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Invariant Polynomials for Multiplicity Free Actions

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Abstract. This work concerns linear multiplicity free actions of the complex groups $G_{\mathbb{C}} = GL(n,\mathbb{C})$, $GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ and $GL(2n,\mathbb{C})$ on the vector spaces $V = Sym(n,\mathbb{C})$, $M_n(\mathbb{C})$ and $Skew(2n,\mathbb{C})$. We relate the canonical invariants in $\mathbb{C}[V \oplus V^*]$ to spherical functions for Riemannian symmetric pairs (G,K) where $G = GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$ or $GL(n,\mathbb{H})$ respectively. These in turn can be expressed using three families of classical zonal polynomials. We use this fact to derive a combinatorial algorithm for the generalized binomial coefficients in each case. Many of these results were obtained previously by Knop and Sahi using different methods.
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1. Introduction

Suppose that V is a finite dimensional complex vector space and that $G_{\mathbb{C}}$ is a reductive complex algebraic group acting linearly on V by some rational representation

$$G_{\mathbb{C}} \times V \to V, \quad (g,z) \mapsto g \cdot z.$$

One obtains an associated representation ρ of $G_{\mathbb{C}}$ in the algebra $\mathbb{C}[V]$ of holomorphic polynomials on V via

$$\rho(g)p(z) = (g \cdot p)(z) = p(g^{-1} \cdot z).$$

When ρ is a multiplicity free representation one says that the action $G_{\mathbb{C}} : V$ is *multiplicity free*. In this case we have a canonical decomposition

$$\mathbb{C}[V] = \bigoplus_{\lambda \in \Lambda} P_\lambda \tag{1.1}$$

of $\mathbb{C}[V]$ into $G_{\mathbb{C}}$-irreducible subspaces P_λ. Here Λ denotes a countable index set, appropriate to the example at hand. For background on multiplicity free actions we refer the reader to the survey articles [How95] and [BR04].

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When $G_{\mathbb{C}} : V$ is multiplicity free the product action $G_{\mathbb{C}} : (V \oplus V^*)$ is of particular interest. Here $G_{\mathbb{C}}$ acts on V^* by the contragredient of $G_{\mathbb{C}} : V$. One can construct a canonical basis for the space $\mathbb{C}[V \oplus V^*]^{G_{\mathbb{C}}}$ of $G_{\mathbb{C}}$-invariant polynomials on $V \oplus V^*$, with one basis element from each subspace $P_\lambda \otimes P_\lambda^*$. Equivalently, one can look at the action of the maximal compact subgroup of $G_{\mathbb{C}}$ on polynomials on the underlying real vector space.

There are no (non-constant) $G_{\mathbb{C}}$-invariant holomorphic polynomials on V. If U is the maximal compact subgroup of $G_{\mathbb{C}}$, there are, however, $U-$invariant polynomials on the real vector space $V_{\mathbb{R}}$. We can describe a canonical set of invariants as follows: Let $\{f_j \ : \ j = 1, \ldots d_\lambda\}$ be an orthonormal basis for P_λ with respect to some U-invariant inner product. The polynomial

$$p_\lambda(z) = \sum_{j=1}^{d_\lambda} f_j(z)\overline{f_j(z)} \tag{1.2}$$

is a non-zero U-invariant. In fact, $\{p_\lambda \ : \ \lambda \in \Lambda\}$ is a canonical basis for the space $\mathcal{P}(V_{\mathbb{R}})^U$ of U-invariant polynomials on $V_{\mathbb{R}}$. Equation 1.2 does not depend on the basis $\{f_j\}$ used. The p_λ's are called the *canonical invariants.* In some works, including [BR04], the p_λ's are normalized via division by $d_\lambda = dim(P_\lambda)$. For our purposes, however, it seems preferable to use the un-normalized p_λ's given by (1.2). One can identify $\mathbb{C}[V \oplus V^*]$ with $\mathcal{P}(V_{\mathbb{R}})$ in such a way that the canonical $G_{\mathbb{C}}$-invariants and U-invariants coincide.

It is often quite difficult to obtain explicit formulas for the canonical invariants in the context of specific examples. In this paper we examine three classical multiplicity free actions. In each case, V is a space of complex matrices. Letting $M_n(\mathbb{C})$ denote the set of all $n \times n$ complex matrices, these are the following:

(*i*) $G_{\mathbb{C}} = GL(n, \mathbb{C})$ acts on the space $V = Sym(n, \mathbb{C}) = \{z \in M_n(\mathbb{C}) \ : \ z^t = z\}$ of $n \times n$ symmetric matrices via

$$g \cdot z = g^{-t} z g^{-1}.$$

(Here g^{-t} is shorthand for $(g^{-1})^t$.) The associated representation in $\mathbb{C}[V]$ becomes

$$(g \cdot p)(z) = p(g^t z g).$$

(*ii*) $G_{\mathbb{C}} = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ acts on $V = M_n(\mathbb{C})$ and $\mathbb{C}[V]$ as

$$g \cdot z = g_1^{-t} z g_2^{-1}, \quad (g \cdot p)(z) = p(g_1^t z g_2)$$

for $g = (g_1, g_2)$.

(*iii*) $G_{\mathbb{C}} = GL(2n, \mathbb{C})$ acts on $V = Skew(2n, \mathbb{C}) = \{z \in M_{2n}(\mathbb{C}) \ : \ z^t = -z\}$, the space of $(2n) \times (2n)$ skew symmetric matrices, by the formula in (*i*).

For each of the actions (*i*)-(*iii*), the multiplicity free decomposition (1.1) can be described using highest weights and indexed by the partitions with at most n parts:

$$\Lambda = \{\lambda = (\lambda_1, \lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \ : \ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}. \tag{1.3}$$

These decompositions are well known (see [How89, How95, GW98, BR04]) and discussed below in Section 2.

