Heinz-Kato’s Inequalities for Lie Groups

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Abstract. Extensions of Heinz-Kato’s inequalities and related inequalities are obtained for semisimple connected noncompact Lie groups. {Mathematics Subject Index 2000:Primary 22E46; Secondary 15A45.
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1. Introduction

Let \( \mathbb{C}^{n \times n} \) be the set of \( n \times n \) complex matrices and let
\[
\|X\| := \max_{\|v\|=1} \|Xv\|_2
\]
denote the spectral norm of \( X \in \mathbb{C}^{n \times n} \). We have the following norm inequalities.

Theorem 1.1. 1. (Heinz-Kato [16, Theorem 3]) If \( A, B \in \mathbb{C}^{n \times n} \) are positive semi-definite and \( X \in \mathbb{C}^{n \times n} \), then
\[
\|A^tXB^t\| \leq \|X\|^{1-t}\|AXB\|^t, \quad 0 \leq t \leq 1.
\]

2. (Heinz-Kato [17]) If \( A, B \in \mathbb{C}^{n \times n} \) are positive semi-definite and \( X \in \mathbb{C}^{n \times n} \), then
\[
\|A^tXB^{1-t}\| \leq \|AX\|^t\|XB\|^{1-t}, \quad 0 \leq t \leq 1,
\]

3. (McIntosh [23], Bhatia-Davis [4]) For \( A, B, X \in \mathbb{C}^{n \times n} \),
\[
\|A^*XB\| \leq \|AA^*X\|^{1/2}\|XBB^*\|^{1/2}.
\]

See [1, 9, 10, 15, 16, 17, 18, 25] for the inequalities and related inequalities. By continuity it suffices to consider the general linear group \( GL_n(\mathbb{C}) \) instead of \( \mathbb{C}^{n \times n} \). Moreover by the homogeneous property of \( \| \cdot \| \) we can restrict ourselves to \( SL_n(\mathbb{C}) \).

We will obtain some extensions of the above inequalities and other related inequalities in the context of semisimple connected noncompact Lie groups.

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2. Log majorization

If we order the singular values of $X \in \mathbb{C}_{n \times n}$ in descending order

$$s_1(X) \geq \cdots \geq s_n(X),$$

then (2), for example, can be written as

$$s_1(A^tXB^{1-t}) \leq s_1^t(AX)s_1^{1-t}(XB), \quad 0 \leq t \leq 1. \quad (4)$$

Let $\mathbb{R}^n_+$ denote the set of positive $n$-tuples and let $a,b \in \mathbb{R}^n_+$. Then $a$ is said to be log majorized by $b$, denoted by $a <_{\log} b$ if

$$\max_{\sigma \in S_n} \prod_{i=1}^k a_{\sigma(i)} \leq \max_{\sigma \in S_n} \prod_{i=1}^k b_{\sigma(i)}, \quad k = 1, \ldots, n-1,$n

$$\prod_{i=1}^n a_i = \prod_{i=1}^n b_i,$$

where $S_n$ denotes the symmetric group on $\{1, \ldots, n\}$. Write

$$s(X) := (s_1(X), \ldots, s_n(X)), \quad s^t(X) := (s_1^t(X), \ldots, s_n^t(X)), \quad t \geq 0.$$

Suppose $1 \leq k \leq n$. The $k$th compound of $X \in \mathbb{C}_{n \times n}$ is defined to be the \((n) \times (n)\) complex matrix $C_k(X)$ \([22]\) whose elements are defined by

$$C_k(X)_{\alpha,\beta} = \det X[\alpha|\beta],$$

where $\alpha, \beta \in Q_{k,n}$ and $Q_{k,n} = \{\omega = (\omega(1), \ldots, \omega(k)) : 1 \leq \omega(1) < \cdots < \omega(k) \leq n\}$ is the set of increasing sequences of length $k$ chosen from $1, \ldots, n$. For example, if $n = 3$ and $k = 2$, then

$$C_2(X) = \begin{pmatrix} 
\det X[1,2|1,2] & \det X[1,2|1,3] & \det X[1,2|2,3] \\
\det X[1,3|1,2] & \det X[1,3|1,3] & \det X[1,3|2,3] \\
\det X[2,3|1,2] & \det X[2,3|1,3] & \det X[2,3|2,3]
\end{pmatrix}.$$ In general $C_1(X) = X$ and $C_n(X) = \det X$. Compound matrix has very nice properties: (i) $C_k(AB) = C_k(A)C_k(B)$ for all $A, B \in \mathbb{C}_{n \times n}$, (ii) the eigenvalues of $C_k(X)$ are $\prod_{j=1}^k \lambda_{\omega(j)}(X)$, $\omega \in Q_{k,n}$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $X$, (iii) the singular values of $C_k(X)$ are $\prod_{j=1}^k s_{\omega(j)}(X)$, $\omega \in Q_{k,n}$. The compound matrix $C_k(X)$ is indeed the matrix representation (with respect to some induced basis) of the induced operator $C_k(T)$ on the exterior space $\wedge^k \mathbb{C}^n$ where $T : \mathbb{C}^n \to \mathbb{C}^n$ is an operator: $C_k(T)v_1 \wedge \cdots \wedge v_k = Tv_1 \wedge \cdots \wedge Tv_k$, $v_1, \ldots, v_k \in \mathbb{C}^n$.

