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Diamond Representations for Rank Two Semisimple Lie Algebras

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Abstract. The present work is a part of a larger program to construct explicit combinatorial models for the (indecomposable) regular representation of the nilpotent factor N in the Iwasawa decomposition of a semisimple Lie algebra $\mathfrak{g}$, using the restrictions to N of the simple finite dimensional modules of $\mathfrak{g}$. Such a description is given in Arnal, D., N. Bel Baraka, and N.-J. Wildberger, *Diamond representations of* $\mathfrak{sl}(n)$, Annales Mathématiques Blaise Pascal **13** (2006), 381–429 for the case $\mathfrak{g} = \mathfrak{sl}(n)$. Here, we perform the same construction for the rank 2 semisimple Lie algebras (of type $A_1 \times A_1$, A_2, C_2 and G_2). The algebra $\mathbb{C}[N]$ of polynomial functions on N is a quotient, called the reduced shape algebra, of the shape algebra for $\mathfrak{g}$. Bases for the shape algebra are known, for instance the so-called semistandard Young tableaux give an explicit basis (see Alverson, L.-W., R.-G. Donnelly, S.-J. Lewis, M. McClard, R. Pervine, R.-A. Proctor, and N.-J. Wildberger, *Distributive lattice defined for representations of rank two semisimple Lie algebras*, SIAM J. Discrete Math. 23 (2008/09), no. 1, 527–559). We select among the semistandard tableaux, the so-called quasistandard ones which define a kind basis for the reduced shape algebra.
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1. Introduction

We study the diamond cone of representations for the nilpotent factor N^+ of any rank 2 semisimple Lie algebra $\mathfrak{g}$. This is the indecomposable regular representation onto $\mathbb{C}[N^-]$, described from explicit realizations of the restrictions to N^+ of the simple $\mathfrak{g}$-modules V^λ.

In [ABW06], this description is explicitly given in the case $\mathfrak{g} = \mathfrak{sl}(n)$, using the notion of quasistandard Young tableaux. Roughly speaking, a quasistandard Young tableau is an usual semistandard Young tableau such that, it is impossible

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to extract the top of the first column, either because this top of column is not 'trivial', *i.e.* it does not consist of numbers $1, 2, \dots, k$, or because, when we extract this top by pushing to the left the k first rows of the tableau, we do not get a semistandard tableau.

Let us come back for the case of rank 2 Lie algebra $\mathfrak{g}$. The modules V^λ have well known explicit realizations (see for instance [FH91]). They are characterized by their highest weight $\lambda = a\omega_1 + b\omega_2$, integral combination of fundamental weights. In [ADLMPPrW07], there is a construction for a basis for each V^λ, as the collection of all semistandard tableaux with shape (a, b). The definition and construction of semistandard tableaux for $\mathfrak{g}$ uses the notion of grid poset and their ideals. It is possible to perform compositions of grid posets, the ideals of these compositions (of a grid posets associated to V^{ω_1} and b grid posets associated to V^{ω_2}) give a basis for V^λ if $\lambda = a\omega_1 + b\omega_2$.

Here, we realize the Lie algebra $\mathfrak{g}$ as a subalgebra of $\mathfrak{sl}(n)$ (with $n = 4, 3, 4, 7$), and we recall the notion of shape algebra for $\mathfrak{g}$, it is the direct sum of all the simple modules V^λ, but we see it as the algebra $\mathbb{C}[G]^{N^+}$ of all the polynomial functions on the group G (corresponding to $\mathfrak{g}$), which are invariant under right action by elements in N^+. This gives a very concrete interpretation of the semistandard tableaux for $\mathfrak{g}$ as a product of determinant functions for submatrices.

The algebra $\mathbb{C}[N^-]$ is the restriction to N^- of the functions in $\mathbb{C}[G]$. But it is also a quotient of the shape algebra by the ideal generated by $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} - 1$, $\boxed{1} - 1$. We call this quotient the reduced shape algebra for $\mathfrak{g}$. To give a basis for this quotient, we define, case by case, the quasistandard tableaux for $\mathfrak{g}$. They are semistandard Young tableaux, with an extra condition, which is very similar to the condition given in the $\mathfrak{sl}(n)$ case. We prove that the quasistandard Young tableaux give a kind basis for the reduced shape algebra.

2. Semistandard and quasistandard Young tableaux for $SL(n)$

Semistandard Young tableaux Recall that the Lie algebra $\mathfrak{sl}(n) = \mathfrak{sl}(n, \mathbb{C})$ is the set of $n \times n$ traceless matrices, it is the Lie algebra of the Lie group $SL(n)$ of $n \times n$ matrices, with determinant 1.

Denote N^+ the subgroup of all the upper triangular matrices $n^+ = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

Let us consider the algebra $\mathbb{C}[SL(n)]^{N^+}$ of polynomial functions on the group $SL(n)$, which are invariant under the right multiplication by the subgroup N^+. The group $SL(n)$ acts on this space by multiplication on the left by the transpose of g: $(g.f)(g_1) = f({}^t g g_1)$, for any f in $\mathbb{C}[SL(n)]^{N^+}$, any g in $SL(n)$.

Example 2.1. Let $k < n$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. We define:

$$\delta_{i_1,\dots,i_k} = \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_k \\ \hline \end{array} : SL(n) \longrightarrow \quad \mathbb{C}$$

$$g \longmapsto \det(submatrix(g, (i_1 ... i_k, 1 ... k)))$$

i.e. for an element $g \in SL(n)$, we associate the polynomial function which is the determinant of the submatrix of g obtained by considering the k first columns of g and the rows $i_1, \dots, i_k$.

If k is fixed, $SL(n)$ acts on the vector space spanned by all columns $\delta_{i_1,\dots,i_k}$ as on $\wedge^k \mathbb{C}^n$.

Thus we look for $Sym^\bullet(\bigwedge \mathbb{C}^n) = Sym^\bullet(\mathbb{C}^n \oplus \wedge^2\mathbb{C}^n \oplus \cdots \oplus \wedge^{n-1}\mathbb{C}^n)$. A natural basis for this algebra is given by the Young tableaux

$$\begin{array}{|c|c|c|c|}\hline i_1^1 & i_1^2 & \cdots & i_1^r \\ \hline \vdots & \vdots & \vdots & \multicolumn{1}{c}{} \\ \cline{1-3} & i_{k_2}^2 & \multicolumn{2}{c}{} \\ \cline{1-2} i_{k_1}^1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

such that $k_1 \geq k_2 \geq ... \geq k_r$ and $\begin{pmatrix} i_1^j \\ \vdots \\ i_{k_j}^j \end{pmatrix} \leq \begin{pmatrix} i_1^{j+1} \\ \vdots \\ i_{k_j}^{j+1} \end{pmatrix}$ for the lexicographic ordering if $k_j = k_{j+1}$.

Recall now that the fundamental representations of $\mathfrak{sl}(n)$ are the natural ones on $\mathbb{C}^n, \dots, \wedge^{n-1}\mathbb{C}^n$ with highest weights $\omega_1, \dots, \omega_{n-1}$.

It is well known that each simple $\mathfrak{sl}(n)$-module has a highest weight λ and the module is characterized by its highest weight. The highest weights are exactly the elements

$$\lambda = a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}$$

where $a_1, \dots, a_{n-1}$ are nonnegative integral numbers. Note $\mathbb{S}^\lambda$ (or $\Gamma_{a_1,\dots,a_{n-1}}$) this module, it is a submodule of the tensor product

$$Sym^{a_1}(\mathbb{C}^n) \otimes Sym^{a_2}(\wedge^2\mathbb{C}^n) \otimes \cdots \otimes Sym^{a_{n-1}}(\wedge^{n-1}\mathbb{C}^n).$$

The direct sum $\mathbb{S}^\bullet$ of all the simple modules $\mathbb{S}^\lambda$ is the shape algebra of $SL(n)$. As an algebra, it is isomorphic to $\mathbb{C}[SL(n)]^{N^+}$ (see [FH91]).

Now, we have a natural mapping from $Sym^\bullet(\mathbb{C}^n \oplus \cdots \oplus \wedge^{n-1}\mathbb{C}^n)$ to $\mathbb{C}[SL(n)]^{N^+}$ which is just the evaluation map:

$$\begin{array}{|c|c|c|c|}\hline i_1^1 & i_1^2 & \cdots & i_1^r \\ \hline \vdots & \vdots & \vdots & \multicolumn{1}{c}{} \\ \cline{1-3} & i_{k_2}^2 & \multicolumn{2}{c}{} \\ \cline{1-2} i_{k_1}^1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \longmapsto \left(g \longmapsto \delta_{i_1^1,\dots,i_{k_1}^1}(g) \dots \delta^r_{i_1^r,\dots,i_{k_r}^r}(g) \right).$$

But, thanks to the Gauss method, each N^+ right invariant monomial function on $SL(n)$ is a product of functions $\delta_{i_1,\dots,i_k}$, thus:

Proposition 2.2. *The map from* $Sym^\bullet(\bigwedge \mathbb{C}^n) = Sym^\bullet(\mathbb{C}^n \oplus \cdots \oplus \wedge^{n-1}\mathbb{C}^n)$ *to* $\mathbb{S}^\bullet = \mathbb{C}[SL(n)]^{N^+}$ *is onto.*

Definition 2.3. Let T be a Young tableau. If T contains a_i columns with height i $(i = 1, ..., n-1)$, we call shape of T the $(n-1)$-uplet $\lambda(T) = (a_1, ..., a_{n-1})$. We consider the partial ordering on the family of shapes defined by:

$$\mu = (b_1, \dots, b_{n-1}) \leq \lambda = (a_1, \dots, a_{n-1}) \text{ if and only if } b_1 \leq a_1, \quad \dots, \quad b_{n-1} \leq a_{n-1}.$$

Definition 2.4. A Young tableau of shape λ is semistandard if its entries are increasing along each row (and strictly increasing along each column).

Theorem 2.5. *1) The algebra* $\mathbb{S}^\bullet = \bigoplus_\lambda \mathbb{S}^\lambda$, *is isomorphic to the quotient of* $Sym^\bullet(\bigwedge \mathbb{C}^n)$ *by the kernel* $\mathcal{PL}$ *of the evaluation mapping. This ideal is generated by the Plücker relations*

$$\delta_{i_1\dots i_p}\delta_{j_1\dots j_q} - \sum_{s=1}^{p} \delta_{i_1\dots j_1\dots i_p}\delta_{i_s j_2\dots j_q} = 0 \qquad (p \geq q).$$

2) If $\lambda = a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}$, *a basis for* $\mathbb{S}^\lambda$ *is given by the set of semistandard Young tableaux* T *of shape* λ.

Example 2.6. The $\mathfrak{sl}(3)$ case

We have one and only one Plücker relation:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = 0.$$

Then to obtain a basis for the algebra $\mathbb{S}^\bullet$, we reject exactly the nonsemistandard Young tableaux: the tableaux which contain $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$ as a subtableau.

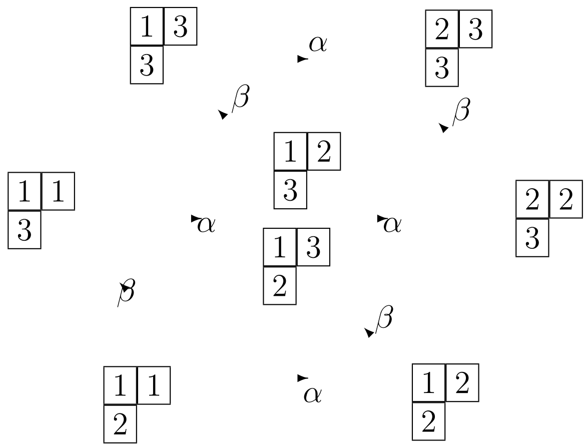
We look at the action of the nilpotent group N^+ onto the lowest weight vector v_λ in $\mathbb{S}^\lambda$. This action generates the representation space $\mathbb{S}^\lambda$. The semistandard Young tableaux with shape λ is a weight vector basis for $\mathbb{S}^\lambda$.

