

Families of Equivariant Differential Operators and Anti-de Sitter Spaces

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Abstract. We prove the existence and uniqueness of a sequence of differential intertwining operators for principal series representations, which are realized on boundaries of anti-de Sitter spaces. Algebraically, these operators correspond to homomorphisms of generalized Verma modules. We relate these families to the asymptotics of eigenfunctions on anti-de Sitter spaces.

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1. Introduction

Let $\widehat{\text{Ein}}_n$ be the compact Lorentzian manifold $S^1 \times S^{n-1}$ with the metric $(-g_{S^1}) \oplus g_{S^{n-1}}$. The conformal group of $\widehat{\text{Ein}}_n$, i.e., the group of all diffeomorphisms which induce conformally equivalent metrics, can be identified with $O(2, n)$. Let

$$i : \widehat{\text{Ein}}_{n-1} \hookrightarrow \widehat{\text{Ein}}_n$$

be the isometric embedding which is induced by the equatorial embedding $S^{n-2} \rightarrow S^{n-1}$ of spheres. We construct a sequence $\hat{D}_N^c(\lambda)$, $N \geq 0$ of polynomial families

$$\hat{D}_N^c(\lambda) : C^\infty(\widehat{\text{Ein}}_n) \rightarrow C^\infty(\widehat{\text{Ein}}_{n-1})$$

of $O(2, n-1)$ -equivariant differential operators. Here equivariance is understood with respect to respective principal series representations π_λ^c on $C^\infty(\widehat{\text{Ein}}_n)$ and $C^\infty(\widehat{\text{Ein}}_{n-1})$.

The results extend corresponding results of [8] on $SO(1, n)^\circ$ -equivariant families $D_N^c(\lambda) : C^\infty(S^n) \rightarrow C^\infty(S^{n-1})$. In [8] the families $D_N^c(\lambda)$ serve as conformally flat models of corresponding conformally covariant “curved analogs” $C^\infty(M) \rightarrow C^\infty(\Sigma)$ which are canonically associated to any hypersurface Σ in a Riemannian manifold M . In [8] it is shown how such families contain information

on Branson's Q -curvature. In particular, combining the holographic formula for Q -curvature ([5]) with the structure of the intertwining families, provides recursive relations for Q -curvature.

In the present situation, the conformal action of $O(2, n)$ on $\widehat{\text{Ein}}_n$ replaces the conformal action of $O(1, n)$ on S^{n-1} . Whereas S^{n-1} is the boundary at infinity of the hyperbolic space \mathbb{H}^n with the isometry group $O(1, n)$, here $\widehat{\text{Ein}}_n$ appears as the conformal boundary of the $n + 1$ -dimensional anti-de Sitter space AdS_{n+1} with the isometry group $O(2, n)$ (see Section 2). The universal covering space of $\widehat{\text{Ein}}_n$ is given by

$$(\mathbb{R} \times S^{n-1}, -dr^2 + g_{S^{n-1}})$$

and is also known as Einstein's static universe; [3].

For even N , the family $\hat{D}_N^c(\lambda)$ drops down to a polynomial family

$$D_N^c(\lambda) : C^\infty(\text{Ein}_n) \rightarrow C^\infty(\text{Ein}_{n-1}) \quad (1)$$

of $O(2, n - 1)$ -equivariant differential operators, where

$$\text{Ein}_n = (S^1 \times S^{n-1})/\mathbb{Z}_2,$$

and the \mathbb{Z}_2 -action is coming from the involution $S^1 \times S^{n-1} \rightarrow S^1 \times S^{n-1}$ given by $(x, y) \mapsto (-x, -y)$.

For odd N , the family $\hat{D}_N^c(\lambda)$ does not drop down to a well-defined $O(2, n - 1)$ -equivariant map $C^\infty(\text{Ein}_n) \rightarrow C^\infty(\text{Ein}_{n-1})$. The non-existence of odd-order families has a geometric reason. In [8] it is shown that for any (orientable) hypersurface Σ of an (orientable) manifold M , the family

$$D_1(\lambda) = i^* \nabla_N + \lambda H i^* : C^\infty(M) \rightarrow C^\infty(\Sigma)$$

is conformally covariant. Here N denotes a unit normal field on Σ and H is the corresponding mean curvature. $\widehat{\text{Ein}}_n$ is an (orientable) hypersurface in $\widehat{\text{Ein}}_{n+1}$ with $H = 0$, and the family $D_1(\lambda)$ reduces to the equivariant family $D_1^c = i^* \nabla_N$ given by the normal derivative. Now for odd n , Ein_n is a non-orientable codimension one submanifold of the orientable manifold Ein_{n+1} , and the above construction does not descend to $\text{Ein}_n \hookrightarrow \text{Ein}_{n+1}$. Similarly, for even n , Ein_{n+1} is non-orientable, and the above construction does not descend.

The space $\widehat{\text{Ein}}_{n-1}$ can alternatively be described as the set of rays through the origin in the light cone

$$C_n = \{-t_1^2 - t_2^2 + x_1^2 + \cdots + x_{n-1}^2 = 0\}. \quad (2)$$

$G^n = O(2, n - 1)$ acts transitively on $\widehat{\text{Ein}}_{n-1}$, and we let $\hat{P}^n \subset G^n$ be the isotropy group of the ray generated by $(1, 0, 0, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$. In a similar way, Ein_{n-1} is given by the set of lines in C_n through the origin, and we let $P^n \subset G^n$ denote the isotropy group of the line through $(1, 0, 0, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$.

As in [8] we discuss two different types of constructions and prove that both lead to the same result. One construction is in terms of representation theory and one is in terms of spectral theory of an associated Laplacian.

We start with the description of the Lie-theoretic construction. For $n \geq 4$ let $G^n = O(2, n - 1)$ with Lie algebra $\mathfrak{g}_n = \mathfrak{o}(2, n - 1)$. The isotropy group P^n is a maximal parabolic subgroup with Langlands decomposition $P^n = M^n A(N^+)^n$. The Lie algebra \mathfrak{p}_n of P^n has the Langlands decomposition $\mathfrak{p}_n = \mathfrak{m}_n \oplus \mathfrak{a} \oplus \mathfrak{n}_n^+$ with a one-dimensional space \mathfrak{a} . Here \mathfrak{a} is spanned by H and the Lie subalgebras \mathfrak{n}_n^\pm are the respective \pm -eigenspaces of $\text{ad}(H)$. Let $(N^-)^n = \exp(\mathfrak{n}_n^-)$.

For $\lambda \in \mathbb{C}$ we define the character $\xi_\lambda : \mathfrak{p}_n \rightarrow \mathbb{C} = \mathbb{C}_\lambda$ by $\xi_\lambda(tH) = t\lambda$ (ξ_λ acts by 0 on $\mathfrak{m}_n \oplus \mathfrak{n}_n^+$). We shall use the same notation ξ_λ for the character of P given by $\xi_\lambda(\exp(tH)) = e^{\lambda t}$.

Let $\mathcal{U}(\mathfrak{g}_n)$ be the universal enveloping algebra of \mathfrak{g}_n . For $\lambda \in \mathbb{C}$ the character ξ_λ of \mathfrak{p}_n gives rise to the generalized Verma module (see (55))

$$\mathcal{M}_\lambda(\mathfrak{g}_n) = \mathcal{U}(\mathfrak{g}_n) \otimes_{\mathcal{U}(\mathfrak{p}_n)} \mathbb{C}_\lambda.$$

Similarly, for the character ξ_λ of P we consider the induced representation $\text{Ind}_P^G(\xi_\lambda)$ of G .

Let $i : \mathcal{U}(\mathfrak{g}_n) \rightarrow \mathcal{U}(\mathfrak{g}_{n+1})$ be the map induced by the inclusion $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ given by (9).

Theorem 1.1. *For any non-negative integer N there exists a polynomial family $\mathcal{D}_N^0(\lambda) \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ such that the map*

$$\mathcal{U}(\mathfrak{g}_n) \otimes \mathbb{C}_{\lambda-N} \ni T \otimes 1 \mapsto i(T)\mathcal{D}_N^0(\lambda) \otimes 1 \in \mathcal{U}(\mathfrak{g}_{n+1}) \otimes \mathbb{C}_\lambda$$

induces a homomorphism

$$\mathcal{M}_{\lambda-N}(\mathfrak{g}_n) \rightarrow \mathcal{M}_\lambda(\mathfrak{g}_{n+1})$$

of $\mathcal{U}(\mathfrak{g}_n)$ -modules for all λ . Furthermore, for any $\lambda \in \mathbb{C}$, $\mathcal{D}_N^0(\lambda)$ spans the space of all homomorphisms $\mathcal{M}_{\lambda-N}(\mathfrak{g}_n) \rightarrow \mathcal{M}_\lambda(\mathfrak{g}_{n+1})$ of $\mathcal{U}(\mathfrak{g}_n)$ -modules.

In addition to the existence, we will actually show how to find explicit formulas for the intertwining families $\mathcal{D}_N^0(\lambda)$. Now viewing $\mathcal{D}_N^0(\lambda) \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ as a left-invariant differential operator acting on $C^\infty(G^{n+1})$ from the right, we obtain the following result.

Theorem 1.2. *(a) For any non-negative integer N the polynomial family $\mathcal{D}_N^0(\lambda) \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a family of left G^{n+1} -equivariant operators*

$$D'_N(\lambda) : \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{P^n}^{G^{n+1}}(\xi_{\lambda-N} \otimes \sigma),$$

where for even N , σ is the trivial representation of M , and for odd N , $\sigma = \sigma_-$ (see section 2). The composition $D_N(\lambda) = i^ \circ D'_N(\lambda)$ of $D'_N(\lambda)$ with the restriction map $i^* : C^\infty(G^{n+1}) \rightarrow C^\infty(G^n)$ defines a family of left G^n -equivariant operators*

$$D_N(\lambda) : \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{P^n}^{G^n}(\xi_{\lambda-N} \otimes \sigma). \tag{3}$$

(b) For any non-negative integer N the polynomial family $\mathcal{D}_N^0(\lambda) \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a family of left G^{n+1} -equivariant operators

$$\hat{D}'_N(\lambda) : \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{P^n}^{G^{n+1}}(\xi_{\lambda-N}).$$

The composition $\hat{D}_N(\lambda) = i^* \circ \hat{D}'_N(\lambda)$ of $\hat{D}'_N(\lambda)$ with the restriction map $i^* : C^\infty(G^{n+1}) \rightarrow C^\infty(G^n)$ defines a family of left G^n -equivariant operators

$$\hat{D}_N(\lambda) : \text{Ind}_{\hat{P}_{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{\hat{P}_n}^{G^n}(\xi_{\lambda-N}). \quad (4)$$

Using

$$\text{Ein}_{m-1} \simeq G^m/P^m \quad \text{and} \quad \widehat{\text{Ein}}_{m-1} \simeq G^m/\hat{P}^m,$$

the representation spaces of principal series can be identified with

$$C^\infty(\text{Ein}_{m-1}) \quad \text{and} \quad C^\infty(\widehat{\text{Ein}}_{m-1}),$$

respectively. The notation $C^\infty(\text{Ein}_m)_\lambda$ and $C^\infty(\widehat{\text{Ein}}_m)_\lambda$ emphasizes the respective module structure. In these terms, Theorem 1.1 admits the following interpretation.

Theorem 1.3. (a) For any even integer $N \geq 0$ the polynomial family (3) defines a family of G^n -equivariant operators

$$D_N^c(\lambda) : C^\infty(\text{Ein}_n)_\lambda \rightarrow C^\infty(\text{Ein}_{n-1})_{\lambda-N}.$$

(b) For any non-negative integer N the polynomial family (4) defines a family of G^n -equivariant operators

$$\hat{D}_N^c(\lambda) : C^\infty(\widehat{\text{Ein}}_n)_\lambda \rightarrow C^\infty(\widehat{\text{Ein}}_{n-1})_{\lambda-N}.$$

Now let \mathbb{M}^n be the Minkowski space with the Lorentzian metric

$$ds^2 = -dx_1^2 + dx_2^2 + \cdots + dx_n^2. \quad (5)$$

The families $D_N^c(\lambda)$ in Theorem 1.3 give rise to families

$$D_N^{\text{nc}}(\lambda) : C^\infty(\mathbb{M}^n) \rightarrow C^\infty(\mathbb{M}^{n-1})$$

as follows. Ein_m is conformally flat. More precisely, the composition of the inclusion $(N^-)^m \rightarrow G^m$ with the projection $G^m \rightarrow G^m/P^m$ defines a conformal embedding $j : \mathbb{M}^{m-1} \rightarrow \text{Ein}_{m-1}$. Here we identify $\mathbb{M}^{m-1} \simeq (N^-)^m$ (see (20)) and $\text{Ein}_{m-1} \simeq G^m/P^m$. The embedding j yields a well-known conformal compactification of Minkowski space; see also [1]. Now the families $D_N^{\text{nc}}(\lambda)$ are defined by restriction to the open sets $j(\mathbb{M}^m) \subset \text{Ein}_m$.

The family $D_{2N}^{\text{nc}}(\lambda)$ is a polynomial in the Laplacian $\Delta_{\mathbb{M}^{n-1}}$ and $\partial^2/\partial x_n^2$ with polynomial coefficients in λ . In particular,

$$D_{2N}^{\text{nc}}\left(-\frac{n}{2}+N\right) = i^* \Delta_{\mathbb{M}^n}^N \quad \text{and} \quad D_{2N}^{\text{nc}}\left(-\frac{n-1}{2}+N\right) = \Delta_{\mathbb{M}^{n-1}}^N i^*.$$

where i is the inclusion map $\mathbb{M}^{n-1} \hookrightarrow \mathbb{M}^n$.

The powers $(\Delta_{\mathbb{M}^m})^k$ of the Laplacian $\Delta_{\mathbb{M}^m}$ on flat Minkowski space are very special cases of the conformally covariant powers of the Laplacian of pseudo-Riemannian manifolds constructed in [4]. Their $O(2, m)$ -equivariance is a consequence of their conformal covariance; see also [13], [7].

Now we turn to an alternative construction of the families $D_N^{\text{nc}}(\lambda)$ in terms of the asymptotics of eigenfunctions of the Laplacian on anti-de Sitter space AdS_n . In the language of [8] these are the residue families $D_N^{\text{res}}(g, \lambda)$, where g is the metric (5).

