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Metacurvature of Riemannian Poisson-Lie groups

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Abstract. We study the triple $(G, \pi, \langle\ ,\ \rangle)$ where G is a connected and simply connected Lie group, π and $\langle\ ,\ \rangle$ are, respectively, a multiplicative Poisson tensor and a left invariant Riemannian metric on G such that the necessary conditions, introduced by Hawkins, to the existence of a non commutative deformation (in the direction of π) of the spectral triple associated to $\langle\ ,\ \rangle$ are satisfied. We show that the geometric problem of the classification of such triple $(G, \pi, \langle\ ,\ \rangle)$ is equivalent to an algebraic one. We solve this algebraic problem in low dimensions and we give the list of all $(G, \pi, \langle\ ,\ \rangle)$ satisfying Hawkins's conditions, up to dimension four.
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1. Introduction

In [8] and [9], Hawkins showed that if a deformation of the graded algebra of differential forms on a Riemannian manifold $(M, \langle\ ,\ \rangle)$ comes from a deformation of the spectral triple describing the Riemannian manifold M, then the Poisson tensor π (which characterizes the deformation) and the Riemannian metric satisfy the following conditions:

1. The associated metric contravariant connection $\mathcal{D}$ is *flat.*
2. The metacurvature of $\mathcal{D}$ vanishes, ($\mathcal{D}$ is *metaflat*).
3. The Poisson tensor π is compatible with the Riemannian volume μ:

$$d(i_\pi \mu) = 0.$$

The metric contravariant connection associated naturally to any couple of pseudo-Riemannian metric and Poisson tensor is an analogue of the Levi-Civita connection. It has appeared first in [3]. Sometimes we call this connection *Levi-Civita contravariant connection.* The metacurvature, introduced by Hawkins in [9], is

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a $(2,3)$-tensor field (symmetric in the contravariant indices and antisymmetric in the covariant indices) associated naturally to any torsion-free and flat contravariant connection.

In [9], Hawkins studied completely the geometry of the triples $(M, \langle\ ,\ \rangle, \pi)$ satisfying 1-3 when M is compact and $\langle\ ,\ \rangle$ is Riemannian. In [4], the second author gave a method which permit the construction of a large class of triples $(M, \langle\ ,\ \rangle, \pi)$ satisfying 1-3. We call the conditions 1-3 *Hawkins's conditions* and a couple $(\pi, \langle\ ,\ \rangle)$ satisfying 1-2 will be called *flat* and *metaflat*.

In this paper, we study the triples $(G, \pi, \langle\ ,\ \rangle)$ satisfying Hawkins's conditions, where G is a connected and simply connected Lie group endowed with a multiplicative Poisson tensor π and a left invariant Riemannian metric $\langle\ ,\ \rangle$. We reduce the geometric problem of classifying such triples to an algebraic one and we solve it when the dimension of the Lie group is ≤ 4. In [1], the authors gave the complete description of the triples $(G, \pi, \langle\ ,\ \rangle)$ satisfying Hawkins's conditions when G is the $2n+1$-dimensional Heisenberg group.

To state our main results, let us introduce the notion of *Milnor Lie algebra* which will be central in this paper and recall briefly some classical facts about Poisson-Lie groups. The notion of Poisson-Lie group was first introduced by Drinfel'd [5] and studied by Semenov-Tian-Shansky [14] (see also [12]).

1. A *Milnor Lie algebra* is a finite dimensional real Lie algebra $\mathcal{G}$ endowed with a scalar product $\langle\ ,\ \rangle$ (positive-definite) such that:
   (a) the Lie subalgebra $S = \{u \in \mathcal{G}, \mathrm{ad}_u + \mathrm{ad}_u^t = 0\}$ is abelian (ad_u^t denotes the adjoint of ad_u w.r.t. $\langle\ ,\ \rangle$),
   (b) the derived ideal $[\mathcal{G}, \mathcal{G}]$ is abelian and $S^\perp = [\mathcal{G}, \mathcal{G}]$ ($S^\perp$ is the orthogonal of S).
   This terminology is justified by a classical result of Milnor. Indeed, in [13], Milnor showed that a left invariant Riemannian metric on a Lie group is flat if and only if its Lie algebra is a semi-direct product of an abelian algebra $\mathfrak{b}$ with an abelian ideal $\mathfrak{u}$ and, for any $u \in \mathfrak{b}$, ad_u is skew-symmetric. This result can be formulated in a more precise way and, in Proposition 2.2, we will show that a left invariant Riemannian metric on a Lie group is flat if and only if its Lie algebra is a Milnor Lie algebra.

2. Let G be a Lie group and $\mathcal{G}$ its Lie algebra. A Poisson tensor π on G is called multiplicative if, for any $a, b \in G$,
$$\pi(ab) = (L_a)_*\pi(b) + (R_b)_*\pi(a),$$
   where $(L_a)_*$ (resp. $(R_b)_*$) denotes the tangent map of the left translation of G by a (resp. the right translation of G by b). Pulling π back to the identity element e of G by left translations, we get a map $\pi_l : G \longrightarrow \mathcal{G} \wedge \mathcal{G}$ defined by $\pi_l(g) = (L_{g^{-1}})_*\pi(g)$. Let $\xi := d_e\pi_l : \mathcal{G} \longrightarrow \mathcal{G} \wedge \mathcal{G}$ be the intrinsic derivative of π_l at e. It is well-known that $(\mathcal{G}, [\ ,\ ], \xi)$ is a Lie bialgebra, i.e.,

ξ is a 1-cocycle relative to the adjoint representation of $\mathcal{G}$ on $\mathcal{G}\wedge\mathcal{G}$, and the dual map of ξ, $[\ ,\ ]^* : \mathcal{G}^*\times\mathcal{G}^* \longrightarrow \mathcal{G}^*$, is a Lie bracket on $\mathcal{G}^*$. It is well-known also that $(\mathcal{G}^*, [\ ,\ ]^*, \rho)$ is also a Lie bialgebra, where $\rho : \mathcal{G}^* \longrightarrow \mathcal{G}^*\wedge\mathcal{G}^*$ is the dual of the Lie bracket on $\mathcal{G}$. Note that $\rho = -d$ where d is the restriction of the differential to left invariant 1-forms.
A Poisson-Lie group endowed with a left invariant Riemannian metric will be called *Riemannian Poisson-Lie group.*

For any scalar product $\langle\ ,\ \rangle$ on a Lie algebra $\mathcal{G}$, we denote by $\langle\ ,\ \rangle^*$ the associated scalar product on $\mathcal{G}^*$.

Let us state our main results:

Theorem 1.1. *Let $(G, \pi, \langle\ ,\ \rangle)$ be a Riemannian Poisson-Lie group and $(\mathcal{G}^*, [\ ,\ ]^*, \rho)$ its dual Lie bialgebra. Then $(\pi, \langle\ ,\ \rangle)$ is flat and metaflat if and only if:*

1. *$(\mathcal{G}^*, [\ ,\ ]^*, \langle\ ,\ \rangle_e^*)$ is a Milnor Lie algebra,*
2. *for any $\alpha, \beta, \gamma \in S = \{\alpha \in \mathcal{G}^*,\ \mathrm{ad}_\alpha + \mathrm{ad}_\alpha^t = 0\}$,*
$$\mathrm{ad}_\alpha \mathrm{ad}_\beta \rho(\gamma) = 0. \tag{1}$$

Theorem 1.2. *Let (G, π) be a connected and unimodular Poisson-Lie group and let μ be a left invariant volume form on G. Then $di_\pi\mu = 0$ if and only if:*

1. *$(\mathcal{G}^*, [\ ,\ ]^*)$ is an unimodular Lie algebra,*
2. *for any $u \in \mathcal{G}$,*
$$\rho(i_{\xi(u)}\mu_e) = 0, \tag{2}$$
where ξ is the 1-cocycle associated to π and $\rho = -d$ is the dual 1-cocycle extended as a differential to $\wedge^{\dim\mathcal{G}-2}\mathcal{G}^*$.

We will see (*cf.* Proposition 3.2) that for a general connected Poisson-Lie group the condition $di_\pi\mu = 0$ implies (2).
If G is abelian then $\rho = 0$ and one can deduce easily from Theorems 1.1-1.2 the following result.

Corollary 1.3. *Let $(\mathcal{G}, \langle\ ,\ \rangle)$ be a Lie algebra endowed with a scalar product and denote by π_l the canonical linear Poisson structure on $\mathcal{G}^*$. Then $(\pi_l, \langle\ ,\ \rangle^*)$ satisfies Hawkins's conditions if and only if $(\mathcal{G}, \langle\ ,\ \rangle)$ is a Milnor Lie algebra.*

The following theorem is an interesting consequence of Theorem 1.1-1.2.

Theorem 1.4. *Let $(G, \pi, \langle\ ,\ \rangle)$ be a Riemannian Poisson-Lie group. Suppose that G is compact semi-simple, $\langle\ ,\ \rangle$ is bi-invariant and $\pi = r^- - r^+$ where r^+ (resp. r^-) is the left invariant (resp. the right invariant) bivector field associated to $r \in \wedge^2\mathcal{G}$. Then $(G, \pi, \langle\ ,\ \rangle)$ satisfies Hawkins's conditions if and only if $[r, r] = 0$.*

There are some interesting consequences of Theorems 1.1-1.2:

1. The classification of connected and simply connected Riemannian Poisson-Lie groups which are flat and metaflat is equivalent to the classification of the Lie bialgebra structures on Milnor Lie algebras for which (1) holds.

2. The classification of unimodular connected and simply connected Riemannian Poisson-Lie groups satisfying Hawkins's conditions is equivalent to the classification of the Lie bialgebra structures on Milnor Lie algebras for which (1) and (2) hold.

3. The Lie bialgebras structures on Milnor Lie algebras of dimension ≤ 4 can be computed (see Section 4) and hence the Riemannian Poisson-Lie groups of dimension ≤ 4 satisfying Hawkins's conditions can be deduced (see Theorems 4.2 and the paragraph devoted to the 4-dimensional case in Section 4).

