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A Note on Howe Duality Correspondence and Isotropy Representations for Unitary Lowest Weight Modules of Mp(n, R)

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Abstract. We give a new proof of the Howe duality theorem for the reductive dual pair $(Sp(n,\mathbb{R}), O(k))$ by using the isotropy representations for unitary lowest weight modules of the metaplectic group $Mp(n,\mathbb{R})$. The irreducible representations of $O(k)$ appearing in the Howe duality correspondence are specified explicitly by means of the branching rule of the representations of $O(k)$ restricted to orthogonal groups of smaller size.
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1. Introduction

This is a continuation of the articles [20] and [23] by the second named author. The purpose of this note is to make a small observation on classical invariant theory [8] in connection with geometric invariants for Harish-Chandra modules.

Let G be a connected reductive Lie group and K a maximal compact subgroup of G. We write $K_{\mathbb{C}}$ for the complexification of K. We denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the complexified Cartan decomposition of the Lie algebra $\mathfrak{g} = (\mathrm{Lie}\ G) \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathfrak{k} = \mathrm{Lie}\ K_{\mathbb{C}}$. Let $\mathbb{X}$ be a $(\mathfrak{g}, K_{\mathbb{C}})$-module of finite length. By means of a graded module gr $\mathbb{X}$ associated with $\mathbb{X}$, one can attach to $\mathbb{X}$ a $K_{\mathbb{C}}$-stable affine algebraic variety $\mathcal{V}(\mathbb{X})$ called the associated variety of $\mathbb{X}$. $\mathcal{V}(\mathbb{X})$ is a union of (finitely many) nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$.

The isotropy representation, attributed to David Vogan ([16], [17]), gives a kind of *refined* geometric invariant for $\mathbb{X}$. More precisely, it is a finite dimensional representation of the isotropy subgroup $K_{\mathbb{C}}(\xi) = Z_{K_{\mathbb{C}}}(\xi)$ of a generic element ξ in each irreducible component of $\mathcal{V}(\mathbb{X})$. The dimension of this representation equals the multiplicity of gr $\mathbb{X}$ at the irreducible component $\overline{\mathrm{Ad}(K_{\mathbb{C}})\xi}$ (see also [23, Section 2]). These kinds of geometric invariants play an essential role to understand infinite dimensional representations of G in connection with nilpotent orbits in the Lie algebra (see e.g., [1], [2], [5]). Especially, we expect strong relationships

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among the isotropy representations, Howe duality correspondence and generalized Whittaker models for $\mathbb{X}$, as seen partially in [21] for the case of unitary highest (or lowest) weight representations.

In this paper, we consider the Howe duality correspondence ([8], [9]) for the classical reductive dual pair $(G, G') = (Sp(n, \mathbb{R}), O(k))$. Let $\widetilde{G} = Mp(n, \mathbb{R})$ be the metaplectic group of rank n, the double cover of G. It is shown that the Howe duality theorem (Theorem 4.1) for $(\widetilde{G}, G')$ can be reproduced by using the idea of isotropy representations for the irreducible constituents of tensor products of the Weil representation studied in [21]. This provides us with a new proof of some results of Kashiwara and Vergne [12] which are essential for the classification of unitary lowest weight representations of $\widetilde{G}$. Note that the compact member $G' = O(k)$ naturally arises in our theory of isotropy representations through a group homomorphism $p\colon \widetilde{K_{\mathbb{C}}}(\xi_m) \to G'$ given in (8). Our argument allows to identify the irreducible representations of G' appearing in the Howe duality correspondence (Theorem 4.4) explicitly by means of the branching rule of the representations of G' restricted to orthogonal groups of smaller size.

2. Isotropy representations

We start with a quick review on associated variety, associated cycle and isotropy representations for Harish-Chandra modules. See [16] (and also [23]) for the detail.

Let G be a connected reductive Lie group. We assume that G is linear or at most finite covering group of a linear group (see [17, Definition 2.5]). Let K be a maximal compact subgroup of G. Keep the notation in Introduction. We denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and let $\{U_n(\mathfrak{g})\}_{n\in\mathbb{Z}_{\geq 0}}$ be the standard increasing filtration of $U(\mathfrak{g})$. The associated graded algebra $\mathrm{gr}\, U(\mathfrak{g}) = \bigoplus_n U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ (with $U(\mathfrak{g})_{-1} = 0$) is naturally isomorphic to the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$. We identity $S(\mathfrak{g})$ with the ring $\mathbb{C}[\mathfrak{g}]$ of polynomial functions on $\mathfrak{g}$ by means of a $\mathfrak{g}$-invariant nondegenerate symmetric bilinear form B on $\mathfrak{g}$.

Now, let $\mathbb{X}$ be a $(\mathfrak{g}, K_{\mathbb{C}})$-module of finite length. Take a finite dimensional $K_{\mathbb{C}}$-stable subspace V of $\mathbb{X}$ such that $\mathbb{X} = U(\mathfrak{g})V$. Put $\mathbb{X}_n = U_n(\mathfrak{g})V$. Then $\{\mathbb{X}_n\}_{n\in\mathbb{Z}_{\geq 0}}$ gives a good filtration of $\mathbb{X}$. The associated graded module $\mathrm{gr}\, \mathbb{X} = \bigoplus_n \mathbb{X}_n/\mathbb{X}_{n-1}$ (with $\mathbb{X}_{-1} = 0$) has a natural structure of $(S(\mathfrak{g}), K_{\mathbb{C}})$-module. We set

$$\mathcal{V}(\mathbb{X}) = \{\xi \in \mathfrak{g} \mid f(\xi) = 0 \text{ for all } f \in \mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{gr}\, \mathbb{X})\}, \tag{1}$$

where $\mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{gr}\, \mathbb{X})$ denotes the annihilator of $\mathrm{gr}\, \mathbb{X}$ in $S(\mathfrak{g})$. Note that the affine variety $\mathcal{V}(\mathbb{X})$ is equal to the support of finitely generated module $\mathrm{gr}\, \mathbb{X}$ over $S(\mathfrak{g})$. $\mathcal{V}(\mathbb{X})$ is called the *associated variety* of $\mathbb{X}$.