The following is an extension of Theorem 1.1. We will prove the second inequality and the rest are similar.

**Theorem 2.1.** 1. If $A, B \in \mathbb{C}_{n \times n}$ are positive semi-definite and $X \in \mathbb{C}_{n \times n}$ and $0 \leq t \leq 1$, then

$$s(A^tXB^t) <_{\log} s^{1-t}(X)s^t(AXB).$$
2. If \( A, B \in \mathbb{C}_{n \times n} \) are positive semi-definite and \( X \in \mathbb{C}_{n \times n} \) and \( 0 \leq t \leq 1 \), then 
\[
s(A^tXB^{1-t}) \preceq_{\log} s^t(AX)s^{1-t}(XB).
\]

3. For \( A, B, X \in \mathbb{C}_{n \times n} \),
\[
s(A^*XB) \preceq_{\log} s^{1/2}(AA^*X)s^{1/2}(XBB^*).
\]

**Proof.**  Let \( C_k(X) \in \mathbb{C}_{(n_k) \times (n_k)} \) denote the \( k \)th compound of \( X \), \( k = 1, \ldots, n \).
Notice that \( s_1(C_k(X)) = \prod_{i=1}^{k} s_i(X), \ C_k(XY) = C_k(X)C_k(Y), \ X, Y \in \mathbb{C}_{n \times n} \), and \( C_k(X) \) is positive semi-definite if \( X \) is positive semi-definite. So
\[
\prod_{i=1}^{k} s_i(A^tXB^{1-t}) = s_1(C_k(A^tXB^{1-t})) \\
= s_1(C_k(A)^tC_k(X)C_k(B)^{1-t}) \\
\leq s_1(C_k(AX))s_1^{1-t}(C_k(XB)) \quad \text{by (4)} \\
= \prod_{i=1}^{k} s_i^{1-t}(AX) \prod_{i=1}^{k} s_i^{1-t}(XB).
\]

When \( k = n \),
\[
\prod_{i=1}^{n} s_i(A^tXB^{1-t}) = |\det(A^tXB^{1-t})| = (\det A)^t |\det X| (\det B)^{1-t}
\]
and
\[
\prod_{i=1}^{n} s_i^{1-t}(AX) \prod_{i=1}^{n} s_i^{1-t}(XB) = |\det(AX)|^t |\det(XB)|^{1-t} \\
= (\det A)^t |\det X| (\det B)^{1-t}.
\]

3. **A pre-order of Kostant**

We now take a close look of Theorem 2.1 for \( X \in \mathrm{GL}_n(\mathbb{C}) \). Let \( A_+ \subset \mathrm{GL}_n(\mathbb{C}) \) denote the set of all positive diagonal matrices with diagonal entries in descending order. Recall that the singular value decomposition of \( X \in \mathrm{GL}_n(\mathbb{C}) \) asserts that there exist unitary matrices \( U, V \) such that
\[
X = Ua_+(X)V, \tag{5}
\]
where \( a_+(X) = \text{diag}(s_1(X), \ldots, s_n(X)) \in A_+ \). Though \( U \) and \( V \) in the decomposition (5) are not unique, \( a_+(X) \in A_+ \) is uniquely defined.

Let \( G \) be a semisimple noncompact connected Lie group having \( \mathfrak{g} \) as its Lie algebra. Let \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) be a fixed Cartan decomposition of \( \mathfrak{g} \). Let \( K \subset G \) be the analytic subgroup with Lie algebra \( \mathfrak{t} \). Then \( \text{Ad}K \) is a maximal compact subgroup of \( \text{Ad}G \). Let \( \mathfrak{a} \subset \mathfrak{p} \) be a maximal abelian subspace. The exponential map \( \text{exp} : \mathfrak{a} \to A \) is bijective.
Set
\[ P := \exp \mathfrak{p}. \]
The map \( K \times P \to G, (k, p) \mapsto kp \) is a diffeomorphism. In particular
\[ G = KP \]
and every element \( g \in G \) can be uniquely written as
\[ g = kp, \quad k \in K, \ p \in P. \tag{6} \]
The map \( \Theta : G \to G \)
\[ \Theta(kp) = kp^{-1}, \quad k \in K, \ p \in P, \]
is an automorphism of \( G \) \cite[p.387]{19}. The map \( * : G \to G \) defined by
\[ g^* := \Theta(g^{-1}) = pk^{-1}, \quad g \in G \]
is clearly a diffeomorphism. Let \( W \) be the Weyl group of \( (a, g) \) which may be defined as the quotient of the normalizer of \( A \) in \( K \) modulo the centralizer of \( A \) in \( K \). The Weyl group operates naturally in \( A \) and \( A \) and the isomorphism \( \exp : a \to A \) is a \( W \)-isomorphism.