The paper is organized as follows. In Section 2, we present a reformulation of a classical result of Milnor and we recall some standard facts about Levi-Civita contravariant connections and about the metacurvature of flat and torsion-free contravariant connections. In section 3, we prove Theorems 1.1, 1.2 and 1.4 and finally, Section 4 is devoted to the determination of Riemannian Poisson-Lie groups satisfying Hawkins's conditions in dimension 2, 3 and 4.

2. Preliminaries

2.1. Milnor Lie algebras. The following lemma is interesting in itself:

Lemma 2.1. *Let $(G, \langle\ ,\ \rangle)$ be a Lie group with a left invariant Riemannian metric. If the sectional curvature of $\langle\ ,\ \rangle$ is nonpositive then the Lie subalgebra $S = \{u \in \mathcal{G},\ \mathrm{ad}_u + \mathrm{ad}_u^t = 0\}$ is abelian.*

Proof. For any $u \in \mathcal{G}$, we denote by u^+ the left invariant vector field associated to u. Remark that $S^+ = \{u^+,\ u \in S\}$ is the Lie algebra of left invariant Killing vector fields. Now, since for any $u \in S$, $\langle u^+, u^+\rangle$ is constant then, for any left invariant vector field X we have:

$$\langle \nabla_X \nabla_X u^+, u^+\rangle + \langle \nabla_X u^+, \nabla_X u^+\rangle = 0, \tag{3}$$

where ∇ is the Levi-Civita connection associated to $\langle\ ,\ \rangle$.
The vector field u^+ is Killing, thus we have the well-known formula (see [2], Theorem 1.81)

$$\nabla_X \nabla_X u^+ - \nabla_{\nabla_X X} u^+ = R(u^+, X)X$$

where $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ is the tensor curvature. Moreover $\langle \nabla_{\nabla_X X} u^+, u^+\rangle = 0$, hence the formula (3) becomes:

$$\langle R(u^+, X)X, u^+\rangle + \langle \nabla_X u^+, \nabla_X u^+\rangle = 0.$$

This implies, since the curvature is nonpositive, that $\langle \nabla_X u^+, \nabla_X u^+\rangle = 0$. So u^+ is a parallel vector field and the lemma follows. ∎

The following proposition is a reformulation of a classical result of Milnor (see [13] Theorem 1.5).

Proposition 2.2. *Let $(G,\langle\ ,\ \rangle)$ be a Lie group endowed with a left invariant Riemannian metric. Then the curvature of $\langle\ ,\ \rangle$ vanishes if and only if the Lie algebra $\mathcal{G}$ of G endowed with the scalar product $\langle\ ,\ \rangle_e$ is a Milnor Lie algebra.*

Proof. Note first that the Levi-Civita connection of $\langle\ ,\ \rangle$ is entirely determined by the product $A:\mathcal{G}\times\mathcal{G}\longrightarrow\mathcal{G}$ given by

$$2\langle A_u v, w\rangle_e = \langle [u,v],w\rangle_e + \langle [w,u],v\rangle_e + \langle [w,v],u\rangle_e, \tag{4}$$

and the curvature vanishes if and only if, for any $u,v\in\mathcal{G}$, $A_{[u,v]}=[A_u,A_v]$.

If $\mathcal{G}=S\oplus[\mathcal{G},\mathcal{G}]$ is a Milnor Lie algebra, then one can deduce easily from (4) that

$$A_u=\begin{cases} 0 & \text{if } u\in[\mathcal{G},\mathcal{G}]\\ \mathrm{ad}_u & \text{if } u\in S,\end{cases}$$

and hence the curvature vanishes identically.

Suppose now that the curvature vanishes. In the proof of his result, Milnor considered $\mathfrak{u}=\{u\in\mathcal{G},\ A_u=0\}$ and showed that $\mathfrak{u}$ is an abelian ideal, its orthogonal $\mathfrak{b}$ is an abelian subalgebra and for all $u\in\mathfrak{b}$, ad_u is skew-symmetric. Hence $\mathfrak{b}\subset S$ and $[\mathcal{G},\mathcal{G}]=[\mathfrak{b},\mathfrak{u}]$.

Now, for any $u\in\mathfrak{u}$, $v\in\mathfrak{b}$ and $w\in\mathcal{G}$, we have $A_u=0$ and then

$$\langle w,[u,v]\rangle + \langle ad_w u, v\rangle + \langle u, ad_w v\rangle = 0.$$

This relation implies that $S=[\mathcal{G},\mathcal{G}]^\perp$ and Lemma 2.1 implies that S is abelian. We deduce that $[\mathcal{G},\mathcal{G}]\subset\mathfrak{u}$ and $[\mathcal{G},\mathcal{G}]$ is abelian and the proposition follows. ■

Proposition 2.3. *Let $\mathcal{G}$ be a Milnor Lie algebra. If $\dim S\geq 1$ then the derived ideal $[\mathcal{G},\mathcal{G}]$ is of even dimension.*

Proof. Let $(s_1,\dots,s_p)$ be a basis of S. The restriction of ad_{s_1} to $[\mathcal{G},\mathcal{G}]$ is a skew-symmetric endomorphism, thus its kernel K_1 is of even codimension in $[\mathcal{G},\mathcal{G}]$. Now, ad_{s_2} commutes with ad_{s_1} and K_1 is invariant by ad_{s_2}. By using the same argument as above, we deduce then $K_1\cap\ker\mathrm{ad}_{s_2}$ is of even codimension in K_1. Finally K_1 is of even codimension in $[\mathcal{G},\mathcal{G}]$. Thus, by induction, we show that

$$K_p=[\mathcal{G},\mathcal{G}]\cap(\cap_{i=1}^p\ker\mathrm{ad}_{s_i})$$

is an even codimensional subspace of $[\mathcal{G},\mathcal{G}]$. Now from its definition K_p is contained in the center of $\mathcal{G}$ which is contained in S and then $K_p=\{0\}$ and the result follows. ■

2.2. Contravariant connections and metacurvature. Contravariant connections associated to a Poisson structure have recently turned out to be useful in several areas of Poisson geometry. Contravariant connections were defined by

Vaisman [15] and were analyzed in detail by Fernandes [7]. This notion appears extensively in the context of noncommutative deformations (see [8, 9]).

Let (P,π) be a Poisson manifold. We consider $\pi_{\#} : T^*P \longrightarrow TP$ the anchor map given by $\beta(\pi_{\#}(\alpha)) = \pi(\alpha,\beta)$, and we denote by $[\ ,\ ]_\pi$ the Koszul bracket on differential 1-forms given by

$$[\alpha,\beta]_\pi = \mathscr{L}_{\pi_{\#}(\alpha)}\beta - \mathscr{L}_{\pi_{\#}(\beta)}\alpha - d(\pi(\alpha,\beta)). \tag{5}$$

This bracket can be extended naturally to $\Omega^*(P)$ and gives rise to a bracket which we denote also by $[\ ,\ ]_\pi$.

A contravariant connection on P, with respect to π, is a $\mathbb{R}$-bilinear map

$$\begin{array}{rccc} \mathcal{D}: & \Omega^1(P)\times\Omega^1(P) & \longrightarrow & \Omega^1(P) \\ & (\alpha,\beta) & \longmapsto & \mathcal{D}_\alpha\beta \end{array}$$

satisfying the following properties:

1. $\alpha \mapsto \mathcal{D}_\alpha\beta$ is $C^\infty(P)$-linear, that is:

$$\mathcal{D}_{f\alpha}\beta = f\mathcal{D}_\alpha\beta, \text{ for all } f\in C^\infty(P).$$

2. $\beta \mapsto \mathcal{D}_\alpha\beta$ is a derivation, in the sense:

$$\mathcal{D}_\alpha(f\beta) = f\mathcal{D}_\alpha\beta + \pi_{\#}(\alpha)(f)\beta, \text{ for all } f\in C^\infty(P).$$

The torsion and the curvature of a contravariant connection $\mathcal{D}$ is formally identical to the usual definitions

$$T(\alpha,\beta) = \mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha - [\alpha,\beta]_\pi \quad \text{and} \quad K(\alpha,\beta) = \mathcal{D}_\alpha\mathcal{D}_\beta - \mathcal{D}_\beta\mathcal{D}_\alpha - \mathcal{D}_{[\alpha,\beta]_\pi}.$$

The connection $\mathcal{D}$ is called *flat* if K vanishes identically.
Let us define now an interesting class of contravariant connections, namely Levi-Civita contravariant connections.
Let (P,π) be a Poisson manifold and $\langle\ ,\ \rangle$ a pseudo-Riemannian scalar product on T^*P. The metric contravariant connection associated to $(\pi,\langle\ ,\ \rangle)$ is the unique contravariant connection $\mathcal{D}$ such that $\mathcal{D}$ is torsion-free and the metric $\langle\ ,\ \rangle$ is parallel with respect to $\mathcal{D}$, i.e.,

$$\pi_{\#}(\alpha).\langle\beta,\gamma\rangle = \langle\mathcal{D}_\alpha\beta,\gamma\rangle + \langle\beta,\mathcal{D}_\alpha\gamma\rangle.$$

The connection $\mathcal{D}$ is the contravariant analogue of the Levi-Civita connection and can be defined by the Koszul formula:

$$\begin{array}{rcl} 2\langle\mathcal{D}_\alpha\beta,\gamma\rangle & = & \pi_{\#}(\alpha).\langle\beta,\gamma\rangle + \pi_{\#}(\beta).\langle\alpha,\gamma\rangle - \pi_{\#}(\gamma).\langle\alpha,\beta\rangle \\ & + & \langle[\gamma,\alpha]_\pi,\beta\rangle + \langle[\gamma,\beta]_\pi,\alpha\rangle + \langle[\alpha,\beta]_\pi,\gamma\rangle. \end{array} \tag{6}$$

We call $\mathcal{D}$ the *Levi-Civita contravariant connection* associated to $(\pi,\langle\ ,\ \rangle)$.

The metacurvature. We recall now the definition of the metacurvature introduced by Hawkins in [9].

