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Heinz-Kato's Inequalities for Lie Groups

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Abstract. Extensions of Heinz-Kato's inequalities and related inequalities are obtained for semisimple connected noncompact Lie groups.
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1. Introduction

Let $\mathbb{C}_{n\times n}$ be the set of $n\times n$ complex matrices and let

$$\|X\| := \max_{\|v\|_2=1} \|Xv\|_2$$

denote the spectral norm of $X\in\mathbb{C}_{n\times n}$. We have the following norm inequalities.

Theorem 1.1.

1. (Heinz-Kato [16, Theorem 3]) If $A,B\in\mathbb{C}_{n\times n}$ are positive semi-definite and $X\in\mathbb{C}_{n\times n}$, then

$$\|A^tXB^t\| \le \|X\|^{1-t}\|AXB\|^t, \qquad 0\le t\le 1. \tag{1}$$

2. (Heinz-Kato [17]) If $A,B\in\mathbb{C}_{n\times n}$ are positive semi-definite and $X\in\mathbb{C}_{n\times n}$, then

$$\|A^tXB^{1-t}\| \le \|AX\|^t\|XB\|^{1-t}, \qquad 0\le t\le 1, \tag{2}$$

3. (McIntosh [23], Bhatia-Davis [4]) For $A,B,X\in\mathbb{C}_{n\times n}$,

$$\|A^*XB\| \le \|AA^*X\|^{1/2}\|XBB^*\|^{1/2}. \tag{3}$$

See [1, 9, 10, 15, 16, 17, 18, 25] for the inequalities and related inequalities.

By continuity it suffices to consider the general linear group $\mathrm{GL}_n(\mathbb{C})$ instead of $\mathbb{C}_{n\times n}$. Moreover by the homogeneous property of $\|\cdot\|$ we can restrict ourselves to $\mathrm{SL}_n(\mathbb{C})$.

We will obtain some extensions of the above inequalities and other related inequalities in the context of semisimple connected noncompact Lie groups.

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2. Log majorization

If we order the singular values of $X \in \mathbb{C}_{n\times n}$ in descending order

$$s_1(X) \geq \cdots \geq s_n(X),$$

then (2), for example, can be written as

$$s_1(A^tXB^{1-t}) \leq s_1^t(AX)s_1^{1-t}(XB), \qquad 0 \leq t \leq 1. \tag{4}$$

Let $\mathbb{R}_+^n$ denote the set of positive n-tuples and let $a, b \in \mathbb{R}_+^n$. Then a is said to be *log majorized* by b, denoted by $a \prec_{\log} b$ if

$$\begin{aligned} \max_{\sigma \in S_n} \prod_{i=1}^k a_{\sigma(i)} &\leq \max_{\sigma \in S_n} \prod_{i=1}^k b_{\sigma(i)}, \quad k = 1, \ldots, n-1, \\ \prod_{i=1}^n a_i &= \prod_{i=1}^n b_i, \end{aligned}$$

where S_n denotes the symmetric group on $\{1, \ldots, n\}$. Write

$$s(X) := (s_1(X), \ldots, s_n(X)), \quad s^t(X) := (s_1^t(X), \ldots, s_n^t(X)), \quad t \geq 0.$$

Suppose $1 \leq k \leq n$. The kth compound of $X \in \mathbb{C}_{n\times n}$ is defined to be the $\binom{n}{k} \times \binom{n}{k}$ complex matrix $C_k(X)$ [22] whose elements are defined by

$$C_k(X)_{\alpha,\beta} = \det X[\alpha|\beta],$$

where $\alpha, \beta \in Q_{k,n}$ and $Q_{k,n} = \{\omega = (\omega(1), \ldots, \omega(k)) : 1 \leq \omega(1) < \cdots < \omega(k) \leq n\}$ is the set of increasing sequences of length k chosen from $1, \ldots, n$. For example, if $n = 3$ and $k = 2$, then

$$C_2(X) = \begin{pmatrix} \det X[1,2|1,2] & \det X[1,2|1,3] & \det X[1,2|2,3] \\ \det X[1,3|1,2] & \det X[1,3|1,3] & \det X[1,3|2,3] \\ \det X[2,3|1,2] & \det X[2,3|1,3] & \det X[2,3|2,3] \end{pmatrix}.$$

In general $C_1(X) = X$ and $C_n(X) = \det X$. Compound matrix has very nice properties: (i) $C_k(AB) = C_k(A)C_k(B)$ for all $A, B \in \mathbb{C}_{n\times n}$, (ii) the eigenvalues of $C_k(X)$ are $\prod_{j=1}^k \lambda_{\omega(j)}(X)$, $\omega \in Q_{k,n}$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X, (iii) the singular values of $C_k(X)$ are $\prod_{j=1}^k s_{\omega(j)}(X)$, $\omega \in Q_{k,n}$. The compound matrix $C_k(X)$ is indeed the matrix representation (with respect to some induced basis) of the induced operator $C_k(T)$ on the exterior space $\wedge^k\mathbb{C}^n$ where $T : \mathbb{C}^n \to \mathbb{C}^n$ is an operator: $C_k(T)v_1 \wedge \cdots \wedge v_k = Tv_1 \wedge \cdots \wedge Tv_k$, $v_1, \ldots, v_k \in \mathbb{C}^n$.