On the n -dimensional Lorentzian upper half space with the metric

$$x_n^{-2} \left(-dx_1^2 + \sum_{i=2}^n dx_i^2 \right)$$

we consider formal approximate eigenfunctions of the Laplacian with eigenvalue $\lambda(n-1-\lambda)$. The ansatz

$$u(x) \sim \sum_{j \geq 0} x_n^{\lambda+2j} c_{2j}(x'), \quad x = (x', x_n),$$

yields recursive relations for the coefficients c_{2j} so that all coefficients are determined by the leading one c_0 . More precisely, there are differential operators

$$T_{2j}(\lambda) : C^\infty(\mathbb{R}^{n-1}) \rightarrow C^\infty(\mathbb{R}^{n-1})$$

of order $2j$ (depending on λ) such that $T_{2j}(\lambda)c_0 = c_{2j}$. It is easily seen that

$$T_{2j}(\lambda) = A_{2j}(\lambda)(\Delta_{\mathbb{M}^{n-1}})^j, \tag{6}$$

where the coefficients A_{2j} satisfy the recursive relations

$$A_{2j-2}(\lambda) + 2j(2j + 2\lambda + 1 - n)A_{2j}(\lambda) = 0, \quad A_0(\lambda) = 1.$$

Here the d'Alembertian

$$\Delta_{\mathbb{M}^{n-1}} = -\frac{\partial^2}{\partial x_1^2} + \sum_{i=2}^{n-1} \frac{\partial^2}{\partial x_i^2}$$

is the Laplacian of Minkowski space. Let

$$S_{2N}(\lambda) : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{n-1})$$

be defined by

$$S_{2N}(\lambda) = \sum_{j=0}^N \frac{1}{(2N-2j)!} T_{2j}(\lambda) i^* \left(\frac{\partial}{\partial x_n} \right)^{2N-2j}. \tag{7}$$

The following theorem shows that both constructions yield the same results (up to normalization).

Theorem 1.4. *The families $S_{2N}(\lambda + n - 1 - 2N)$ and $D_{2N}^{\text{nc}}(\lambda)$ coincide, up to a rational function in λ .*

Moreover, if we define

$$S_{2N+1}(\lambda) = \sum_{j=0}^N \frac{1}{(2N+1-2j)!} T_{2j}(\lambda) t^* \left(\frac{\partial}{\partial x_n} \right)^{2N+1-2j},$$

we get the following analogous result:

Theorem 1.5. *The families $S_{2N+1}(\lambda+n-1-(2N+1))$ and $D_{2N+1}^{\text{nc}}(\lambda)$ coincide, up to a rational function in λ .*

We expect that the analogous construction of equivariant differential operators generalizes to pseudo-Riemannian spaces with constant negative curvature of any signature. This would give differential intertwining operators between principal series representations of $O(p, q)$ and $O(p, q+1)$ induced from maximal parabolic subgroups, for all $q > p \geq 1$, analogous to the operator families treated in this paper. Such equivariant families would be the analogs for non-trivial signatures of the intertwining families discussed in [8] in connection with Branson's Q -curvature. The present results in the case of Lorentzian signature naturally suggest that the methods of [8] extend beyond the Riemannian setting. In particular, we expect that the recursive structure of Q -curvatures does not depend on the signature. For related work concerning the indefinite orthogonal group we refer to [11] and [12].

The paper is organized as follows. In Section 2 we describe the geometric situation, the structure of the Lie groups involved and the principal series representations induced by representations of the maximal parabolic subgroup P^m . In Section 3 we treat the invariance of the d'Alembertian and describes algebraic constructions which are necessary for the proof of equivariance of the differential operators $D_N(\lambda)$. Section 4 contains the proof of Theorem 1.1 and Section 5 contains the proof of Theorem 1.2. Section 6 describe the construction of the families $D_N^{\text{nc}}(\lambda)$ in terms of the asymptotics of eigenfunctions of the Laplacian on anti-de Sitter space.

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2. Geometric preliminaries and principal series representations

The n -dimensional anti-de Sitter space, AdS_n , is the one-sheeted hyperboloid

$$-t_1^2 - t_2^2 + x_1^2 + \cdots + x_{n-1}^2 = -1 \tag{8}$$

in the space \mathbb{R}^{n+1} equipped with the pseudo-Riemannian metric

$$ds^2 = -dt_1^2 - dt_2^2 + dx_1^2 + \cdots + dx_{n-1}^2.$$

The group of isometries of AdS_n is $O(2, n-1)$, that is, all linear transformations of \mathbb{R}^{n+1} that preserve the hyperboloid (8). We let $G^n = O(2, n-1)$. Furthermore, we assume that $n \geq 4$.

The Lie algebra $\mathfrak{g}_n = \mathfrak{g} = \mathfrak{o}(2, n-1)$ consists of all traceless matrices of the form

$$\begin{pmatrix} A & B \\ B^t & D \end{pmatrix},$$

where the 2×2 -matrix A and the $(n-1) \times (n-1)$ -matrix D are skew-symmetric. For $n \geq 4$ we will use the inclusion

$$i : \mathfrak{o}(2, n-1) \rightarrow \mathfrak{o}(2, n)$$

given by

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

On the Lie group level we use the corresponding inclusion

$$i : O(2, n-1) \rightarrow O(2, n)$$

given by

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

The involution $\theta(A) = -A^t$ on $\mathfrak{o}(2, n-1)$ yields a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, where \mathfrak{k} and \mathfrak{q} are the eigenspaces for the eigenvalue 1 and -1 , respectively. We choose a maximal abelian subspace of \mathfrak{q} by

$$\mathfrak{a}_q = \text{span}_{\mathbb{R}}(H, H_2),$$

where

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ 1 & 0 & 0 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 0 & \mathbf{0} \\ 0 & 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Here and henceforth, we write elements in $\mathfrak{o}(2, n-1)$ as matrices of the block forms

$$\begin{pmatrix} 4 \times 4 & 4 \times (n-3) \\ (n-3) \times 4 & (n-3) \times (n-3) \end{pmatrix}.$$

We define $f_i \in (\mathfrak{a}_q)^*$ by $f_i(H_j) = \delta_{ij}$. We choose $\Sigma^+ = \{f_1, f_2, f_1 + f_2, f_1 - f_2\}$ as the set of positive restricted roots. The simple restricted roots are f_1, f_2 and $f_1 - f_2$. The set of restricted roots is $\Sigma = \Sigma^+ \cup -\Sigma^+$, and we have the root decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha, \quad \mathfrak{g}_0 = \mathfrak{a}_q \oplus \mathfrak{m}_q,$$

where \mathfrak{m}_q is the centralizer of \mathfrak{a}_q in \mathfrak{k} . We have

$$\mathfrak{m}_q = \text{span}_{\mathbb{R}}(M_{ij} \mid 1 \leq i, j \leq n-3),$$

where for $1 \leq i, j \leq n-3$ we set

$$M_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & R_{ij} \end{pmatrix} \text{ with } (R_{ij})_{rs} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}.$$

We choose a basis of root vectors as follows. For $1 \leq j \leq n-3$, $e_j \in \mathbb{R}^{n-3}$, let

$$Y_j^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & e_j \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & e_j \\ e_j^t & \mathbf{0} & \mathbf{0} & -e_j^t & \mathbf{0} \end{pmatrix} \in \mathfrak{g}_{f_1}, \quad Z_j^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & e_j \\ 0 & 0 & 0 & 0 & e_j \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ \mathbf{0} & e_j^t & -e_j^t & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathfrak{g}_{f_2},$$

and let $Y_j^- = (Y_j^+)^t \in \mathfrak{g}_{-f_1}$, $Z_j^- = (Z_j^+)^t \in \mathfrak{g}_{-f_2}$. We define W_1 and W_2 by

$$W_1 = \begin{pmatrix} 0 & 1 & -1 & 0 & \mathbf{0} \\ -1 & 0 & 0 & 1 & \mathbf{0} \\ -1 & 0 & 0 & 1 & \mathbf{0} \\ 0 & 1 & -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathfrak{g}_{f_1+f_2}, \quad W_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & \mathbf{0} \\ -1 & 0 & 0 & 1 & \mathbf{0} \\ 1 & 0 & 0 & -1 & \mathbf{0} \\ 0 & 1 & 1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathfrak{g}_{f_1-f_2}.$$

We have $[Y_j^+, Z_k^+] = \delta_{jk}W_1$ and $[Y_j^+, Z_k^-] = \delta_{jk}W_2$. Moreover, we set

$$\begin{aligned} [Y_j^-, Z_k^+] &= \delta_{jk}W_3 = \delta_{jk}\theta(W_2) \in \mathfrak{g}_{-f_1+f_2}, \\ [Y_j^-, Z_k^-] &= \delta_{jk}W_4 = \delta_{jk}\theta(W_1) \in \mathfrak{g}_{-f_1-f_2}. \end{aligned}$$

It will be convenient to introduce the following elements.

$$Q_1^+ = \frac{W_1 + W_2}{2}, \quad Q_1^- = (Q_1^+)^t = \frac{-W_3 - W_4}{2}, \quad (11)$$

$$Q_2^+ = \frac{W_2 - W_1}{2}, \quad Q_2^- = (Q_2^+)^t = \frac{W_4 - W_3}{2}. \quad (12)$$

We define $\mathfrak{n}_q^+ = \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$. A minimal parabolic subalgebra is given by $\mathfrak{p}_{\min} = \mathfrak{m}_q \oplus \mathfrak{a}_q \oplus \mathfrak{n}_q^+$, a maximal parabolic subalgebra is given by $\mathfrak{p} = \mathfrak{p}_{\min} \oplus \mathfrak{g}_{-f_2}$, with Langlands decomposition $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$, where

$$\mathfrak{m} = \mathfrak{m}_q \oplus \mathfrak{g}_{f_2} \oplus \mathfrak{g}_{-f_2} \oplus \text{span}_{\mathbb{R}}(H_2), \quad \mathfrak{a} = \text{span}_{\mathbb{R}}(H), \quad \mathfrak{n}^+ = \mathfrak{g}_{f_1} \oplus \mathfrak{g}_{f_1+f_2} \oplus \mathfrak{g}_{f_1-f_2}.$$

We have $\mathfrak{n}^- = \theta(\mathfrak{n}^+) = \mathfrak{g}_{-f_1} \oplus \mathfrak{g}_{-f_1-f_2} \oplus \mathfrak{g}_{-f_1+f_2}$. We have the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$. The commutator relations for $\mathfrak{o}(2, n-1)$ can be found in the appendix. We have

$$[\mathfrak{m}, \mathfrak{n}^+] \subset \mathfrak{n}^+, [\mathfrak{m}, \mathfrak{n}^-] \subset \mathfrak{n}^-, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}, [\mathfrak{m}, \mathfrak{a}] = 0, [\mathfrak{n}^-, \mathfrak{n}^-] = 0. \quad (13)$$

We will also use the following basis for \mathfrak{n}_n^- :

$$I_j = \begin{cases} Y_j^- & \text{for } 1 \leq j \leq n-3, \\ Q_1^- & \text{for } j = n-2, \\ Q_2^- & \text{for } j = n-1. \end{cases} \quad (14)$$

Following [10, p. 133], we define K_0 , A , N^+ and M_0 as the analytic subgroups of $G^n = O(2, n - 1)$ with Lie algebras \mathfrak{k} , \mathfrak{a} , \mathfrak{n}^+ and \mathfrak{m} , respectively. We let K be the maximal compact subgroup $O(2) \times O(n - 1)$. We will also use the notation K_0^n , K^n , $(N^+)^n$ and M_0^n for these subgroups of G^n . The Lie algebras \mathfrak{n}^+ and \mathfrak{a} are abelian, and hence $N^+ = \exp(\mathfrak{n}^+)$, $A = \exp(\mathfrak{a})$.

We have a bijection $\mathfrak{so}(1, n - 2) \simeq \mathfrak{m}$ which shows that $M_0 \simeq SO_0(1, n - 2)$. In [14] it is proved that, for $0 \leq p \leq q$, we have $\exp(\mathfrak{so}(p, q)) = SO_0(p, q)$ if and only if $p = 0$ or $p = 1$. This shows that $\exp(\mathfrak{m}) = M_0$.

We have

$$Z_K(\mathfrak{a}) = \{k \in K \mid \text{Ad}(k) = 1 \text{ on } \mathfrak{a}\} = \{k \in O(2) \times O(n - 1) \mid kH = Hk\}.$$

Every $m \in Z_K(\mathfrak{a})$ satisfies

$$m = \begin{pmatrix} a & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & b & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & m_{11} & 0 & m_{13} & \dots & m_{1(n-1)} \\ 0 & 0 & 0 & a & 0 & \dots & 0 \\ 0 & 0 & m_{31} & 0 & m_{33} & \dots & m_{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & m_{(n-1)1} & 0 & m_{(n-1)3} & \dots & m_{(n-1)(n-1)} \end{pmatrix} \in O(2) \times O(n - 1),$$

where $a = \pm 1, b = \pm 1$. We know that $\exp(Z_{\mathfrak{k}}(\mathfrak{a}))$ consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & m_{11} & 0 & m_{13} & \dots & m_{1(n-1)} \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & m_{31} & 0 & m_{33} & \dots & m_{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & m_{(n-1)1} & 0 & m_{(n-1)3} & \dots & m_{(n-1)(n-1)} \end{pmatrix} \in SO(2) \times SO(n - 1),$$

and we conclude that

$$Z_K(\mathfrak{a}) = \{1_{(n)}, J_{(n)}\} \langle w_1, w_2 \rangle \exp(Z_{\mathfrak{k}}(\mathfrak{a})),$$

where

$$w_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \mathbf{0} \\ 0 & -1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & -1 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad (15)$$

$$J_{(n)} = \begin{pmatrix} -1 & 0 & 0 & 0 & \mathbf{0} \\ 0 & -1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & -1 & 0 & \mathbf{0} \\ 0 & 0 & 0 & -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{pmatrix},$$

and $\langle w_1, w_2 \rangle$ is the subgroup of $O(2, n - 1)$ generated by w_1 and w_2 . Here $1_{(n)}$ is the identity element in G^n .

We observe that $Z_{\mathfrak{k}}(\mathfrak{a}) \subset \mathfrak{m}$ and $\exp(Z_{\mathfrak{k}}(\mathfrak{a})) \subset M_0$. By definition, $M = Z_K(\mathfrak{a})M_0$ and hence

$$\begin{aligned} M &= Z_K(\mathfrak{a})M_0 = \left(\{1_{(n)}, J_{(n)}\} \langle w_1, w_2 \rangle \exp(Z_{\mathfrak{k}}(\mathfrak{a})) \right) M_0 \\ &= \{1_{(n)}, J_{(n)}\} \langle w_1, w_2 \rangle M_0 \subset G^n = O(2, n - 1). \end{aligned} \tag{16}$$

We have $M_0 \simeq SO_0(1, n - 2)$ and hence $\langle w_1, w_2 \rangle M_0 \simeq O(1, n - 2)$. The maximal parabolic subgroup is given by $P = MAN^+$.