Some further notation is required in order to state our main theorem. Let U denote the usual maximal compact subgroup in $G_{\mathbb{C}}$, namely

$$U = U(n) \ \text{ or } \ U(n)\times U(n) \ \text{ or } \ U(2n)$$

in cases (i),(ii),(iii) respectively. Using a U-invariant hermitian inner product on V one can identify V^* with $\overline{V}$ and $\mathbb{C}[V\oplus V^*]^{G_{\mathbb{C}}} = \mathbb{C}[V\oplus V^*]^U$ with $\mathcal{P}(V\oplus\overline{V})^U = \mathcal{P}(V_{\mathbb{R}})^U$, the complex-valued U-invariant polynomials on the underlying real space $V_{\mathbb{R}}$ for V. Moreover, for $t=(t_1,\ldots,t_n)\in\mathbb{R}^n$ let $x_t\in V$ denote the point

$$x_t = \begin{cases} diag(t_1,\ldots,t_n) & \text{in cases } (i) \text{ and } (ii) \\ diag(t_1J,\ldots,t_nJ) & \text{in case } (iii) \end{cases} \tag{1.4}$$

where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ as usual. It is known that

$$\mathcal{X} = \{x_t \ : \ t_j \geq 0 \text{ for } 1\leq j\leq n\} \tag{1.5}$$

is a cross-section to the U-orbits in V. (See Section 7 in [BJLR97].) Thus our canonical invariants $p_\lambda\in\mathcal{P}(V_{\mathbb{R}})^U$ are completely determined by their values $p_\lambda(x_t)$ on $\mathcal{X}$. We will see that $p_\lambda(x_t)$ is a symmetric function in $t=(t_1,\ldots,t_n)$.

We show that the p_λ's are related to the *zonal polynomials* Z_λ for certain Riemannian symmetric pairs (G,K), namely [Mac95, Mac87, Sta89]:

$$\Big(G = GL(n,\mathbb{F}), K = \{k\in G \ : \ k^*k = I\}\Big) \text{ with } \mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H}.$$

(Here $k^* = \overline{k}^t$ is the conjugate transpose.) The subgroup K is maximal compact in the real Lie group G. For our purposes, it is essential to observe that G is a non-compact real form for the complex group $G_{\mathbb{C}}$ in each case.

Theorem 1.1. For the multiplicity free actions $G_{\mathbb{C}} : V$ given by (i)-(iii) the canonical invariants $p_\lambda\in\mathcal{P}(V_{\mathbb{R}})^U$ are completely determined by their values $p_\lambda(x_t)$ on the cross section $\mathcal{X}$. The maps $t\mapsto p_\lambda(x_t)$ are symmetric functions in t satisfying

$$p_\lambda(x_t) = c_\lambda Z_\lambda(t^2).$$

Here Z_λ is a zonal polynomial for the pair (G,K) in each case, $t^2 = (t_1^2,...,t_n^2)$ and c_λ is a non-zero constant given by Equation 1.6 below.

Our approach is strongly influenced by [GR87], which treats these three cases in parallel. A version of Theorem 1.1 was first proved by Knop and Sahi in [KS96]. They showed that, up to a multiplicative factor, $p_\lambda(x_t)$ is is given by $J_\lambda(t^2;\alpha)$, where $J_\lambda(\cdot;\alpha)$ is a Jack polynomial with parameter $\alpha = 2,1,1/2$ in cases (i),(ii),(iii) respectively. We will provide a new proof and viewpoint on this result. A brief discussion of the methods used by Knop and Sahi appears at the end of this paper.

The literature contains assorted conventions regarding normalization for zonal and Jack polynomials. In this paper we adopt the convention from [Sta89] and [Mac87] for normalization of the Jack polynomial $J_\lambda(\cdot;\alpha)$. (The convention just fixes the coefficient for a specified term in J_λ. It is equivalent to Lemma 3.2 below.) The zonal polynomials in Theorem 1.1 are then $Z_\lambda = J(\cdot;\alpha)$ with parameter $\alpha = 2, 1, 1/2$ for actions (i)-(iii). For action (i) this normalization agrees with that from [Jam61]. Section 2 includes a brief summary of the background material we require concerning zonal polynomials. We refer the reader to [Mac95, Mac87] and [GR87] for further details.

In Section 3 we prove that

$$c_\lambda = \begin{cases} 2^{|\lambda|}/(H_*(\lambda;2)H^*(\lambda;2)) & \text{for action } (i) \\ 1/H(\lambda)^2 & \text{for action } (ii) \\ 1/(H_*(\lambda;1/2)H^*(\lambda;1/2)) & \text{for action } (iii) \end{cases}. \tag{1.6}$$

Here $|\lambda| = \lambda_1 + \cdots + \lambda_n$, $H(\lambda)$ is the product of hook lengths for λ, and $H_*(\lambda;\alpha)$, $H^*(\lambda;\alpha)$ are the products of the lower and upper hook lengths weighted by α. These factors are defined below in Section 3 following [Sta89].

For action (ii), our normalization implies $Z_\lambda = H(\lambda)s_\lambda$ where s_λ is a Schur polynomial. In this case Theorem 1.1 says

$$p_\lambda(x_t) = \frac{1}{H(\lambda)} s_\lambda(t^2),$$

a result that was also derived, up to the normalization factor, in [BR98] using different methods.

The canonical invariant $p_\mu \in \mathcal{P}(V_\mathbb{R})^U$ yields a $G_\mathbb{C}$-invariant polynomial coefficient differential operator $p_\mu(z,\partial) = \sum_{j=1}^{d_\mu} f_j(z)\overline{f_j}(\partial)$ on $\mathbb{C}[V]$. Schur's Lemma ensures that $p_\mu(z,\partial)$ acts by a scalar on each irreducible subspace P_λ in decomposition (1.1):

$$p_\mu(z,\partial)|_{P_\lambda} = \begin{bmatrix} \lambda \\ \mu \end{bmatrix} I_{P_\lambda}. \tag{1.7}$$

The eigenvalue $\left[\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix}\right]$ is called a *generalized binomial coefficient* for the action. As in the case of the canonical invariants, explicit formulas for the generalized binomial coefficients are difficult to obtain.

