for a given ground field $\mathbb{K}$ and a given pair of dimensions $(\dim V, \dim Z)$, we may fix V and Z. After 6.5, it remains to study the effect of the actions of $\mathrm{GL}(V)$ on $V\wedge V$ and $\mathrm{GL}(Z)$ on Z on the set of linear surjections $\hat\beta\colon V\wedge V\to Z$ that satisfy condition $(*)$. Since we consider surjections, this reduces to the problem of determination of the orbits of $\mathrm{GL}(V)$ on the set of possible kernels: that is, on the Grassmann space of all subspaces of codimension $\dim Z$ in $V\wedge V$.

Recall that $\dim(V\wedge V)=\frac12(\dim V)(\dim V-1)$, that $\dim Z\le\dim(V\wedge V)$, and note also that $Z=\{0\}\ne V$ makes it impossible to satisfy condition $(*)$. In particular, it suffices to consider the case where $\dim V\ge 2$.

7.1 Theorem. *We consider reduced Heisenberg algebras $\mathfrak{gh}(V,Z,\beta)$ such that $\dim V=n\ge 2$.*

1. *If n is even, there is exactly one isomorphism type with $\dim Z=1$.*
2. *If n is odd, the case $\dim Z=1$ does not occur.*
3. *For any n, there is exactly one isomorphism type with $\dim Z=\frac12 n(n-1)$.*
4. *If $n>2$ then there are exactly $\lfloor\frac n2\rfloor$ types with $\dim Z=\frac12 n(n-1)-1$.*

Proof. In the case $\dim Z=1$, the map $\beta\colon V\times V\to Z\cong\mathbb{K}^1$ may be interpreted as a non-degenerate alternating form on V. The classification of alternating forms (cf. [3] Thm. 3.3 and Thm. 3.7 or [9] § 11 and § 6) gives the first assertion. For $\dim Z=\frac12 n(n-1)$, the map $\hat\beta$ is an isomorphism, and $\ker\hat\beta=\{0\}$.

Now assume $\dim Z=\frac12 n(n-1)-1$, then the kernel of $\hat\beta$ has dimension 1. According to 2.1, there are $\lfloor\frac n2\rfloor$ different $\mathrm{GL}_4\mathbb{K}$-orbits on $\mathbb{K}^n\wedge\mathbb{K}^n$, represented by the elements of the set $\{S_1^0, S_1^0+S_3^2,\dots,\sum_{m<\frac n2}S_{2m+1}^{2m}\}$. In order to verify condition $(*)$ we note that $\hat\eta(\{v\}\times V)$ has dimension $n-1$ whenever $v\ne0$, and cannot be contained in $\ker\hat\beta$ if $n>2$. ∎

7.2 Remarks. Let V be a vector space of finite dimension n with dual space V^*. Mapping $\lambda\wedge\mu$ to the linear form f defined by $f(x\wedge y)=\det\begin{pmatrix}\lambda(x)&\mu(x)\\ \lambda(y)&\mu(y)\end{pmatrix}$ extends to an isomorphism from $V^*\wedge V^*$ onto $(V\wedge V)^*$, cf.[3] [5] § 8, Thme. 1, p. 102. This isomorphism yields a non-degenerate pairing $(\cdot|\cdot)\colon(V\times V)\times(V^*\times V^*)\to\mathbb{K}$, extending $(v\wedge w\mid\lambda\wedge\mu)=f(x\wedge y)$. We let $\sigma\in\mathrm{GL}(V)$ act on the dual space V^* as $\sigma^*\colon V^*\to V^*\colon\lambda\mapsto\lambda\circ\sigma^{-1}$. Then the pairing is $\mathrm{GL}(V)$-equivariant (cf. [11] Prop. 3.1), and one may translate the orbit decomposition of $\mathrm{Gr}^\wedge_{d,n}$ to that of $\mathrm{Gr}^\wedge_{n-d,n}$. However, condition $(*)$ in 6.4 is not preserved by the pairing, cf. 7.3.

We discuss the cases where $\dim V\le 4$:

7.3 Theorem. 1. *There are no reduced Heisenberg algebras with $\dim V=1$.*

2. *If $\dim V=2$ then $\dim Z=1=\dim(V\wedge V)$, and we have exactly one isomorphism type.*

[3] The treatment in [5] is quite different from the presentation in later editions [6], [7].

3. *For* $\dim V = 3$, *there is one type with* $\dim Z = 2$, *and one with* $\dim Z = 3 = \dim(V \wedge V)$.