The following is an extension of Theorem 1.1. We will prove the second inequality and the rest are similar.

Theorem 2.1. 1. If $A, B \in \mathbb{C}_{n\times n}$ are positive semi-definite and $X \in \mathbb{C}_{n\times n}$ and $0 \leq t \leq 1$, then

$$s(A^tXB^t) \prec_{\log} s^{1-t}(X)s^t(AXB).$$

2. If $A, B \in \mathbb{C}_{n\times n}$ are positive semi-definite and $X \in \mathbb{C}_{n\times n}$ and $0 \le t \le 1$, then
$$s(A^t X B^{1-t}) \prec_{\log} s^t(AX)s^{1-t}(XB).$$

3. For $A, B, X \in \mathbb{C}_{n\times n}$,
$$s(A^* X B) \prec_{\log} s^{1/2}(AA^* X)s^{1/2}(XBB^*).$$

Proof. Let $C_k(X) \in \mathbb{C}_{\binom{n}{k}\times\binom{n}{k}}$ denote the kth compound of X, $k = 1, \ldots, n$. Notice that $s_1(C_k(X)) = \prod_{i=1}^k s_i(X)$, $C_k(XY) = C_k(X)C_k(Y)$, $X, Y \in \mathbb{C}_{n\times n}$, and $C_k(X)$ is positive semi-definite if X is positive semi-definite. So

$$\begin{aligned}
\prod_{i=1}^k s_i(A^t X B^{1-t}) &= s_1(C_k(A^t X B^{1-t})) \\
&= s_1(C_k(A)^t C_k(X) C_k(B)^{1-t}) \\
&\le s_1^t(C_k(AX)) s_1^{1-t}(C_k(XB)) \qquad \text{by (4)} \\
&= \prod_{i=1}^k s_i^t(AX) \prod_{i=1}^k s_i^{1-t}(XB).
\end{aligned}$$

When $k = n$,
$$\prod_{i=1}^n s_i(A^t X B^{1-t}) = |\det(A^t X B^{1-t})| = (\det A)^t\, |\det X|\, (\det B)^{1-t}$$
and
$$\begin{aligned}
\prod_{i=1}^n s_i^t(AX) \prod_{i=1}^n s_i^{1-t}(XB) &= |\det(AX)|^t\, |\det(XB)|^{1-t} \\
&= (\det A)^t\, |\det X|\, (\det B)^{1-t}.
\end{aligned}$$

■

3. A pre-order of Kostant

We now take a close look of Theorem 2.1 for $X \in \mathrm{GL}_n(\mathbb{C})$. Let $A_+ \subset \mathrm{GL}_n(\mathbb{C})$ denote the set of all positive diagonal matrices with diagonal entries in descending order. Recall that the singular value decomposition of $X \in \mathrm{GL}_n(\mathbb{C})$ asserts that there exist unitary matrices U, V such that
$$X = U a_+(X) V, \tag{5}$$
where $a_+(X) = \mathrm{diag}\,(s_1(X), \ldots, s_n(X)) \in A_+$. Though U and V in the decomposition (5) are not unique, $a_+(X) \in A_+$ is uniquely defined.

Let G be a semisimple noncompact connected Lie group having $\mathfrak{g}$ as its Lie algebra. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$. Let $K \subset G$ be the analytic subgroup with Lie algebra $\mathfrak{k}$. Then $\mathrm{Ad}\, K$ is a maximal compact subgroup of $\mathrm{Ad}\, G$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. The exponential map $\exp : \mathfrak{a} \to A$ is bijective.

Set

$$P := \exp \mathfrak{p}.$$

The map $K \times P \to G$, $(k, p) \mapsto kp$ is a diffeomorpism. In particular

$$G = KP$$

and every element $g \in G$ can be uniquely written as

$$g = kp, \qquad k \in K, \ p \in P. \tag{6}$$

The map $\Theta : G \to G$

$$\Theta(kp) = kp^{-1}, \quad k \in K, \ p \in P,$$

is an automorphism of G [19, p.387]. The map $* : G \to G$ defined by

$$g^* := \Theta(g^{-1}) = pk^{-1}, \qquad g \in G$$

is clearly a diffeomorphism. Let W be the Weyl group of $(\mathfrak{a}, \mathfrak{g})$ which may be defined as the quotient of the normalizer of A in K modulo the centralizer of A in K. The Weyl group operates naturally in $\mathfrak{a}$ and A and the isomorphism $\exp : \mathfrak{a} \to A$ is a W-isomorphism.