We consider the irreducible decomposition $\mathcal{V}(\mathbb{X}) = \bigcup_{j=1}^{l} \mathcal{V}_j$ of the affine variety $\mathcal{V}(\mathbb{X})$, and let I_j be the prime ideal of $S(\mathfrak{g})$ corresponding to the irreducible component $\mathcal{V}_j$. Then the formal sum

$$\mathcal{C}(\mathbb{X}) = \sum_{j=1}^{l} m_j \cdot [\mathcal{V}_j] \quad \text{with} \quad m_j = \mathrm{length}_{S(\mathfrak{g})_{I_j}} (\mathrm{gr}\, \mathbb{X})_{I_j} \tag{2}$$

is called the *associated cycle* of $\mathbb{X}$. Here $(\mathrm{gr}\, \mathbb{X})_{I_j}$ denotes the localization of $\mathrm{gr}\, \mathbb{X}$ at I_j, which becomes a module over $S(\mathfrak{g})_{I_j}$ of finite length. Let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{g}$, and put $\mathcal{N}_{\mathfrak{p}} = \mathcal{N} \cap \mathfrak{p}$. It is known that $\mathcal{V}(\mathbb{X})$ and $\mathcal{C}(\mathbb{X})$ are independent of the choice of a $K_{\mathbb{C}}$-submodule V, and that $\mathcal{V}(\mathbb{X})$ is a $K_{\mathbb{C}}$-stable cone contained in $\mathcal{N}_{\mathfrak{p}}$.

Now we assume that $\mathcal{V}(\mathbb{X})$ is irreducible, for simplicity. Then there exists a $K_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathcal{N}_{\mathfrak{p}}$ such that $\mathcal{V}(\mathbb{X})$ is the closure of $\mathcal{O}$. Put $I = \sqrt{\mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{gr}\ \mathbb{X})}$ and take a positive integer n_0 such that $I^{n_0}\,\mathrm{gr}\ \mathbb{X} = 0$. By assumption, I is the prime ideal of $S(\mathfrak{g})$ defining $\mathcal{V}(\mathbb{X}) = \overline{\mathcal{O}}$. Take an element $\xi \in \mathcal{O}$ and put $K_{\mathbb{C}}(\xi) = \{k \in K_{\mathbb{C}} \mid \mathrm{Ad}(k)\xi = \xi\}$. Let $\mathfrak{m}(\xi)$ be the maximal ideal of $S(\mathfrak{g})$ defining one point $\{\xi\}$. Namely, $\mathfrak{m}(\xi)$ is the ideal of $S(\mathfrak{g})$ generated by elements $Y - B(\xi, Y)$ with all $Y \in \mathfrak{g}$. Then the module

$$\mathcal{W}(\mathbb{X}) = \bigoplus_{j=0}^{n_0-1} (I^j\,\mathrm{gr}\ \mathbb{X})/(\mathfrak{m}(\xi) I^j\,\mathrm{gr}\ \mathbb{X}) \tag{3}$$

has a natural structure of a finite dimensional $K_{\mathbb{C}}(\xi)$-representation. This is called the *isotropy representation* for $\mathbb{X}$. By Vogan, $\mathcal{W}(\mathbb{X})$ is independent of the choice of V as an element of the Grothendieck group of $K_{\mathbb{C}}(\xi)$-representations, and the dimension of $\mathcal{W}(\mathbb{X})$ coincides with the multiplicity $m = \mathrm{length}_{S(\mathfrak{g})_I}(\mathrm{gr}\ \mathbb{X})_I$ in the associated cycle $\mathcal{C}(\mathbb{X}) = m \cdot [\overline{\mathcal{O}}]$ of $\mathbb{X}$.

3. Weil representations

Let G be the real symplectic group $Sp(n, \mathbb{R})$ of rank n. We realize the group G in $GL(2n, \mathbb{C})$ as $G = \{cgc^{-1} \mid g \in GL(2n, \mathbb{R}),\ {}^t g J g = J\}$ with

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \quad \text{and} \quad c = \frac{1}{2}\begin{pmatrix} 1_n & -\sqrt{-1}\,1_n \\ -\sqrt{-1}\,1_n & 1_n \end{pmatrix},$$

where 1_n is the identity matrix of size n. The complexification $\mathfrak{g}$ of the Lie algebra of G consists of matrices

$$X(A, B, C) = \begin{pmatrix} A & B \\ C & -{}^t A \end{pmatrix} \quad \text{with} \quad A \in M_n(\mathbb{C}) \text{ and } B, C \in \mathrm{Sym}_n(\mathbb{C}).$$

Here $M_{m,n}(\mathbb{C})$ denotes the totality of complex matrices of size (m, n), and we set $M_n(\mathbb{C}) = M_{n,n}(\mathbb{C})$ and $\mathrm{Sym}_n(\mathbb{C}) = \{Z \in M_n(\mathbb{C}) \mid {}^t Z = Z\}$. Define a Cartan involution θ of G by $\theta(g) = {}^t\overline{g}^{-1}$. Then the groups K and $K_{\mathbb{C}}$ turn to be

$$K = \{\mathrm{diag}(g, {}^t g^{-1}) \mid g \in U(n)\} \text{ and } K_{\mathbb{C}} = \{\mathrm{diag}(g, {}^t g^{-1}) \mid g \in GL(n, \mathbb{C})\},$$

respectively. By the map $\mathrm{diag}(g, {}^t g^{-1}) \mapsto g$, we identify $K_{\mathbb{C}}$ with $GL(n, \mathbb{C})$. We put

$$\widetilde{K_{\mathbb{C}}} = \{(g, \varepsilon) \in K_{\mathbb{C}} \times \mathbb{C}^{\times} \mid \varepsilon^2 = \det g\}.$$

Then the map $(g, \varepsilon) \mapsto g$ from $\widetilde{K_{\mathbb{C}}}$ to $K_{\mathbb{C}}$ gives a double covering of $K_{\mathbb{C}}$. The group $\widetilde{K_{\mathbb{C}}}$ is naturally looked upon as the complexification of a maximal compact subgroup $\widetilde{K}$ of $\widetilde{G} = Mp(n, \mathbb{R})$.

The Lie algebra $\mathfrak{k}$ of $K_{\mathbb{C}}$ consists of matrices $X(A, 0, 0)$ with $A \in M_n(\mathbb{C})$. We set

$$\mathfrak{p}_+ = \{X(0, B, 0) \mid B \in \mathrm{Sym}_n(\mathbb{C})\} \text{ and } \mathfrak{p}_- = \{X(0, 0, C) \mid C \in \mathrm{Sym}_n(\mathbb{C})\}.$$

Then $\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$, and $\mathfrak{p}_\pm$ are $K_{\mathbb{C}}$-invariant abelian Lie subalgebras of $\mathfrak{g}$ contained in $\mathcal{N}_{\mathfrak{p}}$.