4. *For* $\dim V = 4$, *there is one isomorphism type with* $\dim Z = 1$, *one with* $\dim Z = 6$, *and two types with* $\dim Z = 5$ *(see 7.1). Apart from these four types, there are exactly* $3+|\mathcal{P}_1|$ *types with* $\dim Z = 4$, *exactly* $4+|\mathcal{P}_2|+|\mathcal{P}_3|$ *types with* $\dim Z = 3$, *and exactly* $2+|\mathcal{P}_1|$ *types with* $\dim Z = 2$. *Here* $\mathcal{P}_j$ *is a suitable subset of* $\mathrm{Gr}^{\wedge}_{j,4}$, *as introduced in 5.6.*

Proof. After 7.1, it only remains to discuss the cases where $\dim V = 4$ and $\dim Z \in \{2,3,4\}$; we use the classification of orbits obtained in 5.6. Because of condition $(*)$ we have to exclude those elements of

$$\{E, T, S, F, J(F), E+T, E+S, T+S, E^{\perp}, T^{\perp}, S^{\perp}\} \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_1^{\perp}$$

that contain $\eta(\{v\} \times V) = \{v\} \wedge V$ for some $v \in V \setminus \{0\}$. We note that $\eta(\{b_0\} \times V) = F$ implies that the elements that we have to exclude are those that contain an element of the $\mathrm{GL}_4\mathbb{K}$-orbit of F. Apart from F itself and $E^{\perp} = F + \langle S_2^1 \rangle_{\mathbb{K}}$, the only candidates for exclusion are the four-dimensional spaces $T^{\perp}$ and $S^{\perp}$.

Since the restriction of the polar form f_q to S is non-degenerate, the restriction to $S^{\perp}$ is non-degenerate, as well. Thus $S^{\perp}$ does not contain any element of $\mathcal{M}_3$. The restriction of f_q to $T^{\perp}$ is degenerate, the radical equals $\langle S_1^0 \rangle_{\mathbb{K}}$ if $\operatorname{char} \mathbb{K} \neq 2$, it equals $\langle S_1^0, S_3^0 - S_2^1 \rangle_{\mathbb{K}}$ if $\operatorname{char} \mathbb{K} = 2$. In the first case the induced form on the three-dimensional space $T^{\perp}/\langle S_1^0 \rangle_{\mathbb{K}}$ is non-degenerate, and $T^{\perp}$ does not contain any element of $\mathcal{M}_3$. In the case where $\operatorname{char} \mathbb{K} = 2$ every element of $\mathcal{M}_3$ inside $T^{\perp}$ would contain the radical, contradicting $q(S_3^0 - S_2^1) = -1 \neq 0$.

Thus we have shown that every element of

$$\{E, T, S\} \cup \mathcal{P}_1\,, \quad \{J(F), E+T, E+S, T+S\} \cup \mathcal{P}_2 \cup \mathcal{P}_3\,, \quad \{T^{\perp}, S^{\perp}\} \cup \mathcal{P}_1^{\perp}\,,$$

represents an isomorphism type of reduced Heisenberg algebras $\mathfrak{gh}(\mathbb{K}^4, Z, \beta)$, where $\dim Z$ equals 4, 3, and 2, respectively. ■

7.4 Remark. Using 6.2, we may interpret 7.3 as a classification of nilpotent Lie algebras of class 2, with small values for the invariant $v := \dim(L/\mathfrak{z}(L))$, but no restriction on $\dim L$.

Applying 5.8 and 7.1, we obtain:

7.5 Corollary. *We consider isomorphism types of reduced Heisenberg algebras* $\mathfrak{gh}(V, Z, \beta)$, *with* $\dim V = 4$ *and* $\dim Z = z$.

1. *In any case, there are* 2 *types with* $z = 5$, *one type with* $z = 6$, *and one type with* $z = 1$.

2. *If* $\mathbb{K}$ *is a euclidian field (e.g.* $\mathbb{K} = \mathbb{R}$*) then there are*

$$\begin{aligned} 3+1 = {} & 4 \quad \text{types with} \quad z = 4, \\ 4+1+1 = {} & 6 \quad \text{types with} \quad z = 3, \\ 2+1 = {} & 3 \quad \text{types with} \quad z = 2. \end{aligned}$$

3. *If $\mathbb{K}$ is a finite field then there are*

$$\begin{aligned} 3+1 = &\ 4 \quad \textit{types with} \quad z=4, \\ 4+0+1 = &\ 5 \quad \textit{types with} \quad z=3, \\ 2+1 = &\ 3 \quad \textit{types with} \quad z=2. \end{aligned}$$

4. *If $\mathbb{K}$ is quadratically closed (i.e., if there are no quadratic field extensions of $\mathbb{K}$ — surely this is the case if $\mathbb{K}$ is algebraically closed) then there are 3, 4, and 2 types with $z=4,3,2$, respectively.* ■

7.6 Remark. The classification in the cases discussed in 7.5 also yields explicit structure constants. However, since our description avoids any choice of structure constants it allows to determine the full group of automorphisms easily, cf. [14], [15].