Let (P, π) be a Poisson manifold and $\mathcal{D}$ a torsion-free and flat contravariant connection with respect to π. In [9], Hawkins showed that such a connection defines a bracket $\{\ ,\ \}$ on the space of differential forms $\Omega^*(P)$ such that:

1. $\{\ ,\ \}$ is $\mathbb{R}$-bilinear, degree 0 and antisymmetric, i.e.,
$$\{\sigma, \rho\} = -(-1)^{\deg\sigma\deg\rho}\{\rho, \sigma\}.$$
2. The differential d is a derivation with respect to $\{\ ,\ \}$, i.e.,
$$d\{\sigma, \rho\} = \{d\sigma, \rho\} + (-1)^{\deg\sigma}\{\sigma, d\rho\}.$$
3. $\{\ ,\ \}$ satisfies the product rule
$$\{\sigma, \rho \wedge \lambda\} = \{\sigma, \rho\} \wedge \lambda + (-1)^{\deg\sigma\deg\rho}\rho \wedge \{\sigma, \lambda\}.$$
4. For any $f, g \in C^\infty(P)$ and for any $\sigma \in \Omega^*(P)$ the bracket $\{f, g\}$ coincides with the initial Poisson bracket and
$$\{f, \sigma\} = \mathcal{D}_{df}\sigma.$$

Hawkins called this bracket a *generalized Poisson bracket* and showed that there exists a $(2,3)$-tensor $\mathcal{M}$ (symmetric in the contravariant indices and antisymmetric in the covariant indices) such that the following assertions are equivalent:

1. The generalized Poisson bracket satisfies the graded Jacobi identity
$$\{\{\sigma, \rho\}, \lambda\} = \{\sigma, \{\rho, \lambda\}\} - (-1)^{deg\sigma deg\rho}\{\rho, \{\sigma, \lambda\}\}.$$
2. The tensor $\mathcal{M}$ vanishes identically.

$\mathcal{M}$ is called the *metacurvature* and is given by
$$\mathcal{M}(df, \alpha, \beta) = \{f, \{\alpha, \beta\}\} - \{\{f, \alpha\}, \beta\} - \{\{f, \beta\}, \alpha\}. \tag{7}$$
The connection $\mathcal{D}$ is called *metaflat* if $\mathcal{M}$ vanishes identically.

The following formulas, due to Hawkins, will be useful later. Indeed, Hawkins pointed out in [9] pp. 394, that for any parallel 1-form α with respect to $\mathcal{D}$ and any 1-form β, the generalized Poisson bracket of α and β is given by
$$\{\alpha, \beta\} = -\mathcal{D}_\beta d\alpha. \tag{8}$$
Thus, one can deduce from (7) that for any parallel 1-forms α, γ and for any 1-form β,
$$\mathcal{M}(\alpha, \beta, \gamma) = -\mathcal{D}_\beta\mathcal{D}_\gamma d\alpha. \tag{9}$$

To finish this section, we give an useful full global formula for Hawkin's generalized Poisson bracket of two 1-forms. Let α and β be two 1-forms on a Poisson manifold P endowed with a torsion-free and flat contravariant connection $\mathcal{D}$. One can suppose that $\beta = gdf$ where $f, g \in C^\infty(P)$. Then, we have

$$\begin{aligned}
\{\alpha, fdg\} =& \{\alpha, f\} \wedge dg + f\{\alpha, dg\} \\
=& -D_{df}\alpha \wedge dg + f\left(dD_{dg}\alpha - D_{dg}d\alpha\right) \\
=& -D_{fdg}d\alpha + dD_{fdg}\alpha - D_{df}\alpha \wedge dg - df \wedge D_{dg}\alpha \\
=& -D_{fdg}d\alpha + dD_{fdg}\alpha - D_\alpha\left(df \wedge dg\right) - [df, \alpha]_\pi \wedge dg - df \wedge [dg, \alpha]_\pi \\
=& -D_{fdg}d\alpha - D_\alpha\left(d(fdg)\right) + dD_{fdg}\alpha - [df, \alpha]_\pi \wedge dg - df \wedge [dg, \alpha]_\pi \\
=& -D_{fdg}d\alpha - D_\alpha\left(d(fdg)\right) + dD_{fdg}\alpha + [\alpha, d(fdg)]_\pi \\
=& -D_\alpha d\beta - D_\beta d\alpha + dD_\beta\alpha + [\alpha, d\beta]_\pi.
\end{aligned}$$

Thus, for any $\alpha, \beta \in \Omega^1(P)$, we have

$$\{\alpha, \beta\} = -\mathcal{D}_\alpha d\beta - \mathcal{D}_\beta d\alpha + d\mathcal{D}_\beta\alpha + [\alpha, d\beta]_\pi. \tag{10}$$

3. Proofs of Theorems 1.1, 1.2 and 1.4

3.1. Proof of Theorem 1.1. Theorem 1.1 is an immediate consequence of the following result.

Theorem 3.1. *Let $(G, \pi, \langle\ ,\ \rangle)$ be a Riemannian Poisson-Lie group. Then:*

1. *$(\pi, \langle\ ,\ \rangle)$ is flat if and only if the dual Lie algebra $(\mathcal{G}^*, \langle\ ,\ \rangle^*)$ is a Milnor Lie algebra.*
2. *If $(\pi, \langle\ ,\ \rangle)$ is flat then, if one identifies $\mathcal{G}^*$ with the space of left invariant 1-forms, the metacurvature $\mathcal{M}$ is given by*

$$\mathcal{M}(\alpha, \beta, \gamma) = \begin{cases} \mathrm{ad}_\alpha \mathrm{ad}_\beta\, \rho(\gamma) & \text{for all } \alpha, \beta, \gamma \in S, \\ 0 & \text{if } \alpha, \beta \text{ or } \gamma \in [\mathcal{G}^*, \mathcal{G}^*], \end{cases} \tag{11}$$

where $S = \{\alpha \in \mathcal{G}^*,\ \mathrm{ad}_\alpha + \mathrm{ad}_\alpha^t = 0\}$ and $\rho : \mathcal{G}^* \longrightarrow \mathcal{G}^* \wedge \mathcal{G}^*$ is the dual 1-cocycle.

Proof. Note first that in a Poisson-Lie group the Koszul bracket of two left invariant 1-form is a left invariant 1-form (see [16]) and, if one identifies $\mathcal{G}^*$ with the space of left invariant 1-forms, the Koszul bracket coincides with the Lie bracket of $\mathcal{G}^*$. Through this proof, we identify $\mathcal{G}^*$ with the space of left invariant 1-forms on G.

1. Denote by $\langle\ ,\ \rangle^*$ the left invariant metric on T^*G associated to $\langle\ ,\ \rangle$ and denote by $\mathcal{D}$ the Levi-Civita contravariant connection associated to $(\pi, \langle\ ,\ \rangle^*)$. Since the Riemannian metric is left invariant, for any $\alpha, \beta, \gamma \in \mathcal{G}^*$, (6) becomes

$$2\langle \mathcal{D}_\alpha\beta, \gamma\rangle^* = \langle [\gamma, \alpha]_\pi, \beta\rangle^* + \langle [\gamma, \beta]_\pi, \alpha\rangle^* + \langle [\alpha, \beta]_\pi, \gamma\rangle^*. \tag{12}$$

Hence the restriction of $\mathcal{D}$ to $\mathcal{G}^* \times \mathcal{G}^*$ defines a product on $\mathcal{G}^*$. The vanishing of the curvature of $\mathcal{D}$ is equivalent to the vanishing of the restriction of the curvature of $\mathcal{D}$ to $\mathcal{G}^*$. Now, one can deduce from (12) that the vanishing of the restriction of the curvature of $\mathcal{D}$ to $\mathcal{G}^*$ is equivalent to the flatness of the left invariant Riemannian metric associated to $\langle\ ,\ \rangle_e^*$ on any Lie group with $\mathcal{G}^*$ as a Lie algebra and one can conclude by using Proposition 2.2.

2. Suppose now that $(\pi, \langle\ ,\ \rangle)$ is flat and, according to the first part, let $\mathcal{G}^* = S \overset{\perp}{\oplus} [\mathcal{G}^*, \mathcal{G}^*]$ where $S = \{\alpha \in \mathcal{G}^*, ad_\alpha + ad_\alpha^* = 0\}$ and both S and $[\mathcal{G}^*, \mathcal{G}^*]$ are abelian. Note that for any $\alpha \in \mathcal{G}^*$, $d\alpha(X, Y) = -\alpha([X, Y])$ and hence $\rho = -d$. Let us establish (11).

First, one can deduce from (12) that, for any $\gamma \in \mathcal{G}^*$,

$$\mathcal{D}_\alpha \gamma = \begin{cases} 0 & \text{if } \alpha \in [\mathcal{G}^*, \mathcal{G}^*] \\ [\alpha, \gamma]_\pi = \mathrm{ad}_\alpha \gamma & \text{if } \alpha \in S, \end{cases} \tag{13}$$

and moreover, for any $\alpha \in S$, $\mathcal{D}\alpha = 0$.

(a) If $\alpha, \beta, \gamma \in S$, since $\mathcal{D}\alpha = \mathcal{D}\beta = \mathcal{D}\gamma = 0$, we deduce from (9) that

$$\mathcal{M}(\alpha, \beta, \gamma) = -\mathcal{D}_\alpha \mathcal{D}_\beta d\gamma \overset{(13)}{=} ad_\alpha ad_\beta \rho(\gamma).$$

(b) If $\alpha, \gamma \in S$ and $\beta \in [\mathcal{G}^*, \mathcal{G}^*]$, since $\mathcal{D}\alpha = \mathcal{D}\gamma = 0$, we deduce from (9) that

$$\mathcal{M}(\alpha, \beta, \gamma) = -\mathcal{D}_\beta \mathcal{D}_\alpha d\gamma \overset{(13)}{=} 0.$$

(c) If $\alpha, \beta \in [\mathcal{G}^*, \mathcal{G}^*]$ and $\gamma \in S$. At least locally, we have $\alpha = \sum f_i dg_i$ and we deduce from (7) that

$$\mathcal{M}(\alpha, \beta, \gamma) = \sum f_i \{g_i, \{\beta, \gamma\}\} - f_i \{\{g_i, \beta\}, \gamma\} - f_i \{\{g_i, \gamma\}, \beta\}.$$

>From (8) and (13), we have $\{\beta, \gamma\} = -\mathcal{D}_\beta d\gamma = 0$. On the other hand, also by using (13), $\{g_i, \gamma\} = \mathcal{D}_{dg_i} \gamma = 0$, thus

$$\mathcal{M}(\alpha, \beta, \gamma) = \sum -f_i \{\{g_i, \beta\}, \gamma\} = \sum f_i \mathcal{D}_{\mathcal{D}_{dg_i} \beta} d\gamma = \mathcal{D}_{\mathcal{D}_\alpha \beta} d\gamma = 0.$$

(d) For $\alpha, \beta \in [\mathcal{G}^*, \mathcal{G}^*]$, the computation of $\mathcal{M}(\alpha, \beta, \beta)$ is more difficult. First, by comparing $\mathcal{M}(\alpha, \beta, \beta)$ and $[\beta, [\beta, d\alpha]_\pi]_\pi$, we will show that they agree up to sign and, next, we will show that $[\beta, [\beta, d\alpha]_\pi]_\pi = 0$ and we get the result.

