Journal of Lie Theory
Volume **19** (2009) 223–230

Local Spectral Radius Formulas on Compact Lie Groups

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Communicated by B. Ørsted

Abstract. We determine the local spectrum of a central element of the complexified universal enveloping algebra of a compact connected Lie group at a smooth function as an element of $L^p(G)$. Based on this result we establish a corresponding local spectral radius formula.
Mathematics Subject Classification 2000: Primary 22E30, secondary 47A11.
Key Words and Phrases: Compact Lie group, universal enveloping algebra, local spectrum, local spectral radius, local spectral radius formula.

1. Introduction and statement of result

Let f be a Schwartz function on $\mathbb{R}^d$ and let $P(\partial)$ be a constant coefficient differential operator with complex coefficients. If $1 \leq p \leq \infty$, then it is known that

$$\lim_{n\to\infty} \|P(\partial)^n f\|_p^{1/n} = \sup\left\{|z| : z \in \{P(i\lambda) : \lambda \in \operatorname{supp} \mathcal{F}f\}^{\mathrm{cl}}\right\}, \tag{1}$$

in the extended positive real numbers, where $\mathcal{F}f$ is the Fourier transform of f and A^{cl} denotes the closure of a subset A of the complex plane. This result was first established by Tuan for real coefficients, see [8, Theorem 2], and later by the authors for the general case, see [1, Theorem 2.5].

In [1] we raised the question whether analogues of (1) hold for other Lie groups, with $P(\partial)$ replaced by an element of the center of the universal enveloping algebra, and whether such results could be interpreted as a local spectral radius formula, analogous to the case $p = 1$ on $\mathbb{R}^d$, see [1, Corollary 5.4]. In order to explain this interpretation we recall a few relevant definitions from local spectral theory, see [2], [3] and [9].

Let X be a Banach space, and $T : \mathcal{D}_T \to X$ a closed operator with domain $\mathcal{D}_T$. Then $z_0 \in \mathbb{C}$ is said to be in the local resolvent set of $x \in X$, denoted by $\rho_T(x)$, if there is an open neighborhood U of z_0 in $\mathbb{C}$, and an analytic function $\phi : U \to \mathcal{D}_T$, sending z to ϕ_z, such that

$$(T - z)\phi_z = x \qquad (z \in U). \tag{2}$$

The local spectrum $\sigma_T(x)$ of T at x is the complement of $\rho_T(x)$ in $\mathbb{C}$.

ISSN 0949–5932 / \$2.50

The operator T is said to have the single-valued extension property (SVEP) if, for every non-empty open subset $U \subset \mathbb{C}$, the only analytic solution $\phi : U \to X$ of the equation $(T - z)\phi_z = 0$ $(z \in U)$ is the zero solution. This is equivalent to requiring that the analytic local resolvent function ϕ in (2) is determined uniquely, so that we can speak of "the" analytic local resolvent function on $\rho_T(x)$.

If $\mathcal{D}_T = X$ and T has SVEP, then, by [3, Proposition 3.3.13], the local spectral radius formula

$$\limsup_{n\to\infty} \|T^n x\|^{1/n} = \max\{|z| : z \in \sigma_T(x)\} \tag{3}$$

holds for all $x \in X$. If $\mathcal{D}_T = X$, but T does not necessarily have SVEP, then by [3, Proposition 3.3.14] the set of $x \in X$ for which (3) holds is still always of the second category in X. If $\mathcal{D}_T = X$ and T has Bishop's property (β) (see [3, Definition 1.2.5]; it is immediate that property (β) implies SVEP), then, by [3, Proposition 3.3.17],

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = \max\{|z| : z \in \sigma_T(x)\}, \tag{4}$$

for all $x \in X$.

Thus there exist general results concerning the validity of local spectral radius formulas, such as (3) and (4), for bounded operators. We are not aware of such a priori guarantees for unbounded operators, and it is one of the main results in [1] that, for $p = 1$, the equality in (1) can, in fact, be interpreted as a local spectral radius formula for a closed unbounded operator.[1]

To be precise, let $T_{P(\partial),1} : C_c^\infty(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ be defined canonically as $T_{P(\partial),1} f = P(\partial) f$, for $f \in C_c^\infty(\mathbb{R}^d)$. It is then easily seen, cf. [5, Section 4.2], that $T_{P(\partial),1}$ has a closed extension $\widetilde{T}_{P(\partial),1}$ on $L^1(\mathbb{R}^d)$, with domain $\mathcal{D}_{\widetilde{T}_{P(\partial),1}}$ consisting of those $f \in L^1(\mathbb{R}^d)$ such that $P(\partial)f$ is in $L^1(\mathbb{R}^d)$, and defined as $\widetilde{T}_{P(\partial),1} f = P(\partial) f$ for $f \in \mathcal{D}_{\widetilde{T}_{P(\partial),1}}$.