Fix a *closed* Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$ and set $A_+ := \exp \mathfrak{a}_+$. We have [11] the Cartan decomposition

$$G = KA_+K.$$

Though $k_1, k_2 \in K$ are not unique in $g = k_1 a k_2$ ($g \in G$, $k_1, k_2 \in K$, $a \in A_+$), the element $a = a_+(g) \in A_+$ is unique.

Proposition 3.1. The following maps are continuous.

1. $a'_+ : \mathfrak{p} \to \mathfrak{a}_+$ where for each $X \in \mathfrak{p}$, $a'_+(X)$ is the unique element in $\mathfrak{a}_+$ such that $a'_+(X) = \mathrm{Ad}(s)X$ for some $s \in K$. Indeed it is a contraction.

2. $a_+ : G \to A_+$ where $a_+(g) \in A_+$ is the unique element in $g = k_1 a_+(g) k_2 \in G$, where $k_1, k_2 \in K$.

Proof. (1) By Berezin-Gelfand's result [2], $a'_+(X+Y) \in a'_+(X) + \mathrm{conv}\, W a'_+(Y)$ for any $X, Y \in \mathfrak{p}$. So $a'_+(X) = a'_+(Y + (X - Y)) \in a'_+(Y) + \mathrm{conv}\, W(a'_+(X - Y))$. Hence $a'_+(X) - a'_+(Y) \in \mathrm{conv}\, W a'_+(X - Y)$. Also see [13, Corollary 3.10]. Let $\|\cdot\|$ be the norm on $\mathfrak{p}$ induced by the Killing form of $\mathfrak{g}$. Since $\|\cdot\|$ is K-invariant and strictly convex, $\|a'_+(X) - a'_+(Y)\| \le \|a'_+(X - Y)\| = \|X - Y\|$. So the map a'_+ is a contraction and thus continuous.

(2) Since the map $G \to P$ such that $g = kp \mapsto p$ is differentiable and $a_+(g) = a_+(p)$, it suffices to establish the continuity of $a_+ : P \to A_+$. The map $\exp : \mathfrak{p} \to P$ is a surjective diffeomorphism and the inverse $\log : P \to \mathfrak{p}$ is well defined. So $a_+ = \exp \circ a'_+ \circ \log$ on P is continuous. ∎

Define a pre-order $\prec$ in A. Given $a, b \in A$, $a \prec b$ means

$$\exp(\mathrm{conv}\, W(\log a)) \subset \exp(\mathrm{conv}\, W(\log b)).$$

The set $\exp(\mathrm{conv}\, W(\log a))$ is multiplicatively the convex hull of the compact convex set having the Weyl group orbit $W(\log a)$ as its extreme points.

Example 3.2. Let $G = \mathrm{SL}(n, \mathbb{C})$. We pick

$$\begin{aligned}
\mathfrak{k} &= \mathfrak{su}(n), \\
K &= \mathrm{SU}(n), \\
\mathfrak{p} &= i\mathfrak{su}(n), \text{ i.e., the set of Hermitian matrices of zero trace} \\
P &= \text{the set of positive definite matrices in } \mathrm{SL}_n(\mathbb{C}) \\
A &= \{\mathrm{diag}\,(a_1, \dots, a_n) : a_1, \dots, a_n > 0, \prod_{i=1}^n a_i = 1\}, \\
A_+ &= \{\mathrm{diag}\,(a_1, \dots, a_n) : a_1 \geq \cdots \geq a_n > 0, \prod_{i=1}^n a_i = 1\}.
\end{aligned}$$

Let $a = \mathrm{diag}\,(a_1, \dots, a_n), b = \mathrm{diag}\,(b_1, \dots, b_n) \in A$. Since the Weyl group is the symmetric group S_n on $\{1, \dots, n\}$, $\mathrm{conv}\, W(\log a) = \mathrm{conv}\, S_n(\log a)$. So $a \prec b$ amounts to $\log a \in \mathrm{conv}\, S_n(\log b)$ and by Hardy-Littlewood-Poyla's theorem, $a \prec b$ is equivalent to the log majorization inequalities

$$\begin{aligned}
\prod_{i=1}^k a_{[i]} &\leq \prod_{i=1}^k b_{[i]}, \quad k = 1, \dots, n-1, \\
\prod_{i=1}^n a_{[i]} &= \prod_{i=1}^n b_{[i]},
\end{aligned}$$

where $a_{[1]} \geq \cdots \geq a_{[n]}$ denote the rearranged $a_1, \dots, a_n$ in descending order.