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(n)$ is the $(n-1)$ dimensional vector space consisting of diagonal, traceless matrices $H = (h_{ij})$. The usual basis $(\alpha_1, \dots, \alpha_{n-1})$ of $\mathfrak{h}^*$ is given by simple roots $\alpha_i = \tau_i - \tau_{i+1}$ where $\tau_i(H) = h_{ii}$.

Now, $\mathfrak{h}^*$ is an Euclidean vector space with a scalar product given by the Killing form. We can thus draw pictures in the real vector space $\mathfrak{h}^*_{\mathbb{R}}$ generated by the α_i.

Let us do that for $\mathfrak{sl}(3)$. We note $\alpha = \alpha_1$ and $\beta = \alpha_2$. The action of X_α (resp. X_β) on a weight vector is pictured by an arrow α (resp. β).

Example 2.7. With the above convention, we get the weight diagrams for the modules $\Gamma_{1,1}$ for $\mathfrak{sl}(3)$:



Quasistandard Young tableaux for $\mathfrak{sl}(n)$ Now we are interested by the restriction of polynomial functions on $SL(n)$ to the subgroup $N^- = {}^tN^+$. This restriction leads to an exact sequence (see [ABW06])

$$0 \longrightarrow \left\langle \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline k \\ \hline \end{array} - 1,\ k = 1,\dots,n-1 \right\rangle \longrightarrow \mathbb{C}[SL(n)]^{N^+} \longrightarrow \mathbb{C}[N^-] \longrightarrow 0.$$

($< w_k >$ denotes the ideal generated by the w_k). Or:

$$0 \longrightarrow \left\langle \ \delta_{1,\dots,k} - 1 \ \right\rangle + \mathcal{PL} = \mathcal{PL}_{red} \longrightarrow Sym^\bullet(\bigwedge \mathbb{C}^n) \longrightarrow \mathbb{C}[N^-] \longrightarrow 0.$$

For instance, in $SL(3)$, the Plücker relation becomes in $\mathcal{PL}_{red}$ a relation among semistandard tableaux:

$$\boxed{3} + \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} = 0.$$

Now, we look for a basis for $\mathbb{C}[N^-]$, by selecting some semistandard Young tableaux.

Definition 2.8. The column $\delta_{1,2,\dots,k}$ is said trivial. Suppose T is a Young tableau whose first column has a trivial top: $\delta_{1,\dots,k,i_{k+1},\dots,i_r}$, and there is a column with height k. We say we push T if we shift the k firsts rows of T to the left and we delete the top of the first column which spill out. Denote $P(T)$ the new tableau obtained. If $P(T)$ is a semistandard Young tableau, we say that T is nonquasistandard. Else, T is quasistandard.

Example 2.9. The $\mathfrak{sl}(3)$ case
The tableaux

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ , \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

are nonquasistandard tableaux, but the tableaux

$$\begin{array}{|c|}\hline 3 \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$$

are quasistandard.

To find a basis of $\mathbb{C}[N^-]$, adapted to its N^+ module structure, we restrict ourselves to quasistandard Young tableaux.

Theorem 2.10. *The set of quasistandard Young tableaux form a basis for the algebra $\mathbb{C}[N^-]$.*

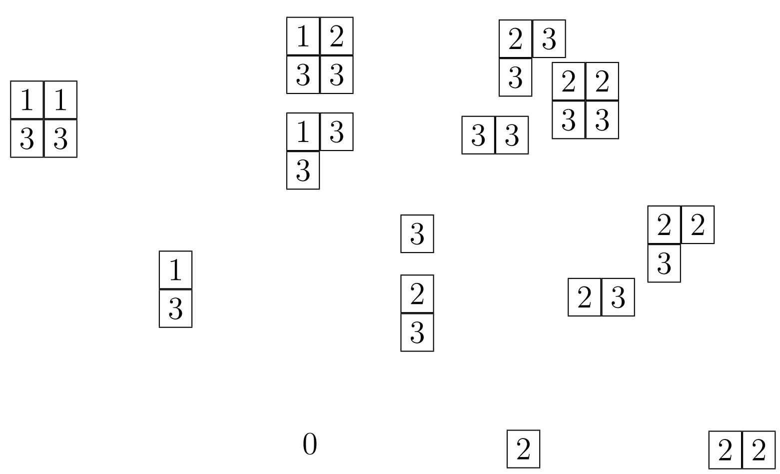
To be more precise, if we denote π the canonical mapping:

$$\pi:\ \mathbb{S}^\bullet = \mathbb{C}[SL(n)]^{N^+} \longrightarrow \mathbb{C}[N^-] = Sym^\bullet(\bigwedge \mathbb{C}^n)/\mathcal{PL}_{red},$$

the algebra of polynomial functions on N^- is an indecomposable N^+-module, called the diamond representation of N^+, each module $\mathbb{S}^\lambda|_{N^+}$ is occurring in $\mathbb{C}[N^-]$ as the image by π of $\mathbb{S}^\lambda$.

Proposition 2.11. *A parametrization of a basis for the quotient $\pi(\mathbb{S}^\lambda) = \mathbb{S}^\lambda|_{N^+}$ is given by the set of quasistandard Young tableaux of shape $\leq \lambda$.*

Example 2.12. For the Lie algebra $\mathfrak{sl}(3)$, we get the picture:



3. Principle of our construction. Fundamental representations

The purpose of this article is to address in the same way the rank two semisimple Lie algebras. Let us first recall the definition of semistandard Young tableaux for the algebras $A_1 \times A_1$, A_2, C_2 and G_2. In the present section, we define the semistandard tableaux with one column.

Let us realize the rank two semisimple Lie algebras as subalgebras of $\mathfrak{sl}(n)$ for $n = 4$,3, 4, 7 in such a way that the simple coroots X_α and X_β (α denotes the 'short' simple root and β denotes the 'long' simple root) are matrices such that:

$$\begin{aligned} t &\longmapsto \text{ first row of } tX_\alpha \\ (t,s) &\longmapsto \text{ two first rows of } tX_\alpha + sX_\beta \end{aligned} \qquad (*)$$

are one-to-one.
Explicitly, we choose the following realizations:
$\underline{\underline{A_1 \times A_1 = \mathfrak{sl}(2) \times \mathfrak{sl}(2)}}$
Let $(g_1, g_2) \in \mathfrak{sl}(2) \times \mathfrak{sl}(2)$ where $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ such that $a_i + d_i = 0$. We thus modify the natural realization of the Lie algebra $A_1 \times A_1$ as:

$$X = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & d_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & c_2 & d_2 \end{pmatrix} \longmapsto \begin{pmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & c_2 & d_2 & 0 \\ c_1 & 0 & 0 & d_1 \end{pmatrix},$$

(we acts on the basis vectors with the permutation (2,4,3)). Then

$$N^- = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & y & 1 & 0 \\ x & 0 & 0 & 1 \end{pmatrix}, \ x, y \in \mathbb{C} \right\}.$$

$\underline{\underline{A_2 = \mathfrak{sl}(3)}}$:
Let $g \in \mathfrak{sl}(3)$ i.e

$$g = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad \text{such that} \quad a_1 + b_2 + c_3 = 0.$$

then

$$N^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}, \ x, y, z \in \mathbb{C} \right\}.$$

With this parametrization, we immediately see the Plücker relation in $\mathcal{PL}_{red}$:

$$\boxed{3}\,(g) + \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}\,(g) - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\,(g) = z + (xy - z) - yx = 0.$$

$\underline{\underline{C_2 = \mathfrak{sp}(4)}}$:

The natural realization of the Lie algebra $\mathfrak{sp}(4)$ is given by $X = \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}$ where A, B, C are 2×2 matrices, and ${}^tB = B$, ${}^tC = C$. We modify this realization by permuting the basis vectors 3 and 4:

$$X = \begin{pmatrix} a & b & u & v \\ c & d & v & w \\ x & y & -a & -c \\ y & z & -b & -d \end{pmatrix} \longmapsto \begin{pmatrix} a & b & v & u \\ c & d & w & v \\ y & z & -d & -b \\ x & y & -c & -a \end{pmatrix}.$$

Then the group N^- becomes:

$$N^- = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ z & u & 1 & 0 \\ y & z - xu & -x & 1 \end{pmatrix}, \quad x,\ y,\ z,\ u \in \mathbb{C} \right\}.$$

$\underline{\underline{G_2}}$:

The natural realization of the Lie algebra G_2 is given by:

$$X = \begin{pmatrix} A & V & -j(\frac{W}{\sqrt{2}}) \\ -{}^tW & 0 & -{}^tV \\ -j(\frac{V}{\sqrt{2}}) & W & -{}^tA \end{pmatrix}$$

where V, W are 3×1 column-matrices, $j(U)$ is the 3×3 matrix of the exterior product in $\mathbb{C}^3$: $j(U)V = U \wedge V$ and A is a 3×3 matrix such that $tr(A) = 0$.

To imbed N^- in the space of lower triangular matrices, we effect the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 1 & 4 & 5 & 6 & 3 \end{pmatrix}$ on the vector basis. Then, we obtain the Lie algebra of N^-:

$$\mathfrak{n}^- = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & 0 & 0 & 0 \\ y & a & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}z & \sqrt{2}y & \sqrt{2}x & 0 & 0 & 0 & 0 \\ -b & -z & 0 & -\sqrt{2}x & 0 & 0 & 0 \\ -c & 0 & z & -\sqrt{2}y & -a & 0 & 0 \\ 0 & c & b & -\sqrt{2}z & -y & x & 0 \end{pmatrix} \right\}$$

and the following corresponding group: N^- is the set of matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 & 0 & 0 \\ y & a & 1 & 0 & 0 & 0 & 0 \\ z & -\sqrt{2}ax + \sqrt{2}y & -\sqrt{2}x & 1 & 0 & 0 & 0 \\ b & -ax^2 + xy - \frac{\sqrt{2}}{2}z & -x^2 & \sqrt{2}x & 1 & 0 & 0 \\ c & axy + \frac{\sqrt{2}}{2}az - y^2 & xy + \frac{\sqrt{2}}{2}z & -\sqrt{2}y & -a & 1 & 0 \\ -yb - xc - \frac{z^2}{2} & \frac{\sqrt{2}}{2}axz - ab - \frac{\sqrt{2}}{2}yz - c & \frac{\sqrt{2}}{2}xz - b & -z & -y + ax & -x & 1 \end{pmatrix},$$

with a, b, c, x, y, z in $\mathbb{C}$.

In each case, we now consider the Young tableaux with 1 column and 1 or 2 rows, corresponding to particular subrepresentations in $\mathbb{C}^n$ $(n = 4, 3, 4, 7)$ and $\wedge^2\mathbb{C}^n$, which are isomorphic to the fundamental representations $\Gamma_{1,0}$ and $\Gamma_{0,1}$ of the Lie algebra. This selection of tableaux can be viewed as the consequence of some 'internal' Plücker relations for our Lie algebra.