For $m = 1, 2, \ldots, n$, define $\xi_m \in \mathfrak{p}_-$ by

$$\xi_m = X(0, 0, C_m) \quad \text{with} \quad C_m = \sum_{i=1}^{m} E_{ii} = \begin{pmatrix} 1_m & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{pmatrix}, \tag{4}$$

where $E_{ij} = (\delta_{ip}\delta_{jq})_{p,q} \in M_n(\mathbb{C})$ is the (i,j) matrix unit. Put $\mathcal{O}_m = \mathrm{Ad}(K_{\mathbb{C}})\xi_m$. The group $GL(n,\mathbb{C})$ acts on $\mathrm{Sym}_n(\mathbb{C})$ by $g \cdot Z = gZ\,{}^t g$ ($g \in GL(n,\mathbb{C}), Z \in \mathrm{Sym}_n(\mathbb{C})$), and the set $\{Z \in \mathrm{Sym}_n(\mathbb{C}) \mid \mathrm{rank}\, Z = m\}$ forms a single $GL(n,\mathbb{C})$-orbit for every $m = 0, 1, \ldots, n$. This implies the following well-known lemma.

Lemma 3.1. *The set of $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_-$ is $\{\mathcal{O}_m \mid m = 0, 1, \ldots, n\}$, where $\mathcal{O}_0 = \{0\}$. Moreover, the closure of $\mathcal{O}_m$ is equal to $\mathcal{O}_0 \cup \cdots \cup \mathcal{O}_m$.*

We now introduce the $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-module of the ($k$-fold tensor product of) Weil representation of $\widetilde{G} = Mp(n,\mathbb{R})$, by using the Fock realization. For a positive integer k, let $\mathbb{P} = \mathbb{C}[M_{n,k}]$ be the ring of polynomials on $M_{n,k} = M_{n,k}(\mathbb{C})$. We write $z = (z_{ip})_{1 \le i \le n, 1 \le p \le k}$ for the variable of $M_{n,k}$. We take a basis of $\mathfrak{g}$ as

$$\begin{cases} A_{ij} = X(E_{ij}, 0, 0) \in \mathfrak{k} & (i, j = 1, 2, \ldots, n), \\ B_{ij} = X(0, E_{ij} + E_{ji}, 0) \in \mathfrak{p}_+ & (1 \le i \le j \le n), \\ C_{ij} = X(0, 0, E_{ij} + E_{ji}) \in \mathfrak{p}_- & (1 \le i \le j \le n). \end{cases}$$

Define an action of $\mathfrak{g}$ and $\widetilde{K_{\mathbb{C}}}$ on $\mathbb{P}$ through

$$\begin{cases} \varpi_k(A_{ij}) = \displaystyle\sum_{p=1}^{k} z_{ip} \frac{\partial}{\partial z_{jp}} + \frac{k}{2}\delta_{ij}, \\ \varpi_k(B_{ij}) = \sqrt{-1} \displaystyle\sum_{p=1}^{k} z_{ip} z_{jp}, \\ \varpi_k(C_{ij}) = \sqrt{-1} \displaystyle\sum_{p=1}^{k} \frac{\partial^2}{\partial z_{ip} \partial z_{jp}}, \end{cases}$$

and

$$(\varpi_k((g, \varepsilon)) f)(z) = \varepsilon^k f({}^t g z) \quad \left(f \in \mathbb{P},\ (g, \varepsilon) \in \widetilde{K_{\mathbb{C}}} \right).$$

Set $G' = O(k)$ and $G'_{\mathbb{C}} = O(k, \mathbb{C})$. We define a representation π' of $G'_{\mathbb{C}}$ on $\mathbb{P}$ by $(\pi'(g') f)(z) = f(zg')$ for $g' \in G'_{\mathbb{C}}$. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{P}$ defined by $\langle f, g \rangle = (f(\partial)\overline{g})(0)$, where $\overline{g}(z) = \overline{g(\overline{z})}$ and $\partial = (\partial / \partial z_{ip})_{i,p}$.

Lemma 3.2. (cf. [3], [7]. See also [6], [24]) *The action ϖ_k defines a unitary $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-module $\mathbb{P}$ with respect to the inner product $\langle \cdot, \cdot \rangle$. Moreover, the representation π' of $G'_{\mathbb{C}}$ commutes with ϖ_k.*

The representation ϖ_k is called the *Weil representation*. This comes from a representation of the double cover $\widetilde{G}$ of G, which does not factor through a representation of G if k is odd.

Let $\widehat{G'_{\mathbb{C}}}$ be the set of equivalence classes of finite dimensional irreducible holomorphic representations of $G'_{\mathbb{C}}$. Through Weyl's unitarian trick, $\widehat{G'_{\mathbb{C}}}$ is identified with the unitary dual of the compact orthogonal group $G' = O(k)$. For $(\sigma, V_\sigma) \in \widehat{G'_{\mathbb{C}}}$, put $L(\sigma) = \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{P})$. This is a $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-module, and $\mathbb{P}$ decomposes as

$$\mathbb{P} \simeq \bigoplus_{\sigma \in \Xi} L(\sigma) \otimes V_\sigma \quad \text{as } (\mathfrak{g}, \widetilde{K_{\mathbb{C}}}) \times G'_{\mathbb{C}}\text{-modules}, \tag{5}$$

where $\Xi = \{\sigma \in \widehat{G'_{\mathbb{C}}} \mid L(\sigma) \neq 0\}$.

4. Howe duality theorem

We now state the Howe duality theorem (cf. [8]) for the dual pair $(G, G') = (Sp(n,\mathbb{R}), O(k))$, which is essentially due to Kashiwara and Vergne [12].

Theorem 4.1. (1) *$L(\sigma)$ is an irreducible $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-module with lowest weight for every $\sigma \in \Xi$.*

(2) *For $\sigma_1, \sigma_2 \in \Xi$, $L(\sigma_1) \simeq L(\sigma_2)$ implies $\sigma_1 \simeq \sigma_2$.*

(3) *Let H be the subgroup of $G'_{\mathbb{C}}$ consisting of matrices* $\mathrm{diag}(1_n, h)$ *with $h \in O(k-n, \mathbb{C})$ for $k > n$, and let H be the identity group for $k \leq n$. Then one gets*

$$\Xi = \{\sigma \in \widehat{G'_{\mathbb{C}}} \mid V_\sigma^H \neq 0\}, \tag{6}$$

where V_σ^H denotes the space of H-fixed vectors in V_σ. In particular, it follows that $\Xi = \widehat{G'_{\mathbb{C}}}$ if $k \leq n$.

We will prove this theorem in Section 7 by using the idea of isotropy representations for $L(\sigma)$.