Put $\alpha = \sum f_i dg_i$. By using (7), we get

$$\begin{aligned}\mathcal{M}(\alpha,\beta,\beta) =& \sum f_i\{g_i,\{\beta,\beta\}\} - 2f_i\{\{g_i,\beta\},\beta\}\\ =& \sum f_i\mathcal{D}_{dg_i}\{\beta,\beta\} - 2\sum f_i\{\mathcal{D}_{dg_i}\beta,\beta\}\\ =& \mathcal{D}_\alpha\{\beta,\beta\} - 2\sum f_i\{\mathcal{D}_{dg_i}\beta,\beta\}\\ \overset{(*)}{=}& -2\sum f_i\{\mathcal{D}_{dg_i}\beta,\beta\}\\ =& -2\sum\left(\{f_i\mathcal{D}_{dg_i}\beta,\beta\} + \mathcal{D}_{df_i}\beta\wedge\mathcal{D}_{dg_i}\beta\right)\\ =& -2\{\mathcal{D}_\alpha\beta,\beta\} - 2\sum \mathcal{D}_{df_i}\beta\wedge\mathcal{D}_{dg_i}\beta\\ =& -2\sum \mathcal{D}_{df_i}\beta\wedge\mathcal{D}_{dg_i}\beta.\end{aligned}$$

In $(*)$ we have used (13) and the fact that $\{\alpha,\beta\}\in\wedge^2\mathcal{G}^*$ which can be deduced from (10). On the other hand,

$$\begin{aligned}[\beta,[\beta,d\alpha]_\pi]_\pi =& \sum[\beta,[\beta,df_i\wedge dg_i]_\pi]_\pi\\ =& \sum[\beta,[\beta,df_i]_\pi\wedge dg_i]_\pi + [\beta,df_i\wedge[\beta,dg_i]_\pi]_\pi\\ =& \sum[\beta,[\beta,df_i]_\pi]_\pi\wedge dg_i + [\beta,df_i]_\pi\wedge[\beta,dg_i]_\pi\\ &+ [\beta,df_i]_\pi\wedge[\beta,dg_i]_\pi + df_i\wedge[\beta,[\beta,dg_i]_\pi]_\pi.\end{aligned}$$

Now, choose an orthonormal basis $\{\alpha_1,...,\alpha_n\}$ of $\mathcal{G}^*$. For any 1-form $\gamma\in\Omega^1(G)$, we have $\gamma = \sum\langle\gamma,\alpha_i\rangle^*\alpha_i$, and

$$\begin{aligned}[\beta,\gamma]_\pi =& \sum \pi_\sharp(\beta)\cdot\langle\gamma,\alpha_i\rangle^*\,\alpha_i + \langle\gamma,\alpha_i\rangle^*[\beta,\alpha_i]_\pi\\ =& \mathcal{D}_\beta\gamma + \sum\langle\gamma,\alpha_i\rangle^*[\beta,\alpha_i]_\pi.\end{aligned}$$

Hence

$$\begin{aligned}[\beta,[\beta,\gamma]_\pi]_\pi =& [\beta,\mathcal{D}_\beta\gamma]_\pi + \sum\pi_\sharp(\beta)\cdot\langle\gamma,\alpha_i\rangle^*[\beta,\alpha_i]_\pi + \sum\langle\gamma,\alpha_i\rangle^*[\beta,[\beta,\alpha_i]_\pi]_\pi\\ \overset{(*)}{=}& [\beta,\mathcal{D}_\beta\gamma]_\pi - \sum\langle\mathcal{D}_\beta\gamma,\alpha_i\rangle^*\mathcal{D}_{\alpha_i}\beta\\ =& [\beta,\mathcal{D}_\beta\gamma]_\pi - \mathcal{D}_{\mathcal{D}_\beta\gamma}\beta\\ =& \mathcal{D}_\beta\mathcal{D}_\beta\gamma - 2\mathcal{D}_{(\mathcal{D}_\beta\gamma)}\beta\\ =& \mathcal{D}_\beta\mathcal{D}_\beta\gamma - 2\mathcal{D}_{[\beta,\gamma]_\pi}\beta + 2\sum\langle\gamma,\alpha_i\rangle^*\mathcal{D}_{[\beta,\alpha_i]_\pi}\beta\\ \overset{(**)}{=}& \mathcal{D}_\beta\mathcal{D}_\beta\gamma - 2\mathcal{D}_{[\beta,\gamma]_\pi}\beta\\ =& \mathcal{D}_\beta\mathcal{D}_\beta\gamma - 2(K(\beta,\gamma)\beta + \mathcal{D}_\beta\mathcal{D}_\gamma\beta - \mathcal{D}_\gamma\mathcal{D}_\beta\beta)\\ =& \mathcal{D}_\beta\mathcal{D}_\beta\gamma - 2\mathcal{D}_\beta\mathcal{D}_\gamma\beta.\end{aligned}$$

We have used in the equalities $(*)$ and $(**)$ the fact that $[\mathcal{G}^*,\mathcal{G}^*]$ is abelian and hence $[\beta,[\beta,\alpha_i]_\pi]_\pi = -\mathcal{D}_{[\beta,\alpha_i]_\pi}\beta = 0$.

By using this formula, we get

$$\begin{aligned}[\beta,[\beta,df_i]_\pi]_\pi\wedge dg_i =&\mathcal{D}_\beta\mathcal{D}_\beta df_i\wedge dg_i-2\mathcal{D}_\beta\mathcal{D}_{df_i}\beta\wedge dg_i\\ =&\mathcal{D}_\beta\left(\mathcal{D}_\beta df_i\wedge dg_i\right)-\mathcal{D}_\beta df_i\wedge\mathcal{D}_\beta dg_i\\ &-2\mathcal{D}_\beta\left(D_{df_i}\beta\wedge dg_i\right)+2D_{df_i}\beta\wedge\mathcal{D}_\beta dg_i,\\ df_i\wedge[\beta,[\beta,dg_i]_\pi]_\pi =&-\mathcal{D}_\beta\left(\mathcal{D}_\beta dg_i\wedge df_i\right)+\mathcal{D}_\beta dg_i\wedge\mathcal{D}_\beta df_i\\ &+2\mathcal{D}_\beta\left(D_{dg_i}\beta\wedge df_i\right)-2D_{dg_i}\beta\wedge\mathcal{D}_\beta df_i.\end{aligned}$$

On the other hand

$$\begin{aligned}2[\beta,df_i]\wedge[\beta,dg_i] =&2\mathcal{D}_\beta df_i\wedge\mathcal{D}_\beta dg_i-2\mathcal{D}_\beta df_i\wedge\mathcal{D}_{dg_i}\beta\\ &-2\mathcal{D}_{df_i}\beta\wedge\mathcal{D}_\beta dg_i+2\mathcal{D}_{df_i}\beta\wedge\mathcal{D}_{dg_i}\beta.\end{aligned}$$

Thus

$$\begin{aligned}[\beta,[\beta,d\alpha]_\pi]_\pi =&\mathcal{D}_\beta\mathcal{D}_\beta d\alpha+2\sum\mathcal{D}_{df_i}\beta\wedge\mathcal{D}_{dg_i}\beta-2\mathcal{D}_\beta\left(\mathcal{D}_{df_i}\beta\wedge dg_i\right)\\ &-2\mathcal{D}_\beta\left(df_i\wedge\mathcal{D}_{dg_i}\beta\right)\\ =&\mathcal{D}_\beta\mathcal{D}_\beta d\alpha+2\sum\mathcal{D}_{df_i}\beta\wedge\mathcal{D}_{dg_i}\beta+2\mathcal{D}_\beta\left([\beta,df_i]_\pi\wedge dg_i\right)\\ &-2\mathcal{D}_\beta\left(\mathcal{D}_\beta df_i\wedge dg_i\right)+2\mathcal{D}_\beta\left(df_i\wedge[\beta,dg_i]_\pi\right)-2\mathcal{D}_\beta\left(df_i\wedge\mathcal{D}_\beta dg_i\right)\\ =&-\mathcal{D}_\beta\mathcal{D}_\beta d\alpha-\mathcal{M}(\alpha,\beta,\beta)+2\mathcal{D}_\beta[\beta,d\alpha]_\pi\\ =&-\mathcal{M}(\alpha,\beta,\beta).\end{aligned}$$

Let us show now that $[\beta,[\beta,d\alpha]_\pi]_\pi=0$. We have $\alpha=\sum[\gamma_i,\gamma_j]_\pi$ and furthermore d is a 1-cocycle with respect to the adjoint action of $\mathcal{G}^*$, then

$$d\alpha=\sum\left([d\gamma_i,\gamma_j]_\pi+[\gamma_i,d\gamma_j]_\pi\right).$$

This relation implies that $d\alpha=\alpha_1+\alpha_2$ where $\alpha_1\in S\wedge[\mathcal{G}^*,\mathcal{G}^*]$ and $\alpha_2\in\wedge^2[\mathcal{G}^*,\mathcal{G}^*]$. From this decomposition and from the fact that $[\mathcal{G}^*,\mathcal{G}^*]$ is abelian, one can deduce easily that $[\beta,[\beta,d\alpha]_\pi]_\pi=0$ and hence $\mathcal{M}(\alpha,\beta,\beta)=0$. Since $\mathcal{M}$ is symmetric, we deduce that, for any $\alpha,\beta,\gamma\in[\mathcal{G}^*,\mathcal{G}^*]$, $\mathcal{M}(\alpha,\beta,\gamma)=0$ which completes the proof.