$\underline{\underline{A_1 \times A_1 = \mathfrak{sl}(2) \times \mathfrak{sl}(2)}}$:

The $\Gamma_{1,0}$ representation occurs in $\mathbb{C}^4$, we find the basis $\boxed{1}$, $\boxed{4}$ and 2 internal Plücker relations

$$\boxed{2} = 0, \ \boxed{3} = 0.$$

The $\Gamma_{0,1}$ representation occurs in $\wedge^2\mathbb{C}^4$, we find the basis $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ and $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$ and 4 internal Plücker relations

$$\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = 0, \quad \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} = 0, \quad \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} = -\boxed{4} \quad \text{and} \quad \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} = -\begin{array}{|c|c|}\hline 1 & 4\\\hline 3 & \multicolumn{1}{c}{}\\\cline{1-1}\end{array}.$$

Thus we get the following Young semistandard tableaux with 1 column, for $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$:

$$\boxed{1}\,, \boxed{4}\,, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \text{ and } \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}.$$

$\underline{\underline{A_2 = \mathfrak{sl}(3)}}$:

By definition, there is no internal Plücker relations for A_2, the semistandard Young tableaux with 1 column are:

$$\boxed{1}\,, \boxed{2}\,, \boxed{3}\,, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\,, \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \text{ and } \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}.$$

$\underline{\underline{C_2 = \mathfrak{sp}(4)}}$:

The $\Gamma_{1,0}$ representation occurs in $\mathbb{C}^4$, we find the basis $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ and $\boxed{4}$.
The $\Gamma_{0,1}$ representation is the quotient of $\wedge^2\mathbb{C}^4$ by the invariant symplectic form. Then we have 1 internal Plücker relation which is written as follows:

$$\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = 0.$$

Thus we choose the Young semistandard tableaux with 1 column, for $\mathfrak{sp}(4)$:

$$\boxed{1}\,, \boxed{2}\,, \boxed{3}\,, \boxed{4}\,, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\,, \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\,, \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\,, \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} \text{ and } \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}.$$

This choice does not coincide with the choice made in [ADLMPPrW07], but it is more coherent with the G_2 construction and more convenient for the description of quasistandard tableaux.

$\underline{\underline{G_2}}$:

The $\Gamma_{1,0}$ representation occurs in $\mathbb{C}^7$, we find the basis $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\boxed{5}$,

$\boxed{6}$ and $\boxed{7}$.
The $\Gamma_{0,1}$ representation is the quotient of $\wedge^2\mathbb{C}^7$ by a seven dimensional module. Then we have 7 internal Plücker relations which are:

$$\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \sqrt{2}\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = 0,\quad \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} - \sqrt{2}\begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array} = 0,\quad \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} + \sqrt{2}\begin{array}{|c|}\hline 1\\\hline 6\\\hline\end{array} = 0,\quad \begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array} + \sqrt{2}\begin{array}{|c|}\hline 2\\\hline 7\\\hline\end{array} = 0,$$

$$\begin{array}{|c|}\hline 4\\\hline 6\\\hline\end{array} - \sqrt{2}\begin{array}{|c|}\hline 3\\\hline 7\\\hline\end{array} = 0,\quad \begin{array}{|c|}\hline 4\\\hline 7\\\hline\end{array} + \sqrt{2}\begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array} = 0 \text{ and } \begin{array}{|c|}\hline 1\\\hline 7\\\hline\end{array} - \begin{array}{|c|}\hline 2\\\hline 6\\\hline\end{array} - \begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} = 0.$$

Indeed, in view of the lower triangular matrices in G_2, with 1 on the diagonal, we directly find these relations for the corresponding functions on N^-. Moreover, these relations are covariant under the action of the diagonal matrices, they are thus holding for the corresponding functions on the lower triangular matrices in G_2, with any nonvanishing diagonal entries, thus by N^+ invariance, they hold on G_2.
Therefore we choose the Young semistandard tableaux with 1 column, for G_2:

$$\boxed{1}\,,\boxed{2}\,,\boxed{3}\,,\boxed{4}\,,\boxed{5}\,,\boxed{6}\,,\boxed{7}\,,$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 6\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 7\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 6\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 7\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 6\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 7\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline 7\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 7\\\hline\end{array} \text{ and } \begin{array}{|c|}\hline 6\\\hline 7\\\hline\end{array}.$$

This choice does coincide with the choice made in [ADLMPPrW07].

4. Semistandard Young tableaux

Following [ADLMPPrW07], there is a construction of semistandard Young tableaux for $\Gamma_{a,b}$, for any a and b, knowing those of $\Gamma_{0,1}$ and $\Gamma_{1,0}$. In fact, by a general result of Kostant (see [FH91] for instance), each nonsemistandard Young tableau contains a nonsemistandard tableau with 2 columns. Thus, it is sufficient to determine all nonsemistandard tableaux with 2 columns. (In fact we shall get conditions for 1 or 2 successive columns $T^{(i)}$ and $T^{(i+1)}$ in the tableau T).

We begin to look the fundamental representations $\Gamma_{0,1}$ and $\Gamma_{1,0}$ for the rank two semisimple Lie algebras as spaces generated by a succession of action of $X_{-\alpha}$ and $X_{-\beta}$ on the highest weight vector.

$\underline{\underline{A_1 \times A_1}}$:

The fundamental representations look like:

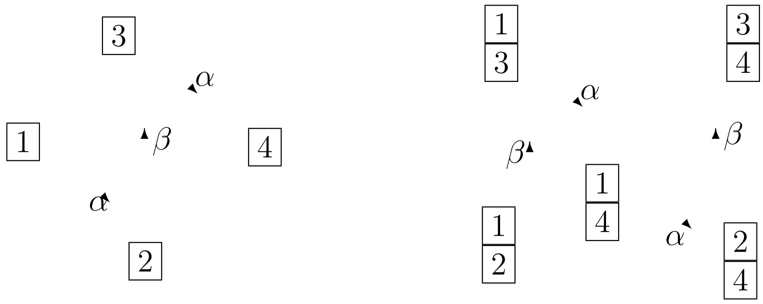
$$\boxed{1} \xrightarrow{\alpha} \boxed{4} \qquad\qquad \begin{array}{c}\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\\ \uparrow \beta \\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{array}$$

We associate to these drawings the following two ordered sets (respectively):

α $\qquad\qquad$ β

$\underline{\underline{C_2}}$:

The fundamental representations look like:



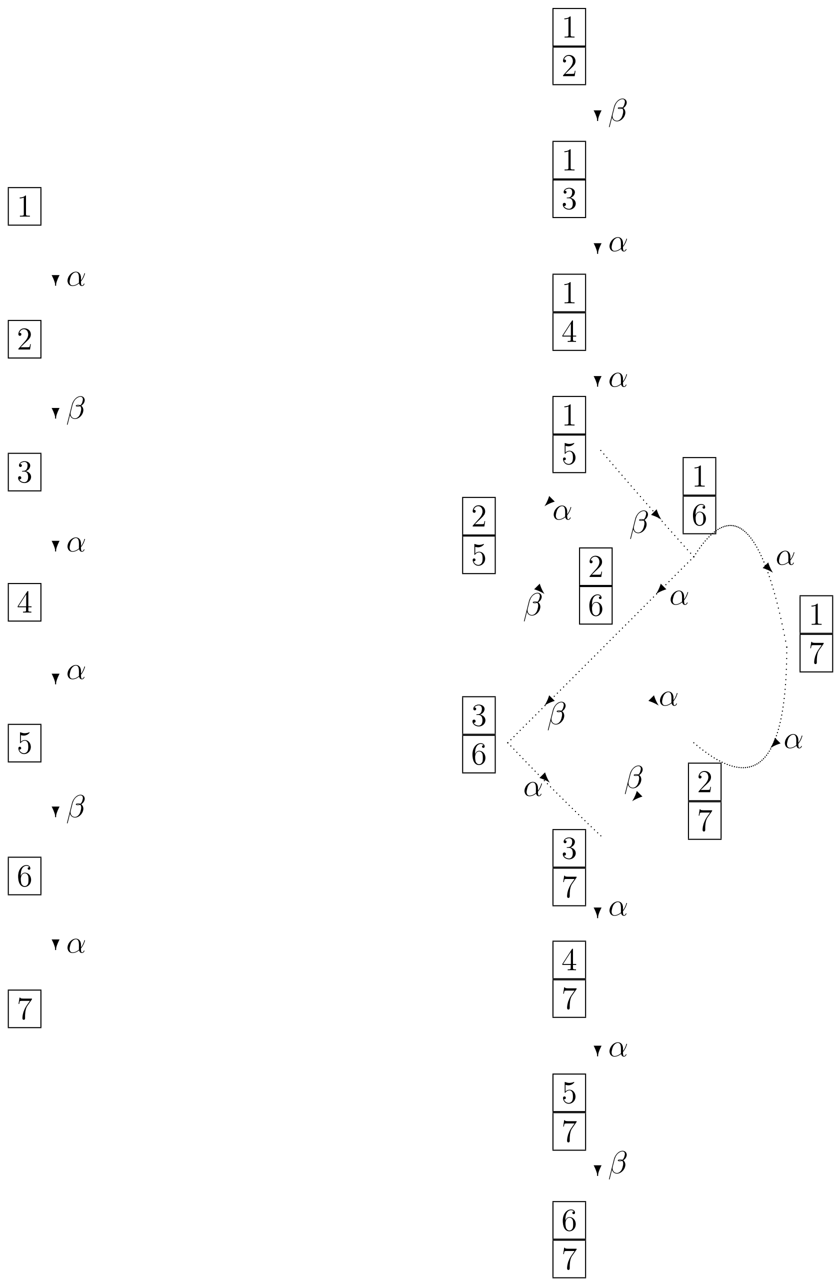
Then, we associate to these drawing the two following ordered sets (respectively):

$$\boxed{1} \xrightarrow{\alpha} \boxed{2} \xrightarrow{\beta} \boxed{3} \xrightarrow{\alpha} \boxed{4}$$

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} \xrightarrow{\beta} \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array} \xrightarrow{\alpha} \begin{array}{|c|}\hline 1\\ \hline 4\\ \hline\end{array} \xrightarrow{\alpha} \begin{array}{|c|}\hline 2\\ \hline 4\\ \hline\end{array} \xrightarrow{\beta} \begin{array}{|c|}\hline 3\\ \hline 4\\ \hline\end{array}$$

$\underline{\underline{G_2}}$:

For the G_2 case, we give just the two following ordered sets associated to the two fundamental representations of G_2:



We can now realize these chosen paths as the family L of ideals of some partially ordered sets P (which are called posets). An ideal in P is a subset $I \subset P$ such that if $u \in P$ and $v \leq u$, then $v \in I$. With our choice, we take the following fundamental posets denoted $P_{1,0}$ and $P_{0,1}$ and we associate for each of them the correspondent distributive lattice of their ideals respectively denoted $L_{1,0}$ and $L_{0,1}$.

Example 4.1.

$\underline{\underline{A_1 \times A_1}}$:

$\underline{P_{1,0}}$:

α

$\underline{L_{1,0}}$:

(α)

α

$(\varnothing)$

$\underline{P_{0,1}}$:

β

$\underline{L_{0,1}}$:

(β)

β

$(\varnothing)$

$\underline{\underline{C_2}}$:

$\underline{P_{1,0}}$:

α

β

α

$\underline{L_{1,0}}$:

α β α

α β α

β

(α)

α

$(\varnothing)$

$\underline{P_{0,1}}$:

β

α

α

β

$\underline{L_{0,1}}$:

β α α β β

α α β α

α β

α

(β)

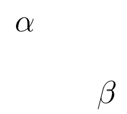
β

$(\varnothing)$

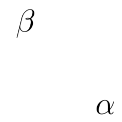
For the A_2 and G_2 cases, we just draw the fundamental posets $P_{0,1}$ and $P_{1,0}$, (for more details, see [ADLMPPrW07]).