The characterization (6) of Ξ (cf. [21, Th. 5.14], see also [4]) allows to specify the parameters of $\sigma \in \Xi$ for $k > n$ in the following way.

Define a subset $\Lambda(k)$ of $\mathbb{Z}^k$ by

$$\Lambda(k) = \{(\lambda_1, \ldots, \lambda_l, 0_{k-l}),\ (\lambda_1, \ldots, \lambda_l, 1_{k-2l}, 0_l) \mid \\ \lambda_1 \geq \cdots \geq \lambda_l > 0,\ 0 \leq l \leq [k/2]\}.$$

Here $\epsilon_q = (\epsilon, \ldots, \epsilon)$ (q copies of ϵ) for $\epsilon = 0, 1$. Then the set $\Lambda(k)$ parametrizes finite dimensional irreducible holomorphic representations of $G'_{\mathbb{C}} = O(k,\mathbb{C})$ (Weyl's construction, see [11, §19.5]). More precisely, we can attach to each $\lambda \in \Lambda(k)$ an irreducible representation $\sigma_{k,\lambda}$ of $O(k,\mathbb{C})$, as follows. Set $V = \mathbb{C}^k$ and, as usual, we regard it as an $O(k,\mathbb{C})$-representation. Then V has an $O(k,\mathbb{C})$-invariant nondegenerate symmetric bilinear form $Q\colon V \otimes V \to \mathbb{C}$. We put $d = \sum_{i=1}^k a_i$ for $\lambda = (a_1, \ldots, a_k) \in \Lambda(k)$. For $1 \leq i < j \leq d$, Q defines a linear map $Q_{i,j}\colon V^{\otimes d} \to V^{\otimes (d-2)}$ by contraction:

$$v_1 \otimes \cdots \otimes v_d \longmapsto Q(v_i, v_j)\, v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes \hat{v}_j \otimes \cdots \otimes v_d.$$

Set $V^{[d]} = \bigcap_{1 \leq i < j \leq d} \mathrm{Ker}\, Q_{i,j}$. Let $\mathbb{S}_\lambda$ be the Schur functor ([11, §6.1]). Then $\mathbb{S}_\lambda V$ gives an irreducible $GL(k,\mathbb{C})$-subrepresentation of $V^{\otimes d}$ with highest weight λ. Set $\sigma_{k,\lambda} = V^{[d]} \cap \mathbb{S}_\lambda V$. This gives an irreducible $O(k,\mathbb{C})$-representation. The trivial representation corresponds to (0_k).

Then one gets the following branching rule.

Lemma 4.2. (cf. [14, Proposition 10.1]. See also [13]) *The restriction of* $\sigma_{k,\lambda} \in \widehat{G'_{\mathbb{C}}}$ *to the subgroup* $O(k-1,\mathbb{C})$ $(\subset G'_{\mathbb{C}}$ *as in Eq.* (7) *in Section 6*) *is of multiplicity free. Moreover, we have* $\mathrm{Hom}_{O(k-1,\mathbb{C})}(\sigma_{k-1,\mu},\sigma_{k,\lambda}) \neq 0$ *for* $\lambda = (a_1,\dots,a_k) \in \Lambda(k)$ *and* $\mu = (c_1,\dots,c_{k-1}) \in \Lambda(k-1)$ *if and only if* $a_1 \geq c_1 \geq \cdots \geq c_{k-1} \geq a_k$.

Remark 4.3. We thank Professor T. Kobayashi for informing us the parametrization of irreducible representations of $O(k)$ described in [13].

This lemma together with Theorem 4.1 (3) deduces the following theorem.

Theorem 4.4. *Assume that* $k > n$. *The module* $L(\sigma_{k,\lambda})$ *is nonzero if and only if*

$$\lambda = (a_1,\dots,a_i,0_{k-i})$$

for $a_1 \geq \cdots \geq a_i > 0$ *with* $i \leq n$, *or*

$$\lambda = (a_1,\dots,a_i,1_{k-2i},0_i)$$

for $a_1 \geq \cdots \geq a_i > 0$ *with* $i \geq k-n$.

Proof. By virtue of (6) we have $L(\sigma_{k,\lambda}) \neq 0$ if and only if $\sigma_{k,\lambda}|_H \supset \sigma_{k-n,(0_{k-n})}$.

First assume that $\lambda = (a_1,\dots,a_i,0_{k-i})$ for some $a_1 \geq \cdots \geq a_i > 0$, $i \leq n$. Put $\lambda_s = (a_1,\dots,a_{i-s},0_{k-i}) \in \Lambda(k-s)$ for $s \leq i$, and $\lambda_s = (0_{k-s}) \in \Lambda(k-s)$ for $s > i$. Then by Lemma 4.2, we have $\sigma_{k-s,\lambda_s}|_{O(k-s-1,\mathbb{C})} \supset \sigma_{k-s-1,\lambda_{s+1}}$ for $s = 0,1,\dots,n-1$. Since $\lambda_0 = \lambda$ and $\lambda_n = (0_{k-n})$, we have $L(\sigma_{k,\lambda}) \neq 0$.

Next assume that $\lambda = (a_1,\dots,a_i,1_{k-2i},0_i)$ for some $a_1 \geq \cdots \geq a_i > 0$, $i \geq k-n$. Define $\lambda_s \in \Lambda(k-s)$ as follows. We put $\lambda_s = (a_1,\dots,a_{i-s},1_{k-2i},0_i)$ for $0 \leq s \leq i$, $\lambda_s = (1_{k-i-s},0_i)$ for $i < s \leq k-i$, and $\lambda_s = (0_{k-s})$ for $k-i < s \leq n$. By Lemma 4.2, we have $\sigma_{k-s,\lambda_s}|_{O(k-s-1,\mathbb{C})} \supset \sigma_{k-s-1,\lambda_{s+1}}$. Since $\lambda_0 = \lambda$ and $\lambda_n = (0_{k-n})$, we find $L(\sigma_{k,\lambda}) \neq 0$.

Conversely, we prove that if $\lambda = (a_1,\dots,a_k) \in \Lambda(k)$ satisfies $\sigma_{k,\lambda}|_H \supset \sigma_{k-n,(0_{k-n})}$, then $a_i = 0$ for $i > n$. For $s \leq n$ there exists $\lambda_s = (a_1^{(s)},\dots,a_{k-s}^{(s)}) \in \Lambda(k-s)$ such that $a_1^{(s)} \geq a_1^{(s+1)} \geq \cdots \geq a_{k-s-1}^{(s+1)} \geq a_{k-s}^{(s)}$, $\lambda_0 = \lambda$ and $\lambda_n = (0_{k-n})$. If $i > n$ then $a_i = a_i^{(0)} \leq a_{i-1}^{(1)} \leq \cdots \leq a_{i-n}^{(n)} = 0$. Hence one gets $a_i = 0$ for $i > n$.