Fix a closed Weyl chamber \( a_+ \) in \( a \) and set \( A_+ := \exp a_+ \). We have \cite{11} the Cartan decomposition
\[ G = KA_+K. \]
Though \( k_1, k_2 \in K \) are not unique in \( g = k_1ak_2 \ (g \in G, \ k_1, k_2 \in K, \ a \in A_+) \), the element \( a = a_+(g) \in A_+ \) is unique.

**Proposition 3.1.** The following maps are continuous.

1. \( a_+^* : \mathfrak{p} \to a_+ \) where for each \( X \in \mathfrak{p}, \ a_+^*(X) \) is the unique element in \( a_+ \) such that \( a_+^*(X) = \text{Ad}(s)X \) for some \( s \in K \). Indeed it is a contraction.

2. \( a_+ : G \to A_+ \) where \( a_+(g) \in A_+ \) is the unique element in \( g = k_1a_+(g)k_2 \in G, \) where \( k_1, k_2 \in K \).

**Proof.** (1) By Berezin-Gelfand’s result \cite{2}, \( a_+^*(X + Y) \in a_+^*(X) + \text{conv} Wa_+^*(Y) \) for any \( X, Y \in \mathfrak{p} \). So \( a_+^*(X) = a_+^*(Y + (X-Y)) \in a_+^*(Y) + \text{conv} W(a_+^*(X-Y)) \). Hence \( a_+^*(X) - a_+^*(Y) \in \text{conv} Wa_+^*(X-Y) \). Also see \cite[Corollary 3.10]{13}. Let \( \| : \| \) be the norm on \( \mathfrak{p} \) induced by the Killing form of \( g \). Since \( \| : \| \) is \( K \)-invariant and strictly convex, \( \| a_+^*(X) - a_+^*(Y) \| \leq \| a_+^*(X-Y) \| = \| X-Y \| \). So the map \( a_+^* \) is a contraction and thus continuous.

(2) Since the map \( G \to P \) such that \( g = kp \mapsto p \) is differentiable and \( a_+(g) = a_+(p) \), it suffices to establish the continuity of \( a_+: P \to A_+ \). The map \( \exp : \mathfrak{p} \to P \) is a surjective diffeomorphism and the inverse \( \log : P \to \mathfrak{p} \) is well defined. So \( a_+ = \exp \circ a_+^* \circ \log \) on \( P \) is continuous. \( \blacksquare \)

Define a pre-order \( \prec \) in \( A \). Given \( a, b \in A, \ a \prec b \) means
\[ \exp(\text{conv} W(\log a)) \subset \exp(\text{conv} W(\log b)). \]
The set \( \exp(\text{conv} W(\log a)) \) is multiplicatively the convex hull of the compact convex set having the Weyl group orbit \( W(\log a) \) as its extreme points.
Example 3.2. Let $G = \text{SL}(n, \mathbb{C})$. We pick
\[ t = \text{su}(n), \]
\[ K = \text{SU}(n), \]
\[ P = \text{isu}(n), \text{ i.e., the set of Hermitian matrices of zero trace} \]
\[ A = \{ \text{diag} (a_1, \ldots, a_n) : a_1, \ldots, a_n > 0, \prod_{i=1}^{n} a_i = 1 \}, \]
\[ A_+ = \{ \text{diag} (a_1, \ldots, a_n) : a_1 \geq \cdots \geq a_n > 0, \prod_{i=1}^{n} a_i = 1 \}. \]

Let $a = \text{diag} (a_1, \ldots, a_n), b = \text{diag} (b_1, \ldots, b_n) \in A$. Since the Weyl group is the symmetric group $S_n$ on $\{1, \ldots, n\}$, $\text{conv} W(\log a) = \text{conv} S_n(\log a)$. So $a \prec b$ amounts to $\log a \in \text{conv} S_n(\log b)$ and by Hardy-Littlewood-Poyla’s theorem, $a \prec b$ is equivalent to the log majorization inequalities
\[ \prod_{i=1}^{k} a_{[i]} \leq \prod_{i=1}^{k} b_{[i]}, \quad k = 1, \ldots, n-1, \]
\[ \prod_{i=1}^{n} a_{[i]} = \prod_{i=1}^{n} b_{[i]}, \]
where $a_{[1]} \geq \cdots \geq a_{[n]}$ denote the rearranged $a_1, \ldots, a_n$ in descending order.