$\underline{\underline{A_2}}$:

$\underline{P_{1,0}}$:

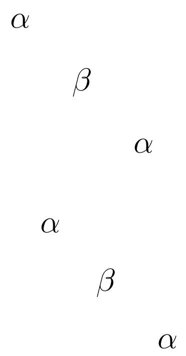


$\underline{P_{0,1}}$:

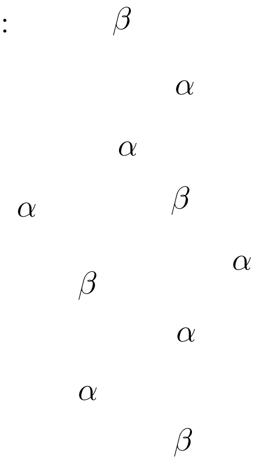


$\underline{\underline{G_2}}$:

$\underline{P_{1,0}}$:



$\underline{P_{0,1}}$:



We shall generalize this construction for all irreducible representations. We want to define the poset $P_{a,b}$ associated to the representation $\Gamma_{a,b}$ in such a way that $L_{a,b}$ gives us the possible paths in $\Gamma_{a,b}$.
We need some definitions (see [ADLMPPrW07]).

Definition 4.2. 1) Let $(P, \leq)$ be a partially ordered set and $v, w \in P$ such that $v \leq w$. We define the interval $[v, w]$ as the set

$$[v, w] = \{x \in P : v \leq x \leq w\}.$$

We say that w covers v if $[v, w] = \{v, w\}$.

2) A two-color poset is a poset P for which we can associate for each vertex in P a color α or β. The function $v \longmapsto \mathit{color}(v)$ is the **color** function .

3) We are going to select and numbered some chains in P. To do this, we define a chain function:

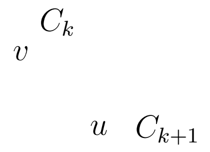
$$\mathbf{chain} : P \longrightarrow [[1, m]]$$

such that:

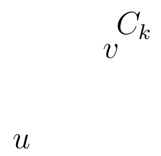
i) for $1 \leq i \leq m$, $\mathbf{chain}^{-1}(i)$ is a (possibly empty) chain in P.

ii) for any u, $v \in P$, if v covers u then either $\mathbf{chain}(u) = \mathbf{chain}(v)$ or $\mathbf{chain}(u) = \mathbf{chain}(v) + 1$.

We represent the function chain as follows:
If $\mathbf{chain}(u) = \mathbf{chain}(v) + 1 = k + 1$ then we draw:

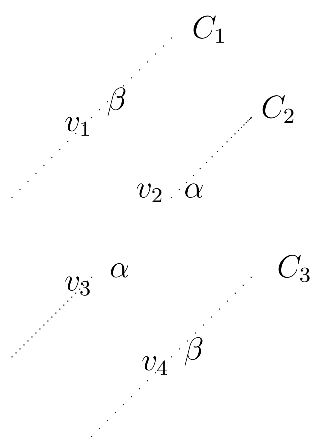


and if $\mathbf{chain}(u) = \mathbf{chain}(v) = k$ then we draw:



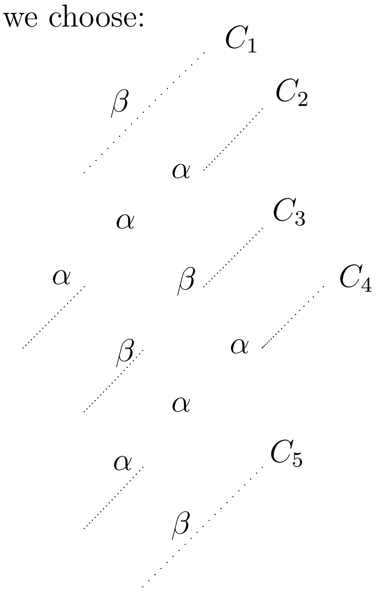
Examples 4.3. For the C_2 case, we shall choose:

$P_{0,1}$:



For the G_2 case, we choose:

$P_{0,1}$:



These pictures represent the fundamental posets with the function color and the function chain. They are uniquely defined with the grid property.

Definition 4.4. A two-color grid poset is a poset $(P, \leq)$ together with a chain function **chain** and a color function **color** such that two vertices, u and v, in the same connected components of P satisfying:

i) if $\mathbf{chain}(u) = \mathbf{chain}(v) + 1$ then $\mathbf{color}(u) \neq \mathbf{color}(v)$,

ii) if $\mathbf{chain}(u) = \mathbf{chain}(v)$ then $\mathbf{color}(u) = \mathbf{color}(v)$.

Remark 4.5. On the fundamental posets, there is an unique chain map, up to a global translation, such that the result is the two-color grid poset. This choice corresponds to our drawing for each $P_{a,b}$ where $a + b = 1$.

Let us now define posets $P_{a,b}$, $a + b \geq 1$.

Definition 4.6. A grid is a two-color grid poset which has moreover the following "max" property:

i) if u is any maximal element in the poset P, then

$$\mathbf{chain}(u) \leq \inf_{x \in P} \ \mathbf{chain}(x) + 1,$$

ii) if $v \neq u$ is another maximal element in P, then

$$\mathbf{color}(u) \neq \mathbf{color}(v).$$

Remark 4.7. The fundamental posets are grid posets.

From now one, we identify two grid posets with the same poset, the same color function and two chain maps: **chain** and $\mathbf{chain}'$, if there exists an integer k such that $\mathbf{chain}'(u) = \mathbf{chain}(u) + k$ for any u.

Definition 4.8. Given two grid posets P and Q, we denote by $P \triangleleft Q$ the grid poset with the following properties:

i) The elements of $P \triangleleft Q$ is the union of elements of P and those of Q.

ii) P is an ideal of $P \triangleleft Q$ i.e if $u \in P$ and $v \leq u$ in $P \triangleleft Q$ then $v \in P$, the functions color and chain of P are the restriction of the functions color and chain of $P \triangleleft Q$ (up to a renumbering of chains).

iii) $(P \triangleleft Q) \backslash P$ with the restriction of functions color and chain on $P \triangleleft Q$ is isomorphic to Q (up to a renumbering of chains).

iv) If u (resp. v) is a maximal element in P (resp. in Q), then

$$\mathbf{chain}(u) \leq \mathbf{chain}(v),$$

and if u (resp. v) is a minimal element in P (resp. in Q), then

$$\mathbf{chain}(u) \leq \mathbf{chain}(v).$$

If $P \lhd Q$ exists, thus $P \lhd Q$ is uniquely determined by these conditions, up to a translation on **chain**.

Remark 4.9. Given three grid posets P, Q, and R then $(P \lhd Q) \lhd R \simeq P \lhd (Q \lhd R)$. We denote this $P \lhd Q \lhd R$.

Starting with the grid posets $P_{1,0}$ and $P_{0,1}$ defined for the rank two semisimple Lie algebra, for any natural numbers a and b, there exists one and only one grid poset

$$P_{a,b} = \underbrace{P_{0,1} \lhd \ldots \lhd P_{0,1}}_{b} \lhd \underbrace{P_{1,0} \lhd \ldots \lhd P_{1,0}}_{a}.$$

Now, given the grid poset $P_{a,b}$, we obtain a basis of $\Gamma_{a,b}$ by building the corresponding distributive lattice $L_{a,b}$ of ideals in $P_{a,b}$ and labeling the vertices of $L_{a,b}$ as follows:

We start with the highest weight Young tableau of shape λ: b columns $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ and a columns $\boxed{1}$, we put this tableau on the vertex of $L_{a,b}$ corresponding to the total ideal $P_{a,b}$. Now, we reach any vertex in $L_{a,b}$ by following a sequence of edges α or β. By construction, we know if this edge corresponds to a vertex in $P_{0,1}$ or in $P_{1,0}$. If the corresponding vertex is in a $P_{1,0}$-component in $P_{a,b}$, we act with the edge on the first possible column with size 1, if it is in a $P_{0,1}$-component in $P_{a,b}$, we act with the edge on the first possible column with size 2. Then we get a basis for $\Gamma_{a,b}$ by Young tableaux, we call these tableaux the semistandard tableaux.

In fact a Young tableau T is semistandard if and only if each subtableau formed with two consecutive columns T^i, T^{i+1} is semistandard. Therefore, we just draw the $L_{2,0}$, $L_{1,1}$ and $L_{0,2}$ pictures for each rank two Lie algebra to describe semistandard tableaux. We summarize the result here:

Proposition 4.10. *Let a, b be 2 natural numbers, and let $\lambda = (a, b)$. The set of semistandard tableaux for the Lie algebra of type 'type' with shape λ is denoted $\mathcal{S}_{type}(\lambda)$. Then we get:*

- $\mathcal{S}_{A_1\times A_1}(\lambda) = \Big\{$ *usual semistandard tableaux T of shape λ with entries in $\{1,2,3,4\}$ such that $\boxed{2}$, $\boxed{3}$, $\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}$, $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$, $\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}$, $\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}$ are not a column of T* $\Big\}$.
- $\mathcal{S}_{A_2}(\lambda) = \Big\{$ *usual semistandard tableaux T of shape λ with entries in $\{1,2,3\}$* $\Big\}$.
- $\mathcal{S}_{C_2}(\lambda) = \Big\{$ *usual semistandard tableaux T of shape λ with entries in $\{1,2,3,4\}$ such that $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$ is not a column of T and the succeeding column of $\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}$ can not be $\boxed{1}$ or $\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}$* $\Big\}$.
- $\mathcal{S}_{G_2}(\lambda) = \Big\{$ *usual semistandard tableaux T of shape λ with entries in $\{1,2,3,4,5,$*

6, 7} such that the column $\boxed{4}$ *appears at most once in* T, $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$, $\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}$, $\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}$, $\begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array}$, $\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$, $\begin{array}{|c|}\hline 4\\\hline 6\\\hline\end{array}$, $\begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array}$ *are not a column in* T *plus the restriction given by the following table* $\Big\}$.