7.7 Theorem. *Let L be a nilpotent Lie algebra of class 2, and abbreviate $v := \dim(L/\mathfrak{z}(L))$, $c := \dim(\mathfrak{z}(L)/L')$, $z := \dim L'$. The number of isomorphism types is as follows:*

$\dim L \le 3$:	*a single type, with*	$(v,z,c) = (2,1,0)$.
$\dim L = 4$:	*a single type, with*	$(v,z,c) = (2,1,1)$.
$\dim L = 5$:	*three types, with*	$(v,z,c) \in \{(2,1,2),(3,2,0),(4,1,0)\}$.
$\dim L = 6$:	$7+\lvert\mathcal{P}_1\rvert$ *types, with*	$(v,z,c) \in \left\{ \begin{array}{ccc} (2,1,3), & (3,2,1), & (3,3,0), \\ & (4,1,1), & (4,2,0) \end{array} \right\}$
		(where only $(4,2,0)$ *occurs more than once).*

For $\dim L = 7$*, we have* $12+\lvert\mathcal{P}_1\rvert+\lvert\mathcal{P}_2\rvert+\lvert\mathcal{P}_3\rvert$ *types with*

$$(v,z,c) \in \{(2,1,4),(3,2,2),(3,3,1),(4,1,2),(4,2,1),(4,3,0),(6,1,0)\}$$

(where only $(4,2,1)$ *and* $(4,3,0)$ *occur more than once), and an unknown number of types with* $(v,z,c)=(5,2,0)$*. See 7.11 below for a discussion of the latter case over an algebraically closed field* $\mathbb{K}$*.* ■

Note that we have excluded the abelian algebras in 7.7.

7.8 Remarks. Recall from 5.8 that $\mathcal{P}_1$ has exactly one element if $\mathbb{K}$ is finite, or a euclidian field.

For $\mathbb{K}=\mathbb{Q}$, the sets $\mathcal{P}_1$ and $\mathcal{P}_3$ become infinite: in fact, the quadratic forms q_a and $c\cdot q_b$ given by $q_a(x,y) := x^2+ay^2$ and $c\cdot q_b(x,y) := c(x^2+by^2)$ are equivalent only if there exists $(u,v)\in\mathbb{Q}^2$ and $f\in\mathbb{Q}\setminus\{0\}$ such that $1 = c\cdot q_b(u,v)$ and $a = c\cdot q_b(fbv,-fu)$. But this means $a = f^2 b$, and a,b belong to the same class modulo squares.

Even if the ground field is algebraically closed, one obtains infinitely many types of Heisenberg algebras $\mathfrak{gh}(V,Z,\beta)$ of fixed dimension, if only $v := \dim V \ge 6$ and $3 \le \dim Z \le \binom{v}{2}-3$. For $\dim V \ge 8$, there are also infinitely many types with $\dim Z = 2$. See [11] 7.8, 7.10.

7.9 Remarks. Table III in [21] claims to list the isomorphism types of those real nilpotent algebras of order six that are not direct products of smaller algebras, the text refers to [20]. The table contains three entries for algebras of class 2, denoted by $A_{6,3}$, $A_{6,4}$, and $A^a_{6,5}$, respectively. The entries $A_{6,3}$ and $A_{6,4}$ describe algebras that are isomorphic to Heisenberg algebras $\mathfrak{gh}(\mathbb{R}^3,\mathbb{R}^3,\gamma)$ and $\mathfrak{gh}(\mathbb{R}^4,\mathbb{R}^2,\beta)$, where $\hat\gamma$ is the identity, and the kernel of $\hat\beta$ is $T^\perp$, see 5.1. The last entry describes a family[4] of algebras (with $a\neq 0$), given by the commutator relations $[e_1,e_3]=e_5$, $[e_1,e_4]=e_6$, $[e_2,e_3]=ae_6$, and $[e_2,e_4]=e_5$, all the remaining commutators between basis elements being zero.

According to our result 7.7, the table should contain three single isomorphism types: one corresponding to $(v,z,c)=(3,3,0)$, and two with $(v,z,c)=(4,2,0)$. (The third algebra with $(v,z,c)=(4,2,0)$ occurring in 7.7 is isomorphic to a direct product of two algebras of type $(2,1,0)$, cf. 8.2.)