The following nice result of Kostant describes the pre-order $\prec$ in $A$ via the representations of $G$. We remark that Kostant’s pre-order [21, p.426] is more general and is defined in $G$ via the complete multiplicative Jordan decomposition and hyperbolic elements (see Section 5 and [21]).

Theorem 3.3. (Kostant [21, Theorem 3.1]) Let $f, g \in A$. Then $f \prec g$ if and only if $|\pi(f)| \leq |\pi(g)|$ for all finite dimensional representations $\pi$ of $G$, where $| \cdot |$ denotes the spectral radius.

One may derive the log majorization in Example 3.2 via Theorem 3.3 and the fundamental representations on the exterior space $\wedge^k \mathbb{C}^n$, $k = 1, \ldots, n-1$, since $|\pi_k(a)| = a_1 \cdots a_k$.

4. Extension of the inequalities

Lemma 4.1. Let $g \in G$. Then $a_+(g) = a_+(g^*) = a_+^{1/2}(gg^*) = a_+^{1/2}(g^*g)$.

Proof. Let $g = kp$ be the Cartan decomposition of $g \in G$, $k \in K$, $p \in P$. Notice that $g^* = pk^{-1}$ so that $a_+(g) = a_+(p) = a_+(g^*)$. Now $g^*g = p^2$ and $gg^* = kp^2k^{-1}$. So $a_+^{1/2}(gg^*) = a_+^{1/2}(g^*g) = a_+^{1/2}(p^2)$. Since each element in $P$ is $K$-conjugate to some element in $A_+$ [19, p.320]. Thus $a_+^{1/2}(p^2) = a_+(p) = a_+(g)$. 

$\blacksquare$
Lemma 4.2. Let $h_1, h_2 \in A_+$. For any finite dimensional representation $\pi : G \rightarrow \text{GL}(V)$, $|\pi(h_1 h_2)| = |\pi(h_1)||\pi(h_2)|$, where $| \cdot |$ denotes the spectral radius.

Proof. Since the spectral radius of an operator is invariant under similarity, by the complete reducibility [5, p.50], [14, p.28] of $\pi$, we may assume that $\pi$ is irreducible. We also use the same notation $d\pi$ to denote the irreducible representation of the complexification $g_C := g \oplus i\mathfrak{p}$ (direct sum) of $g$, induced by $d\pi : g \rightarrow g(V)$, i.e., $d\pi(X + iY) = d\pi(X) + i d\pi(Y)$, $X, Y \in g$.

Let $g = \mathfrak{t} + \mathfrak{p}$ be the Cartan decomposition of $g$. Since $u := \mathfrak{t} + i\mathfrak{p}$ is a compact real form of $g_C$, there is an inner product (unique up to scalar multiple) on $V$ [6, p.217] such that $d\pi(u)$ are skew Hermitian. So $d\pi(t)$ are skew Hermitian and $d\pi(p)$ are Hermitian. Thus the elements of

$$\pi(P) = \pi(\exp p) = \exp d\pi(p)$$

[11, p.110] are positive definite operators. Since $A \subset P$ and is abelian, $\pi(A)$ is an abelian subgroup of positive definite operators. Thus the elements of $\pi(A)$ are positive diagonal operators under an appropriate orthonormal basis (once fixed and for all) of $V$. For each $H \in \mathfrak{a}$, $\exp d\pi(H) = \pi(\exp H) \in \pi(A)$ so that $d\pi(H)$ are real diagonal operators. Let $H_1, H_2 \in \mathfrak{a}_+$ such that $h_1 = \exp H_1, h_2 = \exp H_2 \in A_+$. Then

$$\pi(h_1)\pi(h_2) = \exp d\pi(H_1) \exp d\pi(H_2) = \exp d\pi(H_1 + H_2)$$

since $\mathfrak{a}$ is abelian and $d\pi$ respects the bracket. Notice that $|\pi(h_1)\pi(h_2)|$ is the exponent of the largest diagonal entry of the diagonal operator $d\pi(H_1) + d\pi(H_2)$. To arrive at $|\pi(h_1 h_2)| = |\pi(h_1)||\pi(h_2)|$, it is sufficient to show that the sum of the largest diagonal entries $d\pi(H_1)$ and $d\pi(H_2)$ is also a diagonal entry of $d\pi(H_1 + H_2)$. To this end, we will use the theory of highest weights [14, p.108] on the finite dimensional irreducible representations of the complex semisimple Lie algebra $g_C$ (since $g$ is semisimple).