Column T^i of T	Then the succeeding column T^{i+1} of T cannot be...
$\boxed{4}$	$\boxed{4}$
$\begin{array}{\|c\|}\hline 1\\\hline 4\\\hline\end{array}$	$\boxed{1}$, $\begin{array}{\|c\|}\hline 1\\\hline 4\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 1\\\hline 5\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 1\\\hline 6\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 1\\\hline 7\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 1\\\hline 5\\\hline\end{array}$	$\boxed{1}$, $\begin{array}{\|c\|}\hline 1\\\hline 5\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 1\\\hline 6\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 1\\\hline 7\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 1\\\hline 6\\\hline\end{array}$	$\boxed{1}$, $\boxed{2}$, $\begin{array}{\|c\|}\hline 1\\\hline 6\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 1\\\hline 7\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 2\\\hline 6\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 2\\\hline 7\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 2\\\hline 6\\\hline\end{array}$	$\boxed{2}$, $\begin{array}{\|c\|}\hline 2\\\hline 6\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 2\\\hline 7\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 1\\\hline 7\\\hline\end{array}$	$\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\begin{array}{\|c\|}\hline 1\\\hline 7\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 2\\\hline 7\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 3\\\hline 7\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 4\\\hline 7\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 2\\\hline 7\\\hline\end{array}$	$\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\begin{array}{\|c\|}\hline 2\\\hline 7\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 3\\\hline 7\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 4\\\hline 7\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 3\\\hline 7\\\hline\end{array}$	$\boxed{3}$, $\boxed{4}$, $\begin{array}{\|c\|}\hline 3\\\hline 7\\\hline\end{array}$, $\begin{array}{\|c\|}\hline 4\\\hline 7\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 4\\\hline 7\\\hline\end{array}$	$\boxed{4}$, $\begin{array}{\|c\|}\hline 4\\\hline 7\\\hline\end{array}$

Remark 4.11. For each nonsemistandard tableau with two columns, there is an 'external' Plücker relation, homogeneous with degree 2. With our explicit description for functions δ_i and δ_{ij}, we can write such a system of relations between them: there are

For $A_1 \times A_1$, 6 internal Plücker relations and 0 external ones,

For A_2, 0 internal Plücker relations and 1 external in $\Gamma_{1,0} \otimes \Gamma_{0,1}$,

For C_2, 1 internal Plücker relations and 4 external ones in $\Gamma_{1,0} \otimes \Gamma_{0,1}$ and 1 in $\Gamma_{0,1} \otimes \Gamma_{0,1}$,

For G_2, 7 internal Plücker relations and 1 external ones in $\Gamma_{1,0} \otimes \Gamma_{1,0}$, 34 in $\Gamma_{1,0} \otimes \Gamma_{0,1}$ and 28 in $\Gamma_{0,1} \otimes \Gamma_{0,1}$.

We shall not write here these relations.

5. Shape and reduced shape algebras

For any rang two semisimple Lie algebra, we denote G the corresponding matrix group, we can repeat the argument in [ABW06] for the decomposition of the G module $\mathbb{C}[G]^{N^+}$ (for the left action). This module is completely decomposable as a sum of finite dimensional irreducible modules, the highest weight are biinvariant polynomial functions (from the right by N^+, for the left by N^-) with possible weight $a\omega_1 + b\omega_2$, for each pair (a,b) there is one and only one such function, namely:

$$\delta_1^a \delta_{1,2}^b.$$

¿From this, we deduce that as a G module, $\mathbb{C}[G]^{N^+} = \oplus_{a,b}\Gamma_{a,b}$. Moreover, $\mathbb{C}[G]^{N^+}$ is an algebra, called the shape algebra of G.

Definition 5.1. The shape algebra $\mathbb{S}_G$ of G is by definition the algebra $\mathbb{C}[G]^{N^+}$.

Then by construction, the set of semistandard tableaux forms a basis of the shape algebra and we get:

$$\mathbb{C}[G]^{N^+} = \mathbb{S}_G \simeq Sym^\bullet(\wedge\mathbb{C}^2) \Big/ \mathcal{PL}$$

where $\mathcal{PL}$ is the ideal generated by all the Plücker relations (internal or external).

¿From now one, we consider the restriction of the functions in $\mathbb{S}_G$ to the subgroup N^-. We get a quotient of $\mathbb{S}_G$ which is, as a vector space, the space $\mathbb{C}[N^-]$. Indeed, with the restriction of the functions δ_i and $\delta_{i,j}$ to N^-, it is easy, case by case to get the variables x, y for $A_1 \times A_1$, x, y, z for A_2, x, y, z, u for C_2, a, b, c, x, y, z for G_2.
The quotient has the form

$$\mathbb{C}[G]^{N^+} \Big/ < \delta_1 - 1,\ \delta_{1,2} - 1 > \simeq \mathbb{C}[N^-].$$

Definition 5.2. We call reduced shape algebra and denote $\mathbb{S}_G^{red}$ this quotient, $\mathbb{S}_G^{red} \simeq \mathbb{C}[N^-]$.

Since the ideal defining the quotient is N^+ invariant, we get a structure of N^+ module on this space $\mathbb{C}[N^-]$. This structure is simply the regular action:

$$(n^+.f)(n_1^-) = f(\,{}^t n^+ n_1^-).$$

Starting with the lowest weight vector in any $\Gamma_{a,b} \subset \mathbb{C}[G]^{N^+}$, which is $\delta_n^a \delta_{n-1,n}^b$ and acting with N^+, we generate exactly $\Gamma_{a,b}$ thus the canonical projection mapping

$$\pi : \mathbb{S}_G \longrightarrow \mathbb{S}_G^{red}$$

induces a bijective map of N^+ module from $\Gamma_{a,b}|_{N^+}$ onto $\pi(\Gamma_{a,b})$.

Now, since the highest weight vector $\delta_1^a\delta_{1,2}^b$ is the constant function 1 in $\mathbb{S}_G^{red}$, the N^+ module $\mathbb{S}_G^{red}$ is indecomposable and $\pi(\Gamma_{a',b'}) \subset \pi(\Gamma_{a,b})$ if $a' \leq a$ and $b' \leq b$.

Finally, we have, as N^+ module,

$$\mathbb{S}_G^{red} = \bigcup_{a,b} \pi(\Gamma_{a,b}) \quad \text{and} \quad \pi(\Gamma_{a,b}) = \bigcup_{a'\leq a, \quad b'\leq b} \pi(\Gamma_{a',b'}).$$

This N^+ module is called the diamond cone for G. We now look for a basis for the diamond cone, which will be well adapted to this layering of $\mathbb{C}[N^-] = \mathbb{S}_G^{red}$.

6. Quasistandard Young tableaux

Let us give now the definition of quasistandard Young tableaux for each rank two semisimple Lie algebra, generalizing the $\mathfrak{sl}(n)$ case construction. With our choice of semistandard Young tableaux for the $A_1 \times A_1$ and C_2 case and the choice given in ([ADLMPPrW07]) in the G_2 case, we define the quasistandard Young tableaux in the same way as for $\mathfrak{sl}(n)$:

We start from a semistandard Young tableau for a rank two semi simple Lie algebra and we apply the strategy of pushing the rows to extract case $\boxed{1}$ or column $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ as for $\mathfrak{sl}(n)$. This method gives the wanted basis for $\mathbb{S}_G^{red}$, except for G_2, where we moreover shall replace the column $\begin{array}{|c|}\hline 4\\\hline 4\\\hline\end{array}$ by $\begin{array}{|c|}\hline 1\\\hline 7\\\hline\end{array}$.

The set of quasistandard tableaux for the Lie algebras of type 'type' with shape λ will be denoted $\mathcal{QS}_{type}(\lambda)$. For more details, we use a case-by-case argument. Let us begin by the A_2 case (see section 2).

$\underline{\underline{A_2}}$:

We found in section 2 the following characterization for quasistandard Young tableaux.

Let $T = \begin{array}{|c|c|c|c|c|c|}\hline a_1 & \cdots & a_p & a_{p+1} & \cdots & a_{p+q}\\\hline b_1 & \cdots & b_p \\\cline{1-3}\end{array} \in \mathcal{S}_{A_2}(\lambda)$ for $\lambda = (q,p)$. T is said quasistandard ($T \in \mathcal{QS}_{A_2}(\lambda)$) if and only if:

- $\begin{array}{|c|}\hline a_1\\\hline b_1\\\hline\end{array} \neq \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$

  and

- $a_1 > 1$ or $q = 0$ or there is $i = 1, ..., p$ such that $a_{i+1} \geq b_i$.

$\underline{\underline{A_1 \times A_1}}$:

There is no external Plücker relation in this case, thus we just cancel the trivial columns $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ and $\begin{array}{|c|}\hline 1\\\hline\end{array}$ in the semistandard Young tableaux for $A_1 \times A_1$. Thus we get:

$$\mathcal{QS}_{A_1\times A_1}(\lambda) = \{T \in \mathcal{S}_{A_1\times A_1}(\lambda),\ \text{T without any trivial column}\},$$

or

let $T = \begin{array}{|c|c|c|c|c|c|}\hline a_1 & \cdots & a_p & a_{p+1} & \cdots & a_{p+q}\\\hline b_1 & \cdots & b_p \\\cline{1-3}\end{array} \in \mathcal{S}_{A_1\times A_1}(\lambda)$ for $\lambda = (q,p)$. T is said quasi standard ($T \in \mathcal{QS}_{A_1\times A_1}(\lambda)$) if and only if:

- $\begin{array}{|c|}\hline a_1\\\hline b_1\\\hline\end{array} \neq \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$

  and

- $a_1 > 1$ or $q = 0$ or there is $i = 1, ..., p$ such that $a_{i+1} \geq b_i$.

We can present the diamond cone by the drawing:

$$\begin{array}{ccc}
\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} & \qquad\qquad & \begin{array}{|c|c|}\hline 1 & 4\\\hline 3\\\cline{1-1}\end{array}\\
 & & \\
\emptyset & & \begin{array}{|c|}\hline 4\\\hline\end{array}
\end{array}$$

$\underline{\underline{C_2}}$:

Let us put:

$$\mathcal{QS}_{C_2}(\lambda) = \{T \in \mathcal{S}_{C_2}(\lambda) \text{ and } T \in \mathcal{QS}_{A_3}(\lambda)\}.$$

Or

Let $T = \begin{array}{|c|c|c|c|c|c|}\hline a_1 & \cdots & a_p & a_{p+1} & \cdots & a_{p+q}\\\hline b_1 & \cdots & b_p \\\cline{1-3}\end{array} \in \mathcal{S}_{C_2}(\lambda)$ for $\lambda = (q,p)$. T is said quasistandard ($T \in \mathcal{QS}_{C_2}(\lambda)$) if and only if:

- $\begin{array}{|c|}\hline a_1\\\hline b_1\\\hline\end{array} \neq \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$

  and

- $a_1 > 1$ or $q = 0$ or there is $i = 1, ..., p$ such that $a_{i+1} \geq b_i$.

Example 6.1. For $\lambda = (2,1)$, we get the following family of quasistandard tableaux with shape λ:

$$\mathcal{QS}_{C_2}(2,1)=\left\{\begin{array}{l}
\begin{array}{|c|c|c|}\hline 1&3&3\\\hline 3\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 3\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&4&4\\\hline 3\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&4&4\\\hline 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 2&2&2\\\hline 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 2&2&3\\\hline 4\\\cline{1-1}\end{array}\ ,\\
\begin{array}{|c|c|c|}\hline 2&3&3\\\hline 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 2&2&4\\\hline 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 2&4&4\\\hline 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 3&3&3\\\hline 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 3&3&4\\\hline 4\\\cline{1-1}\end{array}\ ,\\
\begin{array}{|c|c|c|}\hline 3&4&4\\\hline 4\\\cline{1-1}\end{array}
\end{array}\right\}.$$

Theorem 6.2. *For any $\lambda=(a,b)$, a basis for $\pi(\Gamma_{a,b})$ is parameterized by the disjoint union*

$$\bigsqcup_{a'\leq a,\quad b'\leq b}\mathcal{QS}_{C_2}(a',b').$$

The family of quasistandard Young tableaux forms a basis for the reduced shape algebra $\mathbb{S}^{red}_{C_2}$.