A more detailed definition of the operators $p_\lambda(z,\partial)$ appeared in [BR04] together with some interesting properties of the generalized binomial coefficients. These include a Pieri formula, originally due to Z. Yan [Yan92]. In Section 4 we reconcile Yan's Pieri formula with Stanley's Pieri formula for Jack polynomials [Sta89]. This results in a combinatorial algorithm, Theorem 4.2 below, to evaluate the generalized binomial coefficients for actions (i)-(iii) working from the Young's diagrams for λ and μ.

2. The canonical invariants as zonal polynomials

Throughout $G_\mathbb{C} : V$ will denote one of the multiplicity free actions (i)-(iii). Table 1 lists the groups that play a role in our story, along with the space V on which

$G_{\mathbb{C}}$ acts. The groups U and G are compact and non-compact real forms in $G_{\mathbb{C}}$. They are real Lie groups. The complexification $K_{\mathbb{C}}$ of K is a subgroup of the complex group $G_{\mathbb{C}}$.

	$G_{\mathbb{C}}$	U	G	K	$K_{\mathbb{C}}$	V
(i)	$GL(n,\mathbb{C})$	$U(n)$	$GL(n,\mathbb{R})$	$O(n,\mathbb{R})$	$O(n,\mathbb{C})$	$Sym(n,\mathbb{C})$
(ii)	$GL(n,\mathbb{C})\times GL(n,\mathbb{C})$	$U(n)\times U(n)$	$GL(n,\mathbb{C})$	$U(n)$	$GL(n,\mathbb{C})$	$M_n(\mathbb{C})$
(iii)	$GL(2n,\mathbb{C})$	$U(2n)$	$GL(n,\mathbb{H})$	$Sp(n)$	$Sp(n,\mathbb{C})$	$Skew(2n,\mathbb{C})$

Table 1:

In each case the inclusions $U \subset G_{\mathbb{C}}$ and $K \subset K_{\mathbb{C}}$ are clear. The inclusions $K \subset G \subset G_{\mathbb{C}}$ and $K_{\mathbb{C}} \subset G_{\mathbb{C}}$ are equally clear in case (i) but require some explanation in the remaining cases.

- Case (ii): Here $G = GL(n,\mathbb{C})$ is viewed as a real Lie group embedded diagonally in $G_{\mathbb{C}} = GL(n,\mathbb{C})\times GL(n,\mathbb{C})$. That is $G = \{(g,g) \ : \ g \in GL(n,\mathbb{C})\}$. The group $K_{\mathbb{C}} = GL(n,\mathbb{C})$ is then embedded in $G_{\mathbb{C}}$ via $K_{\mathbb{C}} = \{(g,g^{-t}) \ : \ g \in GL(n,\mathbb{C})\}$.
- Case (iii): The quaternions $\mathbb{H}$ are to be viewed as 2×2 complex matrices of the form
$$\begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix}.$$
The group $G = GL(n,\mathbb{H})$ embeds in $G_{\mathbb{C}} = GL(2n,\mathbb{C})$ as the subgroup of matrices consisting of 2×2 blocks of the above sort. It is a real Lie group whose elements are certain $(2n)\times(2n)$ complex matrices. The group $K_{\mathbb{C}} = Sp(n,\mathbb{C})$ embeds in $G_{\mathbb{C}} = GL(2n,\mathbb{C})$ as the subgroup preserving the bilinear form with matrix
$$\mathcal{J} = diag(\underbrace{J,\ldots,J}_{n}). \tag{2.1}$$
That is $Sp(n,\mathbb{C}) = \{g \in GL(2n,\mathbb{C}) \ : \ g^t\mathcal{J}g = \mathcal{J}\}$. The compact symplectic group $K = Sp(n)$ then embeds in G as $K = Sp(n,\mathbb{C})\cap U(2n)$.

Recall from (1.5) that the set $\mathcal{X} = \{x_t \ : \ t \in \mathbb{R}^n, t_j \geq 0\}$ is a cross-section to the U-orbits in V. This is not hard to prove for action (ii). For actions (i) and (iii) the assertion amounts to the fact that any symmetric (resp. skew symmetric) bilinear form over $\mathbb{C}$ can be diagonalized by a unitary transformation.

We let $z_\circ$ denote the point in $\mathcal{X}$ given by
$$z_\circ = \begin{cases} I & \text{in cases } (i) \text{ and } (ii) \\ \mathcal{J} & \text{in case } (iii). \end{cases} \tag{2.2}$$
This base point will play an important role throughout this section. Note that

$$K_{\mathbb{C}} \text{ is the stabilizer of } z_\circ \text{ under the action of } G_{\mathbb{C}} \text{ and} \tag{2.3}$$
$$\text{the } G_{\mathbb{C}}\text{-orbit through } z_\circ \text{ is a Zariski-open dense set in } V. \tag{2.4}$$

As regards (2.4), it suffices to observe that

$$G_{\mathbb{C}} \cdot z_\circ = \{z \in V \ : \ det(z) \neq 0\}.$$

In fact, given $z \in V$ with $det(z) \neq 0$ we know that for some $u \in U$ one has $u \cdot z = x_t \in \mathcal{X}$. As $det(x_t) \neq 0$ we must have $t_j > 0$ for $1 \leq j \leq n$. Now let

$$d = \begin{cases} diag(t_1^{-1/2}, \ldots, t_n^{-1/2}) & \text{in cases } (i), (iii) \\ \big(diag(t_1^{-1/2}, \ldots, t_n^{-1/2}), diag(t_1^{-1/2}, \ldots, t_n^{-1/2})\big) & \text{in case } (ii) \end{cases}$$

and set $g = du$. Then $g \cdot z = z_\circ$.

2.1. ***U*-invariant polynomials as symmetric functions.**

Let $p \in \mathcal{P}(V_{\mathbb{R}})^U$. The polynomial p is determined by its restriction to the cross section $\mathcal{X}$. We claim that

$$t \mapsto p(x_t)$$

is symmetric in $(t_1, \ldots, t_n)$. To prove this, it suffices to show that for any permutation $\sigma \in S_n$, one has $x_{\sigma(t)} = u \cdot x_t$ for some $u \in U$.

