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A Geometric Approach to the Frobenius Unicity Conjecture for the Markoff Equation

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Abstract. The long–standing Frobenius conjecture on the unicity of ordered solutions for the Markoff equation is translated in a very simple way into an arithmetic statement on the existence of integral points on certain hyperbolas. Some previous work of Kang and Melville can then be used for relating the problem to a statement concerning rank 2 symmetric hyperbolic Kac–Moody algebras.
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1. Introduction

The Markoff equation is the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz; \quad x, y, z \in \mathbf{Z}_+;$$

which was studied first by Markoff in [7, 8]. Markoff proved many interesting properties related to the solutions of this equation. Among other things, he proved there were infinitely many solutions and gave a procedure to construct new solutions from old ones.

However, it was Frobenius [4], while studying Markoff equations over Gauss integers, who noticed that, for a given ordered solution $x \leq y \leq z$, there was no other ordered solution $x' \leq y' \leq z$. This conjecture, widely known as the Frobenius unicity conjecture, has remained open since, despite some important partial results have been settled. To mention the most popular ones:

- If either z, $3z-2$ or $3z+2$ has the form $2^a p$, for $a = 0, 1, 2$ and p prime, the conjecture is known to be true [1]. The case $z = p$ was also solved independently in [2, 9].
- If $z = kp^r$, with p prime and $k^4 < m$, the conjecture is also known to hold, as proved in [3].

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These results, combined with brute force computational attacks have proved the conjecture to be true for $z < 10^{140}$ so far. Many other mathematicians (I. Borosh, R.T. Mumby, H. Cohn, M.L. Lang, J.H. Silverman, S.P. Tang, D. Zagier,...) have worked in and around this fascinating problem.

Also it should be mentioned here that the Markoff equation has appeared in a number of apparently disconnected fields as quadratic forms whose values are one-third their discriminant or more, closed geodesics on certain Riemann surfaces or modular groups.

Rather than building over these strong (and, in some cases, quite highbrow) techniques, we decided to approach the problem in a naive, yet not reported, way. Fix a positive integer c, and consider the conic defined by

$$\mathcal{H}_c : x^2 - 3cxy + y^2 + c^2 = 0,$$

by the way a hyperbola. It is then plain than the Frobenius unicity conjectured can be expressed in the following terms:

Conjecture 1.1. There is, at most, one integral point (x, y) in the hyperbola $\mathcal{H}_c$ verifying $x \leq y \leq c$.

This paper is based in the fact that integral points in hyperbolas of the form

$$x^2 - axy + y^2 = k, \quad a \geq 3, \ k \leq 1$$

have been studied by S.J. Kang and D.J. Melville as a by–product of their work on certain Kac–Moody algebras [6]. We will apply Kang and Melville results to our hyperbola $\mathcal{H}_c$ in order to look for a different version of the Frobenius conjecture. In order to do that, we will first review briefly some results from [6].

2. Review of the Kang–Melville procedure

Consider a rank 2 symmetric hyperbolic Kac–Moody algebra $\mathfrak{g}(a)$ given by a Cartan matrix

$$\begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}, \qquad a \geq 3.$$

In [6] Kang and Melville show that the set of roots of $\mathfrak{g}(a)$ is connected to the diophantine problem of finding points with integral coordinates in certain hyperbolas. More specifically:

- There is a one–to–one correspondence between the set of real roots of $\mathfrak{g}(a)$ and the set of integral points on $\mathcal{H}_{real} : x^2 - axy + y^2 = 1$.
- There is a one–to–one correspondence between the set of imaginary roots of length $-2k$ of $\mathfrak{g}(a)$ and the set of integral points on $\mathcal{H}_k : x^2 - axy + y^2 = -k$, for $k \geq 1$.

They went further in order to describe the root system of $\mathfrak{g}(a)$. We will review only the necessary information for what follows, but much more can be found in their paper. Consider the sequence $\{B_j\}_{j\geq 0}$, defined by

$$B_0 = 0, \ \ B_1 = 1, \ \ B_{j+2} = aB_{j+1} - B_j, \ \text{for } j \geq 0.$$

Also let Ω_k be the following set

$$\Omega_k = \Bigg\{ (m,n) \in \mathbf{Z}_+ \times \mathbf{Z}_+ \mid \frac{2\sqrt{k}}{\sqrt{a^2-4}} \le m \le \sqrt{\frac{k}{a-2}},$$

$$n = \frac{am - \sqrt{(a^2-4)m^2 - 4k}}{2} \Bigg\}.$$

Then the set of all imaginary roots of $\mathfrak{g}(a)$ with length $-2k$ is given by

$$\begin{aligned} \Delta_k &= \Omega_k \cup \{(mB_{j+1} - nB_j,\ mB_{j+2} - nB_{j+1}) \mid (m,n) \in \Omega_k\} \cup \\ &\quad \{(nB_{j+1} - mB_j,\ nB_{j+2} - mB_{j+1}) \mid (m,n) \in \Omega_k\} \\ &= \Omega_k \cup \Delta_k^{(1)} \cup \Delta_k^{(2)} \end{aligned}$$

up to permutation of coordinates ($\mathcal{H}_k$ is obviously invariant w.r.t. these transformations) and sign changes (the displayed roots being the positive ones, up to permutation of coordinates).

For a given pair (a,k) the explicit computation of Ω_k is fairly easy, and so is the obtained recurrent formula for the set of integral points on $\mathcal{H}_k$. The elements B_j can also be expressed in terms of a close formula (à la Fibonacci) which we will not be concerned about.

This algorithm is, according to the authors, much faster than the usual number–theoretic option which involves finding a point and then using a stereographic projection. The process will be called KM process from now on.