Then [1, Corollary 5.4] reads as follows:

Theorem 1.1. *The closed operator $\widetilde{T}_{P(\partial),1}$ on $L^1(\mathbb{R}^d)$ has SVEP. Furthermore, if f is a Schwartz function on $\mathbb{R}^d$, then*

$$\sigma_{\widetilde{T}_{P(\partial),1}}(f) = \{P(i\lambda) : \lambda \in \operatorname{supp} \mathcal{F} f\}^{\mathrm{cl}}.$$

Combined with (1) *this implies that the local spectral radius formula*

$$\lim_{n\to\infty} \|\widetilde{T}^n_{P(\partial),1} f\|_1^{1/n} = \sup\left\{|z| : z \in \sigma_{\widetilde{T}_{P(\partial),1}}(f)\right\} \tag{5}$$

holds in the extended positive real numbers.

This paper is concerned with the analogue of Theorem 1.1, for $1 \le p \le \infty$, on a connected compact Lie group G, with Lie algebra $\mathfrak{g}$. We will replace $P(\partial)$

[1] For other values of p the problem is still open, although it is conjectured in [1] that the interpretation then holds as well.

with an element D in the center of the complexified universal enveloping algebra $U(\mathfrak{g})_{\mathbb{C}}$, viewed as the algebra of left-invariant differential operators on G. In order to state the results, we need some preliminaries which will also be used in the proofs in the next section.

We let $^\dagger : U(\mathfrak{g})_{\mathbb{C}} \to U(\mathfrak{g})_{\mathbb{C}}$ be the complex linear anti-homomorphism of $U(\mathfrak{g})_{\mathbb{C}}$ such that $X^\dagger = -X$, for $X \in \mathfrak{g}$. If S is a distribution on G, and $D \in U(\mathfrak{g})_{\mathbb{C}}$, then DS is the distribution defined by

$$\langle DS, \psi\rangle = \langle S, D^\dagger \psi\rangle \qquad (\psi \in C^\infty(G)).$$

Since G is unimodular, this is compatible with the action of G on smooth functions.

For $1 \leq p \leq \infty$, and $D \in U(\mathfrak{g})_{\mathbb{C}}$, we define the operator $T_{D,p} : C^\infty(G) \to L^p(G)$ canonically by $T_{D,p} f = Df$, for $f \in C^\infty(G)$. Then, as in [5, Section 4.2], $T_{D,p}$ has a closed extension $\widetilde{T}_{D,p}$ on $L^p(G)$, with domain $\mathcal{D}_{\widetilde{T}_{D,p}}$ equal to those $f \in L^p(G)$ such that the distribution Df is in $L^p(G)$, and defined as $\widetilde{T}_{D,p} f = Df$, for $f \in \mathcal{D}_{\widetilde{T}_{D,p}}$.

Choose and fix representatives (π, H_π) for the unitary dual $\widehat{G}$ of G. If $\pi \in \widehat{G}$ (we will allow ourselves such abuse of notation), we let $\bar{\pi}$ denote its contragredient representation, and $\chi_\pi : Z(U(\mathfrak{g})_{\mathbb{C}}) \to \mathbb{C}$ its infinitesimal character, defined on the center $Z(U(\mathfrak{g})_{\mathbb{C}})$ of $U(\mathfrak{g})_{\mathbb{C}}$.

Let dg be the normalized Haar measure on G. If $f \in L^1(G)$, and $\pi \in \widehat{G}$, define the Fourier transform $\mathcal{F}f(\pi)$ of f at π as

$$\mathcal{F}f(\pi) = \int_G f(g)\pi(g)dg \in \mathrm{End}_{\mathbb{C}}(H_\pi). \tag{6}$$

Note that $L^p(G) \subset L^1(G)$, for $1 \leq p \leq \infty$, so that the Fourier transform is defined on $L^p(G)$, for all $1 \leq p \leq \infty$.