In fact, our method allows to spot the isomorphism types in the family $(A^a_{6,5}\mid a\neq 0)$ easily, as follows. First of all, we adapt the notation, writing b_{j-1} for e_j. The algebras in question are Heisenberg algebras $\mathfrak{gh}(V,Z,\beta_a)$, where $V=\langle b_0,b_1,b_2,b_3\rangle_\mathbb{R}\cong\mathbb{R}^4$, $Z=\langle b_4,b_5\rangle_\mathbb{R}\cong\mathbb{R}^2$, and the kernel N_a of $\hat\beta_a$ is generated by S^0_1, S^2_3, $S^0_2-S^1_3$, and $aS^0_3-S^1_2$. The orthogonal space $N_a^\perp$ is generated by $S^0_2+S^1_3$ and $S^0_3+aS^1_2$. With respect to these bases, the restrictions of q to N_a and $N_a^\perp$ are described by the matrices

$$\begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&-a\end{pmatrix}\ \text{and}\ \begin{pmatrix} -1&0\\ 0&a\end{pmatrix},\ \text{respectively.}$$

For $a>0$, the space $N_a^\perp$ is in the orbit of S, see 5.1. Therefore, the algebras $\mathfrak{gh}(V,Z,\beta_a)$ with $a>0$ are isomorphic to $\mathfrak{gh}(\mathbb{R}^2,\mathbb{R},\det)\times\mathfrak{gh}(\mathbb{R}^2,\mathbb{R},\det)$, see 8.2.

For $a<0$, the space $N_a^\perp$ is anisotropic, and belongs to the orbit of $P^0_{1,1}=\langle S^0_1+S^2_3,S^0_3+S^1_2\rangle_\mathbb{R}$, see 5.8. In particular, all the algebras $\mathfrak{gh}(V,Z,\beta_a)$ with $a<0$ belong to a single isomorphism type: they are all isomorphic to $\mathfrak{gh}(\mathbb{C}^2,\mathbb{C},\det)$, cf. 8.3.

In order to describe explicit isomorphisms from $\mathfrak{gh}(V,Z,\beta_a)$ onto standard representatives of the isomorphism classes, we put $t:=|a|^{-\frac12}$, and extend $b_0\mapsto b_0$, $b_1\mapsto b_1$, $b_2\mapsto tb_2$, $b_3\mapsto tb_3$, $b_4\mapsto tb_4$, $b_5\mapsto b_5$ linearly: this gives an isomorphism from $\mathfrak{gh}(V,Z,\beta_{\sigma t^{-2}})$ onto $\mathfrak{gh}(V,Z,\beta_\sigma)$, where $\sigma\in\{1,-1\}$.

An explicit isomorphism from the Lie algebra $\mathfrak{gh}(\mathbb{R}^2,\mathbb{R},\det)\times\mathfrak{gh}(\mathbb{R}^2,\mathbb{R},\det)$ onto $\mathfrak{gh}(V,Z,\beta_1)$ is given by

$$\begin{array}{lrlr}
\big(((1,0),0),((0,0),0)\big)\mapsto & b_0+b_1\ , & \big(((0,0),0),((1,0),0)\big)\mapsto & b_0-b_1\ ,\\
\big(((0,1),0),((0,0),0)\big)\mapsto & b_2+b_3\ , & \big(((0,0),0),((0,1),0)\big)\mapsto & b_2-b_3\ ,\\
\big(((0,0),1),((0,0),0)\big)\mapsto & 2(b_4+b_5), & \big(((0,0),0),((0,0),1)\big)\mapsto & 2(b_4-b_5).
\end{array}$$

7.10 Remarks. Gauger's enumeration [11] 7.19 of 4-generator, 3-relation metabelian Lie algebras (i.e., Heisenberg algebras $\mathfrak{gh}(V,Z,\beta)$ with $\dim V=4$ and

[4] In Morozov's list [20], there is a hint that a should not be a square. In fact, the remarks following the list make clear that classes modulo squares describe the isomorphism types in the family $A^a_{6,5}$.

surjective $\hat{\beta}$ such that $\dim(\ker\hat{\beta}) = 3$) over an algebraically closed field $\mathbb{K}$ counts one of the isomorphism types twice. This also entails an error in the counting of types in [11] 7.20.

In fact, Gauger gives the following list of subspaces of $\mathbb{K}^4 \wedge \mathbb{K}^4$:

$$
\begin{aligned}
I_1 &= \langle x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_3 \rangle_{\mathbb{K}}\,, & I_4 &= \langle x_1 \wedge x_2, x_3 \wedge x_4, (x_1 + x_3) \wedge (x_2 + x_4) \rangle_{\mathbb{K}}\,,\\
I_2 &= \langle x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4 \rangle_{\mathbb{K}}\,, & I_5 &= \langle x_1 \wedge x_2 + x_3 \wedge x_4, x_2 \wedge x_4, x_1 \wedge x_4 \rangle_{\mathbb{K}}\,,\\
I_3 &= \langle x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_4 \rangle_{\mathbb{K}}\,, & I_6 &= \langle x_1 \wedge x_2 + x_3 \wedge x_4, x_2 \wedge x_4, x_1 \wedge x_3 \rangle_{\mathbb{K}}\,.
\end{aligned}
$$

In [11] 7.19 it is claimed that these ideals of the universal algebra $N(2,4) \cong \mathfrak{gh}(\mathbb{K}^4, \mathbb{K}^4 \wedge \mathbb{K}^4, \eta)$ yield quotients that form a system of representatives for the isomorphism types in question.