Furthermore, we see that $M_0 \cap K \simeq SO(n - 2)$, the elements of $M_0 \cap K$ are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & d_{11} & 0 & d_{12} & \dots & d_{1(n-2)} \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & d_{21} & 0 & d_{22} & \dots & d_{2(n-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & d_{(n-2)1} & 0 & d_{(n-2)2} & \dots & d_{(n-2)(n-2)} \end{pmatrix} \in SO(2, n - 1),$$

and the elements of $\langle w_1, w_2 \rangle M_0 \cap K \simeq \{\pm 1\} \times O(n - 2)$ are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \pm 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & d_{11} & 0 & d_{12} & \dots & d_{1(n-2)} \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & d_{21} & 0 & d_{22} & \dots & d_{2(n-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & d_{(n-2)1} & 0 & d_{(n-2)2} & \dots & d_{(n-2)(n-2)} \end{pmatrix} \in O(2, n - 1).$$

Lemma 2.1. $K/(\langle w_1, w_2 \rangle M_0 \cap K) = S^1 \times S^{n-2}$, and

$$K/(M \cap K) \simeq (S^1 \times S^{n-2})/\mathbb{Z}_2,$$

where the \mathbb{Z}_2 -action is defined by

$$(x, y) \mapsto (-x, -y) \text{ on } S^1 \times S^{n-2}. \tag{17}$$

Proof. We have $\langle w_1, w_2 \rangle M_0 \cap K \simeq O(1) \times O(n - 2)$ and hence

$$K/(\langle w_1, w_2 \rangle M_0 \cap K) \simeq (O(2) \times O(n - 1))/(O(1) \times O(n - 2)) \simeq S^1 \times S^{n-2}. \tag{18}$$

Using $M \cap K = \{1_{(n)}, J_{(n)}\}(\langle w_1, w_2 \rangle M_0 \cap K)$, we conclude that

$$K/(M \cap K) \simeq (S^1 \times S^{n-2})/\mathbb{Z}_2. \tag{19}$$

This completes the proof. ■

Let C_n be given by (2). We define Ein_{n-1} , the Einstein universe, as the set of all straight lines in C_n through the origin; see [3] and [2]. We define $\widehat{\text{Ein}}_{n-1}$ as the set of all rays in C_n from the origin. The sphere with radius $\sqrt{2}$ in \mathbb{R}^{n+1} with center at the origin intersects C_n in the set $S^1 \times S^{n-2}$, and hence $\widehat{\text{Ein}}_{n-1} \simeq S^1 \times S^{n-2}$ and $\text{Ein}_{n-1} \simeq (S^1 \times S^{n-2})/\mathbb{Z}_2$, where the \mathbb{Z}_2 -action is coming from the involution (17).

Lemma 2.2. $G^n = O(2, n - 1)$ acts transitively on $\widehat{\text{Ein}}_{n-1}$, and

$$\hat{P} = \langle w_1, w_2 \rangle M_0 AN^+$$

is the isotropy group of the ray generated by $(1, 0, 0, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$. It follows that

$$\widehat{\text{Ein}}_{n-1} \simeq G^n / \hat{P}^n.$$

Proof. Let \tilde{P}^n denote the isotropy group of the ray generated by

$$(1, 0, 0, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}.$$

It is immediate that $\langle w_1, w_2 \rangle M_0 AN^+ \subset \tilde{P}^n$. We now prove $\tilde{P}^n \subset \hat{P}^n$. Let $g \in \tilde{P}^n$. First, assume that $g \in \tilde{P}^n \cap SO_0(2, n - 1)$ and write $g = kan \in K_0 \exp(\mathfrak{a}_q) \exp(\mathfrak{n}_q^+)$. We have $\exp(\mathfrak{a}_q) \subset \tilde{P}^n$, $\exp(\mathfrak{n}_q^+) \subset \tilde{P}^n$ and hence $k \in \tilde{P}^n$. It is easy to see that $K_0 \cap \tilde{P}^n \subset M_0$ and hence we conclude $k \in M_0$. Because $\exp(\mathfrak{n}_q^+) \subset N^+$ and $\exp(\mathfrak{a}_q) \subset M_0 A$ we conclude that $g \in M_0 AN^+$. Finally, using $O(2, n - 1) = \langle w_1, w_2 \rangle SO_0(2, n - 1)$ we conclude that $\tilde{P}^n \subset \langle w_1, w_2 \rangle M_0 AN^+$. ■

Lemma 2.3. $G^n = O(2, n - 1)$ acts transitively on Ein_{n-1} , and P^n is the isotropy group of the line generated by $(1, 0, 0, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$. It follows that

$$\text{Ein}_{n-1} \simeq G^n / P^n.$$

Proof. Let \tilde{P}^n denote the isotropy group of the line generated by

$$(1, 0, 0, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}.$$

It is immediate that $P^n \subset \tilde{P}^n$. We now prove $\tilde{P}^n \subset P^n$. Let $g \in \tilde{P}^n$. First, assume that $g \in \tilde{P}^n \cap SO_0(2, n - 1)$, and write $g = kan \in K_0 \exp(\mathfrak{a}_q) \exp(\mathfrak{n}_q^+)$. We have $\exp(\mathfrak{a}_q) \subset \tilde{P}^n$, $\exp(\mathfrak{n}_q^+) \subset \tilde{P}^n$ and hence $k \in \tilde{P}^n$. It is easy to see that $K_0 \cap \tilde{P}^n \subset M$ and hence we conclude $k \in M$. Because $\exp(\mathfrak{n}_q^+) \subset N^+$ and $\exp(\mathfrak{a}_q) \subset MA$ we conclude that $g \in P^n$. Finally, using $O(2, n - 1) = \langle w_1, w_2 \rangle SO_0(2, n - 1)$, $\langle w_1, w_2 \rangle \subset P^n$, we conclude that $\tilde{P}^n \subset P^n$. ■

We will use the notation

$$n_{(x_1, x_2, x_3, \dots, x_{n-1})}^- = \exp(x_1 Q_1^- + x_2 Q_2^- + \sum_{i=3}^{n-1} x_i Y_{i-2}^-),$$

and $(N^-)^n$ is identified with $\mathbb{R}^{n-1} = \mathbb{M}^{n-1}$ by

$$\mathbb{M}^{n-1} \ni (x_1, x_2, x_3, \dots, x_{n-1}) \mapsto n_{(x_1, x_2, x_3, \dots, x_{n-1})}^- \in (N^-)^n. \tag{20}$$

We let $\pi : C_n \setminus \{0\} \rightarrow \text{Ein}_{n-1}$ denote the natural projection and define the map

$$p : G^n \rightarrow \text{Ein}_{n-1}, x \mapsto \pi(x(1, 0, 0, 1, 0_{n-3})).$$

Here $(1, 0, 0, 1, 0_{n-3})$ denotes the vector $(1, 0, 0, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$, and

$$x(1, 0, 0, 1, 0_{n-3})$$

denotes matrix multiplication of $x \in (N^-)^n \subset G^n$ with the column matrix $(1, 0, 0, 1, 0_{n-3})^t$.

Lemma 2.4. *The injection*

$$j : \mathbb{M}^{n-1} \simeq (N^-)^n \rightarrow \text{Ein}_{n-1}, x \mapsto p(x)$$

has image $\text{Ein}_{n-1} \setminus \pi(D_n)$, where

$$D_n = \{(t_1, t_2, x_1, x_2, x_3, \dots, x_{n-1}) \mid t_1 + x_2 = 0\}.$$

The injection j gives a conformal compactification of \mathbb{M}^{n-1} in Ein_{n-1} .

Proof. We compute

$$\begin{aligned} & \exp(w_1 Q_1^- + w_2 Q_2^- + \sum_{j=1}^{n-3} w_{j+2} Y_j^-)(1, 0, 0, 1, 0_{n-3}) = \\ & = (1 - w_1^2 + w_2^2 + |w'|^2, 2w_1, 2w_2, 1 + w_1^2 - w_2^2 - |w'|^2, 2w_3, 2w_4, \dots, 2w_{n-1}), \end{aligned}$$

where we let $w' = (w_3, \dots, w_{n-1})$. The equations

$$\begin{aligned} t_1 &= r(1 - w_1^2 + w_2^2 + |w'|^2), \\ t_2 &= 2rw_1, \\ x_1 &= 2rw_2, \\ x_2 &= r(1 + w_1^2 - w_2^2 - |w'|^2), \\ x_3 &= 2rw_3, \\ x_4 &= 2rw_4, \\ &\dots \\ x_{n-1} &= 2rw_{n-1}, \end{aligned}$$

imply that $r = \frac{t_1 + x_2}{2}$ and hence

$$w_1 = \frac{t_2}{t_1 + x_2}, w_2 = \frac{x_1}{t_1 + x_2}, w_3 = \frac{x_3}{t_1 + x_2}, w_4 = \frac{x_4}{t_1 + x_2}, \dots, w_{n-1} = \frac{x_{n-1}}{t_1 + x_2}. \quad (21)$$

This completes the proof. \blacksquare

If we let $H^n \subset G^n$ denote the elements $g \in G^n$ such that $p(g) \in \pi(D_n)$ we get the Bruhat decomposition

$$G^n = (N^-)^n M^n A(N^+)^n \cup H^n.$$

For $g \in G^n, g \notin H^n$ we write $g = \eta(g)m(g)a(g)n$.

Lemma 2.5. *Let ξ be a character of A . Then every $f \in C_0^\infty((N^-)^n)$ extends to a function $\hat{f} \in C^\infty(G^n)$ such that $\hat{f}(x\tilde{m}\tilde{a}\tilde{n}) = \xi(\tilde{a})\hat{f}(x)$ for all $\tilde{m}\tilde{a}\tilde{n} \in P^n$ and $x \in G^n$.*

Proof. We define $\hat{f}(n^-\tilde{m}\tilde{a}\tilde{n}) = f(n^-)\xi(\tilde{a})$, where $n^- \in N^-$, $\tilde{m} \in M$, $\tilde{a} \in A$ and $\tilde{n} \in N^+$, and $\hat{f}(g) = 0$ for $g \in H^n$. To prove that \hat{f} is smooth, it is sufficient to prove that for every $h \in H^n$ there exists an open neighborhood of h (in G^n), where \hat{f} is identically zero.

Assume that $h \in H^n$. Then $p(h) \in \pi(D_n)$. For any $\epsilon > 0$ the set

$$U(\epsilon) := \{\pi((t_1, t_2, x_1, x_2, x_3, \dots, x_{n-1})) \mid t_1 + x_2 < \epsilon\}$$

is a neighborhood of $p(h)$, and using the continuity of p the set $p^{-1}(U(\epsilon))$ is a neighborhood of h . Now let $g \in G^n$. If $gP^n \in \text{Ein}_{n-1}$ is given by $\pi((t_1, t_2, x_1, \dots, x_{n-1}))$ then using (21) we get

$$\eta(g) = n^- \left(\frac{t_2}{t_1+x_2}, \frac{x_1}{t_1+x_2}, \frac{x_3}{t_1+x_2}, \dots, \frac{x_{n-1}}{t_1+x_2} \right),$$

and using the compact support of f , we see that for sufficiently small $\epsilon > 0$ the function \hat{f} is identically zero in $p^{-1}(U(\epsilon))$. ■

Let R be the right-regular representation of G on $C^\infty(G)$, i.e., $(R(g)u)(x) = u(xg)$. We let $\mathcal{U}(\mathfrak{g})$ operate from the *right* on $C^\infty(G)$ through the extension of its differential

$$(R(X)u)(x) = \left. \frac{d}{dt} \right|_{t=0} u(x \exp(tX)), \quad X \in \mathfrak{g}. \tag{22}$$

Observe that for any $u \in C^\infty(G)$ and $X \in \mathfrak{g}$ we have

$$R(\text{Ad}(g)X)u = R(g) \circ R(X) \circ R(g^{-1})u. \tag{23}$$

Using R , elements of $\mathcal{U}(\mathfrak{g})$ induce left-invariant differential operators on $C^\infty(G)$.

We now define principal series representations of $G^m = O(2, m-1)$ induced from the parabolic subgroups P^m, P_0^m and \hat{P}^m with respective Langlands decompositions

$$P = MAN^+, \quad P_0 = M_0AN^+, \quad \hat{P} = \hat{M}AN^+,$$

where $\hat{M} = \langle w_1, w_2 \rangle M_0$.

We first define principal series as induced representations. For $\lambda \in \mathbb{C}$ with corresponding character ξ_λ and a representation $\sigma : M \rightarrow \mathbb{C}^*$ let

$$\begin{aligned} \text{Ind}_{P^m}^{G^m}(\xi_\lambda \otimes \sigma) = \\ \{u \in C^\infty(G^m) \mid u(g\tilde{m}\tilde{a}\tilde{n}) = \sigma(\tilde{m})\xi_\lambda(\tilde{a})u(g), \tilde{m}\tilde{a}\tilde{n} \in P^m, g \in G^m\}. \end{aligned}$$

G^m acts on $\text{Ind}_{P^m}^{G^m}(\xi_\lambda \otimes \sigma)$ by left translation, i.e.,

$$(\pi_{\xi_\lambda \otimes \sigma}(g)u)(x) = u(g^{-1}x).$$

Note that our definition differs from the convention in [10].

We use the notation $\text{Ind}_{P^m}^{G^m}(\xi_\lambda) = \text{Ind}_{P^m}^{G^m}(\xi_\lambda \otimes \sigma_+)$, where σ_+ is the trivial representation of M . We also use the notation $\pi_{\xi_\lambda}(g) = \pi_{\xi_\lambda \otimes \sigma_+}(g)$. Let the representation $\sigma_- : M^m \rightarrow \mathbb{C}$ be the unique representation such that $\sigma_-(x) = 1$ for $x \in \langle w_1, w_2 \rangle M_0$, $\sigma_-(J_{(m)}) = -1$.

The principal series representations of G^m induced from \hat{P}^m and P_0^m are defined in a similar way.

Let $K_M := K \cap M$. In order to describe the *compact realization* of the principal series representation of G^m (induced from P^m), let

$$C^\infty(K)^{K_M} = \{F \in C^\infty(K) \mid F(k\tilde{m}) = F(k), \tilde{m} \in K_M\}.$$

Let $\delta : \text{Ind}_P^G(\xi_\lambda) \rightarrow C^\infty(K)^{K_M}$ be the restriction map. Using the decomposition $G = KMAN^+$ ([9, prop 7.83(g)]), we write $g = \kappa(g)\mu(g)e^{H(g)}\eta(g)$ and find

$$\delta^{-1}(F)(g) = F(\kappa(g))\xi_\lambda(e^{H(g)}).$$

δ becomes an isomorphism of G -modules if on $C^\infty(K)^{K_M}$ we define

$$(\pi_\lambda^c(g)F)(kK_M) = F(\kappa(g^{-1}k)K_M)\xi_\lambda(e^{H(g^{-1}k)}). \tag{24}$$

Now (18) gives a bijection

$$C^\infty(K^m)^{K_M^m} \rightarrow C^\infty(\text{Ein}_{m-1}), \tag{25}$$

and the equation (24) defines a G -module structure on $C^\infty(\text{Ein}_{m-1})$. If we want to emphasize the parameter $\lambda \in \mathbb{C}$ of π_λ^c we write $C^\infty(\text{Ein}_{m-1})_\lambda$ for the representation space.