Let

$$g = (\mathfrak{a} \oplus \mathfrak{m}) \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

be the restricted root decomposition of $g$ [11, p.263], where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{t}$ and $\Sigma$ is the set of restricted roots of $(\mathfrak{g}, \mathfrak{a})$. Let $\mathfrak{k}$ be the maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{k} \cap \mathfrak{p}$ and we set $\mathfrak{k}_R := \mathfrak{k} \cap \mathfrak{t}$. It is known that $\mathfrak{k}_C := \mathfrak{k} \oplus i\mathfrak{k}$, the complexification of $\mathfrak{k}$, is a Cartan subalgebra of the complex semisimple Lie algebra $g_C$ [11, p.259]. Let $\Delta$ be the set of roots of $(g_C, \mathfrak{k}_C)$ and set $\mathfrak{k}_R := \sum_{\alpha \in \Sigma} \mathbb{R}H_\alpha$, where $H_\alpha \in \mathfrak{k}_C$ is defined by the restriction to $\mathfrak{k}_C$ of the Killing form, i.e., $B(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{k}_C$. Then $\mathfrak{k}_C = \mathfrak{k}_R \oplus i\mathfrak{k}_R$ and $\mathfrak{k}_R = \mathfrak{a} \oplus i\mathfrak{k}_R$. Each root $\alpha \in \Delta$ is real-valued on $\mathfrak{k}_R$ [11, p.170]. Let $\Delta_p \subset \Delta$ be the set of roots which do not vanish identically on $\mathfrak{a}$. It is known that $\Sigma$ is the set of restrictions of $\Delta_p$ to $\mathfrak{a}$ [11, p.263]. Furthermore we can choose a positive root system $\Delta^+ \subset \Delta$ so that $\mathfrak{a}_+$ is in the corresponding Weyl chamber (in $\mathfrak{k}_R$) [21, p.431], that is, $\alpha(H) \geq 0$ for all $H \in \mathfrak{a}_+, \alpha \in \Delta^+$. So any root of $\Delta^+$ restricts to either zero or an element in $\Sigma^+$ as a linear functional on $\mathfrak{a}$ [11, p.263].

The diagonal entries of the diagonal operator $d\pi(H), H \in a_+ \subset \mathfrak{k}_C$ are the eigenvalues of $d\pi(H)$ so that they are of the form $\mu(H)$, where $\mu$ are the weights.
of the representation \( d\pi \) of \( g_{\mathbb{C}} \) [14, p.107-108]. Let \( \lambda \in \mathfrak{h}' \) be the highest weight of \( d\pi \), where \( \mathfrak{h}' \) denotes the dual space of \( \mathfrak{h} \). From the theory of representation \( \lambda - \mu \) is a sum of positive roots, i.e.,

\[
\lambda - \mu = \sum_{\alpha \in \Delta^+} k_\alpha \alpha, \quad k_\alpha \in \mathbb{N}.
\]

Since the restrictions of the positive roots in \( \Delta^+ \) to \( a \) are either zero or elements in \( \Sigma^+ \), we conclude \( \lambda(H) \geq \mu(H) \) for all \( H \in a_+ \). Since \( a_+ \) is a cone, \( H_1 + H_2 \in a_+ \).

Thus \( \lambda(H_1 + H_2) = \lambda(H_1) + \lambda(H_2) \) is the largest diagonal entry (eigenvalue) of the diagonal operator \( d\pi(H_1 + H_2) \) and \( \lambda(H_1) \), and \( \lambda(H_2) \) are the largest diagonal entries (eigenvalues) of \( d\pi(H_1) \) and \( d\pi(H_2) \), respectively. ■

The following theorem is an extension of Theorem 2.1.

**Theorem 4.3.** The following are equivalent and are valid.

\[
a_+(a^t gb^{1-t}) \prec [a_+(ag)]^t[a_+(gb)]^{1-t}, \quad 0 \leq t \leq 1, \quad a, b \in P, \quad g \in G,
\]

\[
a_+(a^t gb^t) \prec [a_+(ag)]^t[a_+(gb)]^{1-t}, \quad 0 \leq t \leq 1, \quad a, b \in P, \quad g \in G,
\]

\[
a_+(a^* gb) \prec [a_+(aa^* g)]^{1/2} [a_+(gbb^*)]^{1/2}, \quad a, b, g \in G.
\]

**Proof.** We will first establish (7) and then the equivalence among the relations. Let \( g \in G \) and write \( g = k_1 a_+(g) k_2 \), where \( a_+(g) \in A_+ \), \( k_1, k_2 \in K \). Let \( \pi \) be any representation of \( G \). Since the elements of \( d\pi(t) \) are skew Hermitian,

\[
\|\pi(g)\| = \|\pi(k_1 a_+(g) k_2)\| = \|\pi(k_1) \pi(a_+(g)) \pi(k_2)\| = \|\pi(a_+(g))\|.
\]