- Case (i): Let $u_\sigma \in U(n)$ be the permutation matrix for σ^{-1}. That is

  $$u_\sigma \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} z_{\sigma^{-1}(1)} \\ \vdots \\ z_{\sigma^{-1}(n)} \end{bmatrix}.$$

  Then $u = u_\sigma$ satisfies

  $$u \cdot x_t = u^{-t} x_t u^{-1} = u x_t u^t = x_{\sigma(t)}.$$

- Case (ii): Let u_σ be as in case (i). Then $u = (u_\sigma, u_\sigma) \in (U = U(n) \times U(n))$ has $u \cdot x_t = x_{\sigma(t)}$.

- Case (iii): Let $\tilde{\sigma} \in S_{2n}$ be the permutation for which

  $$\tilde{\sigma}(2j-1) = 2\sigma(j) - 1 \quad \text{and} \quad \tilde{\sigma}(2j) = 2\sigma(j)$$

  for $j = 1, \ldots, n$. I. e., apply σ simultaneously to the subsets $\{1, 3, \ldots, 2n-1\}$ and $\{2, 4, \ldots, 2n\}$ of odd and even indices. Let $u_{\tilde{\sigma}} \in (U = U(2n))$ be the permutation matrix for $\tilde{\sigma}^{-1}$. Then $u = u_{\tilde{\sigma}}$ satisfies $u \cdot x_t = x_{\sigma(t)}$.

Note that the elements $u \in U$, constructed above so that $u \cdot x_t = x_{\sigma(t)}$, in fact belong to the subgroup $K = U \cap G$ of U. That is, u belongs to $O(n, \mathbb{R})$, $U(n)$ or $Sp(n)$ in each cases (i)-(iii). This observation has the following consequence, which will be used below in our proof of Proposition 2.2.

Lemma 2.1. If $p \in \mathcal{P}(V_{\mathbb{R}})^K$ then $t \mapsto p(x_t)$ is a symmetric function.

2.2. Multiplicity free decompositions.

We recall the decomposition for $\mathbb{C}[V]$ under the action of $G_{\mathbb{C}}$. For details see [How89, How95, GW98, BR04]. As in (1.3), Λ is the set of partitions with at most n parts.

Let $A_n \cong (\mathbb{C}^{\times})^n$ denote the diagonal matrices in $GL(n, \mathbb{C})$ and B_n the set of $n \times n$ upper-triangular matrices. The standard maximal torus A and Borel subgroup B in $G_{\mathbb{C}}$ are then

$$A = \begin{cases} A_n & \text{in case } (i) \\ A_n \times A_n & \text{in case } (ii) \\ A_{2n} & \text{in case } (iii) \end{cases}, \quad B = \begin{cases} B_n & \text{in case } (i) \\ B_n \times B_n & \text{in case } (ii) \\ B_{2n} & \text{in case } (iii) \end{cases}. \tag{2.5}$$

We use torus A and Borel subgroup B when describing weights and their ordering for (finite dimensional) rational representations of $G_{\mathbb{C}}$. For actions (i)-(iii) all weights that occur in $\mathbb{C}[V]$ are non-negative and the highest weights that appear are precisely

$$2\lambda = \begin{cases} (2\lambda_1, \ldots, 2\lambda_n) & \text{in case } (i) \\ (\lambda; \lambda) & \text{in case } (ii) \\ (\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n) & \text{in case } (iii) \end{cases} \tag{2.6}$$

for each $\lambda \in \Lambda$. That is, we have

$$\mathbb{C}[V] = \bigoplus_{\lambda \in \Lambda} P_\lambda \tag{2.7}$$

where $G_{\mathbb{C}}$ acts on P_λ by a copy of the representation with highest weight 2λ. Note that the subspaces P_λ are also irreducible for the action of U on $\mathbb{C}[V]$. A (2λ)-highest weight vector in P_λ is given by

$$\xi_\lambda = \xi_1^{\lambda_1 - \lambda_2} \xi_2^{\lambda_2 - \lambda_3} \cdots \xi_{n-1}^{\lambda_{n-1} - \lambda_n} \xi_n^{\lambda_n} \tag{2.8}$$

where $\xi_j \in \mathbb{C}[V]$ denotes

$$\xi_j(z) = \begin{cases} det \begin{bmatrix} z_{1,1} & \cdots & z_{1,j} \\ \vdots & & \vdots \\ z_{j,1} & \cdots & z_{j,j} \end{bmatrix} & \text{in cases } (i) \text{ and } (ii) \\ Pf \begin{bmatrix} z_{1,1} & \cdots & z_{1,2j} \\ \vdots & & \vdots \\ z_{2j,1} & \cdots & z_{2j,2j} \end{bmatrix} & \text{in case } (iii) \end{cases}.$$

Here $Pf(z)$ is the Pfaffian of the complex skew-symmetric matrix z.

Note that the action of $G_{\mathbb{C}}$ on $\mathbb{C}[V]$ preserves the subspace $\mathcal{P}_m(V)$ of polynomials homogeneous of degree m. As ξ_λ is homogeneous of degree $|\lambda| = \lambda_1 + \cdots + \lambda_n$, it follows that $P_\lambda \subset \mathcal{P}_{|\lambda|}(V)$.

Next consider the space $\mathbb{C}[G_{\mathbb{C}}]$ of holomorphic polynomials on the complex group $G_{\mathbb{C}}$. (These are the restrictions to $G_{\mathbb{C}}$ of polynomials on the complex vector spaces $M_n(\mathbb{C})$, $M_n(\mathbb{C}) \times M_n(\mathbb{C})$ or $M_{2n}(\mathbb{C})$ in cases (i)-(iii) respectively.) We let

$$\mathbb{C}[K_{\mathbb{C}} \backslash G_{\mathbb{C}}] = \{p \in \mathbb{C}[G] \ : \ p(kg) = p(g) \text{ for all } k \in K_{\mathbb{C}}\}$$

denote the subspace of left $K_{\mathbb{C}}$-invariant polynomials. Using (2.3) and (2.4), one sees that $\mathbb{C}[V]$ and $\mathbb{C}[K_{\mathbb{C}}\backslash G_{\mathbb{C}}]$ are isomorphic as algebras via