Let $K_{\hat{M}} = K \cap \hat{M}$. In order to describe the *compact realization* of the principal series representation of G^m (induced from \hat{P}^m) let

$$C^\infty(K)^{K_{\hat{M}}} = \{F \in C^\infty(K) \mid F(k\hat{m}) = F(k), \hat{m} \in K_{\hat{M}}\}.$$

Let $\hat{\delta} : \text{Ind}_{\hat{P}}^G(\xi_\lambda) \rightarrow C^\infty(K)^{K_{\hat{M}}}$ be the restriction map. Using the decomposition $G = K\hat{M}AN^+$, we write $g = \kappa(g)\mu(g)e^{H(g)}\eta(g)$ and find

$$\hat{\delta}^{-1}(F)(g) = F(\kappa(g))\xi_\lambda(e^{H(g)}).$$

$\hat{\delta}$ becomes an isomorphism of G -modules if on $C^\infty(K)^{K_{\hat{M}}}$ we define

$$(\pi_\lambda^c(g)F)(kK_{\hat{M}}) = F(\kappa(g^{-1}k)K_{\hat{M}})\xi_\lambda(e^{H(g^{-1}k)}).$$

Now (19) gives a bijection

$$C^\infty(K^m)^{K_{\hat{M}}^m} \rightarrow C^\infty(\widehat{\text{Ein}}_{m-1}). \tag{26}$$

If we want to emphasize the parameter $\lambda \in \mathbb{C}$ of π_λ^c , we write $C^\infty(\widehat{\text{Ein}}_{m-1})_\lambda$ for the representation space.

We describe the *non-compact realization* of the principal series representation (of G^m induced from P^m). Let $\beta_{\xi_\lambda \otimes \sigma} : \text{Ind}_{P^m}^{G^m}(\xi_\lambda \otimes \sigma) \rightarrow C^\infty((N^-)^m) \simeq C^\infty(\mathbb{R}^{m-1})$ be the restriction map. We define

$$C^\infty((N^-)^m)_{\xi_\lambda \otimes \sigma} := \beta_{\xi_\lambda \otimes \sigma}(\text{Ind}_{P^m}^{G^m}(\xi_\lambda \otimes \sigma)).$$

We know that $G^m = (N^-)^m M^m A(N^+)^m \cup H^m$, where $H^m \subset G^m$ is a lower-dimensional set, and write $g = \eta(g)m(g)a(g)n$. We find

$$(\beta_{\xi_\lambda \otimes \sigma})^{-1}(F)(n^-man) = F(n^-)\sigma(m)\xi_\lambda(a).$$

The definition

$$(\pi_{\xi_\lambda \otimes \sigma}^{\text{nc}}(g)F)(x) = F(\eta(g^{-1}x))\sigma(m(g^{-1}x))\xi_\lambda(a(g^{-1}x)), \quad F \in C^\infty((N^-)^m)_{\xi_\lambda \otimes \sigma}$$

for $x \in (N^-)^m$ makes $\beta_{\xi_\lambda \otimes \sigma} : \text{Ind}_{P^m}^{G^m}(\xi_\lambda) \rightarrow C^\infty((N^-)^m)_{\xi_\lambda \otimes \sigma}$ into a G^m -equivariant map.

Lemma 2.6. *Let $u \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ such that for every $f \in \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda)$ we have*

$$(uf)(e) = 0,$$

where e is the identity element of G^{n+1} . This implies that $u = 0$.

Proof. We use the notation (14). Observe that that $N_{n+1}^- = \exp(\mathfrak{n}_{n+1}^-)$. Let $f(\exp(\sum_{i=1}^n x_i I_i)) = \xi(x_1, \dots, x_n)p(x_1, \dots, x_n)$, where p is a polynomial and ξ a smooth cut-off function such that $\xi(x_1, \dots, x_n) = 1$ for $x_1^2 + \dots + x_n^2 < 1$ and $\xi(x_1, \dots, x_n) = 0$ for $x_1^2 + \dots + x_n^2 > 2$. According to Lemma 2.5 f extends to a smooth function $f \in \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda)$. Note that for any polynomial p and $u \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$, $u = (I_1)^{k_1} \dots (I_n)^{k_n}$ we get

$$(uf)(e) = \left(\left(\frac{\partial}{\partial x_1} \right)^{k_1} \left(\frac{\partial}{\partial x_2} \right)^{k_2} \dots \left(\frac{\partial}{\partial x_n} \right)^{k_n} \right) p(0).$$

We see that if we let $p(x_1, \dots, x_n) = x_1^{l_1} \dots x_n^{l_n}$ then $(uf)(e) = 0$ if and only if $u \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ contains no monomial of the form $I_1^{l_1} \dots I_n^{l_n}$. This completes the proof. ■

3. Powers of the Laplacian and algebraic constructions

Under the identification (20), the d'Alembertian $\Delta_{\mathbb{M}^{n-1}}$ is induced by

$$\Delta_{n-1} = -(Q_1^-)^2 + (Q_2^-)^2 + \sum_{j=1}^{n-3} (Y_j^-)^2 \in \mathcal{U}(\mathfrak{n}_n^-) \subset \mathcal{U}(\mathfrak{g}_n). \quad (27)$$

Δ_{n-1} commutes with \mathfrak{m}_n . Moreover,

Proposition 3.1. *$P \in \mathcal{U}(\mathfrak{n}_n^-)$ satisfies $[X, P] = 0$ for all $X \in \mathfrak{m}_n$ if and only if P is a polynomial in Δ_{n-1} .*

Proof. Although the result is well-known, we give a proof for the sake of completeness; the argument is similar to [6, p. 270-271]. It is easy to see that if P is a polynomial in Δ_{n-1} , then $[X, P] = 0$ for all $X \in \mathfrak{m}_n$. We now prove that if $[X, P] = 0$ for all $X \in \mathfrak{m}_n$, then P is a polynomial in Δ_{n-1} . Observe that

$M_0^n = \exp(\mathfrak{m}_n)$ is generated by $A_j(t), B_j(t), C_j(t), D_{ij}(t)$, $1 \leq i, j \leq n-3$, $t \in \mathbb{R}$, where

$$\begin{aligned} A_j(t) &= \exp\left(\frac{Z_j^+ + Z_j^-}{2}t\right), & B_j(t) &= \exp\left(\frac{Z_j^+ - Z_j^-}{2}t\right), \\ C_j(t) &= \exp(H_2t), & D_{ij}(t) &= \exp(M_{ij}t). \end{aligned}$$

The Lorentzian metric on \mathbb{M}^{n-1} induces the following non-degenerate bilinear form on \mathfrak{n}_n^- :

$$\begin{aligned} \langle x_1 Q_1^- + x_2 Q_2^- + \sum_{i=3}^{n-1} x_i Y_{i-2}^-, y_1 Q_1^- + y_2 Q_2^- + \sum_{i=3}^{n-1} y_i Y_{i-2}^- \rangle \\ = -x_1 y_1 + x_2 y_2 + \sum_{i=3}^{n-1} x_i y_i. \end{aligned} \quad (28)$$

We prove that for any $Y \in \mathfrak{n}_n^-$ such that $\langle Y, Y \rangle = 1$ there exists some $m \in M_0^n$ satisfying

$$\text{Ad}(m)Q_2^- = Y. \quad (29)$$

Let $Y = x_1 Q_1^- + x_2 Q_2^- + x_3 Y_1^- + \cdots + x_{n-1} Y_{n-3}^-$. For some $t \in \mathbb{R}$ we have

$$\text{Ad}(\exp(tH_2))Q_2^- = Q_2^- \cosh t + Q_1^- \sinh t = Q_2^- \cosh t + x_1 Q_1^-. \quad (30)$$

Using

$$\begin{aligned} \text{Ad}(B_j(t))(Q_2^- \sin \alpha + Y_j^- \cos \alpha) &= Q_2^- \sin(\alpha + t) + Y_j^- \cos(\alpha + t), \\ \text{Ad}(\exp(M_{ij}t))(Y_j^- \cos \alpha + Y_i^- \sin \alpha) &= Y_j^- \cos(\alpha + t) + Y_i^- \sin(\alpha + t), \end{aligned} \quad (31)$$

we see that $\text{Ad}(B_j(t))$ and $\text{Ad}(D_{ij}(t))$ acts as rotations in the subspace spanned by $Q_2^-, Y_1^-, \dots, Y_{n-3}^-$ leaving the subspace spanned by Q_1^- invariant, and (29) follows. We conclude that

1. $\text{Ad}(m)(\mathfrak{n}_n^-) \subset \mathfrak{n}_n^-$ for all $m \in M_0^n$,
2. $\langle X, Y \rangle = \langle \text{Ad}(m)X, \text{Ad}(m)Y \rangle$ for all $X, Y \in \mathfrak{n}_n^-$, $m \in M_0^n$,
3. $\text{Ad}(M_0^n)$ acts transitively on the set $\{X \in \mathfrak{n}_n^- \mid X \neq 0, \langle X, X \rangle = c\}$ for each real number $c > 0$.

In the following we choose

$$X_1 = Q_1^-, X_2 = Q_2^-, X_3 = Y_1^-, \dots, X_{n-1} = Y_{n-3}^-$$

as a basis for \mathfrak{n}_n^- . Now assume that $[X, P] = 0$ for all $X \in \mathfrak{m}_n$. This implies that $\text{Ad}(m)P = P$ for all $m \in M_0^n$. Let $P = \sum a_{r_1 \dots r_{n-1}} X_1^{r_1} X_2^{r_2} \cdots X_{n-1}^{r_{n-1}}$. We define $P^* : \mathfrak{n}_n^- \rightarrow \mathbb{C}$ by

$$P^*\left(\sum_{i=1}^{n-1} x_i X_i\right) = \sum a_{r_1 \dots r_{n-1}} x_1^{r_1} x_2^{r_2} \cdots x_{n-1}^{r_{n-1}}. \quad (32)$$

The Lie algebra \mathfrak{n}_n^- is abelian, and hence

$$(PQ)^*\left(\sum x_i X_i\right) = P^*\left(\sum x_i X_i\right)Q^*\left(\sum x_i X_i\right), \tag{33}$$

which implies

$$P^*(\text{Ad}(m)^{-1}X) = (\text{Ad}(m)P)^*(X). \tag{34}$$

Using $\text{Ad}(m)P = P$ for all $m \in M_0^n$ and (34), we conclude that $P^*(\text{Ad}(m)X) = P^*(X)$ for all $m \in M_0^n$ and $X \in \mathfrak{n}_n^-$. Using (2) and (3) above, we see that for each real number $c > 0$ the function P^* is constant on $B_c := \{X \in \mathfrak{n}_n^- \mid \langle X, X \rangle = c\}$.

We get $P^*(x_2 X_2) = P^*(-x_2 X_2)$ and hence for some numbers a_k we have

$$P^*(x_2 X_2) = \sum_{k=0}^N a_k (x_2)^{2k} = \sum_{k=0}^N a_k \langle x_2 X_2, x_2 X_2 \rangle^k.$$

Using that for $c > 0$ the function P^* is constant on B_c , we conclude that for $X \in \mathfrak{n}_n^-$, $\langle X, X \rangle > 0$ we have

$$P^*(X) = \sum_{k=0}^N a_k \langle X, X \rangle^k. \tag{35}$$

If we let $X = \sum_{i=1}^{n-1} x_i X_i$ and use (32), together with $\langle X, X \rangle = -x_1^2 + x_2^2 + \dots + x_{n-1}^2$, we get

$$\sum a_{r_1 \dots r_{n-1}} x_1^{r_1} x_2^{r_2} \dots x_{n-1}^{r_{n-1}} = \sum_{k=0}^N a_k (-x_1^2 + x_2^2 + \dots + x_{n-1}^2)^k$$

on the open set $\{(x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \mid -x_1^2 + x_2^2 + \dots + x_{n-1}^2 > 0\}$. But since two polynomials coincide if they coincide on a non-empty open set, (35) is valid for all $X \in \mathfrak{n}_n^-$. We conclude that $P = \sum_{k=0}^N a_k (-X_1^2 + \dots + X_{n-1}^2)^k$. ■

The following result will be used to construct the induced operator families. It will be convenient to use the notation $T_n = Y_{n-2}^- \in \mathfrak{n}_{n+1}^-$.

Theorem 3.2. *Let $\lambda \in \mathbb{C}$.*

$$\mathcal{D}_{2N}(\lambda) = \sum_{j=0}^N a_j(\lambda) (\Delta_{n-1})^j (T_n)^{2N-2j} \in \mathcal{U}(\mathfrak{n}_{n+1}^-) \tag{36}$$

satisfies

$$[X, \mathcal{D}_{2N}(\lambda)] \in \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda)) \tag{37}$$

for all $X \in \mathfrak{n}_n^+$ if and only if

$$(N - j + 1)(2N - 2j + 1)a_{j-1}(\lambda) + j(n - 1 + 2\lambda - 4N + 2j)a_j(\lambda) = 0 \tag{38}$$

for $j = 1, \dots, N$.

Theorem 3.3. *Let $\lambda \in \mathbb{C}$.*

$$\mathcal{D}_{2N+1}(\lambda) = \sum_{j=0}^N b_j(\lambda)(\Delta_{n-1})^j (T_n)^{2N-2j+1} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$$

satisfies

$$[X, \mathcal{D}_{2N+1}(\lambda)] \in \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda))$$

for all $X \in \mathfrak{n}_n^+$ if and only if

$$(N - j + 1)(2N - 2j + 3)b_{j-1}(\lambda) + j(n - 3 + 2\lambda - 4N + 2j)b_j(\lambda) = 0 \quad (39)$$

for all $j = 1, \dots, N$.

Let $\mathcal{D}_{2N}^0(\lambda) \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ be the unique element satisfying (38) and $a_N(\lambda) = 1$. Let $\mathcal{D}_{2N+1}^0(\lambda) \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ be the unique element satisfying (39) and $b_N(\lambda) = 1$. We have the following algebraic characterization of the family $\mathcal{D}_N^0(\lambda)$.

Theorem 3.4. *Let $N \geq 0, \mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ and let $\lambda \in \mathbb{C}$. The three conditions*

$$[X, \mathcal{E}] = 0 \text{ for all } X \in \mathfrak{m}_n, \quad (40)$$

$$[X, \mathcal{E}] \in \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda)) \text{ for all } X \in \mathfrak{n}_n^+, \quad (41)$$

$$[H, \mathcal{E}] = -N\mathcal{E}, \quad (42)$$

are satisfied if and only if $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ for some $c \in \mathbb{C}$.