Proof. Let us use the Plücker relations. For C_2, since $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}=0$, these external relations are exactly the following:

$$-\begin{array}{|c|c|}\hline 1&1\\\hline 4\\\cline{1-1}\end{array}-\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}+\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}=0,$$

$$\begin{array}{|c|c|}\hline 3&2\\\hline 4\\\cline{1-1}\end{array}-\begin{array}{|c|c|}\hline 2&3\\\hline 4\\\cline{1-1}\end{array}-\begin{array}{|c|c|}\hline 1&4\\\hline 4\\\cline{1-1}\end{array}=0,$$

$$\begin{array}{|c|c|}\hline 3&1\\\hline 4\\\cline{1-1}\end{array}-\begin{array}{|c|c|}\hline 1&3\\\hline 4\\\cline{1-1}\end{array}+\begin{array}{|c|c|}\hline 1&4\\\hline 3\\\cline{1-1}\end{array}=0,$$

$$\begin{array}{|c|c|}\hline 2&1\\\hline 4\\\cline{1-1}\end{array}-\begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}+\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array}=0,$$

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}-\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}-\begin{array}{|c|c|}\hline 1&1\\\hline 4&4\\\hline\end{array}=0,$$

We consider now $\mathbb{S}^{red}_{C_2}$ as the quotient of the polynomial algebra in the variables:

$$X=\boxed{2}\,,Y=\boxed{4}\,,Z=\boxed{3}\,,U=\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\,,V=\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\,,W=\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\ \text{and}\ \ T=\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}$$

by the ideal $\mathcal{PL}_{red}$ generated by the reduced Plücker relations:

$$\mathcal{PL}_{red}=<-W-XU+Z,TX-VZ-WY,T-WZ+UY,V-WX+Y,T-UV-W^2>.$$

Using the monomial ordering given by the lexicographic ordering on (X,Z,Y,W,V,U,T), we get the following Groebner basis for $\mathcal{PL}_{red}$:

$$\Big\{W^2+UV-T\ ,WT+WYU+ZUV-ZT\ ,-T-YU+ZW,-WY+XT-ZV,$$

$W + XU - Z \ , -V - Y + XW \Big\}$.
The leading monomials of these elements, with respect to our ordering are:

$$W^2 \ , \ ZUV \ , \ ZW \ , \ XT \ , \ XU \ , \ XW.$$

Thus a basis for the quotient $\mathbb{S}_G^{red}$ is given by the Young tableaux without any trivial column and not containing one of the following subtableaux:

$$\begin{ytableau} 1 & 1 \\ 4 & 4 \end{ytableau} \ , \ \begin{ytableau} 1 & 2 & 3 \\ 3 & 4 \end{ytableau} \ , \ \begin{ytableau} 1 & 3 \\ 4 \end{ytableau} \ , \ \begin{ytableau} 3 & 2 \\ 4 \end{ytableau} \ , \ \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \ , \ \begin{ytableau} 1 & 2 \\ 4 \end{ytableau} \ .$$

The remaining Young tableaux are exactly the quasistandard Young tableaux.
Indeed, "T is semistandard without any trivial column" is equivalent to "T does not contain any trivial column and does not contain $\begin{ytableau} 1 & 1 \\ 4 & 4 \end{ytableau}$ nor $\begin{ytableau} 3 & 2 \\ 4 \end{ytableau}$ ".
Moreover the remaining tableaux *i.e* $\begin{ytableau} 1 & 2 & 3 \\ 3 & 4 \end{ytableau}$, $\begin{ytableau} 1 & 3 \\ 4 \end{ytableau}$, $\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$, and $\begin{ytableau} 1 & 2 \\ 4 \end{ytableau}$ are by definition nonquasistandard. Now, if T is a semistandard nonquasistandard tableau, without any trivial column, T contains a minimal semistandard nonquasistandard tableau without trivial column. Looking at all the possibilities for such minimal tableau with 2 columns, we get

$$\begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \ , \ \begin{ytableau} 1 & 2 \\ 4 \end{ytableau} \ \text{and} \ \begin{ytableau} 1 & 3 \\ 4 \end{ytableau} \ .$$

But there is also such minimal tableau with three columns. By minimality, such tableau has two columns of size 2 and one column of size 1, T being nonquasistandard, the first column of T is $\begin{ytableau} 1 \\ 3 \end{ytableau}$ or $\begin{ytableau} 1 \\ 4 \end{ytableau}$. If it is $\begin{ytableau} 1 \\ 4 \end{ytableau}$ then we get the nonquasistandard tableaux:

$$\begin{ytableau} 1 & 2 & u \\ 4 & 4 \end{ytableau} \ \text{and} \ \begin{ytableau} 1 & 3 & v \\ 4 & 4 \end{ytableau} \quad \text{with } u \geq 2 \text{ or } v \geq 3.$$

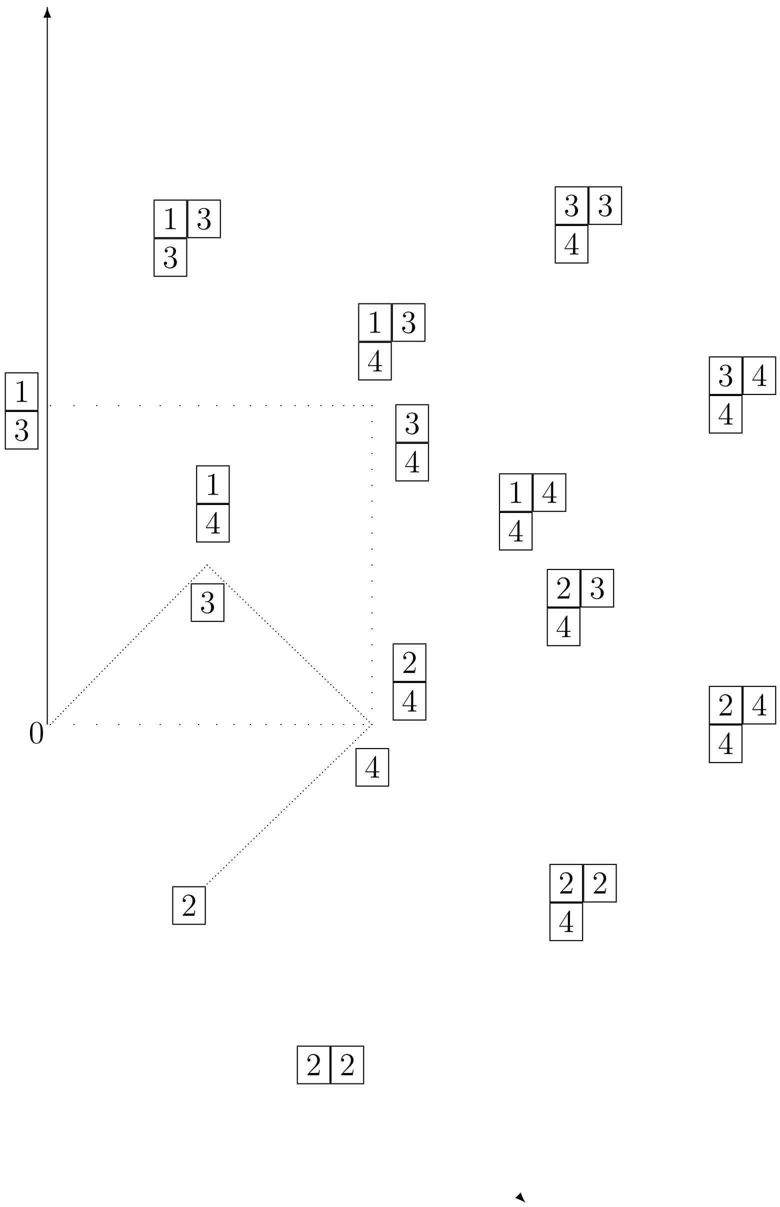
These nonquasistandard tableaux are not minimal. Thus the first column of T is $\begin{ytableau} 1 \\ 3 \end{ytableau}$, since T is minimal, its second column cannot be $\begin{ytableau} 1 \\ 3 \end{ytableau}$ nor $\begin{ytableau} 1 \\ 4 \end{ytableau}$. Therefore T is

$$\begin{ytableau} 1 & 2 & u \\ 3 & 4 \end{ytableau} \ \text{or} \ \begin{ytableau} 1 & 3 & v \\ 3 & 4 \end{ytableau} \ .$$

The tableau $\begin{ytableau} 1 & 3 & v \\ 3 & 4 \end{ytableau}$ is quasistandard for any v. The tableau $\begin{ytableau} 1 & 2 & 2 \\ 3 & 4 \end{ytableau}$ is nonquasistandard, nonminimal, the tableau $\begin{ytableau} 1 & 2 & 3 \\ 3 & 4 \end{ytableau}$ is nonquasistandard minimal.

Finally, if T is any semistandard Young tableau containing a non quasistandard tableau, T is itself nonquasistandard.
This proves that the monomial basis for the quotient coincides with the set of our quasistandard Young tableaux. ∎

Here is the drawing for a part of the diamond cone of $\mathfrak{sp}(4)$



$\underline{\underline{G_2}}$:

Definition 6.3. Let $T = \begin{array}{|c|c|c|c|c|c|}\hline a_1 & \cdots & a_p & a_{p+1} & \cdots & a_{p+q} \\ \hline b_1 & \cdots & b_p \\ \cline{1-3}\end{array}$ be a semistandard Young tableau of shape $\lambda = (q, p)$ for G_2. We say that T is quasistandard if:

- $\begin{array}{|c|}\hline a_1 \\ \hline b_1 \\ \hline\end{array} \neq \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$

  and

- $a_1 > 1$ or $q = 0$ or there is $i = 1, \dots, p$ such that $a_{i+1} > b_i$ or $a_{i+1} = b_i \neq 4$.

Let us denote by $\mathcal{QS}_{G_2}(q,p)$ the set of quasistandard tableaux with shape (q,p), by $\mathcal{SNQS}_{G_2}(q,p)$ the set of semistandard, nonquasistandard tableaux with shape (q,p). We first compute the cardinality of $\mathcal{QS}_{G_2}(q,p)$.

Let us define two operations on $T \in \mathcal{SNQS}_{G_2}(q,p)$.

a) The **'push'** operation:

Let us denote $T = \begin{array}{|c|c|c|c|c|c|}\hline a_1 & \cdots & a_p & a_{p+1} & \cdots & a_{p+q} \\ \hline b_1 & \cdots & b_p \\ \cline{1-3}\end{array} \in \mathcal{SNQS}_{G_2}(q,p)$.