$$\mathbb{C}[V] \to \mathbb{C}[K_{\mathbb{C}}\backslash G_{\mathbb{C}}], \quad p \mapsto \widetilde{p}$$

where

$$\widetilde{p}(g) = p(g^{-1} \cdot z_\circ). \tag{2.9}$$

One checks easily that $p \mapsto \widetilde{p}$ intertwines the representation ρ of $G_{\mathbb{C}}$ on $\mathbb{C}[V]$ with its right-quasi-regular representation

$$r(g)p(h) = p(hg)$$

in $\mathbb{C}[K_{\mathbb{C}}\backslash G_{\mathbb{C}}]$. In view of (2.7) we conclude that $\mathbb{C}[K_{\mathbb{C}}\backslash G_{\mathbb{C}}]$ admits a multiplicity free decomposition

$$\mathbb{C}[K_{\mathbb{C}}\backslash G_{\mathbb{C}}] = \bigoplus_{\lambda\in\Lambda} \widetilde{P}_\lambda, \quad \widetilde{P}_\lambda = \{\widetilde{p} \mid p \in P_\lambda\} \tag{2.10}$$

under the right-quasi-regular representation. As in (2.7), the irreducible $G_{\mathbb{C}}$-module $\widetilde{P}_\lambda$ has highest weight 2λ.

We require one further fact concerning the representations that occur in $\mathbb{C}[V]$ and $\mathbb{C}[K_{\mathbb{C}}\backslash G_{\mathbb{C}}]$.

Proposition 2.2. Let $\sigma : G_{\mathbb{C}} \to GL(W)$ be an irreducible rational representation with highest weight 2λ. Then (σ, W) is $K_{\mathbb{C}}$-spherical. That is, the space of $K_{\mathbb{C}}$-fixed vectors in W is one-dimensional. Likewise, irreducible representations of the real form G with highest weight 2λ are K-spherical.

Proof. Our argument is adapted from the proof of Theorem 4.8 in [GR87]. The assertion in Proposition 2.2 does not depend on the model for the representation (σ, W) since all representations with a given highest weight are equivalent. We will use (ρ, P_λ).

One can obtain a $K_{\mathbb{C}}$-invariant f_λ in P_λ by averaging the highest weight vector (2.8) over K:

$$f_\lambda = \int_K k \cdot \xi_\lambda dk$$

where dk denotes normalized Haar measure on the compact group K. Using the inclusions $K \subset K_{\mathbb{C}} \subset G_{\mathbb{C}}$ described above, this gives

$$f_\lambda(z) = \int_K \xi_\lambda(kzk^t)dk \text{ in cases } (i), (iii) \text{ and } f_\lambda(z) = \int_K \xi_\lambda(kzk^{-1})dk \text{ in case } (ii)$$

where $K = O(n, \mathbb{R})$, $U(n)$ or $Sp(n)$. Evaluating f_λ at the point $z_\circ \in V$ given by (2.2), we see that $f_\lambda(z_\circ) = 1$. Indeed, K stabilizes $z_\circ$ in each case and $\xi_\lambda(z_\circ) = 1$. In particular, f_λ is a *non-zero* $K_{\mathbb{C}}$-invariant in P_λ.

Fix $\lambda \in \Lambda$ and let $m = |\lambda|$. Recall that we have a direct sum decomposition

$$\mathcal{P}_m(V) = \bigoplus_{|\mu|=m} P_\mu.$$

Thus the polynomials $\{f_\mu : |\mu| = m\}$ are necessarily linearly independent. By restriction to $\mathcal{X}$ we obtain a linearly independent set of polynomials $\{t \mapsto f_\mu(x_t) : |\mu| = m\}$ of degree m in $t = (t_1, \ldots, t_m)$. Lemma 2.1 shows that these are symmetric functions. But the space $\mathcal{P}_m(t_1, \ldots, t_n)^{S_n}$ of symmetric polynomials homogeneous of degree m in n variables has dimension $\#(\{\mu \in \Lambda : |\mu| = m\})$, the number of partitions of m with at most n parts. It follows that the space P_λ cannot contain any K-invariant elements linearly independent of f_λ. ■

2.3. Zonal Polynomials.

In [GR87], the pairs $(G = GL(n, \mathbb{F}), K)$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ are treated in parallel. In each case, the authors realize the zonal polynomials Z_λ $(\lambda \in \Lambda)$ as functions on the space

$$\mathcal{H} = Herm(n, \mathbb{F}) = \{x \in M_n(\mathbb{F}) : x^* = x\}$$

of $n \times n$ hermitian matrices over the real division algebra $\mathbb{F}$. (For $\mathbb{F} = \mathbb{R}$, we have $x^* = x^t$ and $\mathcal{H} = Sym(n, \mathbb{R})$.)

There are several identifications that enhance our point of view. The real group G acts on $\mathcal{H}$ and $\mathcal{P}(\mathcal{H})$ via

$$g \cdot x = g^{-*} x g^{-1}, \qquad (g \cdot p)(x) = p(g^{-1} \cdot x) = p(g^* x g). \tag{2.11}$$

We have a G-equivariant embedding of $K \backslash G$ into $\mathcal{H}$ via $\varphi : G \to \mathcal{H}$, $\varphi(g) = g^* g$ with image $\mathcal{H}_{PD}$, the cone of positive definite hermitian matrices. As $\mathcal{H}_{PD}$ is open in $\mathcal{H}$, the polynomials on $\mathcal{H}$ are determined by their restrictions to $\mathcal{H}_{PD}$. Thus we can identify the polynomial spaces $\mathcal{P}(K \backslash G) = {}^K\mathcal{P}(G)$ and $\mathcal{P}(\mathcal{H})$ as G-modules.