Then we have the following result:

Theorem 1.2. *Let G be a connected compact Lie group and $D \in Z(U(\mathfrak{g})_{\mathbb{C}})$. Let $1 \leq p \leq \infty$. Then the closed extension $\widetilde{T}_{D,p}$ of $T_{D,p}$ has SVEP. If $f \in C^\infty(G)$, then the local spectrum of $\widetilde{T}_{D,p}$ at $f \in \mathcal{D}_{\widetilde{T}_{D,p}}$ is given by*

$$\sigma_{\widetilde{T}_{D,p}}(f) = \{\chi_{\bar{\pi}}(D) : \pi \in \mathrm{supp}\,\mathcal{F}f\}^{\mathrm{cl}}, \tag{7}$$

and the local spectral radius formula

$$\lim_{n\to\infty} \|\widetilde{T}^n_{D,p} f\|_p^{1/n} = \sup\left\{|z| : z \in \sigma_{\widetilde{T}_{D,p}}(f)\right\} \tag{8}$$

holds in the extended positive real numbers.

Remark 1.3. Obviously, Theorem 1.2 is an analogue of Theorem 1.1. It would be premature to state a conjecture, but in view of these two results, the material presented in [1] and the proofs below, it is tempting to consider the possibility that Theorem 1.2 and Theorem 1.1 have a common generalization for Schwartz functions on connected reductive groups – or perhaps even symmetric spaces – including appropriate analogues of (7) and (8).

2. Proofs

We now turn to the proof of Theorem 1.2, which will occupy the remainder of the paper. It is based on results in [6] on the Fourier transform of smooth functions on a connected compact Lie group, which we will now recall.

Let G be a connected compact Lie group, with Lie algebra $\mathfrak{g}$. Choose and fix a maximal torus T with Lie algebra $\mathfrak{t}$. Then $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and where $[\mathfrak{g}, \mathfrak{g}]$ is either zero or semisimple. In the latter case, $(\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}])_{\mathbb{C}}$ is a Cartan subalgebra of the semisimple complex Lie algebra $[\mathfrak{g}, \mathfrak{g}]_{\mathbb{C}}$ and we let Δ be the roots of $[\mathfrak{g}, \mathfrak{g}]_{\mathbb{C}}$ relative to $(\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}])_{\mathbb{C}}$. Fix a choice of positive roots, and hence a set of dominant weights on $(\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}])_{\mathbb{C}}$.

Let $\Gamma_G = \{X \in \mathfrak{t} : \exp X = 1\}$, so that $T \cong \mathfrak{t}/\Gamma_G$. Then, according to [10, Theorem 4.6.12], $\widehat{G}$ is in bijective correspondence with the set $\Lambda_{\widehat{G}}$ of complex linear forms λ on $\mathfrak{t}_{\mathbb{C}}$ such that

(1) $\lambda(\Gamma_G) \subset 2\pi i\mathbb{Z}$.

(2) $\lambda_{|(\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}])_{\mathbb{C}}}$ is dominant integral.

The correspondence is via highest weight modules for $(\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}])_{\mathbb{C}}$, but its precise form is not relevant for the present paper. Note that if $[\mathfrak{g}, \mathfrak{g}] = 0$, i.e., if $G = T$, then the above result is still valid if one takes condition (2) to be vacuously fulfilled. As a notation in the sequel, we will let $\lambda \in \Lambda_{\widehat{G}}$ correspond to $(\pi_\lambda, H_\lambda) \in \widehat{G}$.

The space $C^\infty(G)$ is a Fréchet space when equipped with the seminorms $p_D(f) = \|Df\|_\infty$, $D \in U(\mathfrak{g})_{\mathbb{C}}$, for $f \in C^\infty(G)$. As is stated below, its counterpart on the Fourier series side is the space $\mathcal{S}(\widehat{G})$ of rapidly decreasing operator valued functions on $\widehat{G}$, which we now define. Fix a norm on the dual of $\mathfrak{t}_{\mathbb{C}}$. Then $\mathcal{S}(\widehat{G})$ is the space of functions $\phi : \Lambda_{\widehat{G}} \to \bigcup_{\pi \in \widehat{G}} \mathrm{End}_{\mathbb{C}}(H_\pi)$, such that

(a) $\phi(\lambda) \in \mathrm{End}_{\mathbb{C}}(H_{\pi_\lambda})$ for all $\lambda \in \Lambda_{\widehat{G}}$, and

(b) $\sup_{\lambda \in \Lambda_{\widehat{G}}} |\lambda|^s \|\phi(\lambda)\| < \infty$, for all $s \in \mathbb{N} \cup \{0\}$.