Translating from Gauger's notation into our terminology[5], we obtain the algebras $N(2,4)/I_j = \mathfrak{gh}(\mathbb{K}^4, Z_j, \pi_j)$, where $\widehat{\pi_j}\colon \mathbb{K}^4 \wedge \mathbb{K}^4 \to Z_j := (\mathbb{K}^4 \wedge \mathbb{K}^4)/I_j$ is the canonical projection. Determining the (Gram matrices of the) restriction q_j of the quadratic form q to I_j, and using 5.6, we find:

1. The subspace I_1 is equivalent to $J(F)$.

2. The subspace I_2 is equivalent to F.

3. The restriction q_3 has a radical of dimension 1, and I_3 is equivalent to $E{+}S$.

4. The restriction q_4 is non-degenerate, and I_4 is equivalent to $T + S$.

5. The restriction q_5 has a radical of dimension 2, and I_5 is equivalent to $E{+}T$.

6. The restriction q_6 is non-degenerate, and I_6 is equivalent to $T + S$.

Thus I_4 and I_6 yield isomorphic algebras. Explicitly, an isomorphism from $\mathfrak{gh}(\mathbb{K}^4, Z_4, \pi_4)$ onto $\mathfrak{gh}(\mathbb{K}^4, Z_6, \pi_6)$ is induced by linear extension of $b_0 \mapsto b_0$, $b_1 \mapsto b_3$, $b_2 \mapsto b_1$, $b_3 \mapsto b_2$.

It appears that the error in [11] 7.15 is caused by the assumption that I_6 cannot be spanned by decomposables. However, this is wrong; one has $S_1^0 + S_3^2 = S_3^1 - S_2^0 + (b_0 - b_3) \wedge (b_1 + b_2)$.

7.11 Remark. Assume that $\mathbb{K}$ is algebraically closed, and that $\operatorname{char}\mathbb{K} \neq 2$. According to Gauger's results [11] 7.13, the orbits of $\mathrm{GL}_5\mathbb{K}$ on $\mathrm{Gr}^{\wedge}_{2,5}$ are represented by $\langle S_1^0, S_3^2\rangle_{\mathbb{K}}$, $\langle S_3^0 + S_2^1, S_3^1\rangle_{\mathbb{K}}$, $\langle S_1^0 + S_4^3, S_4^2\rangle_{\mathbb{K}}$, $\langle S_4^2, S_4^3\rangle_{\mathbb{K}}$, and $\langle S_4^0 + S_3^1, S_4^1 + S_3^2\rangle_{\mathbb{K}}$. The classification problem appears to be open for general fields.

We are interested in the orbits on $\mathrm{Gr}^{\wedge}_{8,5}$, because these subspaces occur as kernels of maps $\hat{\beta}\colon \mathbb{K}^5 \wedge \mathbb{K}^5 \to Z$ with 2-dimensional image. The pairing that we have described in 7.2 can be used to obtain a set of representatives for the orbits on $\mathrm{Gr}^{\wedge}_{8,5}$ from the representatives for the orbits on $\mathrm{Gr}^{\wedge}_{2,5}$.

However, only the subspaces $\Psi(\langle S_1^0 + S_4^3, S_4^2\rangle_{\mathbb{K}})$ and $\Psi(\langle S_4^0 + S_3^1, S_4^1 + S_3^2\rangle_{\mathbb{K}})$ lead to reduced Heisenberg algebras: we find that $\{b_4\} \wedge \mathbb{K}^5$ is annihilated both by $\Psi(\langle S_1^0, S_3^2\rangle_{\mathbb{K}})$ and by $\Psi(\langle S_3^0 + S_2^1, S_3^1\rangle_{\mathbb{K}})$, and that $\langle b_0, b_1\rangle_{\mathbb{K}} \wedge \mathbb{K}^5$ is annihilated by $\Psi(\langle S_4^2, S_4^3\rangle_{\mathbb{K}})$.

[5] We use the basis $x_j = b_{j-1}$.

8. Examples Involving Field Extensions and Quaternion Algebras

Subspaces that meet the Klein quadric in only few points (or even no points at all) present particular problems when classifying them or when determining the automorphism groups of the corresponding Heisenberg algebras. Therefore, we will now discuss connections between anisotropic subspaces, quadratic extension fields, quaternion fields, and constructions of Heisenberg algebras related to these structures.