We thus complete the proof of the theorem. ∎

Remark 4.5. We can establish the results corresponding to Theorems 4.1 and 4.4 also for the dual pairs $(U(p,q),U(k))$ and $(SO^*(2n),Sp(k))$. But, the argument and the detail are considerably different from the present case $(Sp(n,\mathbb{R}),O(k))$, if G/K is of non-tube type. For this reason, we will treat them elsewhere (see also [24]).

5. Pluriharmonic polynomials and associated graded modules

We put $\mathcal{H} = \{f \in \mathbb{P} \mid \varpi_k(\mathfrak{p}_-)f = 0\}$. A polynomial in $\mathcal{H}$ is called *pluriharmonic*. Since $\mathfrak{p}_-$ is stable under the adjoint action of $\widetilde{K_{\mathbb{C}}}$, the subspace $\mathcal{H}$ is $\widetilde{K_{\mathbb{C}}}$-stable. It is stable under the representation π' of $G'_{\mathbb{C}}$ also. This section introduces graded $(S(\mathfrak{g}),\widetilde{K_{\mathbb{C}}})$-modules associated with irreducible constituents of $\mathbb{P}$, in connection with the pluriharmonic polynomials.

Let us begin with

Proposition 5.1. ([12, Lemma (5.3)]) *We have* $\varpi_k(U(\mathfrak{p}_+))\mathcal{H} = \mathbb{P}$.

We now set $U(\sigma) = \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathcal{H}) \subset L(\sigma)$ for $\sigma \in \widehat{G'_{\mathbb{C}}}$. $U(\sigma)$ is a $\widetilde{K_{\mathbb{C}}}$-submodule of $L(\sigma)$. It follows from the above proposition that

$$L(\sigma) = \varpi_k(U(\mathfrak{p}_+))U(\sigma) = \varpi_k(U(\mathfrak{g}))U(\sigma).$$

Using the space $\mathcal{H}$, we can define an increasing filtration of $\mathbb{P}$ by $\mathbb{P}^{(j)} = U_j(\mathfrak{g})\mathcal{H}$ $(j = 0, 1, \dots)$. Since $U(\mathfrak{p}_-)$ and $U(\mathfrak{k})$ preserve $\mathcal{H}$, we have $\mathbb{P}^{(j)} = U_j(\mathfrak{p}_+)\mathcal{H}$. Moreover, the action of $S(\mathfrak{p}_- \oplus \mathfrak{k})$ on the associated graded module $\mathrm{gr}\ \mathbb{P} = \bigoplus_{j\geq 0} \mathbb{P}^{(j)}/\mathbb{P}^{(j-1)}$ is trivial. More precisely one gets

Proposition 5.2. *The associated graded* $(S(\mathfrak{g}), \widetilde{K_{\mathbb{C}}})$*-module* $\mathrm{gr}\ \mathbb{P} = \bigoplus_{j\geq 0} \mathbb{P}^{(j)}/\mathbb{P}^{(j-1)}$ *is realized on the original* $\mathbb{P}$, *where* $\mathfrak{k}+\mathfrak{p}_-$ *acts on* $\mathbb{P}$ *trivially, and* $S(\mathfrak{p}_+) = U(\mathfrak{p}_+)$, $\widetilde{K_{\mathbb{C}}}$ *act by* ϖ_k.

Lemma 5.3. *Let* L *be an irreducible* $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$*-submodule of* $\mathbb{P}$. *Then for every* $j \geq 0$, $\dim(L \cap \mathbb{P}^{(j)}) < \infty$, *and* $\{L \cap \mathbb{P}^{(j)}\}_{j=0,1,\dots}$ *gives a good filtration of* L.

6. $\widetilde{K_{\mathbb{C}}}(\xi_m)$-modules $L(\sigma)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma)$

In this section, we consider the associated graded $(S(\mathfrak{g}), \widetilde{K_{\mathbb{C}}})$-module $\mathrm{gr}\ \mathbb{P}$ realized on $\mathbb{P}$, and we describe after [21, Section 5] the representation on the right hand side of Eq. (3) for $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-modules $L(\sigma)$. First, we define a map $\psi\colon M_{n,k} \to \mathfrak{p}_-$ by

$$\psi(z) = \begin{pmatrix} 0 & 0 \\ z\,{}^t z & 0 \end{pmatrix} \quad (z \in M_{n,k}),$$

and put $m = \min(n, k)$. The pair (G, G') is called in the *stable range* if $k \leq n$.

The following lemma is easy to prove by noting that the map ψ is $K_{\mathbb{C}}$-equivariant, where $K_{\mathbb{C}} = GL(n, \mathbb{C})$ acts on $M_{n,k}$ by left multiplication.

Lemma 6.1. *We have* $\mathrm{Im}\,\psi = \overline{\mathcal{O}_m}$, *where* $\mathcal{O}_m = \mathrm{Ad}(K_{\mathbb{C}})\xi_m$ *is the nilpotent* $K_{\mathbb{C}}$*-orbit in* $\mathfrak{p}_-$ *in Lemma 3.1.*

Define a $\mathfrak{g}$-invariant nondegenerate symmetric bilinear form B on $\mathfrak{g}$ by $B(X, Y) = (\sqrt{-1}/2) \cdot \mathrm{Tr}(XY)$ for $X, Y \in \mathfrak{g}$. We identify $S(\mathfrak{g})$ with $\mathbb{C}[\mathfrak{g}]$ through B. The following lemma is clear from the definition of ϖ_k.

Lemma 6.2. *We have* $(\varpi_k(D)f)(z) = D(\psi(z))f(z)$ *for all* $D \in S(\mathfrak{p}_+)$.

Set $\mathfrak{m}(\xi_m) = \sum_{X \in \mathfrak{p}_+}(X - B(\xi_m, X))S(\mathfrak{p}_+)$. Then $\mathfrak{m}(\xi_m)$ is the maximal ideal of $S(\mathfrak{p}_+)$ defining the point $\xi_m \in \mathfrak{p}_- = \mathfrak{p}_+^*$.

Proposition 6.3. *Put* $I(\overline{\mathcal{O}_m}) = \{D \in S(\mathfrak{g}) \mid D(\xi) = 0\ (\xi \in \overline{\mathcal{O}_m})\}$.