Here, and in the sequel, the norm of an element of $\mathrm{End}_{\mathbb{C}}(H_\pi)$ will always be its Hilbert–Schmidt norm. The space $\mathcal{S}(\widehat{G})$ becomes a Fréchet space when equipped with the seminorms $q_s(\phi) = \sup_{\lambda \in \Lambda_{\widehat{G}}} |\lambda|^s \|\phi(\lambda)\|$, for $\phi \in \mathcal{S}(\widehat{G})$ and $s \in \mathbb{N} \cup \{0\}$.

Let $f \in L^1(G)$. In view of the description of $\widehat{G}$ above, the Fourier transform $\mathcal{F}f$ of f, as defined in (6), can be regarded as an operator valued function on $\Lambda_{\widehat{G}}$ which satisfies (a). With this in mind we can now give the following alternative formulation of some of the results from [6]:

Theorem 2.1. *If $f \in C^\infty(G)$, then $\mathcal{F}f \in \mathcal{S}(\widehat{G})$. Moreover, the map $\mathcal{F} : C^\infty(G) \to \mathcal{S}(\widehat{G})$ is a topological isomorphism of $C^\infty(G)$ onto $\mathcal{S}(\widehat{G})$. The inverse map is given as*

$$(\mathcal{F}^{-1}\phi)(g) = \sum_{\pi \in \widehat{G}} \dim(\pi) \operatorname{tr}(\phi(\pi)\pi(g^{-1})) \qquad (\phi \in \mathcal{S}(\widehat{G}),\ g \in G), \tag{9}$$

where the series converges absolutely and uniformly on G.

The part on absolute and uniform convergence also follows from [4, 7]. If G is a torus, then this result specializes to a well known statement from classical Fourier analysis.

After these preparations, we can now prove Theorem 1.2 in a number of steps. Let $D \in Z(U(\mathfrak{g})_{\mathbb{C}})$, $1 \leq p \leq \infty$, and define $\widetilde{T}_{D,p}$ as in the introduction. If $f \in \mathcal{D}_{\widetilde{T}_{D,p}}$, then $Df \in L^p(G) \subset L^1(G)$, so that $\mathcal{F}(Df)(\pi)$ is defined for all $\pi \in \widehat{G}$. Since the matrix coefficients of π are smooth, it is easily seen that

$$\mathcal{F}(\widetilde{T}_{D,p}f)(\pi) = \chi_\pi(D^\dagger)\mathcal{F}f(\pi) \qquad (f \in \mathcal{D}_{\widetilde{T}_{D,p}}, \pi \in \widehat{G}),$$

which, since $\chi_\pi(D^\dagger) = \chi_{\bar{\pi}}(D)$, can also be written as

$$\mathcal{F}(\widetilde{T}_{D,p}f)(\pi) = \chi_{\bar{\pi}}(D)\mathcal{F}f(\pi) \qquad (f \in \mathcal{D}_{\widetilde{T}_{D,p}}, \pi \in \widehat{G}).$$

Lemma 2.2. *If $D \in Z(U(\mathfrak{g})_{\mathbb{C}})$ and $1 \leq p \leq \infty$, then $\widetilde{T}_{D,p}$ has SVEP.*

Proof. If $U \subset \mathbb{C}$ is open and non-empty, and $\phi : U \to \mathcal{D}_{\widetilde{T}_{D,p}}$ is analytic and such that $(\widetilde{T}_{D,p} - z)\phi_z = 0$ for $z \in U$, then taking Fourier transforms yields $(\chi_{\bar{\pi}}(D) - z)\mathcal{F}\phi_z(\pi) = 0$, for all $z \in U$ and $\pi \in \widehat{G}$. If $\pi \in \widehat{G}$ is fixed, we conclude that $\mathcal{F}\phi_z(\pi) = 0$ for all $z \in U$ with at most one exception, which could possibly occur at $\chi_{\bar{\pi}}(D)$ if $\chi_{\bar{\pi}}(D) \in U$. However, since $\mathcal{F}\phi_z(\pi)$ depends continuously on z, as a consequence of the continuity of the inclusion $L^p(G) \subset L^1(G)$, such an exception does, in fact, not occur. Hence $\mathcal{F}\phi_z(\pi) = 0$, for all $z \in U$ and $\pi \in \widehat{G}$, so that $\phi_z = 0$ for all $z \in U$ by the injectivity of the Fourier transform on $L^1(G)$. ■