8.1 Example. An explicit model for the (unique) isomorphism type of reduced Heisenberg algebra $\mathfrak{gh}(V,Z,\beta)$ with $\dim V=2$ and $\dim Z=1$ is $\mathfrak{gh}(\mathbb{K}^2,\mathbb{K},\det)$, where $\det(v,w)$ is the usual determinant of the 2×2 matrix with columns v,w. This algebra is *the* Heisenberg algebra used to explain the uncertainty principle.

8.2 Example. Among the reduced Heisenberg algebras $\mathfrak{gh}(V,Z,\beta)$ with $\dim V = 4$ and $\dim Z=2$, we find the direct product $\mathfrak{gh}(\mathbb{K}^2,\mathbb{K},\det)\times\mathfrak{gh}(\mathbb{K}^2,\mathbb{K},\det)$. This algebra is isomorphic to $\mathfrak{gh}(\mathbb{K}^4,\mathbb{K}^2,\beta)$, where the kernel of $\hat\beta$ is $S^\perp = \langle S_2^0,S_3^0,S_2^1,S_3^1\rangle_\mathbb{K}$, see 5.1.

8.3 Examples from quadratic extensions. Let $\mathbb{L}$ be a quadratic extension field of $\mathbb{K}$. As a $\mathbb{K}$-algebra, the Heisenberg algebra $\mathfrak{gh}(\mathbb{L}^2,\mathbb{L},\det)$ is then isomorphic to $\mathfrak{gh}(\mathbb{K}^4,\mathbb{K}^2,\beta)$, with suitable β. In order to show that the kernel of $\hat\beta$ is a member of $\mathcal{P}_1^\perp$, we pick any irreducible polynomial X^2+tX+d over $\mathbb{K}$ that has a root u in $\mathbb{L}$. Note that we may choose $t\in\{0,1\}$, and that $t=1$ is only needed if $\operatorname{char}\mathbb{K}=2$ and the extension is a separable one (cf. [25] 8.11, p. 313). We put $b_0 := \left(\begin{smallmatrix}-d\\0\end{smallmatrix}\right)$, $b_1 := \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, $b_2 := \left(\begin{smallmatrix}u\\0\end{smallmatrix}\right)$, $b_3 := \left(\begin{smallmatrix}0\\u\end{smallmatrix}\right)$ and compute the commutator relations

$$\begin{array}{rclrclrclcl} [b_0,b_1] &=& -d, & [b_0,b_2] &=& 0, & [b_0,b_3] &=& -du &=& d[b_1,b_2],\\ [b_1,b_3] &=& 0, & [b_2,b_3] &=& u^2 &=& -tu-d &=& t[b_1,b_2] + [b_0,b_1]. \end{array}$$

This yields $\ker\hat\beta = \langle S_2^0, S_3^1, S_3^0-dS_2^1, S_3^2-tS_2^1-S_1^0\rangle_\mathbb{K} = (P_{1,d}^t)^\perp$, cf. 5.8.

8.4 Examples from quaternion algebras. Assume that $\mathbb{H}=\mathbb{H}_\mathbb{K}^{u,v}$ is a quaternion algebra over a field $\mathbb{K}$ with $\operatorname{char}\mathbb{K}\ne 2$, i.e. there is a basis 1, i, j, k of $\mathbb{H}$ as a vector space over $\mathbb{K}$ and an associative and $\mathbb{K}$-bilinear multiplication such that 1 is the unit element, $u:=i^2$ and $v:=j^2$ belong to $\mathbb{K}1$, and $ij=k=-ji$ holds. We identify $\mathbb{K}$ and $\mathbb{K}1$ in the sequel.

Such a multiplication also satisfies $ik=uj=-ki$, $jk=-vi=-kj$ and $k^2=-uv$. Mapping $x=x_11+x_ii+x_jj+x_kk\in\mathbb{H}$ (with $x_1,x_i,x_j,x_k\in\mathbb{K}$) to $\bar x := x_11-x_ii-x_jj-x_kk$ defines a $\mathbb{K}$-linear involution such that $\overline{xy}=\bar y\bar x$.

For each $x\in\mathbb{H}$, one has $x\bar x=\bar x x = x_1^2-x_i^2u-x_j^2v+x_k^2uv\in\mathbb{K}$, and $N(x):=x\bar x$ defines a quadratic form N on $\mathbb{H}$. Note that $N(x)^{-1}\bar x$ is an inverse for x whenever $N(x)\ne 0$. If $(\mathbb{H},N)$ is nonsingular then $\mathbb{H}$ is a non-commutative field. (For instance, this happens if we take a subfield of $\mathbb{R}$ for $\mathbb{K}$ and use $u:=-1=:v$; the resulting algebra $\mathbb{H}_\mathbb{R}^{-1,-1}$ is the classical one introduced by Hamilton.)