The following nice result of Kostant describes the pre-order $\prec$ in A via the representations of G. We remark that Kostant's pre-order [21, p.426] is more general and is defined in G via the complete multiplicative Jordan decomposition and hyperbolic elements (see Section 5 and [21]).

Theorem 3.3. (Kostant [21, Theorem 3.1]) Let $f, g \in A$. Then $f \prec g$ if and only if $|\pi(f)| \leq |\pi(g)|$ for all finite dimensional representations π of G, where $|\cdot|$ denotes the spectral radius.

One may derive the log majorization in Example 3.2 via Theorem 3.3 and the fundamental representations on the exterior space $\wedge^k \mathbb{C}^n$, $k = 1, \dots, n-1$, since $|\pi_k(a)| = a_1 \cdots a_k$.

4. Extension of the inequalities

Lemma 4.1. Let $g \in G$. Then $a_+(g) = a_+(g^*) = a_+^{1/2}(gg^*) = a_+^{1/2}(g^*g)$.

Proof. Let $g = kp$ be the Cartan decomposition of $g \in G$, $k \in K$, $p \in P$. Notice that $g^* = pk^{-1}$ so that $a_+(g) = a_+(p) = a_+(g^*)$. Now $g^*g = p^2$ and $gg^* = kp^2k^{-1}$. So $a_+^{1/2}(gg^*) = a_+^{1/2}(g^*g) = a_+^{1/2}(p^2)$. Since each element in P is K-conjugate to some element in A_+ [19, p.320]. Thus $a_+^{1/2}(p^2) = a_+(p) = a_+(g)$. ∎

Lemma 4.2. Let $h_1, h_2 \in A_+$. For any finite dimensional representation $\pi : G \to \mathrm{GL}(V)$, $|\pi(h_1 h_2)| = |\pi(h_1)|\,|\pi(h_2)|$, where $|\cdot|$ denotes the spectral radius.

Proof. Since the spectral radius of an operator is invariant under similarity, by the complete reducibility [5, p.50], [14, p.28] of π, we may assume that π is irreducible. We also use the same notation $d\pi$ to denote the irreducible representation of the complexification $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g}$ (direct sum) of $\mathfrak{g}$, induced by $d\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, i.e., $d\pi(X + iY) = d\pi(X) + i\, d\pi(Y)$, $X, Y \in \mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Since $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$, there is an inner product (unique up to scalar multiple) on V [6, p.217] such that $d\pi(\mathfrak{u})$ are skew Hermitian. So $d\pi(\mathfrak{k})$ are skew Hermitian and $d\pi(\mathfrak{p})$ are Hermitian. Thus the elements of

$$\pi(P) = \pi(\exp \mathfrak{p}) = \exp d\pi(\mathfrak{p})$$

[11, p.110] are positive definite operators. Since $A \subset P$ and is abelian, $\pi(A)$ is an abelian subgroup of positive definite operators. Thus the elements of $\pi(A)$ are positive diagonal operators under an appropriate orthonormal basis (once fixed and for all) of V. For each $H \in \mathfrak{a}$, $\exp d\pi(H) = \pi(\exp H) \in \pi(A)$ so that $d\pi(H)$ are real diagonal operators. Let $H_1, H_2 \in \mathfrak{a}_+$ such that $h_1 = \exp H_1, h_2 = \exp H_2 \in A_+$. Then

$$\pi(h_1)\pi(h_2) = \exp d\pi(H_1) \exp d\pi(H_2) = \exp d\pi(H_1 + H_2)$$

since $\mathfrak{a}$ is abelian and $d\pi$ respects the bracket. Notice that $|\pi(h_1)\pi(h_2)|$ is the exponent of the largest diagonal entry of the diagonal operator $d\pi(H_1) + d\pi(H_2)$. To arrive at $|\pi(h_1 h_2)| = |\pi(h_1)|\,|\pi(h_2)|$, it is sufficient to show that the sum of the largest diagonal entries $d\pi(H_1)$ and $d\pi(H_2)$ is also a diagonal entry of $d\pi(H_1 + H_2)$. To this end, we will use the theory of highest weights [14, p.108] on the finite dimensional irreducible representations of the complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ (since $\mathfrak{g}$ is semisimple).