The space $\mathbb{C}[K_\mathbb{C} \backslash G_\mathbb{C}]$ can be identified with $\mathcal{P}(K \backslash G)$ via restriction of holomorphic polynomials on $G_\mathbb{C}$ to the real form G. Analytic continuation gives the inverse isomorphism. Thus (2.10) can also be viewed as a multiplicity free decomposition of $\mathcal{P}(K \backslash G)$ and $\mathcal{P}(\mathcal{H})$:

$$\mathcal{P}(\mathcal{H}) \cong \mathcal{P}(K \backslash G) = \bigoplus_{\lambda \in \Lambda} \widetilde{P}_\lambda.$$

Proposition 2.2 ensures that the subspace $\widetilde{P}_\lambda$ of $\mathcal{P}(\mathcal{H})$ contains a non-zero K-fixed element Z_λ, unique up to scalar multiples. This defines the zonal polynomial Z_λ, modulo normalization, as a function on $\mathcal{H}$. In fact, however, Z_λ is determined by its values on the (necessarily real) diagonal matrices in $\mathcal{H}$ and is symmetric as a function of the diagonal entries. (This follows from the fact that hermitian matrices over $\mathbb{F}$ are diagonalizable by matrices in K.) Thus, as we do in the statement of Theorem 1.1, the Z_λ's can be thought of as symmetric polynomials in n variables.

The zonal polynomials Z_λ also determine the spherical functions for the Gelfand pair (G, K), as well as its complexification $(G_\mathbb{C}, K_\mathbb{C})$ and compact dual (U, K). Indeed the isomorphism $\mathcal{P}(\mathcal{H}) \cong \mathcal{P}(K \backslash G)$ enables us to regard Z_λ as a function on G. We can be more explicit about this connection: Using the map $\varphi : G \to \mathcal{H}$ to lift Z_λ to G, we obtain the K-bi-invariant spherical function

$$\Omega_\lambda(g) = \frac{1}{Z_\lambda(I)} Z_\lambda(g^* g) \tag{2.12}$$

for (G, K). (By convention one normalizes Ω_λ to ensure $\Omega_\lambda(I) = 1$.)

2.4. $G_{\mathbb{C}}$-invariant Polynomials .

By the multiplicity-free condition, we have (up to multiples) a single $G_{\mathbb{C}}$-fixed (equivalently U-fixed) vector in each product $P_\lambda \otimes P_\lambda^*$. By the same reasoning, there is a single $G_{\mathbb{C}}$-fixed vector in each product $P_\lambda \otimes \mathbb{C}[V^*]$.

Let r_λ be any non-zero $G_{\mathbb{C}}$-fixed vector in $P_\lambda \otimes \mathbb{C}[V^*]$. Using $K_{\mathbb{C}}$-fixed base points $z_\circ \in V$ and $\xi_\circ \in V^*$ in Borel-open orbits, define

$$\widetilde{r}_\lambda(g) = r_\lambda(g^{-1} \cdot z_\circ, \xi_\circ)$$

for $g \in G_{\mathbb{C}}$. (At present we do not require specific base points. Later we will use (2.2) for $z_\circ$ and specify a choice for $\xi_\circ$.)

Lemma 2.3. The function $\widetilde{r}_\lambda$ is a $K_{\mathbb{C}}$-bi-invariant polynomial on $G_{\mathbb{C}}$.

Proof. Left invariance follows from the choice of the $K_{\mathbb{C}}$-invariant base point $z_\circ$. Recalling that r_λ is $G_{\mathbb{C}}$-invariant, we see that

$$\widetilde{r}_\lambda(gk) = r_\lambda(k^{-1}g^{-1} \cdot z_\circ, \xi_\circ) = r_\lambda(g^{-1} \cdot z_\circ, k \cdot \xi_\circ) = r_\lambda(g^{-1} \cdot z_\circ, \xi_\circ) = \widetilde{r}_\lambda(g). \qquad \blacksquare$$

As we saw in (2.10), the right quasi-regular representation of $G_{\mathbb{C}}$ on $\mathbb{C}[K_{\mathbb{C}} \backslash G_{\mathbb{C}}]$ has the multiplicity-free decomposition

$$\mathbb{C}[K_{\mathbb{C}} \backslash G_{\mathbb{C}}] = \bigoplus_{\lambda \in \Lambda} \widetilde{P}_\lambda, \tag{2.13}$$

where λ ranges over the set Λ described in (1.3) and $\widetilde{P}_\lambda$ is an irreducible subspace with highest weight 2λ as defined in (2.6).

Since $r_\lambda \in P_\lambda \otimes \mathbb{C}[V^*]$, we have $\widetilde{r}_\lambda(g) = \sum_{i=1}^d c_i f_i(g^{-1} \cdot z_\circ)$, where $\{f_1, ..., f_d\}$ is a basis for P_λ. The functions $g \mapsto f_i(g^{-1} \cdot z_\circ)$ span a subspace of $\mathbb{C}[G_{\mathbb{C}}]$ in which the right $G_{\mathbb{C}}$-action coincides with the $G_{\mathbb{C}}$-action on P_λ. Since the function $\widetilde{r}_\lambda$ is left $K_{\mathbb{C}}$-invariant, it can be regarded as an element of the irreducible subspace $\widetilde{P}_\lambda$ of $\mathbb{C}[K_{\mathbb{C}} \backslash G_{\mathbb{C}}]$.

We can also view (2.13), via restriction, as a G-decomposition for $\mathcal{P}(K \backslash G)$. Proposition 2.2 implies that $\widetilde{P}_\lambda$ contains a non-zero K-invariant element,

$$\Omega_\lambda \in \widetilde{P}_\lambda^K$$

which is the analytic continuation of the K-spherical function (2.12) on G.

Thus we have

$$\widetilde{r}_\lambda \sim \Omega_\lambda,$$

where the symbol "$\sim$" indicates proportionality.

2.5. Proof of Theorem 1.1 modulo normalization.

The canonical invariant $p_\lambda \in \mathcal{P}(V_{\mathbb{R}})^U$ is given by Equation 1.2. In this subsection we will prove that for $x_t \in \mathcal{X}$

$$p_\lambda(x_t) = c_\lambda Z_\lambda(t^2) \tag{2.14}$$

for some constant c_λ. Later in Section 3 we will compute c_λ to complete the proof of Theorem 1.1.