Now $\beta_\mathbb{H}(x,y) := \bar x y-\bar y x$ defines an alternating map from $\mathbb{H}$ to $P := \{p\in\mathbb{H} \mid \bar p=-p\}$. We compute $\beta_\mathbb{H}(-k,1)=2k=-\beta_\mathbb{H}(i,j)$, $\beta_\mathbb{H}(-k,i)=-2uj=-u\beta_\mathbb{H}(1,j)$, and $\beta_\mathbb{H}(-k,j)=2vi=v\beta_\mathbb{H}(1,i)$. Identifying $\mathbb{H}$ and $\mathbb{K}^4$ via $b_0=-k$, $b_1=1$, $b_2=i$, $b_3=j$, we find $\ker\widehat{\beta_\mathbb{H}} = \langle S_1^0+S_3^2, S_2^0+uS_3^1, S_3^0-vS_2^1\rangle_\mathbb{K} = P_{1,-u,-v}^0$.

In particular, the classical quaternion algebra $\mathbb{H}_{\mathbb{K}}^{-1,-1}$ over $\mathbb{K} \leq \mathbb{R}$ corresponds to the standard representative $P_{1,1,1}^0$ in $\mathcal{P}_2$.

8.5 Lemma. *The multiplicative group of a quaternion algebra $\mathbb{H}$ can be used to construct many automorphisms of $\mathfrak{gh}(\mathbb{H}, P, \beta_{\mathbb{H}})$, as follows. For any pair (a, c) of invertible quaternions, let $\sigma_{a,c}(x) := ax\bar{c}$. Then $\beta_{\mathbb{H}}(ax\bar{c}, ay\bar{c}) = \overline{ax\bar{c}}ay\bar{c} - \overline{ay\bar{c}}ax\bar{c} = \bar{a}a\,c(\bar{x}y - \bar{y}x)\bar{c} = \bar{a}a\,c\beta_{\mathbb{H}}(x, y)\bar{c}$ shows that $\sigma_{a,c}$ is induced by an automorphism of $\mathfrak{gh}(\mathbb{H}, P, \beta_{\mathbb{H}})$.*

If $\mathbb{H} = \mathbb{H}_{\mathbb{R}}^{-1,-1}$ is the classical quaternion field over the field $\mathbb{R}$, these automorphisms show that $\mathrm{Aut}(\mathfrak{gh}(\mathbb{H}, P, \beta_{\mathbb{H}}))$ acts with three orbits on $\mathfrak{gh}(\mathbb{H}, P, \beta_{\mathbb{H}})$. Thus $\mathfrak{gh}(\mathbb{H}, P, \beta_{\mathbb{H}})$ is an *almost homogeneous* Heisenberg algebra (in the sense of [29], [30] and [17]) in that case. The algebra $\mathfrak{gh}(\mathbb{H}, P, \beta_{\mathbb{H}})$ is denoted $\mathrm{H}_{\mathbb{H}}^4$ in those papers.

In general, there may be more than two orbits on P under the action of the maps $\sigma_{a,c}$; in fact, the number of these orbits reflects the arithmetical structure of $\mathbb{K}$.

8.6 Proposition. *Let $a, b, c \in \mathbb{K}$ with $a \neq 0$, and let $t \in \{0, 1\}$.*

1. *The diagonal matrix* $\mathrm{diag}(1, a, 1, 1) \in \mathrm{GL}_4\mathbb{K}$ *maps $P_{a,b,c}^t$ to $P_{1,ab,ac}^t$.*

2. *If* $\mathrm{char}\,\mathbb{K} \neq 2$ *then $\mathbb{H}_{\mathbb{K}}^{-ab,-ac}$ is a quaternion algebra such that the Lie algebra $\mathfrak{gh}(\mathbb{H}_{\mathbb{K}}^{-ab,-ac}, P, \beta_{\mathbb{H}_{\mathbb{K}}^{-ab,-ac}})$ is isomorphic to $\mathfrak{gh}(\mathbb{K}^4, (\mathbb{K}^4 \wedge \mathbb{K}^4)/\ker\hat{\beta}, \beta)$, where* $\ker\hat{\beta} = P_{a,b,c}^0$.

3. *If* $\mathrm{char}\,\mathbb{K} \neq 2$ *and $P_{a,b,c}^0$ is anisotropic then $\mathbb{H}_{\mathbb{K}}^{-ab,-ac}$ is a quaternion* field.