Lemma 2.3. *If $f \in C^\infty(G)$, $D \in Z(U(\mathfrak{g})_{\mathbb{C}})$ and $1 \leq p \leq \infty$, then the local spectrum of $\widetilde{T}_{D,p}$ at f, as an element of $D_{\widetilde{T}_{D,p}}$, is given by*

$$\sigma_{\widetilde{T}_{D,p}}(f) = \{\chi_{\bar{\pi}}(D) : \pi \in \operatorname{supp} \mathcal{F}f\}^{\mathrm{cl}}.$$

Proof. We first establish that

$$\rho_{\widetilde{T}_{D,p}}(f) \subset \mathbb{C} \setminus \{\chi_{\bar{\pi}}(D) : \pi \in \operatorname{supp} \mathcal{F}f\}^{\mathrm{cl}}. \tag{10}$$

To this end, suppose that $\chi_{\bar{\pi}}(D) \in \rho_{\widetilde{T}_{D,p}}(f)$, for some $\pi \in \widehat{G}$. Then there exist a neighborhood U of $\chi_{\bar{\pi}}(D)$, and an analytic function $\phi : U \to \mathcal{D}_{\widetilde{T}_{D,p}}$ such that $(\widetilde{T}_{D,p} - z)\phi_z = f$, for $z \in U$. Taking the Fourier transform at this particular π gives

$$(\chi_{\bar{\pi}}(D) - z)\mathcal{F}\phi_z(\pi) = \mathcal{F}f(\pi) \qquad (z \in U).$$

Since $\chi_{\bar{\pi}}(D)$ is in U, we can specify z at this value and conclude that $\mathcal{F}f(\pi) = 0$ whenever $\chi_{\bar{\pi}}(D) \in \rho_{\widetilde{T}_{D,p}}(f)$. In other words, if $\chi_{\bar{\cdot}}(D) : \widehat{G} \to \mathbb{C}$ denotes the function which sends $\pi \in \widehat{G}$ to $\chi_{\bar{\pi}}(D)$, then

$$\chi_{\bar{\cdot}}(D)^{-1}\left[\rho_{\widetilde{T}_{D,p}}(f)\right] \subset \widehat{G} \setminus \operatorname{supp} \mathcal{F}f,$$

hence

$$\chi_{\bar{\cdot}}(D)[\operatorname{supp}\mathcal{F}f] \subset \mathbb{C} \setminus \rho_{\widetilde{T}_{D,p}}(f).$$

Since the right hand side is closed, we conclude that

$$(\chi_{\bar{\cdot}}(D)[\operatorname{supp}\mathcal{F}f])^{\mathrm{cl}} \subset \mathbb{C} \setminus \rho_{\widetilde{T}_{D,p}}(f),$$

which is equivalent to (10).

Next, we show the reverse inclusion

$$\rho_{\widetilde{T}_{D,p}}(f) \supset \mathbb{C} \setminus \{\chi_{\bar{\pi}}(D) : \pi \in \operatorname{supp}\mathcal{F}f\}^{\mathrm{cl}}\,, \tag{11}$$

which will complete the proof. Suppose $z_0 \notin \{\chi_{\bar{\pi}}(D) : \pi \in \operatorname{supp}\mathcal{F}f\}^{\mathrm{cl}}$, and let $\varepsilon > 0$ be such that $|\chi_{\bar{\pi}}(D) - z_0| > \varepsilon$, for all $\pi \in \operatorname{supp}\mathcal{F}f$. Let $U = \{z \in \mathbb{C} : |z - z_0| < \varepsilon/2\}$, so that, for $z \in U$ and $\pi \in \operatorname{supp}\mathcal{F}f$, one has $|\chi_{\bar{\pi}}(D) - z| > \varepsilon/2$.

Define, for each $z \in U$, the function $\psi_z : \widehat{G} \to \bigcup_{\pi\in\widehat{G}} \operatorname{End}_{\mathbb{C}}(H_\pi)$ by

$$\psi_z(\pi) = \begin{cases} \dfrac{\mathcal{F}f(\pi)}{\chi_{\bar{\pi}}(D) - z} & \text{if } \pi \in \operatorname{supp}\mathcal{F}f; \\ 0 & \text{if } \pi \notin \operatorname{supp}\mathcal{F}f. \end{cases}$$