(1) *For all* $f \in \mathbb{P} \setminus \{0\}$, *we have* $\mathrm{Ann}_{S(\mathfrak{g})} f = I(\overline{\mathcal{O}_m})$. *In particular, the associated variety of every irreducible subrepresentation of* $\mathbb{P}$ *is equal to* $\overline{\mathcal{O}_m}$.

(2) *For every irreducible subrepresentation* L *of* $\mathbb{P}$, *the isotropy representation for* L *is given by* $L/\varpi_k(\mathfrak{m}(\xi_m))L$ *on which* $\widetilde{K_{\mathbb{C}}}(\xi_m) = Z_{\widetilde{K_{\mathbb{C}}}}(\xi_m)$ *acts through* ϖ_k.

Now, Lemma 6.2 implies that

$$\{z \in M_{n,k} \mid f(z) = 0 \text{ for all } f \in \varpi_k(\mathfrak{m}(\xi_m))\mathbb{P}\} = \psi^{-1}(\xi_m).$$

More precisely, Proposition 5.8 in [21], which amounts to a classical result of Weyl [19, Th.(5.2.C)] on the orthogonal ideal for the case $k \leq n$, assures that $\varpi_k(\mathfrak{m}(\xi_m))\mathbb{P}$ is the defining ideal of $\psi^{-1}(\xi_m)$. Noting that $\psi^{-1}(\xi_m)$ is stable under the action of $K_{\mathbb{C}}(\xi_m) \times G'_{\mathbb{C}}$, one thus gets the following lemma.

Lemma 6.4. *The ideal $\varpi_k(\mathfrak{m}(\xi_m))\mathbb{P}$ of $\mathbb{P} = \mathbb{C}[M_{n,k}]$ is reduced. Therefore we have*

$$\mathbb{P}/\varpi_k(\mathfrak{m}(\xi_m))\mathbb{P} \simeq \mathbb{C}[\psi^{-1}(\xi_m)],$$
$$L(\sigma)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma) \simeq \operatorname{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[\psi^{-1}(\xi_m)]).$$

In order to prove the Howe duality theorem, we are going to identify the $\widetilde{K_{\mathbb{C}}}(\xi_m)$-module $L(\sigma)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma)$ that will turn out to be the isotropy representation for $L(\sigma)$ (Corollary 7.4). But it should be noticed that, at present, we do not know yet whether $L(\sigma)$ is of finite length or not. We now describe after [21] the variety $\psi^{-1}(\xi_m)$ and the action of $\widetilde{K_{\mathbb{C}}}(\xi_m)$ on $\operatorname{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[\psi^{-1}(\xi_m)])$. For $l < k$, we regard $O(l, \mathbb{C})$ as a subgroup of $G'_{\mathbb{C}} = O(k, \mathbb{C})$ by

$$O(l, \mathbb{C}) = \left\{ \begin{pmatrix} 1_{k-l} & 0 \\ 0 & h \end{pmatrix} \in G'_{\mathbb{C}} \right\}. \tag{7}$$

If $l \leq 0$, we put $O(l, \mathbb{C}) = \{e\}$.

Lemma 6.5. (1) *The isotropy subgroup $\widetilde{K_{\mathbb{C}}}(\xi_m) = Z_{\widetilde{K_{\mathbb{C}}}}(\xi_m)$ consists of $\tilde{g} = (g, \varepsilon) \in \widetilde{K_{\mathbb{C}}}$ with*

$$g = \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix}, \; g_{11} \in O(m, \mathbb{C}), \; g_{22} \in GL(n-m, \mathbb{C}).$$

Here g stands for $g = g_{11} \in O(n, \mathbb{C})$ if $m = n$ (or equivalently if $k \geq n$).

(2) *We have $\psi^{-1}(\xi_m) \simeq H\backslash G'_{\mathbb{C}}$ as $G'_{\mathbb{C}}$-varieties, where $H = O(k-n, \mathbb{C})$.*

Let $p \colon \widetilde{K_{\mathbb{C}}}(\xi_m) \to O(k, \mathbb{C})$ and $e \colon \widetilde{K_{\mathbb{C}}}(\xi_m) \to \mathbb{C}^\times$ be group homomorphisms defined by

$$p\left(\left(\begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix}, \varepsilon\right)\right) = \begin{pmatrix} g_{11} & 0 \\ 0 & 1_{k-m} \end{pmatrix}, \quad e\left(\left(\begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix}, \varepsilon\right)\right) = \varepsilon^k \tag{8}$$

for $g_{11} \in O(m, \mathbb{C})$, $g_{21} \in M_{n-m,m}$ and $g_{22} \in GL(n-m, \mathbb{C})$.

Lemma 6.6. *The action of $\widetilde{g} \in \widetilde{K_{\mathbb{C}}}(\xi_m)$ on $f \in \mathbb{C}[\psi^{-1}(\xi_m)] \simeq \mathbb{C}[H\backslash G'_{\mathbb{C}}]$ is given by*

$$(\widetilde{g} \cdot f)(Hg') = e(\widetilde{g}) f(H\, {}^t p(\widetilde{g}) g') \quad (g' \in G'_{\mathbb{C}} = O(k, \mathbb{C})).$$

Using the above three lemmas together with the theorem of Peter-Weyl, we deduce the following description of $L(\sigma)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma)$ for every finite dimensional irreducible holomorphic representation (σ, V_σ) of $G'_{\mathbb{C}}$.

Proposition 6.7. *For $(\sigma, V_\sigma) \in \widehat{G'_{\mathbb{C}}}$, we have*

$$L(\sigma)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma) \simeq (V_\sigma^*)^H \quad \text{as } \widetilde{K_{\mathbb{C}}}(\xi_m)\text{-modules.}$$

Here the action of $\widetilde{g} \in \widetilde{K_{\mathbb{C}}}(\xi_m)$ on $(V_\sigma^*)^H$ is given by

$$\widetilde{g} \cdot v = e(\widetilde{g})^t p(\widetilde{g})^{-1} v \quad \text{for } v \in (V_\sigma^*)^H.$$

This immediately implies the following

Corollary 6.8. *Assume that (G, G') is in the stable range. The representation $L(\sigma)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma)$ of $\widetilde{K_{\mathbb{C}}}(\xi_m)$ is nonzero and irreducible for every $\sigma \in \widehat{G'_{\mathbb{C}}}$.*

7. Proof of Howe duality theorem

We are now in a position to prove the Howe duality Theorem (Theorem 4.1).