■

Before giving a proof of Theorem 1.2, let us show first that, in the general case, the condition (2) is a necessary condition.

Proposition 3.2. *Let (G,π) be a Poisson-Lie group and let μ be a left invariant form on G. If $d(i_\pi\mu)=0$ then* (2) *holds.*

Proof. The proof is based on the Koszul formula [10], satisfied by any vector field X and any multivector Q, and given by

$$i_{[X,Q]}\mu=i_Xdi_Q\mu+(-1)^{\deg Q}di_Xi_Q\mu-i_Qdi_X\mu. \qquad (14)$$

Indeed, if $d(i_\pi\mu)=0$ then, for any left invariant vector field X, we get

$$i_{[X,\pi]}\mu = di_X i_\pi\mu - i_\pi di_X\mu.$$

Or $di_X\mu = \mathcal{L}_X\mu = \alpha\mu$, where α is a constant and hence $di_{[X,\pi]}\mu = 0$. One can conclude by using the fact that $[X,\pi]$ is left invariant and $[X,\pi](e)=\xi(X_e)$. ∎

3.2. Proof of Theorem 1.2.

Proof. Let (G,π) be a connected unimodular Poisson-Lie group and let μ be a left invariant volume form on G. Let ξ be the 1-cocycle associated to π and let $(G^*,[\,,\,]^*,\rho)$ be the dual Lie bialgebra. For any tensor T on $\mathcal{G}$, we denote by T^+ the corresponding left invariant tensor field on G. Recall that the divergence of a vector field X with respect to μ is the function $\operatorname{div}_\mu X$ given by

$$\mathscr{L}_X\mu = (\operatorname{div}_\mu X)\mu.$$

Before giving the proof of the theorem, we need to state some properties of the modular vector field on a Poisson Lie-group.

As shown in [17], the operator $X_\mu : f \mapsto \operatorname{div}_\mu X_f$ (X_f being the Hamiltonian vector field associated to f) is a derivation and hence a vector field called the *modular vector field* of (G,π) with respect to the volume form μ. It is well-known (see [17]) that X_μ is given by

$$di_\pi\mu = i_{X_\mu}\mu. \tag{15}$$

We define the modular form $\kappa : \mathcal{G}^* \to \mathbb{R}$ by

$$\kappa(\alpha) = \operatorname{tr}\operatorname{ad}_\alpha, \tag{16}$$

where $\operatorname{ad}_\alpha\beta = [\alpha,\beta]^*$. The modular form κ, which is in $\mathcal{G}^{**}$, defines a vector in $\mathcal{G}$ denoted also by κ. We have

$$X_\mu(e) = \kappa. \tag{17}$$

Indeed, choose a scalar product $\langle\,,\,\rangle$ on $\mathcal{G}$, an orthonormal basis $(u_1,\dots,u_n)$ of $(\mathcal{G},\langle\,,\,\rangle)$ and denote by $(\alpha_1,\dots,\alpha_n)$ its dual basis. We have

$$\pi = \sum_{i<j}\pi_{ij}\,u_i^+\wedge u_j^+$$

and the Hamiltonian vector field associated to $f\in C^\infty(M)$ is given by

$$X_f = \sum_{j=1}^n\left(\sum_{i=1}^n \pi_{ij}\langle df,\alpha_i^+\rangle^*\right)u_j^+.$$

We have

$$\begin{aligned}
X_\mu(f) &= \operatorname{div}_\mu\sum_{j=1}^n\left(\sum_{i=1}^n \pi_{ij}\langle df,\alpha_i\rangle^*\right)u_j^+\\
&= \sum_{j=1}^n\left(\sum_{i=1}^n \pi_{ij}u_i^+(f)\right)\operatorname{div}_\mu u_j^+ + \sum_{j=1}^n\sum_{i=1}^n u_j^+\left(\pi_{ij}u_i^+(f)\right).
\end{aligned}$$

Now, since for any $i, j = 1, \ldots, n$ $\pi_{ij}(e) = 0$ and, because $\mathcal{G}$ is unimodular, $\mathrm{div}_\mu u_j^+ = 0$ for $j = 1, \ldots, n$, we get

$$X_\mu(e) = \sum_{i=1}^n \left(\sum_{j=1}^n \mathscr{L}_{X_j^+} \pi(\alpha_i^+, \alpha_j^+)_e \right) u_i,$$

and

$$\begin{aligned}
\langle \alpha_i, X_\mu(e) \rangle &= \sum_{j=1}^n \mathscr{L}_{X_j^+} \pi(\alpha_i^+, \alpha_j^+)_e = \sum_{j=1}^n \langle \alpha_i \wedge \alpha_j, \xi(X_j) \rangle \\
&= \sum_{j=1}^n [\alpha_i, \alpha_j]^*(X_j) = \sum_{j=1}^n \left\langle \sum_{k=1}^n [\alpha_i, \alpha_j]^*(X_k)\alpha_k, \alpha_j \right\rangle^* \\
&= \sum_{j=1}^n \langle [\alpha_i, \alpha_j]^*, \alpha_j \rangle^* = \mathrm{tr}\, \mathrm{ad}_{\alpha_i} = \kappa(\alpha_i),
\end{aligned}$$

and (17) is established.

Now, we will show that $X_\mu - \kappa^+$ is a multiplicative vector field by using the characterization of multiplicative multivector fields given in [12]. Indeed, by applying (14) and $\mathscr{L}_X \mu = 0$, we get

$$\begin{aligned}
i_{[X,X_\mu]}\mu &= i_X d i_{X_\mu}\mu - d i_X i_{X_\mu}\mu - i_{X_\mu} d i_X \mu \\
&= -d i_X i_{X_\mu}\mu, \\
i_{[X,\pi]}\mu &= i_X d i_\pi \mu + d i_X i_\pi \mu - i_\pi d i_X \mu \\
&= i_X i_{X_\mu}\mu + d i_X i_\pi \mu.
\end{aligned}$$

Thus

$$d\left(i_{[X,\pi]}\mu\right) = -i_{[X,X_\mu]}\mu. \tag{18}$$

Since $[X, \pi]$ and μ are left invariant, we deduce from (18) that $[X, X_\mu]$ is also left invariant. Moreover, $[X, X_\mu - \kappa^+] = [X, X_\mu] - [X, \kappa^+]$ is left invariant and, since $X_\mu(e) = \kappa^+(e)$, we deduce that $X_\mu - \kappa^+$ is a multiplicative vector field. Thus $X_\mu = X_m + \kappa^+$ where X_m is a multiplicative vector field.

To complete the proof, note that $X_\mu = 0$ if and only if $\kappa = 0$ and $X_m = 0$. Now $\kappa = 0$ if and only if $(\mathcal{G}^*, [\ ,\ ]^*)$ is unimodular, and $X_m = 0$ if and only if $[X, X_m](e) = 0$, for all left invariant vector field X (see [12]). Or the last condition is equivalent, according to (18), to $\rho\left(i_{\xi(u)}\mu\right) = 0$, for any $u \in \mathcal{G}$. ■

3.3. Proof of Theorem 1.4.

Proof. Let $(G, \pi, \langle\ ,\ \rangle)$ be a Riemannian Poisson-Lie group such that G is compact semi-simple, $\langle\ ,\ \rangle$ is bi-invariant and π is exact, i.e., there exists $r \in \wedge^2\mathcal{G}$ such that $\pi = r^- - r^+$, where r^+ (resp. r^-) is the left invariant (resp. the right invariant) bivector field associated to r. It is well-known that $[r, r]$ is ad-invariant and the dual Lie algebra structure on $\mathcal{G}^*$ is given by

$$[\alpha, \beta]^* = ad^*_{r_\#(\alpha)}\beta - ad^*_{r_\#(\beta)}\alpha,$$

where $r_\# : \mathcal{G}^* \longrightarrow \mathcal{G}$ is the contraction associated to r. Now, since $\langle\ ,\ \rangle$ is bi-invariant, the Levi-Civita contravariant connection $\mathcal{D}$ associated to $(\pi, \langle\ ,\ \rangle)$ is given by

$$\mathcal{D}_\alpha \beta = ad^*_{r_\#(\alpha)}\beta, \quad \alpha, \beta \in \mathcal{G}^*, \tag{19}$$

and hence its curvature is given by (see [8])

$$K(\alpha, \beta)\gamma = ad^*_{[r,r](\alpha,\beta,.)}\gamma. \tag{20}$$

Since G is semi-simple, we deduce that K vanishes if and only if $[r, r] = 0$. Suppose now that $[r, r] = 0$ and we will show that $(\pi, \langle\ ,\ \rangle)$ is metaflat and satisfies the third Hawkins's condition.
Since $(\pi, \langle\ ,\ \rangle)$ is flat, according to Theorem 1.1, $\mathcal{G}^* = S \oplus [\mathcal{G}^*, \mathcal{G}^*]$ where $S = \{\alpha \in \mathcal{G}^*,\ ad_\alpha + ad^t_\alpha = 0\}$ is abelian and $[\mathcal{G}^*, \mathcal{G}^*]$ is abelian. By using the proof of lemma 2.1 and (19) it is easy to show that

$$S = \{\alpha \in \mathcal{G}^*, ad^*_{r_\#(\beta)}\alpha = 0 \text{ for all } \beta \in \mathcal{G}^*\}. \tag{21}$$