Since the spectral norm \( \|\cdot\| \) is invariant under unitary equivalence, and \( \|X\| = |X| \) for each positive definite operator \( X \), \( \|\pi(a_+(g))\| = |\pi(a_+(g))| \) and thus

\[
\|\pi(g)\| = \|\pi(a_+(g))\| = |\pi(a_+(g))|.
\]

Suppose \( 0 \leq t \leq 1 \). Since the elements of \( d\pi(p) \) are Hermitian operators, \( \pi(a) \) and \( \pi(b) \) are positive definite operators,

\[
|\pi(a_+(a^t gb^{1-t}))| = \|\pi(a^t gb^{1-t})\| \quad \text{by (10)}
\]

\[
\leq |\pi(a)\pi(g)|^{1-t} \|\pi(g)\|^{1-t} \quad \text{by (2)}
\]

\[
= |\pi(a_+(ag))|^{1-t} \|\pi(a_+(gb))|^{1-t} \quad \text{by (10)}
\]

The elements of \( \pi(A) \) are positive diagonal operators under an appropriate orthonormal basis. Since \( a_+^t(ag), a_+^{1-t}(gb) \in A_+ \), \( |\pi(a_+(ag))|^t = |\pi(a_+^t(ag))| \) and \( |\pi(a_+(gb))|^{1-t} = |\pi(a_+^{1-t}(gb))| \). So

\[
|\pi(a_+(ag))|^t |\pi(a_+(gb))|^{1-t} = |\pi(a_+^t(ag))| |\pi(a_+^{1-t}(gb))| = |\pi(a_+^t(ag))\pi(a_+^{1-t}(gb))| \quad \text{by Lemma 4.2}
\]

\[
= |\pi(a_+(ag)a_+^{1-t}(gb))|.
\]
As a result, \(|\pi(a_+(a^tgb^{1-t}))| \leq |\pi(a_+(ag)a_+^{1-t}(gb))|\) for any representation \(\pi\) of \(G\).

By Theorem 3.3 we have (7).

(7) \(\Rightarrow\) (8): If \(0 \leq t \leq 1\), then \(0 \leq 1 - t \leq 1\). If \(a, b \in P\), so are their inverses. From (7)

\[
a_+(a^tgb^t) = a_+((a^{-1})^{1-t}agb^{1-(1-t)})
\]

\[
< a_+^{1-t}(a^{-1}ag)a_+(agb)
\]

\[
= a_+^{1-t}(g)a_+(agb),
\]

i.e., (8) is established.

(8) \(\Rightarrow\) (9): Let \(a, b \in G\). Write \(a^* = kp, b^* = k'p'\) according to their Cartan decompositions. Then \(b = p'k'^{-1}\). By (8) with \(t = 1/2\),

\[
a_+(a^*gb) = a_+(kp)p(k'^{-1})
\]

\[
= a_+(p)p
\]

\[
= a_+((p^{-2})^{1/2}(p^2g)(p'^2)^{1/2})
\]

\[
< a_+^{1/2}(p^2g)a_+^{1/2}(p^{-2}p'^2g'^2)
\]

\[
= a_+^{1/2}(p^2g)a_+^{1/2}(gp'^2).
\]

Since \(aa^* = p^2\) and \(bb^* = p^2\), (9) follows.

(9) \(\Rightarrow\) (7): Let \(a, b \in P\). For \(t = 0, 1\), (7) is trivial and for \(t = 1/2\), it follows from (9). We will prove by induction for all \(t = \frac{k}{2^n}\), where \(k = 0, 1, \ldots, 2^n\) [3]. Let \(t = \frac{2k+1}{2^n}\). Then \(t = s + \rho\), where \(s = \frac{k}{2^n}\) and \(\rho = \frac{1}{2^n}\). Suppose that (7) is valid for all dyadic rationals with denominator \(2^{n-1}\). Then by induction and (9), with \(\lambda := s + 2\rho\), we have

\[
a_+(a^tgb^{1-t}) = a_+(a^s(a^sgb^1\lambda)b^\rho)
\]

\[
< a_+^{1/2}(a^{2\rho}a^sgb^1\lambda)a_+^{1/2}(a^sgb^1\lambda b^\rho)
\]

\[
= a_+^{1/2}(a^\lambda gb^1\lambda)a_+^{1/2}(a^sgb^1\lambda)
\]

\[
< a_+^{1/2}(ag)a_+^{1/2}(ag)a_+^{1/2}(ag)a_+^{1/(s-2)}(gb)
\]

\[
= a_+^{(\lambda+s)^{1/2}}a_+^{1/(\lambda+s)^{1/2}}(gb)
\]

\[
= a_+^{t}(ag)a_+^{1-t}(gb).
\]

The general case follows from continuity of the spectral radius and Theorem 3.3.