The analogs of Theorem 3.2 and Theorem 3.3 for $\mathfrak{o}(1, n)$ have been proved in [8]. Our proof for $\mathfrak{o}(2, n - 1)$ is similar but incorporates some simplifications (as a proof of Lemma 3.9 using an induction argument). We now prove Theorem 3.2 and Theorem 3.4. We omit the proof of Theorem 3.3; it is similar to the proof of the corresponding result for $\mathfrak{o}(1, n)$ in [8].

We formulate some lemmas which are used in the proof of Theorem 3.2.

Lemma 3.5. *We have*

$$[\mathfrak{m}_n, \Delta_{n-1}] = 0, \quad (43)$$

$$[\mathfrak{n}_n^-, \Delta_{n-1}] = 0, \quad (44)$$

$$[\mathfrak{m}_n, T_n] = 0, \quad (45)$$

$$[H, (\Delta_{n-1})^k] = -2k(\Delta_{n-1})^k, \quad (46)$$

$$[H, (T_n)^{2k}] = -2k(T_n)^{2k}. \quad (47)$$

Proof. (43) follows from Proposition 3.1, (44) follows because \mathfrak{n}_n^- is abelian. The other statements follow from a simple computation. ■

Lemma 3.6. *For any $X \in \mathfrak{n}_n^+$ and for any $F \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ we have that*

$$[X, F] \in (\mathcal{U}(\mathfrak{n}_{n+1}^-)\mathfrak{m}_{n+1}) \oplus (\mathcal{U}(\mathfrak{n}_{n+1}^-)\mathfrak{a}) \oplus \mathcal{U}(\mathfrak{n}_{n+1}^-). \quad (48)$$

Proof. Any $F \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ can be written as a linear combination of terms of the form

$$(I_1)^{a_1} \dots (I_n)^{a_n},$$

where a_1, \dots, a_n are natural numbers and I_1, \dots, I_n is a basis for \mathfrak{n}_{n+1}^- ; see (14). Using

$$[A, B_1 \dots B_n] = [A, B_1]B_2 \dots B_n + \dots + B_1 \dots B_{n-1}[A, B_n]$$

and $[\mathfrak{n}_{n+1}^+, \mathfrak{n}_{n+1}^-] \subset \mathfrak{m}_{n+1} \oplus \mathfrak{a}$, we conclude that $[X, F]$ can be written as a linear combination of terms which are monomials with factors from \mathfrak{n}_{n+1}^- and at most one factor from $\mathfrak{m}_{n+1} \oplus \mathfrak{a}_{n+1}$. Using $[\mathfrak{m}_{n+1}, \mathfrak{n}_{n+1}^-] \subset \mathfrak{n}_{n+1}^-$ and $[\mathfrak{a}, \mathfrak{n}_{n+1}^-] \subset \mathfrak{n}_{n+1}^-$, the statement (48) follows. ■

Lemma 3.7.

$$[Y_1^+, (Y_1^-)^{2N}] = -2N(2N-1)(Y_1^-)^{2N-1} + 4N(Y_1^-)^{2N-1}H.$$

For $2 \leq r \leq n$ we have

$$[Y_1^+, (Y_r^-)^{2N}] = 2N(2N-1)Y_1^-(Y_r^-)^{2N-2} + 4N(Y_r^-)^{2N-1}M_{1r}.$$

Proof. See Lemma 5.1.3 in [8]. ■

Lemma 3.8.

$$[Y_1^+, \Delta_{n-1}] = (2n-6)Y_1^- + 4Y_1^-H + v,$$

where $v \in \mathfrak{n}_n^- \otimes \mathfrak{m}_n$.

Proof. We have

$$\begin{aligned} [Y_1^+, -(Q_1^-)^2] &= 2Q_1^-(Z_1^+ + Z_1^-) + 2Y_1^-, \\ [Y_1^+, (Q_2^-)^2] &= 2Q_2^-(Z_1^- - Z_1^+) + 2Y_1^-, \end{aligned}$$

and hence

$$[Y_1^+, -(Q_1^-)^2 + (Q_2^-)^2] = 4Y_1^- + 2Q_1^-(Z_1^+ + Z_1^-) + 2Q_2^-(Z_1^- - Z_1^+).$$

Furthermore

$$[Y_1^+, \sum_{j=1}^{n-3} (Y_j^-)^2] = (2n-10)Y_1^- + 4Y_1^-H + \sum_{j=2}^{n-3} 4Y_j^-M_{1j}. \quad \blacksquare$$

Lemma 3.9. For $j \geq 1$ we have

$$[Y_1^+, (\Delta_{n-1})^j] = 2j(n-1-2j)Y_1^-(\Delta_{n-1})^{j-1} + 4jY_1^-(\Delta_{n-1})^{j-1}H + v, \quad (49)$$

where $v \in \mathcal{U}(\mathfrak{n}_n^-) \otimes \mathfrak{m}_n$.

Proof. Lemma 3.8 shows that (49) is true for $j = 1$. Assume that (49) is valid for $j \leq k$. We want to prove that (49) is valid for $j = k + 1$. We get for some $v_1, v_2 \in \mathcal{U}(\mathfrak{n}_n^-) \otimes \mathfrak{m}_n$ that

$$\begin{aligned} [Y_1^+, (\Delta_{n-1})^{k+1}] &= [Y_1^+, \Delta_{n-1}](\Delta_{n-1})^k + (\Delta_{n-1})[Y_1^+, (\Delta_{n-1})^k] \\ &= \{(2n-6)Y_1^- + 4Y_1^-H + v_1\}(\Delta_{n-1})^k \\ &\quad + (\Delta_{n-1})\{2k(n-1-2k)Y_1^-(\Delta_{n-1})^{k-1} + 4kY_1^-(\Delta_{n-1})^{k-1}H + v_2\}. \end{aligned}$$

From (46) we see that

$$4Y_1^-H(\Delta_{n-1})^k = 4Y_1^-(\Delta_{n-1})^k(H-2k).$$

From (43) we know that $[\mathfrak{m}_n, \Delta_{n-1}] = 0$ and hence for all $R \in \mathcal{U}(\mathfrak{n}_n^-)$, $S \in \mathfrak{m}_n$ we have

$$RS(\Delta_{n-1})^k = R(\Delta_{n-1})^kS = R'S,$$

where $R' \in \mathcal{U}(\mathfrak{n}_n^-)$.

We conclude that for some $v_3, v_4 \in \mathcal{U}(\mathfrak{n}_n^-) \otimes \mathfrak{m}_n$ we have

$$\begin{aligned} [Y_1^+, (\Delta_{n-1})^{k+1}] &= (2n-6)Y_1^-(\Delta_{n-1})^k + 4Y_1^-(\Delta_{n-1})^k(H-2k) + v_3 \\ &\quad + 2k(n-1-2k)Y_1^-(\Delta_{n-1})^k + 4kY_1^-(\Delta_{n-1})^kH + v_4 \\ &= 2(k+1)(n-1-2(k+1))Y_1^-(\Delta_{n-1})^k \\ &\quad + 4(k+1)Y_1^-(\Delta_{n-1})^kH + (v_3 + v_4). \end{aligned} \quad \blacksquare$$

Proof of Theorem 3.2. For some $v \in \mathcal{U}(\mathfrak{n}_n^-) \otimes \mathfrak{m}_n$ we get using $[\mathfrak{m}_n, T_n] = 0$ that

$$\begin{aligned} Y_1^+(\Delta_{n-1})^j(T_n)^{2k} &= [Y_1^+, (\Delta_{n-1})^j](T_n)^{2k} + (\Delta_{n-1})^jY_1^+(T_n)^{2k} \\ &= 2j(n-1-2j)Y_1^-(\Delta_{n-1})^{j-1}(T_n)^{2k} \\ &\quad + 4jY_1^-(\Delta_{n-1})^{j-1}H(T_n)^{2k} + v(T_n)^{2k} + (\Delta_{n-1})^jY_1^+(T_n)^{2k} \\ &= 2j(n-1-2j)Y_1^-(\Delta_{n-1})^{j-1}(T_n)^{2k} \\ &\quad + 4jY_1^-(\Delta_{n-1})^{j-1}H(T_n)^{2k} + w + (\Delta_{n-1})^jY_1^+(T_n)^{2k}, \end{aligned}$$

where $w \in \mathcal{U}(\mathfrak{n}_{n+1}^-) \otimes \mathfrak{m}_n$. Using $H(T_n)^{2k} = (T_n)^{2k}(H-2k)$, we get

$$\begin{aligned} &[Y_1^+, (\Delta_{n-1})^j(T_n)^{2k}] \\ &= 2j(n-1-2j)Y_1^-(\Delta_{n-1})^{j-1}(T_n)^{2k} \\ &\quad + 4jY_1^-(\Delta_{n-1})^{j-1}H(T_n)^{2k} + w + (\Delta_{n-1})^j[Y_1^+, (T_n)^{2k}] \\ &= 2j(n-1-2j)Y_1^-(\Delta_{n-1})^{j-1}(T_n)^{2k} \\ &\quad + 4jY_1^-(\Delta_{n-1})^{j-1}(T_n)^{2k}H - 8jkY_1^-(\Delta_{n-1})^{j-1}(T_n)^{2k} + w \\ &\quad + (\Delta_{n-1})^j2k(2k-1)Y_1^-(T_n)^{2k-2} + (\Delta_{n-1})^j4k(T_n)^{2k-1}S, \quad (\text{where } S \in \mathfrak{m}_{n+1}) \\ &= \{2j(n-1-2j) + 4j\lambda - 8jk\}Y_1^-(\Delta_{n-1})^{j-1}(T_n)^{2k} \\ &\quad + 2k(2k-1)Y_1^-(\Delta_{n-1})^j(T_n)^{2k-2} \pmod{\mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H-\lambda))}, \end{aligned}$$

where we used that $[Y_1^-, \Delta_{n-1}] = 0$, that is (13).

We obtain $(\text{mod } \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda)))$

$$\begin{aligned} & [Y_1^+, \mathcal{D}_{2N}(\lambda)] \\ &= \sum_{j=0}^N a_j(\lambda)((2j(n-1-2j) + 4j\lambda - 8j(N-j))Y_1^-(\Delta_{n-1})^{j-1}(T_n)^{2N-2j} \\ & \quad + \sum_{j=0}^N 2(N-j)(2N-2j-1)Y_1^-(\Delta_{n-1})^j(T_n)^{2N-2j-2}). \end{aligned}$$

We conclude that $[Y_1^+, \mathcal{D}_{2N}(\lambda)] = 0 \pmod{\mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda))}$ if and only if

$$a_r(\lambda)\{2r(n-1-2r) + 4r\lambda - 8r(N-r)\} + a_{r-1}(\lambda)2(N-r+1)(2N-2r+1) = 0$$

for $r = 1, \dots, N$.

The *only if* statement follows because no non-trivial linear combination of

$$Y_1^-(\Delta_{n-1})^{k-1}(T_n^-)^{2N-2k} = Y_1^-(\Delta_{n-1})^{k-1}(T_n^-)^{2N-2k} \in \mathcal{U}(\mathfrak{g}_{n+1})$$

for $k = 1, \dots, N$ lies in $\mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda))$. This follows expanding $(\Delta_{n-1})^{k-1}$ into monomials and using the Poincaré-Birkhoff-Witt theorem.

Now suppose that the coefficients $a_j(\lambda)$ satisfy (38). Under this assumption, we know that (37) is true for $X = Y_1^+$, that is, we know that

$$[Y_1^+, \mathcal{D}_{2N}(\lambda)] \in \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda)).$$

We prove that this implies (37) for *all* $X \in \mathfrak{n}_n^+$.

First, observe that using (13), we see that for all $\tilde{M} \in M_0^n$ we have

$$\text{Ad}(\tilde{M})\mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda)) \subset \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda)). \quad (50)$$

Using $[\mathfrak{m}_n, \Delta_{n-1}] = 0$ and $[\mathfrak{m}_n, T_n] = 0$, we get

$$\begin{aligned} \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda)) & \ni \text{Ad}(\tilde{M})[Y_1^+, \mathcal{D}_{2N}(\lambda)] \\ &= [\text{Ad}(\tilde{M})Y_1^+, \text{Ad}(\tilde{M})\mathcal{D}_{2N}(\lambda)] \\ &= [\text{Ad}(\tilde{M})Y_1^+, \mathcal{D}_{2N}(\lambda)] \\ &= \begin{cases} [Y_j^+, \mathcal{D}_{2N}(\lambda)] & \text{for } \tilde{M} = \exp(\frac{-\pi M_{1j}}{2}), \\ [Y_1^+ + Q_2^+ - Q_1^+, \mathcal{D}_{2N}(\lambda)] & \text{for } \tilde{M} = Z_1^+, \\ [Y_1^+ - Q_1^+ - Q_2^+, \mathcal{D}_{2N}(\lambda)] & \text{for } \tilde{M} = Z_1^-. \end{cases} \quad \blacksquare \end{aligned}$$

Proof of Theorem 3.4. We only give the proof for the case of even N . Assume $\mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ satisfies (40), (41) and (42). Using the notation (14), the Poincaré-Birkhoff-Witt theorem shows that \mathcal{E} can be written as a linear combination of

$$\{(I_1)^{a_1} \dots (I_n)^{a_n} \mid a_1, \dots, a_n \in \mathbb{N}\}. \quad (51)$$

Now (42) and the fact that $[H, X] = -X$ for all $X \in \mathfrak{n}_n^-$ show that \mathcal{E} is a linear combination of basis elements from (51), where

$$a_1 + \cdots + a_n = 2N. \quad (52)$$

We see that

$$\mathcal{E} = \sum_{j=0}^{2N} p_j T_n^{2N-j},$$

where $p_j \in \mathcal{U}(\mathfrak{n}_n^-)$ is a homogeneous polynomial of degree j in the variables I_1, \dots, I_{n-1} given by (14).

For all $R \in \mathfrak{m}_n$ we have $[R, T_n] = 0$ and hence

$$[R, \mathcal{E}] = [R, p_1]T_n^{2N-1} + \cdots + [R, p_{2N-1}]T_n + [R, p_{2N}] = 0.$$

We have $[\mathfrak{m}_n, \mathfrak{n}_n^-] \subset \mathfrak{n}_n^-$ and hence $[R, p_j] \in \mathcal{U}(\mathfrak{n}_n^-)$. Using the Poincaré-Birkhoff-Witt theorem, we conclude that for $j = 1, \dots, 2N$ we have that $[R, p_j] = 0$.