Obviously $\psi_z \in \mathcal{S}(\widehat{G})$, since $\mathcal{F}f \in \mathcal{S}(\widehat{G})$. It is easy to verify that the map $z \mapsto \psi_z$ is an analytic function from U to $\mathcal{S}(\widehat{G})$, hence, as a consequence of Theorem 2.1, the map $z \mapsto \mathcal{F}^{-1}\psi_z$ is an analytic function from U to $C^\infty(G)$. Composing it with the continuous inclusion of $C^\infty(G)$ in $L^p(G)$, we obtain an analytic map $\phi : U \to \mathcal{D}_{\widetilde{T}_{D,p}}$ defined as $\phi_z = \mathcal{F}^{-1}\psi_z$, for $z \in U$. Since $\mathcal{F}[(\widetilde{T}_{D,p} - z)\phi_z] = \mathcal{F}f$ by construction, we conclude that $(\widetilde{T}_{D,p} - z)\phi_z = f$, for $z \in U$. Hence $z_0 \in \rho_{\widetilde{T}_{D,p}}(f)$ as requested. ∎

The proof of Theorem 1.2 is now completed by the following result:

Lemma 2.4. *If $D \in Z(U(\mathfrak{g})_{\mathbb{C}})$, $1 \le p \le \infty$ and $f \in C^\infty(G)$, then in the extended positive real numbers*

$$\lim_{n\to\infty} \|\widetilde{T}^n_{D,p} f\|_p^{1/n} = \sup\{|\chi_{\bar{\pi}}(D)| : \pi \in \operatorname{supp}\mathcal{F}f\}\,.$$

Proof. Suppose $\pi \in \operatorname{supp}\mathcal{F}f$. Then

$$\begin{aligned} |\chi_{\bar{\pi}}(D)|^n \,\|\mathcal{F}f(\pi)\| &= \|\mathcal{F}(\widetilde{T}^n_{D,p} f)(\pi)\| \\ &\le \int_G |\widetilde{T}^n_{D,p} f(g)| \|\pi(g)\| dg \\ &= \dim(\pi)^{1/2}\, \|\widetilde{T}^n_{D,p} f(g)\|_1 \\ &\le \dim(\pi)^{1/2}\, \|\widetilde{T}^n_{D,p} f(g)\|_p. \end{aligned}$$

Since $\|\mathcal{F}f(\pi)\| \neq 0$, we conclude that

$$|\chi_{\bar{\pi}}(D)| \le \liminf_{n\to\infty} \|\widetilde{T}^n_{D,p} f(g)\|_p^{1/n},$$

hence

$$\sup\{|\chi_{\bar\pi}(D)| : \pi \in \operatorname{supp}\mathcal{F}f\} \leq \liminf_{n\to\infty} \|\widetilde{T}^n_{D,p}f\|_p.$$

We will now proceed to show that

$$\limsup_{n\to\infty} \|\widetilde{T}^n_{D,p}f\|_p \leq \sup\{|\chi_{\bar\pi}(D)| : \pi \in \operatorname{supp}\mathcal{F}f\}, \tag{12}$$

which will conclude the proof of the lemma. We may assume that the right hand side is finite. Let $g \in G$, then

$$\begin{aligned}\widetilde{T}^n_{D,p}f(g) &= (\mathcal{F}^{-1}\mathcal{F}\widetilde{T}^n_{D,p}f)(g)\\ &= \sum_{\pi\in\widehat{G}} \dim(\pi)\operatorname{tr}[\mathcal{F}\widetilde{T}^n_{D,p}f(\pi)\pi(g^{-1})]\\ &= \sum_{\pi\in\widehat{G}} \dim(\pi)\,\chi_{\bar\pi}(D)^n\operatorname{tr}[\mathcal{F}f(\pi)\pi(g^{-1})].\end{aligned}$$

Hence

$$|\widetilde{T}^n_{D,p}f(g)| \leq \sup\{|\chi_{\bar\pi}(D)^n| : \pi \in \operatorname{supp}\mathcal{F}f\} \cdot \sum_{\pi\in\widehat{G}} \dim(\pi)\,|\operatorname{tr}[\mathcal{F}(\pi)\pi(g^{-1})]|.$$

By Theorem 2.1, the series is bounded by a constant M, uniformly in g, so that

$$\|\widetilde{T}^n_{D,p}f\|_p \leq M\,[\sup\{|\chi_{\bar\pi}(D)| : \pi \in \operatorname{supp}\mathcal{F}f\}]^n,$$

and (12) follows. ∎

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Received September 16, 2008
and in final form May 12, 2009