Furuta’s inequality [8] asserts that if \(A, B \in \mathbb{C}_{n \times n}\) are positive semi-definite, then

\[
\|A^tB^s\| \leq \|AB\|^t, \quad 0 \leq t \leq 1.
\]

(11)

It is equivalent to say that

\[
\|A^sB^s\| \geq \|AB\|^s, \quad s \geq 1.
\]

(12)

See [1, 7, 24]. We have the following extension of Furuta’s inequality.
Corollary 4.4. Let $a, b \in P$. Then

1. $a_+(a^t b^t) \prec a_+(ab), \quad 0 \leq t \leq 1.$

2. $a_+(ab) \prec a_+(a^t b^t), \quad t \geq 1.$

Hence $\varphi(t) = [a_+(a^{1/t} b^{1/t})]^t$ is a decreasing function on $t > 0$ with respect to the partial order $\prec$, i.e., $\varphi(s) \prec \varphi(t)$ if $s \geq t > 0$.

Proof. By setting $g$ to be the identity in (8) we have $a_+(a^t b^t) \prec a_+(ab), \quad 0 \leq t \leq 1.$ When $t \geq 1$, $a_+(a^{1/t} b^{1/t}) \prec a_+(ab).$ Then replace $a, b$ by $a^t$ and $b^t$ respectively to have $a_+(ab) \prec a_+(a^t b^t).$ Let $s \geq t > 0.$ Then $s/t > 1$ and

$$a_+(a^s b^s) = a_+((a^s)^{s/t} (b^s)^{s/t}) = a_+(a^t b^t)$$

so that $\varphi(t)$ is decreasing on $t > 0.$

Corollary 4.5. For $f, g \in G$, $a_+(fg) \prec a_+(f)a_+(g)$.

Proof. By (9),

$$a_+(fg) \prec a_+^{1/2}(f^*f)a_+^{1/2}(gg^*).$$

Use Lemma 4.1 to obtain $a_+(fg) \prec a_+(f)a_+(g)$.

Remark 4.6. When $G = \text{GL}_n(\mathbb{C})$, by Corollary 4.5 the singular values of a product $AB$ is log majorized by the product of the singular values of $A, B \in \mathbb{C}_{n \times n}$, assuming that singular values are all arranged in descending order.

Nakamoto [9] showed that (12) holds for normal matrices $A, B \in \mathbb{C}_{n \times n}$ and natural numbers $s$. An element $g \in G$ is said to be normal if $gg^* = g^*g$. It is equivalent to say that $kp = pk$, where $g = kp$ is the Cartan decomposition of $g$. Since Cartan decomposition is unique up to conjugation [11, p.183], normality is independent of the choice of $K$ and $P$. Clearly the elements of $P$ are normal. Normality is reduced to the usual normality when $G = \text{SL}_n(\mathbb{C})$. Now we extend Nakamoto’s result.

Corollary 4.7. Let $f, g \in G$ be normal. Then $a_+^n(fg) \prec a_+(f^ng^n), n \in \mathbb{N}$.

Proof. Let $f = kp, \quad g = k'p'$ be the Cartan decompositions of $f, g \in G = KP$. Since $f, g$ are normal, we have $kp = pk$ and $k'p' = p'k'$. Then

$$a_+^n(fg) = a_+^n(kpk'p') = a_+^n(kpp'k') = a_+^n(pp').$$

By Corollary 4.4,

$$a_+^n(pp') \prec a_+(p^np^{nm}) = a_+(k^n p^n p^{nm} k^m) = a_+(k^n (k'p')^n) = a_+(f^ng^n).$$
5. Inequalities for hyperbolic components

Furuta’s inequality (11) is equivalent to the following inequality: for any positive
semi-definite $A, B \in \mathbb{C}^{n \times n},$
\[
\lambda_1(A^t B^t) \leq \lambda_1^t(AB), \quad 0 \leq t \leq 1.
\] (13)

One can deduce the equivalence by
\[
|AB| = \lambda_1(AB),
\] (14)
where $\lambda_1(AB)$ is the largest eigenvalue of the matrix $AB$ whose eigenvalues are
those of the positive semi-definite $B^{1/2}AB^{1/2}.$

Inequality (13) concerns about the largest eigenvalues of $AB$ and $A^t B^t$
when $A, B \in \mathbb{C}^{n \times n}$ are positive semi-definite. Since the eigenvalues of $AB$ are
nonnegative, (13) can be viewed as a result on the largest eigenvalue modulus.

So we will consider the hyperbolic component of $g \in G$ of a semisimple connected
noncompact Lie group $G$ for an appropriate extension.