Proposition 3.1 shows that p_j is a polynomial in Δ_{n-1} for every $j = 1, \dots, 2N$. Using that p_j is a homogeneous polynomial of degree j , we deduce that $p_j = 0$ for odd j , and $p_{2i} = c_i (\Delta_{n-1})^i$ for some complex constant c_i .

We conclude that

$$\mathcal{E} = \sum_{j=0}^N a_j(\lambda) (\Delta_{n-1})^j (T_n)^{2N-2j},$$

and we see that \mathcal{E} is of the form (36). Now the *only if* statement in Theorem 3.2 completes the proof. \blacksquare

4. $\mathcal{D}_N^0(\lambda)$ as families of homomorphisms of Verma modules

Let W be a $\mathcal{U}(\mathfrak{p}_m)$ -module. The algebra $\mathcal{U}(\mathfrak{g}_m)$ acts on the vector space $\mathcal{U}(\mathfrak{g}_m) \otimes W$ by

$$u_1(u_2 \otimes w) = (u_1 u_2) \otimes w. \quad (53)$$

Let $I_W(\mathfrak{g}_m) \subset \mathcal{U}(\mathfrak{g}_m) \otimes W$ be the left $\mathcal{U}(\mathfrak{g}_m)$ -ideal generated by the elements $X \otimes w - 1 \otimes (X \cdot w)$, $X \in \mathfrak{p}_m$. $I_W(\mathfrak{g}_m)$ equals the subspace of $\mathcal{U}(\mathfrak{g}_m) \otimes W$ spanned by the elements

$$(np) \otimes w - n \otimes (p \cdot w), \quad (54)$$

where $n \in \mathcal{N}$, $p \in \mathcal{P}$, $w \in W$. Here \mathcal{N} is a basis for $\mathcal{U}(\mathfrak{n}_m^-)$ and \mathcal{P} is a basis for $\mathcal{U}(\mathfrak{p}_m)$. In fact, for any $u \in \mathcal{U}(\mathfrak{g}_m)$, $p \in \mathcal{U}(\mathfrak{p}_m)$ and $w \in W$

$$up \otimes w - u \otimes (p \cdot w)$$

can be written as a linear combination of terms of the form (54). Using the Poincaré-Birkhoff-Witt theorem, we know that u can be written as a linear combination of terms of the form $n_0 p_0$ where $n_0 \in \mathcal{U}(\mathfrak{n}_m^-)$, $p_0 \in \mathcal{U}(\mathfrak{p}_m)$. We see that

$$\begin{aligned} n_0 p_0 p \otimes w - n_0 p_0 \otimes (p \cdot w) \\ = \left(n_0 p_0 p \otimes w - n_0 \otimes (p_0 p) \cdot w \right) - \left(n_0 p_0 \otimes (p \cdot w) + n_0 \otimes (p_0 \cdot (p \cdot w)) \right). \end{aligned}$$

The *generalized Verma module* induced from W is defined by the vector space quotient

$$\mathcal{M}_W(\mathfrak{g}_m) = (\mathcal{U}(\mathfrak{g}_m) \otimes W) / I_W(\mathfrak{g}_m) \tag{55}$$

with the left $\mathcal{U}(\mathfrak{g}_m)$ -action. We also use the notation $\mathcal{U}(\mathfrak{g}_m) \otimes_{\mathcal{U}(\mathfrak{p}_m)} W$ for $\mathcal{M}_W(\mathfrak{g}_m)$.

For $\lambda \in \mathbb{C}$ let $\xi_\lambda : \mathfrak{a} \rightarrow \mathbb{C}$ be the character $\xi_\lambda(tH) = t\lambda$. The vector space $W = \mathbb{C}_\lambda$ is made into a left $\mathcal{U}(\mathfrak{p}_m)$ -module by

$$\begin{aligned} Xz &= \xi_\lambda(X)z && \text{for } X \in \mathfrak{a}, \\ Xz &= 0 && \text{for } X \in \mathfrak{m}_m \oplus \mathfrak{n}_m^+. \end{aligned}$$

Let

$$I_\lambda(\mathfrak{g}_m) = I_{\mathbb{C}_\lambda}(\mathfrak{g}_m) \quad \text{and} \quad \mathcal{M}_\lambda(\mathfrak{g}_m) = \mathcal{M}_{\mathbb{C}_\lambda}(\mathfrak{g}_m).$$

We view $\mathcal{U}(\mathfrak{g}_{n+1}) \otimes \mathbb{C}_\lambda$ as a $\mathcal{U}(\mathfrak{g}_n)$ -module with the action

$$u_1(u \otimes v) = (i(u_1)u) \otimes v$$

for $u_1 \in \mathcal{U}(\mathfrak{g}_n)$, where $i : \mathcal{U}(\mathfrak{g}_n) \rightarrow \mathcal{U}(\mathfrak{g}_{n+1})$ is the inclusion induced by (9).

Lemma 4.1. $\lambda \in \mathbb{C}$, $W = \mathbb{C}_\lambda$, $u \otimes 1 \in I_W(\mathfrak{g}_m)$, $u \in \mathcal{U}(\mathfrak{n}_m^-)$ implies that $u = 0$.

Proof. Observe that any element in $I_{\mathbb{C}_\lambda}(\mathfrak{g}_m)$ is spanned by elements of the form (54), and hence $u \otimes 1$ can be written as a linear combination of

$$(n_i p_i) \otimes 1 - \xi_\lambda(p_i) n_i \otimes 1 = (n_i p_i - \xi_\lambda(p_i) n_i) \otimes 1, \quad i = 1, \dots, r \tag{56}$$

for some elements $p_1, p_2, \dots, p_r \in \mathcal{P}$ and elements $n_1, \dots, n_r \in \mathcal{N}$ such that $n_i p_i$, $i = 1, \dots, r$ are linearly independent. We conclude that u is a linear combination of

$$n_i p_i - \xi_\lambda(p_i) n_i, \quad i = 1, \dots, r. \tag{57}$$

Now assume that $p_i \notin \mathbb{C}$ for all $1 \leq i \leq r$. We see that vector space spanned by (57) has empty intersection with $\mathcal{U}(\mathfrak{n}_m^-)$. This implies that $u = 0$.

Now assume that $p_i \in \mathbb{C}$ for some $1 \leq i \leq r$. Reordering the terms in (57), we can assume that $p_1, \dots, p_s \in \mathbb{C}$, and $p_{s+1}, \dots, p_r \notin \mathbb{C}$. Now $p_i \in \mathbb{C}$ for $1 \leq i \leq s$ implies that $n_i p_i - \xi_\lambda(p_i) n_i = 0$ for $1 \leq i \leq s$. We conclude that u is a linear combination of (57) for $s+1 \leq i \leq r$. But the vector space spanned by the terms in (57) with $s+1 \leq i \leq r$ has empty intersection with $\mathcal{U}(\mathfrak{n}_m^-)$, and hence $u = 0$. ■

Theorem 4.2. For any $\lambda \in \mathbb{C}$, the map

$$\mathcal{U}(\mathfrak{g}_n) \otimes \mathbb{C}_{\lambda-N} \ni T \otimes 1 \mapsto i(T) \mathcal{D}_N^0(\lambda) \otimes 1 \in \mathcal{U}(\mathfrak{g}_{n+1}) \otimes \mathbb{C}_\lambda \tag{58}$$

induces a homomorphism

$$\varphi_0 : \mathcal{M}_{\lambda-N}(\mathfrak{g}_n) \rightarrow \mathcal{M}_\lambda(\mathfrak{g}_{n+1}) \tag{59}$$

of $\mathcal{U}(\mathfrak{g}_n)$ -modules. Furthermore,

$$\text{Hom}_{\mathcal{U}(\mathfrak{g}_n)}(\mathcal{M}_{\lambda-N}(\mathfrak{g}_n), \mathcal{M}_\lambda(\mathfrak{g}_{n+1})) = \text{span}_{\mathbb{C}}(\varphi_0).$$

Proof. First, we prove that $\text{Hom}_{\mathcal{U}(\mathfrak{g}_n)}(\mathcal{M}_{\lambda-N}(\mathfrak{g}_n), \mathcal{M}_\lambda(\mathfrak{g}_{n+1})) \subset \text{span}_{\mathbb{C}}(\varphi_0)$. Assume that

$$\varphi \in \text{Hom}_{\mathcal{U}(\mathfrak{g}_n)}(\mathcal{M}_{\lambda-N}(\mathfrak{g}_n), \mathcal{M}_\lambda(\mathfrak{g}_{n+1})).$$

We have

$$\varphi(\{1 \otimes 1\}) = \{F \otimes 1\},$$

for some $F \in \mathcal{U}(\mathfrak{g}_{n+1})$. Here we use $\{\}$ to denote equivalence classes. Since

$$\varphi(\{u \otimes 1\}) = \varphi(u\{1 \otimes 1\}) = i(u)\varphi(\{1 \otimes 1\}) = i(u)\{F \otimes 1\} = \{(i(u)F) \otimes 1\}$$

for all $u \in \mathcal{U}(\mathfrak{g}_n)$, we conclude that φ is induced from a map

$$\zeta : \mathcal{U}(\mathfrak{g}_n) \otimes \mathbb{C}_{\lambda-N} \ni T \otimes 1 \mapsto i(T)F \otimes 1 \in \mathcal{U}(\mathfrak{g}_{n+1}) \otimes \mathbb{C}_\lambda, \quad (60)$$

where $F \in \mathcal{U}(\mathfrak{g}_{n+1})$. Furthermore, we can assume without loss of generality that $F \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$: By the Poincaré-Birkhoff-Witt theorem $F \in \mathcal{U}(\mathfrak{g}_{n+1})$ can be written as a linear combination of terms of the form np , where $n \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ and $p \in \mathcal{U}(\mathfrak{p}_{n+1})$. For $n \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ and $p \in \mathcal{U}(\mathfrak{p}_{n+1})$ we have for some complex number C that

$$\{i(T)np \otimes 1\} = \{i(T)n \otimes p \cdot 1\} = \{i(T)n \otimes C\} = \{i(T)(Cn) \otimes 1\},$$

and hence we conclude that

$$\{i(T)F \otimes 1\} = \{i(T)\tilde{F} \otimes 1\}$$

for some $\tilde{F} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$.

It is also easy to see that a map $f : \mathcal{U}(\mathfrak{g}_n) \otimes \mathbb{C}_{\lambda-N} \rightarrow \mathcal{U}(\mathfrak{g}_{n+1}) \otimes \mathbb{C}_\lambda$ induces a homomorphism $\mathcal{M}_{\lambda-N}(\mathfrak{g}_n) \rightarrow \mathcal{M}_\lambda(\mathfrak{g}_{n+1})$ if and only if

$$f(I_{\lambda-N}(\mathfrak{g}_n)) \subset I_\lambda(\mathfrak{g}_{n+1}). \quad (61)$$

This implies that

$$\zeta(I_{\lambda-N}(\mathfrak{g}_n)) \subset I_\lambda(\mathfrak{g}_{n+1}).$$

We now find conditions for F which are equivalent to that φ induces a $\mathcal{U}(\mathfrak{g}_n)$ -homomorphism (59). Observe that

$$H \otimes 1 - 1 \otimes \xi_{\lambda-N}(H)1 = H \otimes 1 - 1 \otimes (\lambda - N) \in I_{\lambda-N}(\mathfrak{g}_n). \quad (62)$$

We compute

$$\begin{aligned} \zeta(H \otimes 1 - 1 \otimes (\lambda - N)) &= HF \otimes 1 - F \otimes (\lambda - N) \\ &= ([H, F] + FH) \otimes 1 - F \otimes (\lambda - N) \\ &= [H, F] \otimes 1 + F(H \otimes 1 - 1 \otimes \xi_\lambda(H)1) + NF \otimes 1. \end{aligned} \quad (63)$$

We see that

$$([H, F] + NF) \otimes 1 \in I_\lambda(\mathfrak{g}_{n+1}). \quad (64)$$

Now $F \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ and hence $[H, F] \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$. Using (64) and Lemma 4.1, we see that

$$\boxed{[H, F] + NF = 0.} \quad (65)$$

For $M \in \mathfrak{m}_n$ we have

$$M \otimes 1 - 1 \otimes \xi_{\lambda-N}(M)1 = M \otimes 1 \in I_{\lambda-N}(\mathfrak{g}_n). \quad (66)$$

We compute

$$\zeta(M \otimes 1) = MF \otimes 1 = FM \otimes 1 + ([M, F] \otimes 1). \quad (67)$$

We see that

$$[M, F] \otimes 1 \in I_{\lambda}(\mathfrak{g}_{n+1}). \quad (68)$$

Now $F \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ and hence $[M, F] \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$. Using (68) and Lemma 4.1, we see that

$$\boxed{[M, F] = 0.} \quad (69)$$

For $X \in \mathfrak{n}_n^+$ we have

$$X \otimes 1 - 1 \otimes \xi_{\lambda-N}(X) = X \otimes 1 \in I_{\lambda-N}(\mathfrak{g}_n). \quad (70)$$

We compute

$$\zeta(X \otimes 1) = XF \otimes 1 = [X, F] \otimes 1 + FX \otimes 1. \quad (71)$$

We see that

$$[X, F] \otimes 1 \in I_{\lambda}(\mathfrak{g}_{n+1}).$$

Using the notation (14), $F \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ and Lemma 3.6 shows that

$$\begin{aligned} [X, F] \otimes 1 &= \sum_{a_1, \dots, a_n \in \mathbb{N}} c_{a_1 \dots a_n} (I_1)^{a_1} \dots (I_n)^{a_n} \otimes 1 \\ &+ \sum_{a_1, \dots, a_n \in \mathbb{N}} d_{a_1 \dots a_n} (I_1)^{a_1} \dots (I_n)^{a_n} H \otimes 1 \\ &+ \sum_{a_1, \dots, a_n \in \mathbb{N}, 1 \leq f, g \leq n} e_{a_1 \dots a_n, f, g} (I_1)^{a_1} \dots (I_n)^{a_n} M_{f, g} \otimes 1. \end{aligned}$$

Using

$$(I_1)^{a_1} \dots (I_n)^{a_n} M_{f, g} \otimes 1 \in I_{\lambda}(\mathfrak{g}_{n+1}),$$

we get

$$\begin{aligned} &\sum_{a_1, \dots, a_n \in \mathbb{N}} c_{a_1 \dots a_n} (I_1)^{a_1} \dots (I_n)^{a_n} \otimes 1 \\ &+ \sum_{a_1, \dots, a_n \in \mathbb{N}} d_{a_1 \dots a_n} (I_1)^{a_1} \dots (I_n)^{a_n} H \otimes 1 \in I_{\lambda}(\mathfrak{g}_{n+1}), \end{aligned}$$

and hence

$$\sum_{a_1, \dots, a_n \in \mathbb{N}} (c_{a_1 \dots a_n} - \lambda d_{a_1 \dots a_n}) (I_1)^{a_1} \dots (I_n)^{a_n} \otimes 1 \in I_{\lambda}(\mathfrak{g}_{n+1}).$$

Using Lemma 4.1, we conclude that $c_{a_1 \dots a_n} = \lambda d_{a_1 \dots a_n}$ and hence

$$\boxed{[X, F] \in \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda)).} \quad (72)$$

Theorem 3.4 and (65), (69) and (72) show that $X \in \text{span}_{\mathbb{C}}(\varphi_0)$.