7.1. First, let us prepare the following basic proposition resulted from the unitarity of the Weil representation.

Proposition 7.1. *Let L be any nonzero $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodule of $\mathbb{P} = \mathbb{C}[M_{n,k}]$.*

(1) *If N is a $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodule of L, then one gets $L = N \oplus N'$ as $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-modules. Here $N' = \{f \in L \mid \langle f, g\rangle = 0 \ (g \in N)\}$ denotes the subspace of L orthogonal to N with respect to the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$.*

(2) *L decomposes into a direct sum of (at most countably many) irreducible $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodules of L with lowest weights.*

Proof. (1) The $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodule L of $\mathbb{P} = \mathbb{C}[M_{n,k}]$ is stable under the action of the Euler operator:

$$\sum_{i,p} z_{ip} \frac{\partial}{\partial z_{ip}} = \sum_i \varpi_k(A_{ii}) - \frac{nk}{2}\mathrm{Id}.$$

This implies $L = \bigoplus_{j \geq 0} L_{(j)}$ with $L_{(j)} = L \cap \mathbb{P}_{(j)}$, where $\mathbb{P}_{(j)}$ denotes the (finite dimensional) subspace of $\mathbb{P}$ consisting of homogeneous polynomials of degree j. Similarly one sees $N = \bigoplus_{j \geq 0} N_{(j)}$ with $N_{(j)} = N \cap \mathbb{P}_{(j)} \subset L_{(j)}$.

Noting that the subspaces $\mathbb{P}_{(j)}$ $(j = 0, 1, \ldots)$ are orthogonal with respect to the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ in question, the subspace N' of L orthogonal to N turns to be a direct sum of the orthogonal complements $N'_{(j)}$ of $N_{(j)}$ in the finite dimensional Hilbert spaces $L_{(j)}$:

$$N' = \bigoplus_{j \geq 0} N'_{(j)}, \quad L_{(j)} = N_{(j)} \oplus N'_{(j)},$$

which implies $L = N \oplus N'$. Moreover, N' is a $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodule of L by the unitarity of the Weil representation ϖ_k.

(2) First, let j_1 be the smallest integer such that $L_{(j_1)} \neq 0$, and consider a finitely generated $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodule $N_{j_1} = U(\mathfrak{g})L_{(j_1)}$ of L generated by the homogeneous component $L_{(j_1)}$ of degree j_1. Let

$$L_{(j_1)} = \bigoplus_{r=1}^{t} V_r$$

an irreducible decomposition of the $\widetilde{K_{\mathbb{C}}}$-representation $L_{(j_1)}$, where the irreducible constituents V_r are orthogonal to each other. Take a lowest weight vector v_r of the $\widetilde{K_{\mathbb{C}}}$-module V_r. Then v_r becomes a lowest weight vector with respect to $\mathfrak{g}$, since $\mathfrak{p}_- L_{(j_1)} \subset L_{(j_1-2)} = 0$. We thus get

$$N_{j_1} = \sum_{r=1}^{t} U(\mathfrak{g}) V_r \quad \text{with} \quad U(\mathfrak{g}) V_r = U(\mathfrak{p}_+) V_r = U(\mathfrak{g}) v_r.$$

By virtue of the unitarity of ϖ_k, the sum in the right hand side is an orthogonal direct sum, and each lowest weight submodule $U(\mathfrak{g}) v_r$ is irreducible (in general, any unitary $\mathfrak{g}$-module generated by a single lowest weight vector is irreducible). In this way, N_{j_1} is decomposed into an orthogonal direct sum of finitely many (irreducible) unitary lowest weight modules.

In view of the assertion (1), the $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-module L decomposes as

$$L = N_{j_1} \oplus N'_{j_1} \quad \text{with} \quad N'_{j_1} \subset \bigoplus_{j > j_1} L_{(j)}.$$

Repeating the above argument (for L replaced by N'_{j_1}) successively, we find an increasing sequence of nonnegative integers $j_1 < j_2 < \cdots < j_k < \cdots$ and mutually orthogonal $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodules N_{j_k} of L such that

$$L = \bigoplus_{j_k} N_{j_k}, \quad N_{j_k} \subset \bigoplus_{j \geq j_k} L_{(j)},$$

and that each N_{j_k} is a direct sum of finite number of irreducible unitary $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodules with lowest weights. We thus complete the proof of the proposition. ∎

7.2. Now one obtains Theorem 4.1 (3) from the following lemma together with Proposition 6.7. (Notice that $V_\sigma^H \neq 0$ if and only if $(V_\sigma^*)^H \neq 0$.)

Lemma 7.2. *Let L be a $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodule of $\mathbb{P}$. The space $L/\mathfrak{m}(\xi_m)L$ is zero if and only if L is zero.*

Proof. Assume that $L/\mathfrak{m}(\xi_m)L$ is zero. By Proposition 7.1, L decomposes into a direct sum of irreducible $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodules. Then we get $L_{\mathfrak{m}(\xi_m)} = 0$ by applying Nakayama's lemma to each irreducible constituent of L. Hence for all $f \in L$ there exists $D \in S(\mathfrak{p}_+) \simeq \mathbb{C}[\mathfrak{p}_-]$ such that $D(\xi_m) \neq 0$ and $Df = 0$. Then $D \notin I(\overline{\mathcal{O}_m})$. If $f \neq 0$ then we have $\mathrm{Ann}_{S(\mathfrak{g})} f = I(\overline{\mathcal{O}_m})$ by Proposition 6.3 (1). This is a contradiction. One thus concludes $L = 0$. ∎

7.3. We proceed to the proof of the assertions (1) and (2). First, let us consider the case of stable range.

Proof. (of Theorem 4.1 (1) and (2): case of stable range) Assume that (G, G') is in the stable range with G' a smaller member, i.e., $k \leq n$. Then by Corollary 6.8, the $\widetilde{K_{\mathbb{C}}}(\xi_m)$-representation $L(\sigma)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma)$ is irreducible for every $\sigma \in \widehat{G'_{\mathbb{C}}}$. Assume that $L(\sigma)$ is reducible. Since $L(\sigma)$ is isomorphic to a

$(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-submodule of $\mathbb{P}$, there exist nonzero subrepresentations L_1 and L_2 of $L(\sigma)$ such that $L(\sigma) = L_1 \oplus L_2$. Then $L(\sigma)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma) = (L_1/\varpi_k(\mathfrak{m}(\xi_m))L_1) \oplus (L_2/\varpi_k(\mathfrak{m}(\xi_m))L_2)$. Hence we have $L_1/\varpi_k(\mathfrak{m}(\xi_m))L_1 = 0$ or $L_2/\varpi_k(\mathfrak{m}(\xi_m))L_2 = 0$ by the irreducibility of $L(\sigma)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma)$. By Lemma 7.2, we get $L_1 = 0$ or $L_2 = 0$. This is a contradiction. We have proved (1).