Let

$$\mathfrak{g} = (\mathfrak{a} \oplus \mathfrak{m}) \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

be the restricted root decomposition of $\mathfrak{g}$ [11, p.263], where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and Σ is the set of restricted roots of $(\mathfrak{g}, \mathfrak{a})$. Let $\mathfrak{h}$ be the maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ and we set $\mathfrak{h}_{\mathfrak{k}} := \mathfrak{h} \cap \mathfrak{k}$. It is known that $\mathfrak{h}_{\mathbb{C}} := \mathfrak{h} \oplus i\mathfrak{h}$, the complexification of $\mathfrak{h}$, is a Cartan subalgebra of the complex semisimple $\mathfrak{g}_{\mathbb{C}}$ [11, p.259]. Let Δ be the set of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and set $\mathfrak{h}_{\mathbb{R}} := \sum_{\alpha \in \Delta} \mathbb{R} H_\alpha$, where $H_\alpha \in \mathfrak{h}_{\mathbb{C}}$ is defined by the restriction to $\mathfrak{h}_{\mathbb{C}}$ of the Killing form, i.e., $B(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}_{\mathbb{C}}$. Then $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a} \oplus i\mathfrak{h}_{\mathfrak{k}}$. Each root $\alpha \in \Delta$ is real-valued on $\mathfrak{h}_{\mathbb{R}}$ [11, p.170]. Let $\Delta_{\mathfrak{p}} \subset \Delta$ be the set of roots which do not vanish identically on $\mathfrak{a}$. It is known that Σ is the set of restrictions of $\Delta_{\mathfrak{p}}$ to $\mathfrak{a}$ [11, p.263]. Furthermore we can choose a positive root system $\Delta^+ \subset \Delta$ so that $\mathfrak{a}_+$ is in the corresponding Weyl chamber (in $\mathfrak{h}_{\mathbb{R}}$) [21, p.431], that is, $\alpha(H) \geq 0$ for all $H \in \mathfrak{a}_+$, $\alpha \in \Delta^+$. So any root of Δ^+ restricts to either zero or an element in Σ^+ as a linear functional on $\mathfrak{a}$ [11, p.263].

The diagonal entries of the diagonal operator $d\pi(H)$, $H \in \mathfrak{a}_+ \subset \mathfrak{h}_{\mathbb{C}}$ are the eigenvalues of $d\pi(H)$ so that they are of the form $\mu(H)$, where μ are the weights of the representation $d\pi$ of $\mathfrak{g}_{\mathbb{C}}$ [14, p.107-108]. Let $\lambda \in \mathfrak{h}'$ be the highest weight of $d\pi$, where $\mathfrak{h}'$ denotes the dual space of $\mathfrak{h}$. From the theory of representation $\lambda - \mu$ is a sum of positive roots, i.e.,

$$\lambda - \mu = \sum_{\alpha \in \Delta^+} k_\alpha \alpha, \qquad k_\alpha \in \mathbb{N}.$$

Since the restrictions of the positive roots in Δ^+ to $\mathfrak{a}$ are either zero or elements in Σ^+, we conclude $\lambda(H) \geq \mu(H)$ for all $H \in \mathfrak{a}_+$. Since $\mathfrak{a}_+$ is a cone, $H_1 + H_2 \in \mathfrak{a}_+$. Thus $\lambda(H_1+H_2) = \lambda(H_1)+\lambda(H_2)$ is the largest diagonal entry (eigenvalue) of the diagonal operator $d\pi(H_1 + H_2)$ and $\lambda(H_1)$, and $\lambda(H_2)$ are the largest diagonal entries (eigenvalues) of $d\pi(H_1)$ and $d\pi(H_2)$, respectively. ■

The following theorem is an extension of Theorem 2.1.

Theorem 4.3. The following are equivalent and are valid.

$$\begin{aligned} a_+(a^t g b^{1-t}) &\prec [a_+(ag)]^t [a_+(gb)]^{1-t}, \ 0 \leq t \leq 1, \ a, b \in P, \ g \in G, &(7)\\ a_+(a^t g b^t) &\prec [a_+(g)]^{1-t} [a_+(agb)]^t, \ 0 \leq t \leq 1, \ a, b \in P, \ g \in G, &(8)\\ a_+(a^* g b) &\prec [a_+(aa^* g)]^{1/2} \, [a_+(gbb^*)]^{1/2}, \quad a, b, g \in G. &(9) \end{aligned}$$

Proof. We will first establish (7) and then the equivalence among the relations. Let $g \in G$ and write $g = k_1 a_+(g) k_2$, where $a_+(g) \in A_+$, $k_1, k_2 \in K$. Let π be any representation of G. Since the elements of $d\pi(\mathfrak{k})$ are skew Hermitian,

$$\|\pi(g)\| = \|\pi(k_1 a_+(g) k_2)\| = \|\pi(k_1)\pi(a_+(g))\pi(k_2)\| = \|\pi(a_+(g))\|.$$