Second, we prove that (58) satisfy (61), which implies that

$$\text{span}_{\mathbb{C}}(\varphi_0) \subset \text{Hom}_{\mathcal{U}(\mathfrak{g}_n)}(\mathcal{M}_{\lambda-N}(\mathfrak{g}_n), \mathcal{M}_{\lambda}(\mathfrak{g}_{n+1})).$$

From Theorem 3.4 we see that (40), (41) and (42) are satisfied for $\mathcal{E} = \mathcal{D}_N^0(\lambda)$.

Using (62), (63) and (42), we conclude that

$$\varphi_0(H \otimes 1 - 1 \otimes \xi_{\lambda-N}(H)1) \in I_{\lambda}(\mathfrak{g}_{n+1}).$$

Using (66), (67) and (40), we conclude that

$$\varphi_0(X \otimes 1 - 1 \otimes \xi_{\lambda-N}(X)1) \in I_{\lambda}(\mathfrak{g}_{n+1})$$

for all $X \in \mathfrak{m}_n$. Using (70), (71) and (41), we conclude that

$$\varphi_0(X \otimes 1 - 1 \otimes \xi_{\lambda-N}(X)1) \in I_{\lambda}(\mathfrak{g}_{n+1})$$

for all $X \in \mathfrak{n}_n^+$. ■

5. Induced families of differential intertwining operators

In this section we use the algebraic results of Section 3 to prove equivariance of the polynomial families induced by $\mathcal{D}_N^0(\lambda) \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ with respect to certain principal series representations. We view $\mathcal{D}_N^0(\lambda) \in \mathcal{U}(\mathfrak{g}_{n+1})$ as a left-invariant differential operator $C^\infty(G^{n+1}) \rightarrow C^\infty(G^{n+1})$ using R .

Theorem 5.1. *Let $\lambda \in \mathbb{C}$. Then $\mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a left G^{n+1} -equivariant map*

$$\text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_{\lambda}) \rightarrow \text{Ind}_{P_0^n}^{G^{n+1}}(\xi_{\lambda-N})$$

if and only if $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ for some $c \in \mathbb{C}$.

Theorem 5.2. *For $\lambda \in \mathbb{C}$ the element $\mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a left G^{n+1} -equivariant map*

$$\text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_{\lambda}) \rightarrow \text{Ind}_{P^n}^{G^{n+1}}(\xi_{\lambda-N})$$

if and only if $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ for some $c \in \mathbb{C}$ and N is even.

Theorem 5.3. *For $\lambda \in \mathbb{C}$ the element $\mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a left G^{n+1} -equivariant map*

$$\text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_{\lambda}) \rightarrow \text{Ind}_{P^n}^{G^{n+1}}(\xi_{\lambda-N} \otimes \sigma_-)$$

if and only if $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ for some $c \in \mathbb{C}$ and N is odd.

For even N , we define

$$D_N(\lambda) : \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_{\lambda}) \rightarrow \text{Ind}_{P^n}^{G^n}(\xi_{\lambda-N})$$

by $D_N(\lambda) = i^* \circ \widetilde{\mathcal{D}_N^0(\lambda)}$, where $\widetilde{\mathcal{D}_N^0(\lambda)} : C^\infty(G^{n+1}) \rightarrow C^\infty(G^{n+1})$ is the operator induced by $\mathcal{D}_N^0(\lambda)$.

For odd N , we define

$$D_N(\lambda) : \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{P^n}^{G^n}(\xi_{\lambda-N} \otimes \sigma_-)$$

by $D_N(\lambda) = i^* \circ \widetilde{\mathcal{D}_N^0(\lambda)}$, where $\widetilde{\mathcal{D}_N^0(\lambda)} : C^\infty(G^{n+1}) \rightarrow C^\infty(G^{n+1})$ is the operator induced by $\mathcal{D}_N^0(\lambda)$.

Theorem 5.4. For $\lambda \in \mathbb{C}$ the element $\mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a left G^{n+1} -equivariant map

$$\text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{P^n}^{G^{n+1}}(\xi_{\lambda-N})$$

if and only if $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ for some $c \in \mathbb{C}$.

We define

$$\hat{D}_N(\lambda) : \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{P^n}^{G^n}(\xi_{\lambda-N})$$

by $\hat{D}_N(\lambda) = i^* \circ \widetilde{\mathcal{D}_N^0(\lambda)}$, where $\widetilde{\mathcal{D}_N^0(\lambda)} : C^\infty(G^{n+1}) \rightarrow C^\infty(G^{n+1})$ is the operator induced by $\mathcal{D}_N^0(\lambda)$.

Proof of Theorem 5.1. Using Theorem 3.4, it follows that we need to prove that $\mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a map

$$\text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{P_0^n}^{G^{n+1}}(\xi_{\lambda-N})$$

if and only if \mathcal{E} satisfies the following three conditions:

$$\begin{aligned} [X, \mathcal{E}] &= 0 \text{ for all } X \in \mathfrak{m}_n, \\ [X, \mathcal{E}] &\in \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda)) \text{ for all } X \in \mathfrak{n}_n^+, \\ [H, \mathcal{E}] &= -N\mathcal{E}. \end{aligned}$$

Observe that \mathcal{E} induces a map if and only if

$$(\mathcal{E}u)(x\tilde{m}\tilde{a}\tilde{n}) = \xi_{\lambda-N}(\tilde{a})(\mathcal{E}u)(x)$$

for all $\tilde{m}\tilde{a}\tilde{n} \in P_0^n$ and $u \in \text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda)$. First, we prove that for all $u \in \text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda)$, $x \in G^{n+1}$ and $\tilde{m} \in M_0^n$ we have

$$(\mathcal{E}u)(x\tilde{m}) = (\mathcal{E}u)(x) \tag{73}$$

if and only if $[X, \mathcal{E}] = 0$ for all $X \in \mathfrak{m}_n$.

Assume that $[X, \mathcal{E}] = 0$ for all $X \in \mathfrak{m}_n$. Using $\exp(\mathfrak{m}_n) = M_0^n$, it follows that $\text{Ad}(\tilde{m})\mathcal{E} = \mathcal{E}$ for $\tilde{m} \in M_0^n$. Now for all $\tilde{m} \in M_0^n$ and all $u \in \text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda)$ we get using (23) that

$$(\mathcal{E}u)(x\tilde{m}) = (R(\tilde{m})\mathcal{E}u)(x) = (\text{Ad}(\tilde{m})\mathcal{E})u(x). \tag{74}$$

Conversely, assume that (73) is true. Then (74) shows that $\text{Ad}(\tilde{m})\mathcal{E}u = \mathcal{E}u$ for all $u \in \text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda)$ and all $\tilde{m} \in M_0^n$, and Lemma 2.6 shows that $\text{Ad}(\tilde{m})\mathcal{E} = \mathcal{E}$ for $\tilde{m} \in M_0^n$. Hence $[X, \mathcal{E}] = 0$ for all $X \in \mathfrak{m}_n$.

Second, we prove that for all $u \in \text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda)$, $x \in G^{n+1}$ and $\tilde{n} \in N^n$ we have

$$(\mathcal{E}u)(x\tilde{n}) = (\mathcal{E}u)(x)$$

if and only if $[X, \mathcal{E}] \in \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda))$ for all $X \in \mathfrak{n}_n^+$.

Assume that $[X, \mathcal{E}] \in \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda))$ for all $X \in \mathfrak{n}_n^+$. For all $X \in \mathfrak{n}_n^+$, $u \in \text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda)$ we get for $x \in G^{n+1}$ that

$$([X, \mathcal{E}]u)(x) = 0,$$

and hence using that $u(x\tilde{n}) = u(x)$ for all $\tilde{n} \in N^n$ we get

$$(X\mathcal{E}u)(x) = ([X, \mathcal{E}]u)(x) + (\mathcal{E}X)u(x) = 0.$$

Conversely, assume that for all $u \in \text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda)$ we have

$$([X, \mathcal{E}]u)(x) = 0, \quad x \in G^{n+1}, \quad X \in \mathfrak{n}_n. \tag{75}$$

From Lemma 3.6 we know that $[X, \mathcal{E}] \in (\mathcal{U}(\mathfrak{n}_{n+1}^-) \otimes \mathfrak{m}_{n+1}) \oplus (\mathcal{U}(\mathfrak{n}_{n+1}^-) \otimes \mathfrak{a}) \oplus \mathcal{U}(\mathfrak{n}_{n+1}^-)$. Using $u \in \text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda)$, we see that if

$$[X, \mathcal{E}]u = \alpha(I_1)^{a_1} \dots (I_n)^{a_n} Hu + \beta(I_1)^{a_1} \dots (I_n)^{a_n} u + \dots$$

we get

$$[X, \mathcal{E}]u = \alpha(I_1)^{a_1} \dots (I_n)^{a_n} \lambda u + \beta(I_1)^{a_1} \dots (I_n)^{a_n} u + \dots,$$

and using Lemma 2.6 and (75) we get $\beta = -\lambda\alpha$. We conclude that $[X, \mathcal{E}] \in \mathcal{U}(\mathfrak{n}_{n+1}^-)(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H - \lambda))$.

Third we show that for all $u \in \text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda)$, $x \in G^{n+1}$, $\tilde{a} \in A$

$$(\mathcal{E}u)(x\tilde{a}) = \xi_{\lambda-N}(\tilde{a})(\mathcal{E}u)(x)$$

if and only if $[H, \mathcal{E}] = -N\mathcal{E}$. Assume that $[H, \mathcal{E}] = -N\mathcal{E}$. We get using (42)

$$\begin{aligned} (H\mathcal{E})u(g) &= ([H, \mathcal{E}]u)(g) + (\mathcal{E}Hu)(g) = (-N\mathcal{E}u)(g) + \lambda(\mathcal{E}u)(g) \\ &= (\lambda - N)(\mathcal{E}u)(g). \end{aligned}$$

Conversely, assume that $(H\mathcal{E}u)(x) = (\lambda - N)(\mathcal{E}u)(x)$ for all $u \in \text{Ind}_{P_0^{n+1}}^{G^{n+1}}(\xi_\lambda)$.

Using

$$H\mathcal{E}u = [H, \mathcal{E}]u + \mathcal{E}Hu = [H, \mathcal{E}]u + \lambda\mathcal{E}u,$$

we get

$$([H, \mathcal{E}] + N\mathcal{E})u = 0.$$

Now $[H, \mathfrak{n}_{n+1}^-] \subset \mathfrak{n}_{n+1}^-$ and hence Lemma 2.6 shows that $[H, \mathcal{E}] + N\mathcal{E} = 0$. ■

Proof of Theorem 5.2. From (16) we know that $M^n = \{1_{(n)}, J_{(n)}\} \langle w_1, w_2 \rangle M_0^n$, and hence $\mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a map

$$\text{Ind}_{M^{n+1}A(N^+)^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{M^nA(N^+)^n}^{G^{n+1}}(\xi_{\lambda-N})$$

if and only if \mathcal{E} induces a map

$$\mathrm{Ind}_{M_0^{n+1}A(N^+)^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \mathrm{Ind}_{M_0^n A(N^+)^n}^{G^{n+1}}(\xi_{\lambda-N})$$

and

$$(\mathcal{E}u)(xi(w)) = (\mathcal{E}u)(x), \quad x \in G^{n+1},$$

for $u \in \mathrm{Ind}_{P_{n+1}}^{G^{n+1}}(\xi_\lambda)$ and $w \in \{w_1, w_2, J_{(n)}\}$. Here $i : G^n \rightarrow G^{n+1}$ is given by (10).

Using Theorem 5.1, we conclude that $\mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a map

$$\mathrm{Ind}_{M^{n+1}A(N^+)^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \mathrm{Ind}_{M^n A(N^+)^n}^{G^{n+1}}(\xi_{\lambda-N})$$

if and only if for some $c \in \mathbb{C}$ we have $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ and

$$(\mathcal{E}u)(xi(w)) = (\mathcal{E}u)(x), \quad x \in G^{n+1},$$

for $u \in \mathrm{Ind}_{P_{n+1}}^{G^{n+1}}(\xi_\lambda)$ and $w \in \{w_1, w_2, J_{(n)}\}$.

It follows that we need to prove that $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ for some $c \in \mathbb{C}$ and N is even if and only if $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ for $c \in \mathbb{C}$ and

$$(\mathcal{E}u)(xi(w)) = (\mathcal{E}u)(x), \quad x \in G^{n+1} \tag{76}$$

for all $u \in \mathrm{Ind}_{P_{n+1}}^{G^{n+1}}(\xi_\lambda)$ and $w \in \{w_1, w_2, J_{(n)}\}$.