An element $X \in \mathfrak{g}$ is called real semisimple if $\text{ad} \, (X)$ is diagonalizable over $\mathbb{R}$. It is equivalent to say that $\text{ad} \, (X)$ is diagonalizable over $\mathbb{C}$ and the eigenvalues of $\text{ad} \, (X)$ are real. An element $X \in \mathfrak{g}$ is called nilpotent if $\text{ad} \, (X)$ is nilpotent. An element $g \in G$ is called hyperbolic if $g = \exp (X)$ where $X \in \mathfrak{g}$ is
real semisimple and is called unipotent if $g = \exp (X)$ where $X \in \mathfrak{g}$ is nilpotent.
An element $g \in G$ is elliptic if $\text{Ad}(g) \in \text{Aut} (\mathfrak{g})$ is diagonalizable over $\mathbb{C}$ with
eigenvalues of modulus 1. The complete multiplicative Jordan decomposition \cite[Proposition 2.1]{21} for $G$ asserts that each $g \in G$ can be uniquely written as $g = ehu,$
where $e$ is elliptic, $h$ is hyperbolic and $u$ is unipotent and the three elements $e,$ $h,$ $u$ commute. We write
\[
g = e(g)h(g)u(g).
\]

It turns out that $h \in G$ is hyperbolic if and only if it is conjugate to a
unique element $b(h) \in A_+$ \cite[Proposition 2.4]{21}. Denote
\[
b(g) := b(h(g)).
\]
It is known that \cite[Proposition 6.2]{21} $P^2$ is the set of all hyperbolic elements and
$b(g) < a_+(g)$ for all $g \in G$ \cite[Theorem 5.4]{21}.

Example 5.1. When $G = \text{SL}_n(\mathbb{C}), g = ehu$ is the usual complete multiplicative Jordan decomposition \cite[p.431]{11} and
\[
b(g) = \text{diag} (|\lambda_1|, \ldots, |\lambda_n|),
\]
where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $g$ with descending moduli.

Lemma 5.2.

1. Let $f, g \in G.$ Then $h(fg) = g^{-1} h(gf) g$ and $b(fg) = b(gf)$.

2. If $f \in P,$ then $b(f) = a_+(f)$. So $a_+(g^* g) = a_+(gg^*) = b(g^* g) = b(gg^*)$ for
   all $g \in G$.

3. Let $f, g \in P.$ Then $b(f^2 g^2) = a_+^2 (fg) = a_+^2 (gf)$.
Proof. (1) Let $fg = ehu$ be the CMJD of $fg$ where $f, g \in G$. Then $gf = g(fg)g^{-1} = (geg^{-1})(ghg^{-1})(gug^{-1})$. By the uniqueness of CMJD, $h(fg) = g^{-1}h(gf)g$ follows immediately. Now $b(fg) = b(h(fg)) = b(g^{-1}h(gf)g) = b(gf)$.

(2) Since $P$ is $K$-conjugate to some element in $A_+$, $b(f) = a_+(f)$.

(3) By (1) $h(f^2g^2) = g^{-1}h(gf^2g)g = b(gf)(gf)^*$.

The element $(gf)(gf)^*$ is in $P$ so that by (2) and Lemma 4.1

$$b(f^2g^2) = b((gf)(gf)^*) = a_+(gf)(gf)^* = a_+^2(fg).$$

Theorem 5.3. Let $a, b \in P$. The following are equivalent and are valid.

1. $a_+(a^t b^t) \prec a_+(ab)$, $0 \leq t \leq 1$.

2. $b(f^tg^t) \prec b(fg)$, $0 \leq t \leq 1$.

In other words, for any hyperbolic element $\ell \in G$, if we write $\ell = fg$, where $f, g \in P$, then $b(f^tg^t) \prec b^t(\ell)$, $0 \leq t \leq 1$.

Proof. Statement (1) is Corollary 4.4 (1). The set of all hyperbolic elements is $P^2$. Since $f, g \in P$, $f^t, g^t \in P$ and thus $fg, f^tg^t \in P^2$ are hyperbolic for all $t \in \mathbb{R}$. By Lemma 5.2 and Corollary 4.4

$$b^{t/2}(f^tg^t) = a_+(f^tg^t) \prec a_+(fg) = b^{t/2}(f^tg^2).$$

So $b(f^tg^2) \prec b(f^tg^2)$. Then replace $f^2$ and $g^2$ by $f$ and $g$, respectively, to obtain the desired result.

Conversely, suppose that $b(f^tg^t) \prec b^t(fg)$ for all $0 \leq t \leq 1$, where $f, g \in P$. Then

$$a_+(f^tg^t) = b^{t/2}(f^tg^2) \prec b^{t/2}(f^tg^2) = a_+(fg).$$
References


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