Since the spectral norm $\|\cdot\|$ is invariant under unitary equivalence, and $\|X\| = |X|$ for each positive definite operator X, $\|\pi(a_+(g))\| = |\pi(a_+(g))|$ and thus

$$\|\pi(g)\| = \|\pi(a_+(g))\| = |\pi(a_+(g))|. \tag{10}$$

Suppose $0 \leq t \leq 1$. Since the elements of $d\pi(\mathfrak{p})$ are Hermitian operators, $\pi(a)$ and $\pi(b)$ are positive definite operators,

$$\begin{aligned} |\pi(a_+(a^t g b^{1-t}))| &= \|\pi(a^t g b^{1-t})\| && \text{by (10)}\\ &= \|\pi^t(a)\pi(g)\pi^{1-t}(b)\| \\ &\leq \|\pi(a)\pi(g)\|^t \|\pi(g)\pi(b)\|^{1-t} && \text{by (2)}\\ &= \|\pi(ag)\|^t \|\pi(gb)\|^{1-t} \\ &= |\pi(a_+(ag))|^t \, |\pi(a_+(gb))|^{1-t} && \text{by (10)} \end{aligned}$$

The elements of $\pi(A)$ are positive diagonal operators under an appropriate orthonormal basis. Since $a_+^t(ag), a_+^{1-t}(gb) \in A_+$, $|\pi(a_+(ag))|^t = |\pi(a_+^t(ag))|$ and $|\pi(a_+(gb))|^{1-t} = |\pi(a_+^{1-t}(gb))|$. So

$$\begin{aligned} |\pi(a_+(ag))|^t \, |\pi(a_+(gb))|^{1-t} &= |\pi(a_+^t(ag))| \, |\pi(a_+^{1-t}(gb))| \\ &= |\pi(a_+^t(ag) \, \pi(a_+^{1-t}(gb))| && \text{by Lemma 4.2}\\ &= |\pi(a_+^t(ag) a_+^{1-t}(gb))|. \end{aligned}$$

As a result, $|\pi(a_+(a^t g b^{1-t}))| \le |\pi(a_+^t(ag)a_+^{1-t}(gb))|$ for any representation π of G. By Theorem 3.3 we have (7).

(7) $\Rightarrow$ (8): If $0 \le t \le 1$, then $0 \le 1-t \le 1$. If $a, b \in P$, so are their inverses. From (7)

$$\begin{aligned} a_+(a^t g b^t) &= a_+((a^{-1})^{1-t} agb^{1-(1-t)}) \\ &\prec a_+^{1-t}(a^{-1}ag)a_+^t(agb) \\ &= a_+^{1-t}(g)a_+^t(agb), \end{aligned}$$

i.e., (8) is established.

(8) $\Rightarrow$ (9): Let $a, b \in G$. Write $a^* = kp$, $b^* = k'p'$ according to their Cartan decompositions. Then $b = p'k'^{-1}$. By (8) with $t = 1/2$,

$$\begin{aligned} a_+(a^* g b) &= a_+(kpgp'k'^{-1}) \\ &= a_+(pgp') \\ &= a_+((p^{-2})^{1/2}(p^2 g)(p'^2)^{1/2}) \\ &\prec a_+^{1/2}(p^2 g)a_+^{1/2}(p^{-2}p^2 g p'^2) \\ &= a_+^{1/2}(p^2 g)a_+^{1/2}(g p'^2). \end{aligned}$$

Since $aa^* = p^2$ and $bb^* = p'^2$, (9) follows.

(9) $\Rightarrow$ (7): Let $a, b \in P$. For $t = 0, 1$, (7) is trivial and for $t = 1/2$, it follows from (9). We will prove by induction for all $t = \frac{k}{2^n}$, where $k = 0, 1, \ldots, 2^n$ [3]. Let $t = \frac{2k+1}{2^n}$. Then $t = s + \rho$, where $s = \frac{k}{2^{n-1}}$ and $\rho = \frac{1}{2^n}$. Suppose that (7) is valid for all dyadic rationals with denominator 2^{n-1}. Then by induction and (9), with $\lambda := s + 2\rho$, we have

$$\begin{aligned} a_+(a^t g b^{1-t}) &= a_+(a^\rho(a^s g b^{1-\lambda})b^\rho) \\ &\prec a_+^{1/2}(a^{2\rho}a^s g b^{1-\lambda})a_+^{1/2}(a^s g b^{1-\lambda}b^{2\rho}) \\ &= a_+^{1/2}(a^\lambda g b^{1-\lambda})a_+^{1/2}(a^s g b^{1-s}) \\ &\prec a_+^{\lambda/2}(ag)a_+^{(1-\lambda)/2}(gb)a_+^{s/2}(ag)a_+^{(1-s)/2}(gb) \\ &= a_+^{(\lambda+s)/2}(ag)a_+^{1-(\lambda+s)/2}(gb) \\ &= a_+^t(ag)a_+^{1-t}(gb). \end{aligned}$$

The general case follows from continuity of the spectral radius and Theorem 3.3. ∎

Furuta's inequality [8] asserts that if $A, B \in \mathbb{C}_{n\times n}$ are positive semi-definite, then

$$\|A^t B^t\| \le \|AB\|^t, \qquad 0 \le t \le 1. \tag{11}$$

It is equivalent to say that

$$\|A^s B^s\| \ge \|AB\|^s, \qquad s \ge 1. \tag{12}$$

See [1, 7, 24]. We have the following extension of Furuta's inequality.