The complex vector space V carries a standard hermitian inner product

$$\langle z, w\rangle = tr(zw^*),$$

invariant under the action of U. For $g \in G_{\mathbb{C}}$, we have

$$\langle g \cdot z, w\rangle = \langle z, g^* \cdot w\rangle. \tag{2.15}$$

For $z \in V$, let $z^\# \in V^*$ be defined by

$$z^\#(w) = \langle w, \overline{z}\rangle. \tag{2.16}$$

Thus $z \mapsto z^\#$ is an isomorphism $V \to V^*$ of complex vector spaces. Using (2.16) we obtain

$$(g \cdot z)^\# = g^{-t} \cdot z^\# \tag{2.17}$$

because

$$(g \cdot z)^\#(w) = \langle w, \overline{g} \cdot \overline{z}\rangle = \langle \overline{g}^* \cdot w, \overline{z}\rangle = z^\#(g^t \cdot w) = (g^{-t} \cdot z^\#)(w).$$

We let $\xi_\circ \in V^*$ denote the base point given by

$$\xi_\circ = z_\circ^\#.$$

Note that $\xi_\circ$ is $K_{\mathbb{C}}$-invariant, in view of (2.3) and (2.17).

The space $\mathbb{C}[V]$ carries a hermitian inner product compatible with the inner product $\langle \cdot, \cdot \rangle$ on V. This is the Fock (or Fischer) inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ defined as

$$\langle f_1, f_2\rangle_{\mathcal{F}} = \frac{1}{\pi^d} \int f_1(z)\overline{f_2(z)} e^{-\langle z,z\rangle}\, dz d\overline{z} \tag{2.18}$$

where $d = dim_{\mathbb{C}}(V)$ (that is $n(n+1)/2$, n^2 or $n(2n-1)$ in cases (*i*)-(*iii*)) and $dzd\overline{z}$ is Lebesgue measure on $V_{\mathbb{R}}$ normalized using $\langle \cdot, \cdot \rangle$. The formula (2.18) shows that $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ is U-invariant.

Let $\{f_j \ : \ j = 1, \dots d_\lambda\}$ be an orthonormal basis for the irreducible subspace P_λ of $\mathbb{C}[V]$ and $\overline{f_j}$ denote the holomorphic polynomial on V obtained by conjugating the coefficients in f_j. The polynomial

$$(z, w^\#) \mapsto \sum f_j(z)\overline{f_j(w)}.$$

is a U-invariant element of $P_\lambda \otimes \mathbb{C}[V^*]$, and hence it is a multiple of the $G_{\mathbb{C}}$-invariant polynomial r_λ defined in Section 2.

Recall that A is the standard maximal torus in $G_{\mathbb{C}}$ given by (2.5). For $t \in (\mathbb{R}^+)^n$ let $a_t \in A$ denote the point

$$a_t = \begin{cases} diag(t_1, \dots, t_n) & \text{in case } (i) \\ \big(diag(t_1, \dots, t_n), diag(t_1, \dots, t_n)\big) & \text{in case } (ii) \\ diag(t_1, t_1, \dots, t_n, t_n) & \text{in case } (iii) \end{cases}.$$

In each case a_t lies in the real form G and one has

$$a_t^{-1} \cdot z_\circ = a_t z_\circ a_t = x_{t^2} \tag{2.19}$$

where x_t is given by (1.4) and $t^k = (t_1^k, \dots, t_n^k)$.

When we restrict the canonical invariant p_λ to "diagonal" elements, we find:

$$\begin{aligned} p_\lambda(x_t^2) &= \sum f_j(x_t^2)\overline{f_j(x_t^2)} \\ &\sim r_\lambda(x_t^2, (x_t^2)^\#) \\ &= r_\lambda(a_t^{-1} \cdot z_\circ, a_t \cdot \xi_\circ) \\ &= r_\lambda(a_t^{-2} \cdot z_\circ, \xi_\circ) \\ &= \widetilde{r}_\lambda(a_t^2) \\ &\sim \Omega_\lambda(a_t^2). \end{aligned}$$

Thus we see that

$$p_\lambda(x_t) \sim \Omega_\lambda(a_t).$$

Recall that the spherical function Ω_λ and zonal polynomial Z_λ are related by the equation $\Omega_\lambda(g) = Z_\lambda(g^*g)/Z_\lambda(I)$. As a_t belongs to G and $a_t^* a_t = diag(t_1^2, \dots, t_n^2)$ we can now write

$$p_\lambda(x_t) \sim Z_\lambda(t^2).$$

3. Normalization constants

For partitions $\mu \in \Lambda$, the *monomial symmetric function* m_μ is given by

$$m_\mu(s_1, \dots, s_n) = \sum_\alpha s_1^{\alpha_1} \cdots s_n^{\alpha_n}$$

where the sum is over all *distinct* permutations $(\alpha_1, \dots, \alpha_n)$ of $\mu = (\mu_1, \dots, \mu_n)$. The m_μ's form a basis for the symmetric polynomials in n variables. Hence both $p_\lambda(x_t)$ and $Z_\lambda(t^2)$ can be expressed uniquely as linear combinations of $\{m_\mu(t^2) \ : \ |\mu| = |\lambda|\}$. To evaluate the constant c_λ in Equation 2.14 we will compare the coefficient of $m_\lambda(t^2)$ in $p_\lambda(x_t)$ with its coefficient in $Z_\lambda(t^2)$.

3.1. Coefficient of $m_\lambda(t^2)$ in $p_\lambda(x_t)$.

Recall that ξ_λ denotes the highest weight vector in P_λ given by (2.8).

Lemma 3.1. The coefficient of $m_\lambda(t^2)$ in $p_\lambda(x_t)$ is $1/||\xi_\lambda||^2_{\mathcal{F}}$.