- If $\begin{array}{|c|}\hline a_1 \\ \hline b_1 \\ \hline\end{array} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$, we put

$$P(T) = \begin{array}{|c|c|c|c|c|c|}\hline a_2 & \cdots & a_p & a_{p+1} & \cdots & a_{p+q} \\ \hline b_2 & \cdots & b_p \\ \cline{1-3}\end{array}$$

- If $a_1 = 1$, $q > 0$ and for any $i = 1,...,p$, $a_{i+1} < b_i$ or $a_{i+1} = b_i = 4$, we put

$$P(T) = \begin{array}{|c|c|c|c|c|c|}\hline a_2 & \cdots & a_p & a_{p+1} & \cdots & a_{p+q} \\ \hline b_1 & \cdots & b_p \\ \cline{1-3}\end{array}.$$

b) The **'rectification'** operation:

The tableau $P(T)$ is generally nonsemistandard. We define the rectification $R(P(T))$ of $P(T)$ as follows:
we read each 2 column of $P(T)$ and we replace any wrong 2 column by a corresponding acceptable one, following the table (1):

Wrong column	acceptable column
$\begin{array}{\|c\|}\hline 4\\\hline 4\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 7\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 2\\\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 4\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 4\\\hline 6\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 7\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 3\\\hline 5\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 6\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 3\\\hline 4\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 6\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 5\\\hline 6\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 4\\\hline 7\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 2\\\hline 4\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 5\\\hline\end{array}$
$\begin{array}{\|c\|}\hline 4\\\hline 5\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 6\\\hline\end{array}$

(1)

Proposition 6.4. *For any $T \in \mathcal{SNQS}_{G_2}(q,p)$, $R(P(T))$ belongs to $\mathcal{S}_{G_2}(q,p-1) \sqcup \mathcal{S}_{G_2}(q-1,p)$.*

Proof. If $\begin{array}{|c|}\hline a_1\\\hline b_1\\\hline\end{array} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$, it is evident that $R(P(T))$ belongs to $\mathcal{S}_{G_2}(q,p-1)$. For the second case, using a computer, we consider case by case, all the possibilities for 3 successive columns in T and the corresponding result in $P(T)$. We have to consider 3 cases:

- $\begin{array}{|c|c|c|c|c|}\hline \cdots & a_i & a_{i+1} & a_{i+2} & \cdots\\\hline \cdots & b_i & b_{i+1} & b_{i+2} & \cdots\\\hline\end{array} \xrightarrow{P} \begin{array}{|c|c|c|c|}\hline \cdots & a_{i+1} & a_{i+2} & \cdots\\\hline \cdots & b_i & b_{i+1} & \cdots\\\hline\end{array} \xrightarrow{R} \begin{array}{|c|c|c|c|}\hline \cdots & a'_{i+1} & a'_{i+2} & \cdots\\\hline \cdots & b'_i & b'_{i+1} & \cdots\\\hline\end{array}$

- $\begin{array}{|c|c|c|c|c|}\hline \cdots & a_i & a_{i+1} & a_{i+2} & \cdots\\\hline \cdots & b_i & b_{i+1} \\\cline{1-3}\end{array} \xrightarrow{P} \begin{array}{|c|c|c|c|}\hline \cdots & a_{i+1} & a_{i+2} & \cdots\\\hline \cdots & b_i & b_{i+1} \\\cline{1-3}\end{array} \xrightarrow{R} \begin{array}{|c|c|c|c|}\hline \cdots & a'_{i+1} & a'_{i+2} & \cdots\\\hline \cdots & b'_i & b'_{i+1} \\\cline{1-3}\end{array}$

- $\begin{array}{|c|c|c|c|c|}\hline \cdots & a_i & a_{i+1} & a_{i+2} & \cdots\\\hline \cdots & b_i \\\cline{1-2}\end{array} \xrightarrow{P} \begin{array}{|c|c|c|c|}\hline \cdots & a_{i+1} & a_{i+2} & \cdots\\\hline \cdots & b_i \\\cline{1-2}\end{array} \xrightarrow{R} \begin{array}{|c|c|c|c|}\hline \cdots & a'_{i+1} & a'_{i+2} & \cdots\\\hline \cdots & b'_i \\\cline{1-2}\end{array}$

We verify, in each case, that the result is: $R(P(T)) \in \mathcal{S}_{G_2}(q-1,p)$.
Indeed, for example in the third case, all tableaux T in $\mathcal{S}_{G_2}(2,1)$ such that $a_2 < b_1$ define the following tableaux $R(P(T))$:

$$\ytableaushort{12,3}\,,\ \ytableaushort{11,3}\,,\ \ytableaushort{12,4}\,,\ \ytableaushort{13,3}\,,\ \ytableaushort{13,4}\,,\ \ytableaushort{14,3}\,,\ \ytableaushort{14,4}\,,\ \ytableaushort{15,3}\,,\ \ytableaushort{15,4}\,,\ \ytableaushort{16,3}\,,\ \ytableaushort{16,4}\,,$$
$$\ytableaushort{17,3}\,,\ \ytableaushort{17,4}\,,\ \ytableaushort{12,5}\,,\ \ytableaushort{13,5}\,,\ \ytableaushort{13,6}\,,\ \ytableaushort{14,5}\,,\ \ytableaushort{14,6}\,,\ \ytableaushort{15,5}\,,\ \ytableaushort{15,6}\,,\ \ytableaushort{16,5}\,,\ \ytableaushort{16,6}\,,$$
$$\ytableaushort{17,5}\,,\ \ytableaushort{17,6}\,,\ \ytableaushort{22,5}\,,\ \ytableaushort{23,5}\,,\ \ytableaushort{23,6}\,,\ \ytableaushort{24,5}\,,\ \ytableaushort{24,6}\,,\ \ytableaushort{25,5}\,,\ \ytableaushort{25,6}\,,\ \ytableaushort{25,7}\,,\ \ytableaushort{26,5}\,,$$
$$\ytableaushort{26,6}\,,\ \ytableaushort{26,7}\,,\ \ytableaushort{27,5}\,,\ \ytableaushort{27,6}\,,\ \ytableaushort{27,7}\,,\ \ytableaushort{33,6}\,,\ \ytableaushort{34,6}\,,\ \ytableaushort{35,6}\,,\ \ytableaushort{35,7}\,,\ \ytableaushort{45,7}\,,\ \ytableaushort{36,6}\,,$$
$$\ytableaushort{36,7}\,,\ \ytableaushort{46,7}\,,\ \ytableaushort{37,6}\,,\ \ytableaushort{37,7}\,,\ \ytableaushort{47,7}\,,\ \ytableaushort{55,7}\,,\ \ytableaushort{56,7}\,,\ \ytableaushort{66,7}\,,\ \ytableaushort{57,7}\,,\ \ytableaushort{67,7}\,,\ \ytableaushort{12,2}\,,$$
$$\ytableaushort{11,2}\,,\ \ytableaushort{13,2}\,,\ \ytableaushort{14,2}\,,\ \ytableaushort{15,2}\,,\ \ytableaushort{16,2}\,,\ \ytableaushort{17,2}$$

All these tableaux are in $\mathcal{S}_{G_2}(1,1)$.

■

Now we define a mapping f from $\mathcal{S}_{G_2}(q,p)$ into $\displaystyle\bigsqcup_{\substack{p'\leq p\\ q'\leq q}} \mathcal{QS}_{G_2}(q',p')$ as follows.

Let T be in $\mathcal{S}_{G_2}(q,p)$, if T is quasistandard, we put $f(T)=T$, if T is not quasistandard, we put $T'=R(P(T))$. If T' is quasistandard, we define $f(T)=T'$. If it is not the case, we put $T''=R(P(T'))$, if T'' is quasistandard, we put $f(T)=T''$ and so one...

Proposition 6.5. *The map f is a one-to-one mapping from $\mathcal{S}_{G_2}(q,p)$ onto* $\displaystyle\bigsqcup_{\substack{p'\leq p\\ q'\leq q}} \mathcal{QS}_{G_2}(q',p')$.

Proof. We just define the inverse mapping of f. Let T be in $\mathcal{S}_{G_2}(q',p')$. Suppose that $q'\leq q$. We first compute $R^{-1}(T)$ i.e we replace each 2-column of T in the "acceptable columns" in the table (1) by the corresponding wrong columns. Let

$$R^{-1}(T)=\begin{array}{|c|c|c|c|c|c|}\hline a_1&\cdots&a_{p'}&a_{p'+1}&\cdots&a_{p'+q'}\\\hline b_1&\cdots&b_{p'}\\\cline{1-3}\end{array}$$

the resulting tableau. Then we 'pull' the resulting tableau, that is we define:

$$P^{-1}(R^{-1}(T))=T'=\begin{array}{|c|c|c|c|c|c|c|}\hline 1&a_1&\cdots&a_{p'-1}&a_{p'}&\cdots&a_{p'+q'}\\\hline b_1&b_2&\cdots&b_{p'}\\\cline{1-4}\end{array}.$$

We verify, case by case as above, that the resulting tableau T' is in $\mathcal{S}_{G_2}(q'+1,p')$. If $q'+1<q$, we repeat this operation.

Finally, we get a tableau $T'' = (P^{-1} \circ R^{-1}) \circ ... \circ (P^{-1} \circ R^{-1})(T) \in \mathcal{S}_{G_2}(q, p')$. If $p' < p$, we add to T'' $p - p'$ trivial 2-columns $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$. By construction, the mapping g so defined from $\bigsqcup_{\substack{p' \leq p \\ q' \leq q}} \mathcal{QS}_{G_2}(q', p')$ to $\mathcal{S}_{G_2}(q, p)$ is the inverse mapping of f. ■

Let us recall the projection mapping $\pi : \mathbb{S}_{G_2} = \oplus_{p,q} \ \Gamma_{q,p} \longrightarrow \mathbb{S}^{red}_{G_2}$. We show that if $p' \leq p$, $q' \leq q$, then $\pi(\Gamma_{q',p'}) \subset \Gamma_{q,p}$. Now, our proposition proves by induction on p and q that:

$$\sharp\mathcal{QS}_{G_2}(q, p) = dim\big(\pi(\Gamma_{q,p}) \Big/ \sum_{(p',q') < (p,q)} \pi(\Gamma_{q',p'})\big)$$

where $(p', q') < (p, q)$ means $p' \leq p$, $q' \leq q$ and $(p', q') \neq (p, q)$.

Proposition 6.6. *The set $\mathcal{QS}_{G_2}(q, p)$ is a basis for a supplementary space in $\pi(\Gamma_{q,p})$ to the space $\sum_{(p',q') < (p,q)} \pi(\Gamma_{q',p'})$.*

Proof. Since the number of quasistandard tableaux is the dimension of our space, it is enough to prove that the family $\mathcal{QS}_{G_2}(q, p)$ is independent in the quotient $\pi(\Gamma_{q,p}) \Big/ \sum_{(p',q') < (p,q)} \pi(\Gamma_{q',p'})$.

Suppose this is not the case, there is a linear relation $\sum_i a_i T_i$ between some T_i in $\mathcal{QS}_{G_2}(q, p)$ which belongs to $\sum_{(p',q') < (p,q)} \pi(\Gamma_{q',p'})$ that means, there is a S in the ideal $\mathcal{PL}_{red}$ of reduced Plücker relations, a family (T'_j) of tableaux in $\cup_{(p',q') < (p,q)} \ \mathcal{S}_{G_2}(q', p')$ and $b_j \in \mathbb{R}$ such that: $\sum_i a_i T_i = \sum_j b_j T'_j + S$. This means

$$\big(\sum_i a_i T_i - \sum_j b_j T'_j\big) \ |_{N^-} = 0. \tag{1}$$

But now the action of the diagonal matrices $H \in \mathfrak{h}$ in G_2 are diagonal in $\mathbb{C}[\delta_{i,j}, \delta_i]$. Thus we decompose the preceding expression in a finite sum of weight vectors with weight $\mu \in \mathfrak{h}^*$. The relation (1) holds for any weight vector, thus we get a nontrivial relation:

$$\big(\sum_i a_i T_i - \sum_j b_j T'_j\big) \ |_{N^-} = 0, \quad (H - \mu(H)).\big(\sum_i a_i T_i - \sum_j b_j T'_j\big) = 0.$$