For $u \in \mathrm{Ind}_{P_{n+1}}^{G^{n+1}}(\xi_\lambda)$ and $x \in G^{n+1}$ we have

$$\begin{aligned} (\mathrm{Ad}(i(w))\mathcal{E})u(x) &= (R(i(w)) \circ \mathcal{E} \circ R(i(w)^{-1}))u(x) \\ &= (R(i(w)) \circ \mathcal{E})u(x) = (\mathcal{E}u)(xi(w)). \end{aligned} \tag{77}$$

Using (77) and Lemma 2.6, we conclude that (76) is satisfied if and only if

$$\mathrm{Ad}(i(w))\mathcal{E} = \mathcal{E}$$

for $w \in \{w_1, w_2, J_{(n)}\}$. Using $\mathrm{Ad}(w)X = wXw^{-1}$, we see that

$$\begin{aligned} \mathrm{Ad}(i(w_1))Q_1^- &= -Q_1^-, \\ \mathrm{Ad}(i(w_1))Q_2^- &= Q_2^-, \\ \mathrm{Ad}(i(w_1))Y_j^- &= Y_j^-, \quad \text{for } j = 1, \dots, n-2, \\ \mathrm{Ad}(i(w_2))Q_1^- &= Q_1^-, \\ \mathrm{Ad}(i(w_2))Q_2^- &= -Q_2^-, \\ \mathrm{Ad}(i(w_2))Y_j^- &= Y_j^-, \quad \text{for } j = 1, \dots, n-2, \\ \mathrm{Ad}(i(J_{(n)}))Q_1^- &= Q_1^-, \\ \mathrm{Ad}(i(J_{(n)}))Q_2^- &= Q_2^-, \\ \mathrm{Ad}(i(J_{(n)}))Y_{n-2}^- &= -Y_{n-2}^-, \\ \mathrm{Ad}(i(J_{(n)}))Y_j^- &= Y_j^- \quad \text{for } j = 1, \dots, n-3. \end{aligned}$$

Here we used that

$$Q_1^- = \begin{pmatrix} 0 & -1 & 0 & 0 & \mathbf{0} \\ 1 & 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 0 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, Q_2^- = \begin{pmatrix} 0 & 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \\ 1 & 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

We get

$$\begin{aligned} \text{Ad}(i(J_{(n)}))\mathcal{D}_N^0(\lambda) &= (-1)^N \mathcal{D}_N^0(\lambda), \\ \text{Ad}(i(w_1))\mathcal{D}_N^0(\lambda) &= \mathcal{D}_N^0(\lambda), \\ \text{Ad}(i(w_2))\mathcal{D}_N^0(\lambda) &= \mathcal{D}_N^0(\lambda). \end{aligned} \tag{78}$$

This completes the proof. ■

Proof of Theorem 5.3. From (16) we know that

$$M^n = \{1_{(n)}, J_{(n)}\} \langle w_1, w_2 \rangle M_0^n,$$

and hence $\mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a map

$$\text{Ind}_{M^{n+1}A(N^+)^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{M^n A(N^+)^n}^{G^{n+1}}(\xi_{\lambda-N} \otimes \sigma_-)$$

if and only if \mathcal{E} induces a map

$$\text{Ind}_{M_0^{n+1}A(N^+)^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{M_0^n A(N^+)^n}^{G^{n+1}}(\xi_{\lambda-N} \otimes \sigma_- |_{M_0^n}) = \text{Ind}_{M_0^n A(N^+)^n}^{G^{n+1}}(\xi_{\lambda-N})$$

and

$$(\mathcal{E}u)(xi(w)) = \sigma_-(w)(\mathcal{E}u)(x), \quad x \in G^{n+1},$$

for $u \in \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda)$ and $w \in \{w_1, w_2, J_{(n)}\}$. Here $i: G^n \rightarrow G^{n+1}$ is given by (10).

Using Theorem 5.1, we conclude that $\mathcal{E} \in \mathcal{U}(\mathfrak{n}_{n+1}^-)$ induces a map

$$\text{Ind}_{M^{n+1}A(N^+)^{n+1}}^{G^{n+1}}(\xi_\lambda) \rightarrow \text{Ind}_{M^n A(N^+)^n}^{G^{n+1}}(\xi_{\lambda-N} \otimes \sigma_-)$$

if and only if for some $c \in \mathbb{C}$ we have $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ and

$$(\mathcal{E}u)(xi(w)) = \sigma_-(w)(\mathcal{E}u)(x), \quad x \in G^{n+1},$$

for $u \in \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda)$ and $w \in \{w_1, w_2, J_{(n)}\}$.

It follows that we need to prove that $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ for some $c \in \mathbb{C}$ and N is odd if and only if $\mathcal{E} = c\mathcal{D}_N^0(\lambda)$ for $c \in \mathbb{C}$ and

$$(\mathcal{E}u)(xi(w)) = \sigma_-(w)(\mathcal{E}u)(x), \quad x \in G^{n+1}, \tag{79}$$

for all $u \in \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda)$ and $w \in \{w_1, w_2, J_{(n)}\}$.

For $u \in \text{Ind}_{P^{n+1}}^{G^{n+1}}(\xi_\lambda)$ and $x \in G^{n+1}$ we have

$$\begin{aligned} (\text{Ad}(i(w))\mathcal{E})u(x) &= (R(i(w)) \circ \mathcal{E} \circ R(i(w)^{-1}))u(x) \\ &= (R(i(w)) \circ \mathcal{E})u(x) = (\mathcal{E}u)(xi(w)). \end{aligned} \tag{80}$$

Using (80) and Lemma 2.6, we conclude that (79) is satisfied if and only if

$$\begin{aligned} \text{Ad}(i(w_1))\mathcal{E} &= \sigma_-(w_1)\mathcal{E} = \mathcal{E}, \\ \text{Ad}(i(w_2))\mathcal{E} &= \sigma_-(w_2)\mathcal{E} = \mathcal{E}, \\ \text{Ad}(i(J_{(n)}))\mathcal{E} &= \sigma_-(J_{(n)})\mathcal{E} = -\mathcal{E}. \end{aligned}$$

Using (78), the result follows. ■

Proof of Theorem 5.4. The proof is analogous to the proof of Theorem 5.2 using that $\hat{M}^n = \langle w_1, w_2 \rangle M_0^n$. ■

The differential operator families in the non-compact model are defined by

$$\begin{aligned} D_N^{\text{nc}}(\lambda) &: C^\infty((N^-)^{n+1})_{\xi_\lambda} \rightarrow C^\infty((N^-)^n)_{\xi_{\lambda-N} \otimes \sigma}, \\ D_N^{\text{nc}}(\lambda) &= \beta_{\xi_{\lambda-N} \otimes \sigma}^n \circ D_N(\lambda) \circ (\beta_{\xi_\lambda}^{n+1})^{-1}, \end{aligned}$$

where

$$\sigma = \begin{cases} \sigma_+ & \text{if } N \text{ is even,} \\ \sigma_- & \text{if } N \text{ is odd.} \end{cases} \tag{81}$$

$\beta_{\xi_\lambda \otimes \sigma}^m$ is the restriction map $\text{Ind}_{P^m}^{G^m}(\xi_\lambda \otimes \sigma) \rightarrow C^\infty((N^-)^m)_{\xi_\lambda \otimes \sigma} \subset C^\infty(\mathbb{R}^{m-1})$ and σ is given by (81).

Using the identification (20), we have

$$\begin{aligned} D_{2N}^{\text{nc}}(\lambda) &= \sum_{j=0}^N a_j(\lambda) (\Delta_{\mathbb{M}^{n-1}})^j i^* \left(\frac{\partial}{\partial x_n} \right)^{2N-2j}, \\ D_{2N+1}^{\text{nc}}(\lambda) &= \sum_{j=0}^N b_j(\lambda) (\Delta_{\mathbb{M}^{n-1}})^j i^* \left(\frac{\partial}{\partial x_n} \right)^{2N-2j+1}, \end{aligned} \tag{82}$$

where

$$\Delta_{\mathbb{M}^{n-1}} = -\frac{\partial^2}{\partial x_1^2} + \sum_{i=2}^{n-1} \frac{\partial^2}{\partial x_i^2}$$

and $(i^*\varphi)(x') = \varphi(x', 0)$.

6. Intertwining families and asymptotics of eigenfunctions

In the present section we describe an alternative construction of the families $D_N^{\text{nc}}(\lambda)$ in terms of the asymptotics of eigenfunctions of the Laplacian on anti-de Sitter space (called the *residue family* in [8]).

Let

$$\mathbb{U}_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

be the Lorentzian upper half-space with the Lorentzian metric

$$x_n^{-2} \left(-dx_1^2 + \sum_{i=2}^n dx_i^2 \right)$$

and the Laplacian

$$\Delta_{\mathbb{U}_n} = x_n^2 \Delta_{\mathbb{M}^{n-1}} + x_n^2 \frac{\partial^2}{\partial x_n^2} - (n-2)x_n \frac{\partial}{\partial x_n}.$$

We seek *formal* solutions $u \in C^\infty(\mathbb{U}_n)$ to the equation

$$-\Delta_{\mathbb{U}_n} u = \lambda(n-1-\lambda)u$$

in the form

$$u(x) \sim \sum_{j \geq 0} x_n^{\lambda+j} c_j(x'), \quad x = (x', x_n), \quad c_{\text{odd}} = 0.$$

The coefficients c_j satisfy recursive relations such that the coefficients c_{2j} , $j \geq 1$, are determined by c_0 . More precisely, we have maps

$$T_{2j}(\lambda) : c_0(x') \mapsto c_{2j}(\lambda, x')$$

which are differential operators (depending on λ) of order $2j$ on \mathbb{R}^{n-1} , where $T_0(\lambda) = \text{id}$.

We define $S_{2N}(\lambda) : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{n-1})$ by

$$S_{2N}(\lambda) = \sum_{j=0}^N \frac{1}{(2N-2j)!} T_{2j}(\lambda) i^* \left(\frac{\partial}{\partial x_n} \right)^{2N-2j}.$$

Theorem 6.1. *The families $S_{2N}(\lambda + n - 1 - 2N)$ and $D_{2N}^{\text{nc}}(\lambda)$ coincide, up to a rational function in λ .*

Proof. A computation shows that $T_{2j}(\lambda) = A_{2j}(\lambda)(\Delta_{\mathbb{M}^{n-1}})^j$, where

$$A_{2j-2}(\lambda) + 2j(2j+2\lambda+1-n)A_{2j}(\lambda) = 0 \tag{83}$$

and $A_0(\lambda) = 1$. We see that $S_{2N}(\lambda)$ can be written in the form

$$\begin{aligned} & \sum_{j=0}^N \frac{1}{(2N-2j)!} A_{2j}(\lambda) (\Delta_{\mathbb{M}^{n-1}})^j i^* (\partial/\partial x_n)^{2N-2j} \\ &= \sum_{j=0}^N B_{2j}(\lambda) (\Delta_{\mathbb{M}^{n-1}})^j i^* (\partial/\partial x_n)^{2N-2j}, \end{aligned}$$

where the coefficients $B_{2j}(\lambda)$ satisfy

$$B_0(\lambda) = \frac{1}{(2N)!},$$

and

$$B_{2j-2}(\lambda) + \frac{2j(2j+2\lambda+1-n)}{(2N-2j+2)(2N-2j+1)} B_{2j}(\lambda) = 0. \tag{84}$$

We recall that $D_{2N}^{\text{nc}}(\lambda)$ is given by (82), where

$$a_{j-1}(\lambda) + \frac{2j(2j+2\lambda+n-1-4N)}{(2N-2j+2)(2N-2j+1)} a_j(\lambda) = 0.$$

Now we observe that

$$\frac{B_{2j}(\lambda+n-1-2N)}{a_j(\lambda)}$$

does not depend on j . Using $a_N(\lambda) = 1$, (84) shows

$$\begin{aligned} B_{2N}(\lambda) &= \frac{1}{(2 \cdot 4 \cdot \dots \cdot 2N)(n-1-2\lambda-2)(n-1-2\lambda-4) \dots (n-1-2\lambda-2N)} \\ &= \frac{1}{2^N N! (n-1-2\lambda-2)(n-1-2\lambda-4) \dots (n-1-2\lambda-2N)}. \end{aligned}$$

We conclude that $S_{2N}(\lambda+n-1-2N) = B_{2N}(\lambda+n-1-2N)D_{2N}^{\text{nc}}(\lambda)$. ■

If we define

$$S_{2N+1}(\lambda) = \sum_{j=0}^N \frac{1}{(2N+1-2j)!} T_{2j}(\lambda) i^* \left(\frac{\partial}{\partial x_n} \right)^{2N+1-2j},$$

we get the following analogous result:

Theorem 6.2. *The families $S_{2N+1}(\lambda+n-1-(2N+1))$ and $D_{2N+1}^{\text{nc}}(\lambda)$ coincide, up to a rational function in λ .*

Proof. The proof is similar to the proof of Theorem 6.1; we refer to Theorem 5.2.6 in [8]. ■

Appendix

Summary of commutator relations for $\mathfrak{g}_n = \mathfrak{o}(2, n-1)$.

$$\begin{aligned} [Q_1^+, Y_j^+] &= 0, & [Q_1^-, Y_j^+] &= Z_j^+ + Z_j^-, \\ [Q_1^+, Y_j^-] &= -Z_j^+ - Z_j^-, & [Q_1^-, Y_j^-] &= 0, \\ [Q_1^+, Z_j^+] &= Y_j^+, & [Q_1^-, Z_j^+] &= -Y_j^-, \\ [Q_1^+, Z_j^-] &= Y_j^+, & [Q_1^-, Z_j^-] &= -Y_j^-, \end{aligned}$$

$$\begin{aligned} [Q_2^+, Y_j^+] &= 0, & [Q_2^-, Y_j^+] &= Z_j^+ - Z_j^-, \\ [Q_2^+, Y_j^-] &= Z_j^+ - Z_j^-, & [Q_2^-, Y_j^-] &= 0, \\ [Q_2^+, Z_j^+] &= Y_j^+, & [Q_2^-, Z_j^+] &= Y_j^-, \\ [Q_2^+, Z_j^-] &= -Y_j^+, & [Q_2^-, Z_j^-] &= -Y_j^-, \end{aligned}$$

$$\begin{aligned} [Q_1^+, Q_2^+] &= 0, & [H, Q_1^+] &= Q_1^+, \\ [Q_1^-, Q_2^-] &= 0, & [H, Q_1^-] &= -Q_1^-, \\ [Q_1^+, Q_2^-] &= -2H_2, & [H, Q_2^+] &= Q_2^+, \\ [Q_1^-, Q_2^+] &= 2H_2, & [H, Q_2^-] &= -Q_2^-, \\ [Q_1^+, Q_1^-] &= 2H, & [H_2, Q_1^+] &= -Q_2^+, \\ [Q_2^+, Q_2^-] &= 2H, & [H_2, Q_1^-] &= Q_2^-, \\ [Q_i^\pm, M_{jk}] &= 0, & [H_2, Q_2^+] &= -Q_1^+, \\ & & [H_2, Q_2^-] &= Q_1^-, \end{aligned}$$

$$\begin{aligned} [Y_i^+, Y_j^-] &= 2\delta_{ij}H + 2M_{ij}, & [Z_i^+, Z_j^-] &= 2\delta_{ij}H_2 + 2M_{ij}, \\ [M_{ij}, Y_r^\pm] &= \delta_{jr}Y_i^\pm - \delta_{ir}Y_j^\pm, & [M_{ij}, Z_r^\pm] &= \delta_{jr}Z_i^\pm - \delta_{ir}Z_j^\pm, \\ [H, Y_j^\pm] &= \pm Y_j^\pm, & [H_2, Z_j^\pm] &= \pm Z_j^\pm, \\ [H, M_{ij}] &= 0, & [H_2, M_{ij}] &= 0, \end{aligned}$$

$$\begin{aligned} [Y_j^+, Z_k^+] &= \delta_{jk}(Q_1^+ - Q_2^+), & [Y_j^+, Z_k^-] &= \delta_{jk}(Q_1^+ + Q_2^+), \\ [Y_j^-, Z_k^+] &= \delta_{jk}(-Q_1^- - Q_2^-), & [Y_j^-, Z_k^-] &= \delta_{jk}(-Q_1^- + Q_2^-). \end{aligned}$$

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