On the other hand, one can see easily that $\ker r_\# \subset S$. On the other hand $\mathcal{G} = \mathrm{Im} r_\# \oplus \mathrm{Im} r_\#^\perp$ and $\mathrm{Im} r_\#$ is unimodular and symplectic and then solvable (see [11]). Also $\mathrm{Im} r_\#$ carries a bi-invariant scalar product so it must be abelian (see [13]). Let us show now that (1) holds. Choose an othonormal basis $\mathbb{B}_1 = \{e_1, \ldots, e_{2p}\}$ of $\mathrm{Im} r_\#$ and an orthonormal basis $\mathbb{B}_2 = \{f_1, \ldots, f_{n-2p}\}$ of $\mathrm{Im} r_\#^\perp$ and let $\{\alpha_1, \ldots, \alpha_{2p}, \beta_1, \ldots, \beta_{n-2p}\}$ the dual basis of $\mathbb{B}_1 \cup \mathbb{B}_2$. Let $\alpha, \gamma \in S$. For any $r_\#(\mu_1), r_\#(\mu_2) \in \mathrm{Im} r_\#$ and for any $u \in \mathrm{Im} r_\#^\perp$, we have

$$\begin{aligned} d\gamma(r_\#(\mu_1), r_\#(\mu_2)) &= -\gamma([r_\#(\mu_1), r_\#(\mu_2)]) \\ &= 0, \\ d\gamma(r_\#(\mu_1), u) &= -ad^*_{r_\#(\mu_1)}\gamma(u) \\ &\overset{(21)}{=} 0, \end{aligned}$$

and hence

$$d\gamma = \sum_{i,j} a_{i,j} \beta_i \wedge \beta_j,$$

where $a_{i,j} \in \mathbb{R}$. Now,

$$ad_\alpha d\gamma = \sum_{i,j} a_{i,j} \left(ad_\alpha \beta_i \wedge \beta_j + \beta_i \wedge ad_\alpha \beta_j \right).$$

Or

$$\begin{aligned} ad_\alpha \beta_i &= [\alpha, \beta_i]^* \\ &= ad^*_{r_\#(\alpha)}\beta_i - ad^*_{r_\#(\beta_i)}\alpha \\ &\overset{(21)}{=} ad^*_{r_\#(\alpha)}\beta_i. \end{aligned}$$

Now β_i is in the annihilator of $\mathrm{Im} r_\#$ which is equal to $\ker r_\#$. Or we have shown that $\ker r_\# \subset S$ and, according to (21), $ad^*_{r_\#(\alpha)}\beta_i = 0$ so $ad_\alpha d\gamma = 0$ and (1) holds. To conclude, we will show that (2) holds and we get the result, according to Theorem 1.2 ($\mathcal{G}$ is unimodular). Note first that in our case G is $\xi(u) = [u, r]$ and, by using (14), we get

$$i_{[u,r]}\mu = i_u d i_r \mu + d i_u i_r \mu - i_r d i_u \mu. \tag{22}$$

Or since G is unimodular $di_u\mu = 0$. On the other hand $r = \sum_{i,j} b_{ij} e_i \wedge e_j$ and then

$$\begin{aligned} d(i_r\mu) &= d\left(\sum_{i,j} b_{ij} i_{e_i\wedge e_j}\mu\right) \\ &= \sum_{i,j} b_{ij}\left(i_{[e_i,e_j]}\mu - i_{e_i}\mathcal{L}_{e_j}\mu - i_{e_j)}\mathcal{L}_{e_i}\mu\right) \\ &= 0, \end{aligned}$$

which completes the proof.

■

4. Examples

This Section is devoted to the determination of Riemannian Poisson-Lie groups satisfying Hawkins's conditions in the linear case, in dimension 2, 3 and 4. Note first that when the dual Lie algebra is abelian the Poisson tensor is zero and, in what follows, we will omit this trivial case.

The linear case. Let $\mathcal{G} = S \oplus [\mathcal{G}, \mathcal{G}]$ be a Milnor Lie algebra. Since S is abelian and acts on $[\mathcal{G}, \mathcal{G}]$ by skew-symmetric endomorphisms, there exists a family of non vanishing vectors $u_1, \ldots, u_r \in S$ and an orthonormal basis $(f_1, \ldots, f_{2r})$ of $[\mathcal{G}, \mathcal{G}]$ such that, for any $j = 1, \ldots, r$ and for all $s \in S$,

$$[s, f_{2j-1}] = \langle s, u_j \rangle f_{2j} \quad \text{and} \quad [s, f_{2j}] = -\langle s, u_j \rangle f_{2j-1}. \tag{23}$$

According to Corollary 1.3, the triple $(\mathcal{G}^*, \pi_l, \langle\ ,\ \rangle^*)$ satisfies Hawkins's conditions. It is easy to show that there exists a family of constants $(a_{ij})_{1\leq i,j\leq q}$ such that $(\mathcal{G}^*, \pi_l, \langle\ ,\ \rangle^*)$ is isomorphic to $(\mathbb{R}^{q+2r}, \pi_0, \langle\ ,\ \rangle_0)$ where $\langle\ ,\ \rangle_0$ is the canonical Euclidian metric and

$$\pi_0 = \sum_{i=1}^{r} \left(a_{1i}\partial_{x_1} + \ldots + a_{qi}\partial_{x_q}\right) \wedge \left(y_{2i}\partial_{y_{2i-1}} - y_{2i-1}\partial_{y_{2i}}\right).$$

The 2-dimensional case. According to Theorems 1.1-1.2 and since any 2-dimensional Milnor Lie algebra is abelian, a 2-dimensional connected and simply connected Riemannian Poisson-Lie group $(G, \pi, \langle\ ,\ \rangle)$ satisfies Hawkins's conditions if and only if the Poisson tensor is trivial.

The 3-dimensional case. In this paragraph we will determine, up to isomorphism, all the 3-dimensional connected and simply connected Riemannian Poisson-Lie groups satisfying Hawkins's conditions. According to Theorems 1.1-1.2 and Proposition 3.2, the first step is to determine all the Lie bialgebra structures on 3-dimensional Milnor Lie algebras satisfying (1) and (2).

Let $\mathcal{H}$ be a 3-dimensional Milnor Lie algebra. By virtue of (23), there exists a real number $\lambda \neq 0$ and an orthonormal basis (e_1, e_2, e_3) of $\mathcal{H}$ such that

$$[e_2, e_3] = 0, \quad [e_1, e_2] = \lambda e_3 \quad \text{and} \quad [e_1, e_3] = -\lambda e_2.$$

We are looking for the 1-cocycles $\rho : \mathcal{H} \longrightarrow \mathcal{H} \wedge \mathcal{H}$ defining a Lie bialgebra structure on $\mathcal{H}$ and satisfying (1) and (2). Put

$$\rho(e_1) = ae_1 \wedge e_2 + be_1 \wedge e_3 + ce_2 \wedge e_3.$$

The condition (1) is equivalent to

$$ad_{e_1} \circ ad_{e_1} \rho(e_1) = 0.$$

We have $ad_{e_1}\rho(e_1) = a\lambda e_1 \wedge e_3 - b\lambda e_1 \wedge e_2$ and hence

$$ad_{e_1} \circ ad_{e_1} \rho(e_1) = -a\lambda^2 e_1 \wedge e_2 - b\lambda^2 e_1 \wedge e_3.$$

Thus ρ satisfies (1) if and only if

$$\rho(e_1) = ce_2 \wedge e_3.$$

Now put

$$\rho(e_2) = a_1 e_1 \wedge e_2 + b_1 e_1 \wedge e_3 + c_1 e_2 \wedge e_3, \quad \rho(e_3) = a_2 e_1 \wedge e_2 + b_2 e_1 \wedge e_3 + c_2 e_2 \wedge e_3,$$

and write down the cocycle condition $\rho([u, v]) = ad_u \rho(v) - ad_v \rho(u)$. We get

$$\begin{aligned}
\rho([e_2, e_3]) &= -\lambda a_2 e_3 \wedge e_2 - \lambda b_1 e_2 \wedge e_3 = \lambda(a_2 - b_1) e_2 \wedge e_3 = 0, \\
\rho([e_1, e_2]) &= \lambda(a_1 e_1 \wedge e_3 - b_1 e_1 \wedge e_2) = \lambda \rho(e_3), \\
\rho([e_1, e_3]) &= \lambda(a_2 e_1 \wedge e_3 - b_2 e_1 \wedge e_2) = -\lambda \rho(e_2).
\end{aligned}$$

These relations are equivalent to

$$b_1 = a_2 = c_1 = c_2 = 0 \quad \text{and} \quad a_1 = b_2.$$

Thus ρ is a 1-cocycle satisfying (1) if and only if

$$\rho(e_1) = ce_2 \wedge e_3, \quad \rho(e_2) = ae_1 \wedge e_2 \quad \text{and} \quad \rho(e_3) = ae_1 \wedge e_3. \tag{24}$$

We consider now $\mathcal{H}^*$ endowed with the bracket associated to ρ, the dual scalar product and the dual of the bracket on $\mathcal{H}$, $\xi : \mathcal{H}^* \longrightarrow \mathcal{H}^* \wedge \mathcal{H}^*$, given by

$$\xi(e_1^*) = 0, \quad \xi(e_2^*) = -\lambda e_1^* \wedge e_3^* \quad \text{and} \quad \xi(e_3^*) = \lambda e_1^* \wedge e_2^*, \tag{25}$$

where (e_1^*, e_2^*, e_3^*) is the dual basis of (e_1, e_2, e_3). The bracket on $\mathcal{H}^*$ associated to ρ is given by

$$[e_1^*, e_2^*] = ae_2^*, \quad [e_1^*, e_3^*] = ae_3^* \quad \text{and} \quad [e_2^*, e_3^*] = ce_1^*. \tag{26}$$

Note that

$$\operatorname{tr} ad_{e_1^*} = 2a, \quad \operatorname{tr} ad_{e_2^*} = \operatorname{tr} ad_{e_3^*} = 0.$$

The Jacobi identity is given by

$$[[e_1^*, e_2^*], e_3^*] + [[e_2^*, e_3^*], e_1^*] + [[e_3^*, e_1^*], e_2^*] = 2ace_1^*.$$

Let us write down (2). Since $\mu = e_1 \wedge e_2 \wedge e_3$ and by virtue of (25), a straightforward calculation using (24) gives

$$\begin{aligned}\rho\left(i_{\xi(e_2^*)}\mu\right) =& \lambda\rho(e_2) = \lambda ae_1 \wedge e_2,\\ \rho\left(i_{\xi(e_3^*)}\mu\right) =& \lambda\rho(e_3) = \lambda ae_1 \wedge e_3.\end{aligned}$$

In conclusion, ρ defines a Lie bialgebra structure on $\mathcal{H}$ and satisfies (1) and (2) if and only if

$$\rho(e_1) = ce_2 \wedge e_3 \quad \text{and} \quad \rho(e_2) = \rho(e_3) = 0. \tag{27}$$

Note that in this case, the Lie algebra $\mathcal{H}^*$ is unimodular. The following Proposition summarize all the discussion above.