The first relation means there is S_μ in the ideal $\mathcal{PL}_{red}$ such that:

$$\sum_i a_i T_i - \sum_j b_j T'_j = S_\mu.$$

S_μ being in $\mathcal{PL}_{red}$ can be written as:

$$S_\mu = \sum_k PL_k + \sum_l T'_l\Big(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \ - 1 \Big) + \sum_m T''_m\Big(\begin{array}{|c|}\hline 1 \\ \hline \end{array} \ - 1 \Big)$$

where PL_k are Plücker relations which are homogeneous, with weight μ, with respect to the $\mathfrak{h}$ action. Let us put

$$U = \sum_i a_i T_i - \sum_j b_j T'_j - \sum_k PL_k.$$

U is a linear combination of Young tableaux $U = \sum_\ell c_\ell U_\ell$, it is homogeneous with weight μ. If we delete the trivial columns of each the U_ℓ tableau, we get a tableau U'_ℓ of weight $\mu - a\omega_1 - b\omega_2$, if there is a columns $\ytableaushort{1}$ and b columns $\ytableaushort{1,2}$. Now to delete these columns corresponds exactly to the restriction of the corresponding polynomial functions to N^-. Denoting by $'$ the restriction to N^-, we get:

$$U' = \sum_\ell c_\ell U'_\ell = 0.$$

For any (a,b), we put $M_{(a,b)} = \{\ell, \text{ such that } U'_\ell \text{ has weight } \mu - a\omega_1 - b\omega_2\}$ then for any (a,b), by homogeneity,

$$\sum_{\ell \in M_{(a,b)}} c_\ell U_\ell = 0.$$

Finally,

$$U = \sum_{a,b} \left(\ytableaushort{1,2} \right)^b \sum_{\ell \in M_{(a,b)}} c_\ell U_\ell \left(\ytableaushort{1} \right)^a = 0.$$

This proves our proposition. ∎

Finally we can compute all the semistandard, nonquasistandard, minimal tableaux for G_2, without any trivial column:

$$\ytableaushort{12,3}\,,\ \ytableaushort{12,4}\,,\ \ytableaushort{12,5}\,,\ \ytableaushort{13,4}\,,\ \ytableaushort{13,5}\,,\ \ytableaushort{13,6}\,,\ \ytableaushort{14,4}\,,\ \ytableaushort{14,5}\,,\ \ytableaushort{14,6}\,,\ \ytableaushort{15,6}\,,$$

$$\ytableaushort{15,7}\,,\ \ytableaushort{16,7}\,,\ \ytableaushort{123,35}\,,\ \ytableaushort{124,35}\,,\ \ytableaushort{123,36}\,,\ \ytableaushort{124,36}\,,\ \ytableaushort{125,33}\,,\ \ytableaushort{125,37}\,,$$

$$\ytableaushort{126,37}\,,\ \ytableaushort{124,45}\,,\ \ytableaushort{124,46}\,,\ \ytableaushort{125,46}\,,\ \ytableaushort{134,46}\,,\ \ytableaushort{135,46}\,,\ \ytableaushort{125,47}\,,\ \ytableaushort{126,47}\,,$$

$$\ytableaushort{135,47}\,,\ \ytableaushort{136,47}\,,\ \ytableaushort{125,56}\,,\ \ytableaushort{135,56}\,,\ \ytableaushort{125,57}\,,\ \ytableaushort{126,57}\,,\ \ytableaushort{135,57}\,,\ \ytableaushort{136,57}\,,$$

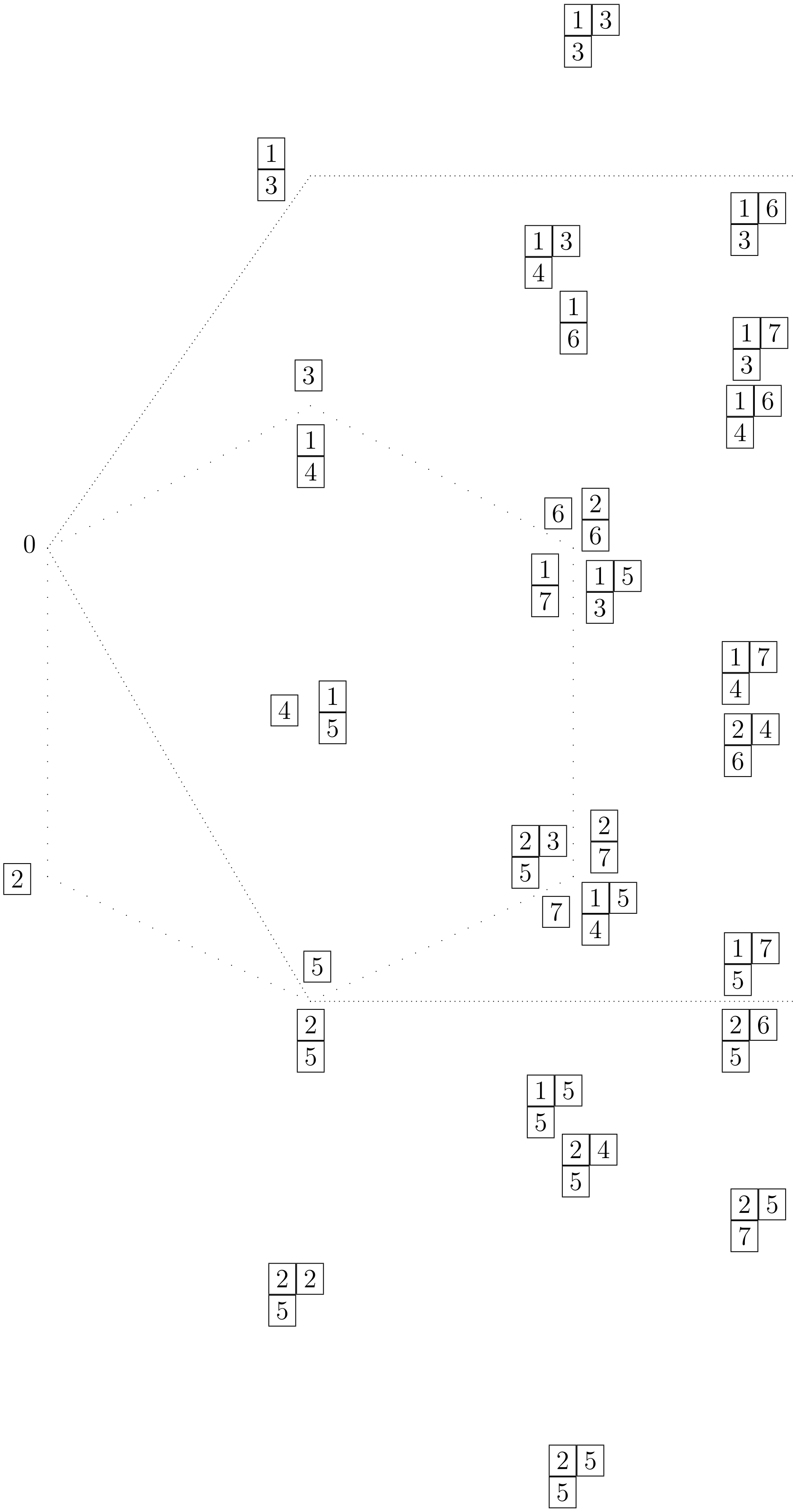
$$\ytableaushort{145,57}\,,\ \ytableaushort{146,57}\,,\ \ytableaushort{136,67}\,,\ \ytableaushort{146,67}\,,\ \ytableaushort{156,67}\,,\ \ytableaushort{1235,356}\,,\ \ytableaushort{1236,367}\,,$$

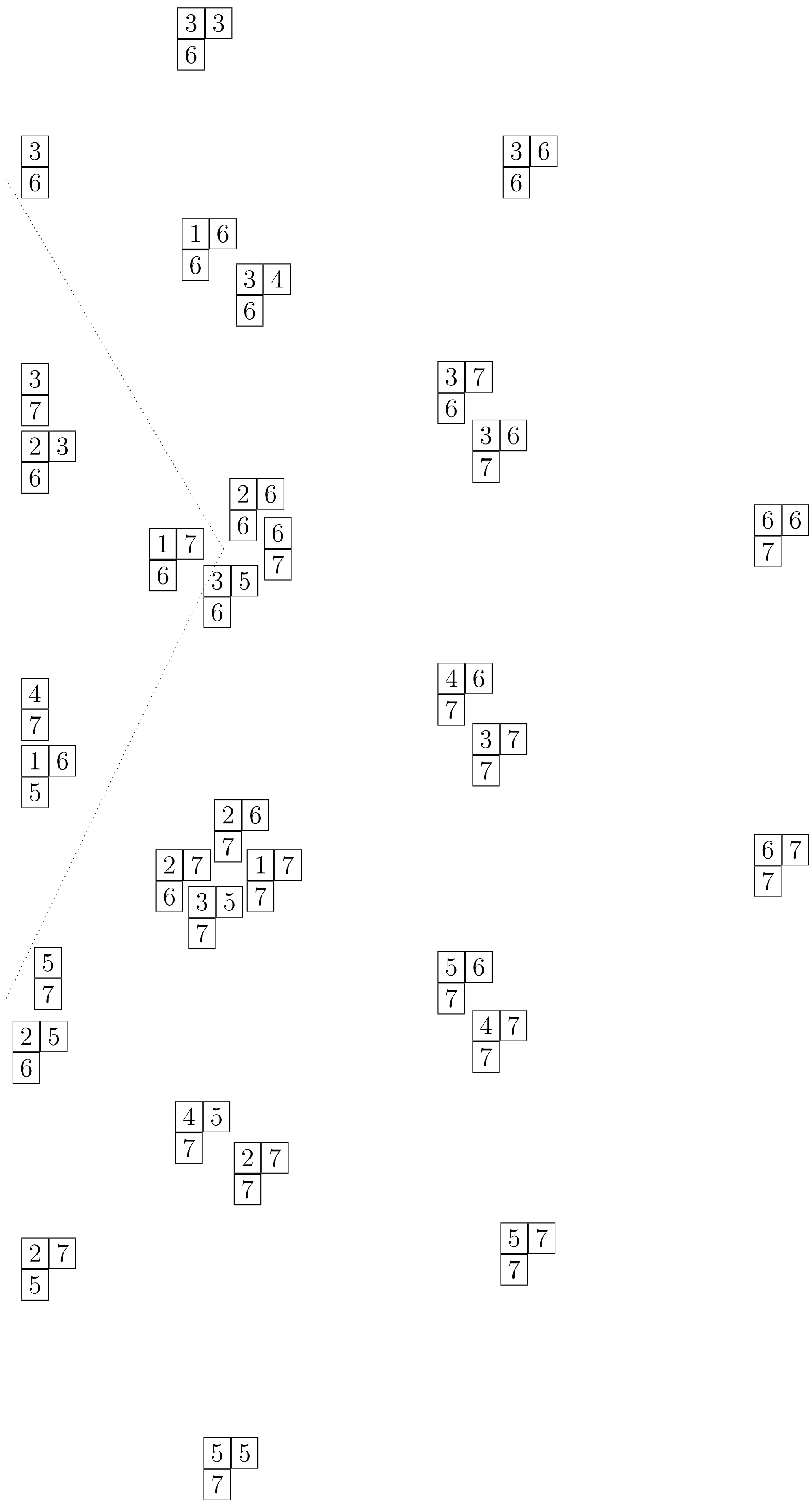
$$\ytableaushort{1246,367}\,,\ \ytableaushort{1256,367}\,,\ \ytableaushort{1235,357}\,,\ \ytableaushort{1245,357}\,,\ \ytableaushort{1236,357}\,,\ \ytableaushort{1246,357}\,,$$

$$\ytableaushort{1245,457}\,,\ \ytableaushort{1246,457}\,,\ \ytableaushort{1247,457}\,,\ \ytableaushort{1246,467}\,,\ \ytableaushort{1256,467}\,,\ \ytableaushort{1346,467}\,,$$

$$\ytableaushort{1356,467}\,,\ \ytableaushort{1256,567}\,,\ \ytableaushort{1356,567}\,,\ \ytableaushort{12356,3567}\,.$$

Now, for G_2, the picture of a part of the diamond cone is as follows (the extreme vertices, at right in the first page, at left in the second one, are common vertices):





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