8.7 Split quaternions in odd characteristic. If $\mathrm{char}\,\mathbb{K} \neq 2$ then an isomorphism φ from the quaternion algebra $\mathbb{H}_{\mathbb{K}}^{-1,1}$ onto the algebra $\mathbb{K}^{2\times 2}$ is obtained by linear extension of $\varphi(1) := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\varphi(i) := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $\varphi(j) := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\varphi(k) := \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$. The involution is then given by $\overline{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. In particular, we have $N(x) = \det x$, and P is the space of matrices with vanishing trace. We find $\ker\hat{\beta} = \langle S_1^0 + S_3^2, S_2^0 - S_3^1, S_3^0 - S_2^1 \rangle_{\mathbb{K}} = P_{1,1,-1}^0$, independent of the ground field. Since we assume $\mathrm{char}\,\mathbb{K} \neq 2$ here, the intersection of $\ker\hat{\beta}$ with the Klein quadric is a non-degenerate conic. Thus $\ker\hat{\beta}$ belongs to the orbit of $T + S$ under the group $\mathrm{GL}_4\mathbb{K}$.

The situation is completely different if $\mathrm{char}\,\mathbb{K} = 2$. In that case, the intersection of $P_{1,1,-1}^0$ with the Klein quadric degenerates: it becomes a (double) line, and the space belongs to the orbit of $E + T$.

8.8 Quaternions in characteristic 2. Quaternion algebras do exist in characteristic 2, but they are quite different from the case where the characteristic is different from 2, cf. [8] 5.4: we have the relations $i^2 = u \in \mathbb{K}$, $j^2 + j = v \in \mathbb{K}$, $ji = ij + i$. The involution and the norm are given as

$$\overline{x_1 1 + x_i i + x_j j + x_k k} = (x_1 + x_j)1 + x_i i + x_j j + x_k k \quad \text{and}$$

$$N(x_1 1 + x_i i + x_j j + x_k k) = x_1^2 + x_i^2 u + x_j^2 v + x_k^2 uv + x_1 x_j + x_i x_k u\,.$$

The bilinear map β becomes an alternating form, with one-dimensional image, and $\hat{\beta}$ has a kernel of dimension 5.

If we try and replace the involution by any other $\mathbb{K}$-linear involutory anti-automorphism σ of this quaternion algebra, nothing new happens: according to the Skolem–Noether Theorem (e.g., cf. [25] 8.4.2), there exists an invertible quaternion s such that $\sigma(x) = s^{-1}\bar{x}s$, and $\bar{s} = s \in \langle 1, i, ij\rangle_{\mathbb{K}}$ is required to ensure that $\sigma^2 = \mathrm{id}$ (cf. [25] 8.7.4). A straightforward (but tedious) computation yields $\beta_\sigma(x,y) := \sigma(x)y - \sigma(y)x = s^{-1}(\bar{x}sy - \bar{y}sx) \in s^{-1}\mathbb{K}$, and β_σ is an alternating form, like β.

8.9 Split quaternions. Independently of the characteristic of the ground field $\mathbb{K}$, each quaternion algebra containing zero divisors (i.e., elements $x \neq 0 = N(x)$) is isomorphic to $\mathbb{K}^{2\times 2}$ (cf. [8] 5.4 or [19] 7.6). For $\operatorname{char}\mathbb{K} \neq 2$, an isomorphism from $\mathbb{H}_{\mathbb{K}}^{-1,1}$ onto $\mathbb{K}^{2\times 2}$ has been given in 8.7. For $\operatorname{char}\mathbb{K} = 2$, the quaternion algebra satisfying $i^2 = 1$, $j^2 = j$ and $ij = ji + i$ is isomorphic to $\mathbb{K}^{2\times 2}$ via $\varphi(i) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\varphi(j) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. An obvious involution on $\mathbb{K}^{2\times 2}$ is given by transposition. According to the Skolem–Noether Theorem, every other anti-automorphism of the (centrally simple) algebra $\mathbb{K}^{2\times 2}$ is of the form $\sigma_S\colon X \mapsto S^{-1}X'S$. We obtain an involution σ_S only if S is symmetric or alternating. We consider $\beta_S\colon (X,Y) \mapsto S^{-1}X'SY - S^{-1}Y'SX = S^{-1}(X'SY - Y'SX)$.

If S is symmetric, the images under β_S are contained in the one-dimensional space spanned by $S^{-1}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and β_S describes an alternating form.

If S is alternating, we may assume $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and end up with the case discussed in 8.7. In particular, we could also obtain the results of 8.8 by extending the field of scalars such that the algebra splits.

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Markus Stroppel
Institut f. Geometrie und Topologie
Universität Stuttgart
70550 Stuttgart
Germany
stroppel@igt.uni-stuttgart.de

Received November 11, 2007
and in final form April 7, 2008