To prove (2), assume that $L(\sigma_1) \simeq L(\sigma_2)$. Then one gets by virtue of Proposition 6.7

$$V_{\sigma_1}^* \simeq L(\sigma_1)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma_1) \simeq L(\sigma_2)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma_2) \simeq V_{\sigma_2}^*$$

as $\widetilde{K_{\mathbb{C}}}(\xi_m)$-modules. This implies $\sigma_1 \simeq \sigma_2$ as desired. ∎

7.4. Second, we assume that $k > n$. Put $\check{\mathfrak{g}} = \mathfrak{sp}(k, \mathbb{C})$ and we regard $\mathfrak{g}$ as a subalgebra of $\check{\mathfrak{g}}$ by

$$\mathfrak{g} \ni \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \longmapsto \begin{pmatrix} A & 0_{n,k-n} & B & 0_{n,k-n} \\ 0_{k-n,n} & 0_{k-n,k-n} & 0_{k-n,n} & 0_{k-n,k-n} \\ C & 0_{n,k-n} & -{}^tA & 0_{n,k-n} \\ 0_{k-n,n} & 0_{k-n,k-n} & 0_{k-n,n} & 0_{k-n,k-n} \end{pmatrix} \in \check{\mathfrak{g}}$$

for $A \in \mathfrak{gl}(n, \mathbb{C})$ and $B, C \in \mathrm{Sym}_n(\mathbb{C})$. Let $\widetilde{K_{\mathbb{C}}}^{\vee}$ be the double cover of $GL(k, \mathbb{C})$ and $\check{\varpi}_k$ be the Weil representation of $(\check{\mathfrak{g}}, \widetilde{K_{\mathbb{C}}}^{\vee})$ on $\check{\mathbb{P}} = \mathbb{C}[M_{k,k}]$. We can define an injective $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}}) \times G'_{\mathbb{C}}$-homomorphism $\mathbb{P} \to \check{\mathbb{P}}$, $f \mapsto \check{f}$, by

$$\check{f}\left(\begin{pmatrix} z \\ w \end{pmatrix}\right) = f(z) \quad \text{for } z \in M_{n,k} \text{ and } w \in M_{k-n,k}.$$

Now, let $\mathfrak{b} = \mathfrak{t} + \mathfrak{n}$ be a Borel subalgebra of $\mathfrak{g}$ with $\mathfrak{t} = \sum_{i=1}^n \mathbb{C}A_{ii}$ and $\mathfrak{n} = \mathfrak{p}_- + \sum_{i<j} \mathbb{C}A_{ij}$. We write $\check{\mathfrak{k}}, \check{\mathfrak{p}}_+, \check{\mathfrak{p}}_-, \check{\mathfrak{b}} = \check{\mathfrak{t}} + \check{\mathfrak{n}}$ for the subalgebras of $\check{\mathfrak{g}}$ corresponding to $\mathfrak{k}, \mathfrak{p}_+, \mathfrak{p}_-, \mathfrak{b} = \mathfrak{t} + \mathfrak{n}$ for $\mathfrak{g}$, respectively. We extend a linear form μ on $\mathfrak{t}$ to $\check{\mu}$ on $\check{\mathfrak{t}}$ by putting $\check{\mu}(A_{ii}) = 0$ for $i > n$. The following lemma is easily proved.

Lemma 7.3. *If f is a $\mathfrak{b}$-lowest weight vector of $\mathbb{P}$ with lowest weight μ, then $\check{f}$ gives a $\check{\mathfrak{b}}$-lowest weight vector of $\check{\mathbb{P}}$ with lowest weight $\check{\mu}$.*

Proof. (of Theorem 4.1 (1) and (2): out of stable range case) We put $\check{L}(\sigma) = \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \check{\mathbb{P}})$. Then $\check{L}(\sigma)$ is an irreducible $(\check{\mathfrak{g}}, \widetilde{K_{\mathbb{C}}}^{\vee})$-module as already proved in Section 7.3, because $(\check{G}, G') = (Sp(k, \mathbb{R}), O(k))$ is in the stable range. Hence $\check{\mathfrak{b}}$-lowest weight vectors of $\check{L}(\sigma)$ are unique up to constant multiples. By virtue of Lemma 7.3, the same holds for $L(\sigma)$. This implies the irreducibility of $L(\sigma)$ in view of Proposition 7.1 (2).

Assume that $L(\sigma_1) \simeq L(\sigma_2)$ for $\sigma_1, \sigma_2 \in \Xi$, and take a $\mathfrak{b}$-lowest weight vector $f_i \in L(\sigma_i)$ $(i = 1, 2)$. Then by Lemma 7.3, $\check{f}_i$ is a $\check{\mathfrak{b}}$-lowest weight vector of $\check{L}(\sigma_i)$ and the weight of $\check{f}_1$ is equal to that of $\check{f}_2$. Hence $\check{L}(\sigma_1) \simeq \check{L}(\sigma_2)$. Therefore we conclude $\sigma_1 \simeq \sigma_2$ as shown in Section 7.3. ∎

Thus we have completed the proof of Theorem 4.1.

7.5. By Proposition 6.3 and the irreducibility of $L(\sigma)$, we have the following corollary that specifies the isotropy representation for $L(\sigma)$.

Corollary 7.4. *Let $L(\sigma)$ be the irreducible $(\mathfrak{g}, \widetilde{K_{\mathbb{C}}})$-module corresponding to $\sigma \in \Xi$.*

(1) *The associated variety of $L(\sigma)$ is $\overline{\mathcal{O}_m}$.*

(2) *The isotropy representation $\mathcal{W}(L(\sigma))$ for $L(\sigma)$ equals*

$$L(\sigma)/\varpi_k(\mathfrak{m}(\xi_m))L(\sigma) \simeq (V_\sigma^*)^H$$

described in Proposition 6.7. *In particular, $\mathcal{W}(L(\sigma))$ is irreducible if (G, G') is in the stable range.*

We can show that the isotropy representation $\mathcal{W}(\mathbb{X})$ is irreducible for every singular unitary highest (or lowest) weight representation $\mathbb{X}$ of a simple Lie group of Hermitian type (including the exceptional case). This will be treated in another paper [18] (see [22] for the announcement of the results).

It should be an interesting and important problem to study isotropy representations for Harish-Chandra modules in connection with the transcending version of the dual pair correspondence (cf. [9]), which we hope to study in near future.

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