Proposition 4.1. *Let $(G, \pi, \langle\ ,\ \rangle)$ be a 3-dimensional connected and simply connected Riemannian Poisson-Lie group and let $(\mathcal{G}, \xi, \langle\ ,\ \rangle_e)$ be its Lie algebra endowed with the cocycle ξ associated to π and the value of the Riemannian metric at the identity. Then $(G, \pi, \langle\ ,\ \rangle)$ satisfies Hawkins's conditions if and only if the triple $(\mathcal{G}, \xi, \langle\ ,\ \rangle_e)$ is isomorphic to one of the following triples:*

1. *$(\mathbb{R}^3, \xi_0, \langle\ ,\ \rangle_0)$ where $\mathbb{R}^3$ is endowed with its abelian Lie algebra structure, ξ_0 is given by*

$$\xi_0(e_1) = 0, \quad \xi(e_2) = -\lambda e_1 \wedge e_3 \quad \textit{and} \quad \xi(e_3) = \lambda e_1 \wedge e_2, \quad \lambda \neq 0,$$

and $\langle\ ,\ \rangle_0$ is the canonical Euclidian scalar product on $\mathbb{R}^3$.

2. *$(\mathcal{H}_3, \xi_0, \langle\ ,\ \rangle_0)$ where $\mathcal{H}_3$ the Heisenberg Lie algebra $\left\{\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y, z, \in \mathbb{R}^3\right\}$,*

ξ_0 is given by

$$\xi_0(e_3) = 0, \quad \xi(e_1) = -\lambda e_3 \wedge e_2 \quad \textit{and} \quad \xi(e_2) = \lambda e_3 \wedge e_1, \quad \lambda \neq 0,$$

and $\langle\ ,\ \rangle_0$ is the scalar product on $\mathcal{H}_3$ whose matrix in (e_1, e_2, e_3) is given by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$, $a > 0$.

The infinitesimal situations in this Proposition can be integrated easily which leads to the following theorem.

Theorem 4.2. *Let $(G, \pi, \langle\ ,\ \rangle)$ be a connected and simply connected 3-dimensional Riemannian Poisson-Lie group. If $(\pi, \langle\ ,\ \rangle)$ satisfies Hawkins's conditions then $(G, \pi, \langle\ ,\ \rangle)$ is isomorphic to:*

1. *$(\mathbb{R}^3, \pi, \langle\ ,\ \rangle)$ where $\mathbb{R}^3$ is endowed with its abelian Lie group structure, $\langle\ ,\ \rangle$ is the canonical Euclidian metric and*
$$\pi = \lambda\partial_x \wedge (z\partial_y - y\partial_z),$$
where $\lambda \in \mathbb{R}$ or,
2. *$(H_3, \pi, \langle\ ,\ \rangle)$ where $H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z, \in \mathbb{R}^3 \right\}$ and*
$$\pi = \lambda(x\partial_y - y\partial_x) \wedge \partial_z,\ \langle\ ,\ \rangle = dx^2 + dy^2 + a(dz - xdy)^2,$$
where $\lambda \in \mathbb{R}$ and $a > 0$.

The 4-dimensional case. In this paragraph we will determine, up to isomorphism, all the 4-dimensional Riemannian Poisson-Lie groups satisfying Hawkins's conditions. According to Theorems 1.1-1.2 Proposition 3.2, the first step is to determine all the Lie bialgebra structures on 4-dimensional Milnor Lie algebras satisfying (1) and (2).

Let $\mathcal{H}$ be a 4-dimensional Milnor Lie algebra. By virtue of (23), there exists non zero real numbers λ_1, λ_2 and an orthonormal basis (s_1, s_2, f_1, f_2) of $\mathcal{H}$ such that
$$[s_1, s_2] = [f_1, f_2] = 0, \quad [s_i, f_1] = \lambda_i f_2 \quad \text{and} \quad [s_i, f_2] = -\lambda_i f_1.$$
Put $e_1 = \frac{\lambda_2 s_1 - \lambda_1 s_2}{\|\lambda_2 s_1 - \lambda_2 s_2\|}$. Then there exists $e_2 \in S$ such that (e_1, e_2, f_1, f_2) is an orthogonal basis,
$$[e_2, f_1] = f_2, \quad [e_2, f_2] = -f_1,$$
and all the other brackets vanish. Note that $\|e_1\| = \|f_1\| = \|f_2\| = 1$.

We are looking for the 1-cocycles $\rho : \mathcal{H} \longrightarrow \mathcal{H} \wedge \mathcal{H}$ defining a Lie bialgebra structure on $\mathcal{H}$ and satisfying (1) and (2). Put
$$\rho(e_i) = a_i e_1 \wedge e_2 + b_i e_1 \wedge f_1 + c_i e_1 \wedge f_2 + d_i e_2 \wedge f_1 + f_i e_2 \wedge f_2 + g_i f_1 \wedge f_2.$$
We have
$$\begin{array}{rcl} ad_{e_2}\rho(e_i) & = & b_i e_1 \wedge f_2 - c_i e_1 \wedge f_1 + d_i e_2 \wedge f_2 - f_i e_2 \wedge f_1, \\ ad_{e_2} \circ ad_{e_2}\rho(e_i) & = & -b_i e_1 \wedge f_1 - c_i e_1 \wedge f_2 - d_i e_2 \wedge f_1 - f_i e_2 \wedge f_2. \end{array}$$
Thus ρ satisfies (1) if and only if, for $i = 1, 2$,
$$\rho(e_i) = \alpha_i e_1 \wedge e_2 + \beta_i f_1 \wedge f_2.$$

Now, put

$$\rho(f_i) \quad = \quad a_i e_1 \wedge e_2 + b_i e_1 \wedge f_1 + c_i e_1 \wedge f_2 + d_i e_2 \wedge f_1 + g_i e_2 \wedge f_2 + h_i f_1 \wedge f_2,$$

and write down the cocycle condition $\rho([u,v]) = ad_u\rho(v) - ad_v\rho(u)$. First, we get

$$\rho([f_1, f_2]) \quad = \quad -a_2 e_1 \wedge f_2 - d_2 f_2 \wedge f_1 - a_1 e_1 \wedge f_1 - g_1 f_1 \wedge f_2 = 0,$$

thus

$$a_1 = a_2 = 0 \quad \text{and} \quad d_2 - g_1 = 0.$$

On the other hand,

$$\begin{array}{rcl} \rho([e_1, f_1]) & = & \alpha_1 e_1 \wedge f_2 = 0, \\ \rho([e_2, f_1]) & = & b_1 e_1 \wedge f_2 - c_1 e_1 \wedge f_1 + d_1 e_2 \wedge f_2 - g_1 e_2 \wedge f_1 + \alpha_2 e_1 \wedge f_2 \\ & = & \rho(f_2), \\ \rho([e_1, f_2]) & = & -\alpha_1 e_1 \wedge f_1 = 0, \\ \rho([e_2, f_2]) & = & b_2 e_1 \wedge f_2 - c_2 e_1 \wedge f_1 + d_2 e_2 \wedge f_2 - g_2 e_2 \wedge f_1 - \alpha_2 e_1 \wedge f_1 \\ & = & -\rho(f_1). \end{array}$$

These relations are equivalent to

$$b_2 = -c_1,\ c_2 = b_1,\ d_2 = -g_1,\ g_2 = d_1 = \alpha_i = h_i = 0.$$

Hence, ρ is a 1-cocycle satisfying (1) if and only if

$$\begin{array}{rcl} \rho(e_i) & = & \beta_i f_1 \wedge f_2, \\ \rho(f_1) & = & b e_1 \wedge f_1 + c e_1 \wedge f_2 + d e_2 \wedge f_1, \\ \rho(f_2) & = & -c e_1 \wedge f_1 + b e_1 \wedge f_2 + d e_2 \wedge f_2. \end{array} \qquad (28)$$

We consider now $\mathcal{H}^*$ endowed with the bracket associated to ρ, the dual scalar product and the dual of the bracket on $\mathcal{H}$, $\xi : \mathcal{H}^* \longrightarrow \mathcal{H}^* \wedge \mathcal{H}^*$, given by

$$\begin{array}{rcl} \xi(e_1^*) & = & \xi(e_2^*) = 0, \\ \xi(f_1^*) & = & -e_2^* \wedge f_2^*, \\ \xi(f_2^*) & = & e_2^* \wedge f_1^*, \end{array} \qquad (29)$$

where $(e_1^*, e_2^*, f_1^*, f_2^*)$ is the dual basis of (e_1, e_2, f_1, f_3). The bracket on $\mathcal{H}^*$ associated to ρ is given by

$$\begin{array}{lll} [e_1^*, e_2^*] = 0, & [e_1^*, f_1^*] = b f_1^* - c f_2^*, & [e_1^*, f_2^*] = c f_1^* + b f_2^*, \\ [e_2^*, f_1^*] = d f_1^*, & [e_2^*, f_2^*] = d f_2^*, & [f_1^*, f_2^*] = \beta_1 e_1^* + \beta_2 e_2^*. \end{array} \qquad (30)$$

Note that

$$\operatorname{tr} ad_{e_1^*} = 2b, \quad \operatorname{tr} ad_{e_2^*} = 2d, \quad \operatorname{tr} ad_{f_1^*} = \operatorname{tr} ad_{f_3^*} = 0. \qquad (31)$$

The Jacobi identities are given by:

$$\begin{array}{rcl} [[e_1^*, e_2^*], f_1^*] + [[e_2^*, f_1^*], e_1^*] + [[f_1^*, e_1^*], e_2^*] & = & 0, \\ [[e_1^*, e_2^*], f_2^*] + [[e_2^*, f_2^*], e_1^*] + [[f_2^*, e_1^*], e_2^*] & = & 0, \\ [[e_1^*, f_1^*], f_2^*] + [[f_1^*, f_2^*], e_1^*] + [[f_2^*, e_1^*], f_1^*] & = & 2b[f_1^*, f_2^*], \\ [[e_2^*, f_1^*], f_2^*] + [[f_1^*, f_2^*], e_2^*] + [[f_2^*, e_2^*], f_1^*] & = & 2d[f_1^*, f_2^*]. \end{array}$$

Let us write down (2). Since $\mu = e_1 \wedge e_2 \wedge f_1 \wedge f_2$ and by virtue of (29), a straightforward computation using (28) gives

$$\begin{aligned} \rho\left(i_{\xi(f_1^*)}\mu\right) &= de_1 \wedge e_2 \wedge f_1, \\ \rho\left(i_{\xi(f_2^*)}\mu\right) &= de_1 \wedge e_2 \wedge f_2. \end{aligned}$$

The following proposition summarize all the computation above.