Corollary 4.4. Let $a, b \in P$. Then

1. $a_+(a^t b^t) \prec a_+^t(ab)$, $0 \le t \le 1$.
2. $a_+^t(ab) \prec a_+(a^t b^t)$, $t \ge 1$.

Hence $\varphi(t) = [a_+(a^{1/t} b^{1/t})]^t$ is a decreasing function on $t > 0$ with respect to the partial order $\prec$, i.e., $\varphi(s) \prec \varphi(t)$ if $s \ge t > 0$.

Proof. By setting g to be the identity in (8) we have $a_+(a^t b^t) \prec a_+^t(ab)$, $0 \le t \le 1$. When $t \ge 1$, $a_+(a^{1/t} b^{1/t}) \prec a_+^{1/t}(ab)$. Then replace a, b by a^t and b^t respectively to have $a_+(ab) \prec a_+^{1/t}(a^t b^t)$. Let $s \ge t > 0$. Then $s/t > 1$ and

$$a_+(a^s b^s) = a_+((a^t)^{s/t} (b^t)^{s/t}) = a_+^{s/t}(a^t b^t)$$

so that $\varphi(t)$ is decreasing on $t > 0$. ∎

Corollary 4.5. *For* $f, g \in G$, $a_+(fg) \prec a_+(f) a_+(g)$.

Proof. By (9),

$$a_+(fg) \prec a_+^{1/2}(f^* f) a_+^{1/2}(g g^*).$$

Use Lemma 4.1 to obtain $a_+(fg) \prec a_+(f) a_+(g)$. ∎

Remark 4.6. When $G = \mathrm{GL}_n(\mathbb{C})$, by Corollary 4.5 the singular values of a product AB is log majorized by the product of the singular values of $A, B \in \mathbb{C}_{n\times n}$, assuming that singular values are all arranged in descending order.

Nakamoto [9] showed that (12) holds for normal matrices $A, B \in \mathbb{C}_{n\times n}$ and natural numbers s. An element $g \in G$ is said to be *normal* if $gg^* = g^* g$. It is equivalent to say that $kp = pk$, where $g = kp$ is the Cartan decomposition of g. Since Cartan decomposition is unique up to conjugation [11, p.183], normality is independent of the choice of K and P. Clearly the elements of P are normal. Normality is reduced to the usual normality when $G = \mathrm{SL}_n(\mathbb{C})$. Now we extend Nakamoto's result.

Corollary 4.7. Let $f, g \in G$ be normal. Then $a_+^n(fg) \prec a_+(f^n g^n), n \in \mathbb{N}$.

Proof. Let $f = kp$, $g = k'p'$ be the Cartan decompositions of $f, g \in G = KP$. Since f, g are normal, we have $kp = pk$ and $k'p' = p'k'$. Then

$$a_+^n(fg) = a_+^n(kpk'p') = a_+^n(kpp'k') = a_+^n(pp').$$

By Corollary 4.4,

$$a_+^n(pp') \prec a_+(p^n p'^n) = a_+(k^n p^n p'^n k'^n) = a_+((kp)^n (k'p')^n) = a_+(f^n g^n).$$

∎

5. Inequalities for hyperbolic components

Furuta's inequality (11) is equivalent to the following inequality: for any positive semi-definite $A, B \in \mathbb{C}_{n\times n}$,

$$\lambda_1(A^tB^t) \le \lambda_1^t(AB), \qquad 0 \le t \le 1. \tag{13}$$

One can deduce the equivalence by

$$|AB| = \lambda_1(AB), \tag{14}$$

where $\lambda_1(AB)$ is the largest eigenvalue of the matrix AB whose eigenvalues are those of the positive semi-definite $B^{1/2}AB^{1/2}$.

Inequality (13) concerns about the largest eigenvalues of AB and A^tB^t when $A, B \in \mathbb{C}_{n\times n}$ are positive semi-definite. Since the eigenvalues of AB are nonnegative, (13) can be viewed as a result on the largest eigenvalue modulus. So we will consider the *hyperbolic component* of $g \in G$ of a semisimple connected noncompact Lie group G for an appropriate extension.