We must remark that the Kang–Melville procedure sketched above is an application of the much more general results on positive imaginary roots one can find in [5], where the set Ω_k is actually introduced. It is also shown that any positive imaginary root lies in the orbit of some element in Ω_k by the Weyl group

From that it is plain that the set Ω_k has an interesting connection with the root multiplicities. In fact, if $|\Omega_k| = 1$ then clearly all roots of length $-2k$ must have the same multiplicity. This might not be the case if $|\Omega_k| > 1$, as [6] show with specific counterexamples.

3. The KM procedure for $\mathcal{H}_c$

Our aim is to apply the KM process to $\mathcal{H}_c$, hence we are considering a rank 2 hyperbolic symmetric Kac–Moody algebra $\mathfrak{g}(3c)$, with $c \in \mathbf{Z}_+$ and we are interested primarily on the imaginary roots of length $-2c^2$, which correspond to integral points on $\mathcal{H}_c$. In our case the situation of the previous section can be now written as:

$$B_0 = 0,\ \ B_1 = 1,\ \ B_{j+2} = 3cB_{j+1} - B_j,\ \text{for } j \ge 0;$$

and

$$\Omega_{c^2} = \Bigg\{ (m,n) \in \mathbf{Z}_+ \times \mathbf{Z}_+ \mid \frac{2c}{\sqrt{9c^2-4}} \le m \le \frac{c}{\sqrt{3c-2}},$$

$$n = \frac{3cm - \sqrt{(9c^2-4)m^2 - 4c^2}}{2} \Bigg\},$$

where some rather popular expressions (as $3c-2$ or $9c^2-4$) from the literature concerning Markoff equation appear.

Now we will devote ourselves some time for showing some special features of Ω_{c^2} and $\{B_j\}$ in this case.

Lemma 3.1. $0 \leq m \leq n \leq c$.

Proof. We will only do $m \leq n$, as the other two inequalities follow easily.

$$\begin{aligned} n \geq m &\iff 3cm - \sqrt{(9c^2-4)m^2-4c^2} \geq 2m \\ &\iff (3c-2)m \geq \sqrt{(9c^2-4)m^2-4c^2} \\ &\iff (9c^2-12c+4)m^2 \geq (9c^2-4)m^2-4c^2 \\ &\iff m^2 \leq \frac{c^2}{3c-2} \end{aligned}$$

which is guaranteed by the definition of Ω_k. ■

In order to make the statements simpler, for a fixed pair $(m,n) \in \Omega_{c^2}$, let us write:

$$\begin{aligned} \alpha_j &= mB_{j+1} - nB_j \\ \beta_j &= nB_{j+1} - mB_j \end{aligned}$$

so that

$$\Delta_{c^2}^{(1)} = \{\, (\alpha_j, \alpha_{j+1}) \mid j \geq 0\,\}, \qquad \Delta_{c^2}^{(2)} = \{\, (\beta_j, \beta_{j+1}) \mid j \geq 0\,\}$$

Lemma 3.2. *The sequence $\{B_j\}_{j\geq 0}$ is a strictly increasing sequence.*

Proof. Straightforward (since $c > 0$). ■

Lemma 3.3. *The sequence $\{\beta_j\}_{j\geq 0}$ is a strictly increasing sequence of positive terms.*

Proof. From Lemmas 1 and 2 it is straightforward

$$\beta_j = nB_{j+1} - mB_j > 0.$$

On the other hand

$$B_{j+2} - B_{j+1} = (3c-1)B_{j+1} - B_j \geq B_{j+1} - B_j;$$

hence

$$\begin{aligned} \beta_j = nB_{j+1} - mB_j &= (n-m)B_{j+1} + m(B_{j+1} - B_j) \\ &< (n-m)B_{j+2} + m(B_{j+2} - B_{j+1} \\ &= nB_{j+2} - mB_{j+1} = \beta_{j+1} \end{aligned}$$

So the result is proved. ■

Lemma 3.4. *The sequence $\{\alpha_j\}_{j\geq 0}$ is a strictly increasing sequence of positive terms.*

Proof. Obviously $\alpha_0 = mB_1 - nB_0 = m > 0$. In the general case

$$\begin{aligned} \alpha_j &= m(3cB_j - B_{j-1}) - B_j \frac{3cm - \sqrt{(9c^2-4)m^2 - 4c^2}}{2} \\ &= B_j \frac{3cm + \sqrt{(9c^2-4)m^2 - 4c^2}}{2} - mB_{j-1} \\ &> mB_j - nB_{j-1} = \alpha_{j-1} \end{aligned}$$

taking into account Lemma 1 and $c > 0$. ∎

Corollary 3.5. *$\Delta_{c^2}^{(1)}$ and $\Delta_{c^2}^{(2)}$ have no points (x, y) with $0 \leq x \leq y \leq c$. That is to say, the interesting points from the point of view of the unicity conjecture lie in Ω_{c^2}.*

Proof. After Lemmas 3 and 4 it is enough proving $\alpha_1 > c$ and $\beta_1 > c$. But:

$$\begin{aligned} \alpha_1 &= 3cm - n \\ &= \frac{3cm + \sqrt{(9c^2-4)m^2 - 4c^2}}{2} \\ &\geq \frac{3c}{2} \cdot \frac{2c}{\sqrt{9c^2-4}} \\ &= \frac{3c}{\sqrt{9c^2-4}} \cdot c > c \\ \\ \beta_1 &= 3cn - m \geq c(3n-1) > c \end{aligned}$$

Hence the result is proved. ∎

The main result of the paper goes then as follows:

Proposition 3.6. *The Frobenius conjecture for the Markoff equation is equivalent to $|\Omega_{c^2}| \leq 1$ for all $c \in \mathbf{N}$.*

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