Proposition 4.3. *Let $(G, \pi, \langle\ ,\ \rangle)$ be a 4-dimensional connected and simply connected Riemannian Poisson-Lie group and let $(\mathcal{G}, \xi, \langle\ ,\ \rangle_e)$ be its Lie algebra endowed with the cocycle ξ associated to π and the value of the Riemannian metric at the identity. If $(G, \pi, \langle\ ,\ \rangle)$ satisfies Hawkins's conditions then the triple $(\mathcal{G}, \xi, \langle\ ,\ \rangle_e)$ is isomorphic to $(\mathbb{R}^4, \xi_0, \langle\ ,\ \rangle_0)$ where:*

1. *in the canonical basis (e_0, e_1, e_2, e_3) of $\mathbb{R}^4$, the Lie bracket is given by*

$$\begin{aligned} &[e_1, e_2] = be_2 - ce_3, \qquad [e_1, e_3] = ce_2 + be_3, \quad [e_2, e_3] = \beta_1 e_0 + \beta_2 e_1, \\ &[e_0, e_i] = 0, \quad i = 1, 2, 3, \end{aligned}$$

   and either $b = 0$, $\beta_1 = 0$ or $\beta_2 = 0$.

2. *the cocycle ξ_0 is given, up to a multiplicative constant, by*

$$\xi_0(e_0) = \xi_0(e_1) = 0, \quad \xi_0(e_2) = e_0 \wedge e_3, \quad \xi_0(e_3) = -e_0 \wedge e_2,$$

3. *the product $\langle\ ,\ \rangle_0$ is the canonical Euclidian scalar product of $\mathbb{R}^4$.*

Remark 4.4. When $b = 0$, the Lie algebra structure of $\mathbb{R}^4$ given in Proposition 4.3 is unimodular and, according to Theorem 1.2, the converse of Proposition 4.3 is true, i.e., the triple $(G, \pi, \langle\ ,\ \rangle)$ integrating $(\mathbb{R}^4, \xi_0, \langle\ ,\ \rangle_0)$ satisfies Hawkins's conditions.

However, when $b \neq 0$, the triple $(G, \pi, \langle\ ,\ \rangle)$ integrating $(\mathbb{R}^4, \xi_0, \langle\ ,\ \rangle_0)$ is flat and metaflat and one must check if the last Hawkins's condition is satisfied. We will see that it does.

The task now is the construction of the triples $(G, \pi, \langle\ ,\ \rangle)$ associated to the different models isomorphic to the triple $(\mathbb{R}^4, \xi_0, \langle\ ,\ \rangle_0)$ given in Proposition 4.3. The computation is very long so we omit it. Note that the determination of the Lie groups is easy since all the models of Lie algebras are product or semi-direct product. The determination of the multiplicative Poisson tensor from the 1-cocycle is a direct calculation using the method exposed in [6] Theorem 5.1.3.

1. **Unimodular case** $b = 0$.

   (a) If $c = \beta_1 = \beta_2 = 0$ then $(G, \pi, \langle\ ,\ \rangle)$ is isomorphic to $(\mathbb{R}^4, \pi_0, \langle\ ,\ \rangle_0)$ where $\mathbb{R}^4$ is endowed with its abelian Lie group structure and

$$\pi_0 = \partial_x \wedge (z\partial_t - t\partial_z) \quad \text{and} \quad \langle\ ,\ \rangle_0 = dx^2 + dy^2 + dz^2 + dt^2.$$

(b) If $c = 0$ and $\beta_1 \neq 0$ then $(G, \pi, \langle\ ,\ \rangle)$ is isomorphic to $(H_0, \pi_0, \langle\ ,\ \rangle_0)$ where

$$H_0 = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1 & y & t \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, x > 0, y, z, t \in \mathbb{R} \right\},$$

$$\beta_1 \pi_0 = (\partial_t - \beta_2 x \partial_x) \wedge (y\partial_z - z\partial_y) + \frac{1}{2}\beta_2(z^2 - y^2)x\partial_x \wedge \partial_t,$$

and

$$\langle\ ,\ \rangle_0 = (x^{-1}dx + \beta_2 dt - \beta_2 y dz)^2 + dy^2 + dz^2 + \beta_1^2(dt - ydz)^2.$$

(c) If $c = 0$ $\beta_1 = 0$ and $\beta_2 \neq 0$ then $(G, \pi, \langle\ ,\ \rangle)$ is isomorphic to $(H_0, \pi_0, \langle\ ,\ \rangle_0)$ where

$$H_0 = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1 & y & t \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, x > 0, y, z, t \in \mathbb{R} \right\},$$

$$\pi_0 = x\partial_x \wedge (y\partial_z - z\partial_y) + \frac{1}{2}(y^2 - z^2)x\partial_x \wedge \partial_t,$$

and

$$\langle\ ,\ \rangle_0 = \frac{1}{x^2}dx^2 + dy^2 + dz^2 + \beta_2^2(dt - ydz)^2.$$

(d) If $c \neq 0$ and $(\beta_1, \beta_2) = (0, 0)$ then $(G, \pi, \langle\ ,\ \rangle)$ is isomorphic to $(\mathbb{R}^4, \pi_0, \langle\ ,\ \rangle_0)$ where $\mathbb{R}^4$ is endowed with the Lie group structure given by

$$u.v = (x + x', y + y', z + z' \cos y + t' \sin y, t - z' \sin y + t' \cos y)$$

when $u = (x, y, z, t)$ and $v = (x', y', z', t')$, and

$$\pi_0 = \partial_x \wedge (z\partial_t - t\partial_z) \quad \text{and} \quad \langle\ ,\ \rangle_0 = dx^2 + ady^2 + dz^2 + dt^2,$$

where $a > 0$.

(e) If $c \neq 0$, $\beta_2 = 0$ and $\beta_1 \neq 0$ then $(G, \pi, \langle\ ,\ \rangle)$ is isomorphic to $(\mathbb{R}^2 \times \mathbb{C}, \pi_0, \langle\ ,\ \rangle_0)$ where $\mathbb{R}^2 \times \mathbb{C}$ is endowed with the structure of oscillator group given by

$$(t, s, z).(t', s', z') = \left(t + t', s + s' + \frac{1}{2}\mathrm{Im}\left(\bar{z} exp(it) z'\right), z + exp(it)z' \right),$$

and

$$\pi_0 = \partial_s \wedge (x\partial_y - y\partial_x), \quad \langle\ ,\ \rangle_0 = adt^2 + bds^2 + ds(ydx - xdy) + \frac{1}{4}(ydx - xdy)^2,$$

where $a > 0$ and $b > 0$.

(f) If $c \neq 0$, $\beta_2 \neq 0$ then $(G, \pi, \langle \ , \ \rangle)$ is isomorphic to $(\mathbb{R} \times G_0, \pi_0, \langle \ , \ \rangle_0)$ where $\mathbb{R} \times G_0$ is the direct product of the abelian group $\mathbb{R}$ with G_0 where G_0 is either $SU(2)$ or $SL(2, \mathbb{R})$ and if $\{E_1, E_2, E_3\}$ is a the basis of the Lie algebra of G_0 satisfying

$$[E_1, E_2] = E_3, \ [E_3, E_1] = E_2 \quad \text{and} \quad [E_2, E_3] = \pm E_1$$

then

$$\pi = \partial_t \wedge (E_1^+ - E_1^-)$$

where E_1^+ (resp. E_1^+) is the left invariant (resp. right invariant) vector field associated to E_1. On the other hand, $\langle \ , \ \rangle_0$ is the left invariant Riemannian metric on $\mathbb{R} \times G_0$ whose value at the identity has the following matrix in the basis $\{E_0, E_1, E_2, E_3\}$

$$\begin{pmatrix} a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$

2. **the non unimodular case:** $b \neq 0$. In this case $(G, \pi, \langle \ , \ \rangle)$ is isomorphic to $(\mathbb{R}^4, \pi_0, \langle \ , \ \rangle_0)$ where $\mathbb{R}^4$ is endowed with the Lie group structure given by

$$uv = \left(x + x', y + y', z + e^{xb}(z' \cos(xc) + t' \sin(xc)), t + e^{xb}(-z' \sin(xc) + t' \cos(xc))\right).$$

when $u = (x, y, z, t)$ and $v = (x', y', z', t')$,

$$\pi_0 = \partial_y \wedge (z\partial_t - t\partial_z) \quad \text{and} \quad \langle \ , \ \rangle_0 = dx^2 + dy^2 + e^{-2bx}(dz^2 + dt^2).$$

The Riemannian volume is given by

$$\mu = e^{-2bx} dx \wedge dy \wedge dz \wedge dt,$$

and

$$i_\pi \mu = -e^{-2bx}(z dx \wedge dz + t dx \wedge dt).$$

Thus $di_\pi \mu = 0$, and the third Hawkins's condition is satisfied.

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