An element $X \in \mathfrak{g}$ is called *real semisimple* if $\operatorname{ad} X \in \operatorname{End}(\mathfrak{g})$ is diagonalizable over $\mathbb{R}$. It is equivalent to say that $\operatorname{ad}(X)$ is diagonalizable over $\mathbb{C}$ and the eigenvalues of $\operatorname{ad}(X)$ are real. An element $X \in \mathfrak{g}$ is called nilpotent if $\operatorname{ad} X$ is nilpotent. An element $g \in G$ is called hyperbolic if $g = \exp(X)$ where $X \in \mathfrak{g}$ is real semisimple and is called unipotent if $g = \exp(X)$ where $X \in g$ is nilpotent. An element $g \in G$ is elliptic if $\operatorname{Ad}(g) \in \operatorname{Aut}(\mathfrak{g})$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. The *complete multiplicative Jordan decomposition* [21, Proposition 2.1] for G asserts that each $g \in G$ can be uniquely written as $g = ehu$, where e is elliptic, h is hyperbolic and u is unipotent and the three elements e, h, u commute. We write

$$g = e(g)h(g)u(g).$$

It turns out that $h \in G$ is hyperbolic if and only if it is conjugate to a unique element $b(h) \in A_+$ [21, Proposition 2.4]. Denote

$$b(g) := b(h(g)).$$

It is known that [21, Proposition 6.2] P^2 is the set of all hyperbolic elements and $b(g) \prec a_+(g)$ for all $g \in G$ [21, Theorem 5.4].

Example 5.1. When $G = \mathrm{SL}_n(\mathbb{C})$, $g = ehu$ is the usual complete multiplicative Jordan decomposition [11, p.431] and

$$b(g) = \operatorname{diag}(|\lambda_1|, \dots, |\lambda_n|),$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of g with descending moduli.

Lemma 5.2. 1. Let $f, g \in G$. Then $h(fg) = g^{-1}h(gf)g$ and $b(fg) = b(gf)$.

2. If $f \in P$, then $b(f) = a_+(f)$. So $a_+(g^*g) = a_+(gg^*) = b(g^*g) = b(gg^*)$ for all $g \in G$.

3. Let $f, g \in P$. Then $b(f^2g^2) = a_+^2(fg) = a_+^2(gf)$.

Proof. (1) Let $fg = ehu$ be the CMJD of fg where $f, g \in G$. Then $gf = g(fg)g^{-1} = (geg^{-1})(ghg^{-1})(gug^{-1})$. By the uniqueness of CMJD, $h(fg) = g^{-1}h(gf)g$ follows immediately. Now $b(fg) = b(h(fg)) = b(g^{-1}h(gf)g) = b(gf)$.

(2) Since P is K-conjugate to some element in A_+, $b(f) = a_+(f)$.

(3) By (1) $h(f^2g^2) = g^{-1}h(gf^2g)g$ so that

$$b(f^2g^2) = b(g^{-1}h(gf^2g)g) = b(gf^2g) = b((gf)(gf)^*).$$

The element $(gf)(gf)^*$ is in P so that by (2) and Lemma 4.1

$$b(f^2g^2) = b((gf)(gf)^*) = a_+((gf)(gf)^*) = a_+^2(fg).$$

■

Theorem 5.3. Let $a, b \in P$. The following are equivalent and are valid.

1. $a_+(a^tb^t) \prec a_+^t(ab)$, $0 \le t \le 1$.
2. $b(f^tg^t) \prec b^t(fg)$, $0 \le t \le 1$.

In other words, for any hyperbolic element $\ell \in G$, if we write $\ell = fg$, where $f, g \in P$, then $b(f^tg^t) \prec b^t(\ell)$, $0 \le t \le 1$.

Proof. Statement (1) is Corollary 4.4 (1). The set of all hyperbolic elements is P^2. Since $f, g \in P$, $f^t, g^t \in P$ and thus $fg, f^tg^t \in P^2$ are hyperbolic for all $t \in \mathbb{R}$. By Lemma 5.2 and Corollary 4.4

$$b^{1/2}(f^{2t}g^{2t}) = a_+(f^tg^t) \prec a_+^t(fg) = b^{t/2}(f^2g^2).$$

So $b(f^{2t}g^{2t}) \prec b^t(f^2g^2)$. Then replace f^2 and g^2 by f and g, respectively, to obtain the desired result.

Conversely, suppose that $b(f^tg^t) \prec b^t(fg)$ for all $0 \le t \le 1$, where $f, g \in P$. Then

$$a_+(f^tg^t) = b^{1/2}(f^{2t}g^{2t}) \prec b^{t/2}(f^2g^2) = a_+^t(fg).$$

■

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