Proof. We seek the coefficient of $t_1^{2\lambda_1} \cdots t_n^{2\lambda_n}$ in $p_\lambda(x_t)$. Let $\{f_j \ : \ 1 \leq j \leq d_\lambda\}$ be an orthonormal basis for P_λ (with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$) consisting of weight vectors for the maximal torus A. We have say $a \cdot f_j = \chi_j(a) f_j$ for $a \in A$. Using equations (1.2) and (2.19) one has

$$p_\lambda(x_t) = \sum_j f_j(a_{\sqrt{t}}^{-1} \cdot z_\circ)\overline{f_j(a_{\sqrt{t}}^{-1} \cdot z_\circ)} = \sum_j |\chi_j(a_{\sqrt{t}})|^2 |f_j(z_\circ)|^2. \tag{3.1}$$

We can suppose that $f_1 = \xi_\lambda/||\xi_\lambda||_{\mathcal{F}}$. For each of actions (*i*)-(*iii*) we have $\chi_1(a_t) = t_1^{2\lambda_1}\cdots t_n^{2\lambda_n}$ since ξ_λ has weight 2λ, as in (2.6). Moreover $\xi_\lambda(z_\circ) = 1$ so that equation (3.1) becomes

$$p_\lambda(x_t) = \frac{1}{||\xi_\lambda||^2_{\mathcal{F}}} t_1^{2\lambda_1}\cdots t_n^{2\lambda_n} + \sum_{j\geq 2} |f_j(z_\circ)|^2 |\chi_j(a_{\sqrt{t}})|^2. \tag{3.2}$$

Each factor $|\chi_j(a_{\sqrt{t}})|^2$ in the preceding sum is a monomial in $(t_1,\ldots,t_n)$. To complete the proof we need only show that $|\chi_j(a_{\sqrt{t}})|^2 \neq t_1^{2\lambda_1}\cdots t_n^{2\lambda_n}$ for $j\geq 2$.

- Case (*i*). Let $\mu^j = (\mu_1^j,\ldots,\mu_n^j) \in \mathbb{Z}^n_{\geq 0}$ be the weight for f_j on $A \cong (\mathbb{C}^\times)^n$. Equation (3.2) now reads

$$p_\lambda(x_t) = \frac{1}{||\xi_\lambda||^2_{\mathcal{F}}} t_1^{2\lambda_1}\cdots t_n^{2\lambda_n} + \sum_{j\geq 2} |f_j(z_\circ)|^2 t_1^{\mu_1^j}\cdots t_n^{\mu_n^j}.$$

  Highest weight theory guarantees that the highest weight 2λ occurs with multiplicity one in the irreducible subspace P_λ. So none of the exponents μ^j $(j\geq 2)$ coincides with 2λ.

- Case (*ii*). Let $(\mu^j;\nu^j) \in \mathbb{Z}^n_{\geq 0} \times Z^n_{\geq 0}$ be the weight for f_j on $A \cong (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$. Now

$$p_\lambda(x_t) = \frac{1}{||\xi_\lambda||^2_{\mathcal{F}}} t_1^{2\lambda_1}\cdots t_n^{2\lambda_n} + \sum_{j\geq 2} |f_j(z_\circ)|^2 t_1^{\mu_1^j+\nu_1^j}\cdots t_n^{\mu_n^j+\nu_n^j}.$$

  Highest weight theory ensures that the highest weight $2\lambda = (\lambda;\lambda)$ *dominates* all of the weights (μ^j,ν^j). That is

$$\lambda_1+\cdots+\lambda_k \geq \mu_1^j+\cdots+\mu_k^j \quad \text{and} \quad \lambda_1+\cdots+\lambda_k \geq \nu_1^j+\cdots+\nu_k^j$$

  for $1\leq k\leq n$. Using these conditions, one sees that if $\mu_k^j+\nu_k^j = 2\lambda_k$ holds for each $k=1,\ldots,n$ then we must have $\mu^j = \lambda = \nu^j$. But this is not possible as the highest weight $(\lambda;\lambda)$ occurs with multiplicity one.

- Case (*iii*). Let $(\mu_1^j,\nu_1^j,\ldots,\mu_n^j,\nu_n^j) \in \mathbb{Z}^{2n}_{\geq 0}$ be the weight for f_j on $A \cong (\mathbb{C}^\times)^{2n}$. As in case (*ii*) we have

$$p_\lambda(x_t) = \frac{1}{||\xi_\lambda||^2_{\mathcal{F}}} t_1^{2\lambda_1}\cdots t_n^{2\lambda_n} + \sum_{j\geq 2} |f_j(z_\circ)|^2 t_1^{\mu_1^j+\nu_1^j}\cdots t_n^{\mu_n^j+\nu_n^j}$$

  and $2\lambda = (\lambda_1,\lambda_1,\ldots,\lambda_n,\lambda_n)$ dominates each $(\mu_1^j,\nu_1^j,\ldots,\mu_n^j,\nu_n^j)$. This means

$$\left\{ \begin{array}{c} 2(\lambda_1+\cdots+\lambda_{k-1})+\lambda_k \geq (\mu_1^j+\cdots+\mu_k^j)+(\nu_1^j+\cdots+\nu_{k-1}^j) \\ 2(\lambda_1+\cdots+\lambda_k) \geq (\mu_1^j+\cdots+\mu_k^j)+(\nu_1^j+\cdots+\nu_k^j) \end{array} \right\}$$

  for $k=1,\ldots,n$. As in case (*ii*) these conditions force $\mu^j = \lambda = \nu^j$ when $\mu_k^j+\nu_k^j = 2\lambda_k$ for each k. ∎

3.2. Coefficient of m_λ in Z_λ.

We now identify partitions λ with their Young's diagrams. Let λ' denote the conjugate partition, obtained by transposing the diagram for λ. Each box $s = (i, j)$ in λ has *hook length*

$$\ell(s) + a(s) + 1$$

where

$$\ell(s) = \lambda'_j - i, \quad a(s) = \lambda_i - j$$

are the *leg* and *arm lengths* for s. For fixed $\alpha > 0$, Stanley defines two weighted versions of the hook length in [Sta89]. The *lower hook length* and *upper hook length* for $s = (i, j)$ are

$$\ell(s) + \alpha a(s) + 1, \qquad \ell(s) + \alpha a(s) + \alpha.$$

$$\begin{array}{|c|c|c|c|}\hline 1 & \alpha & \cdots & \alpha \\ \hline 1 \\ \cline{1-1} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline \alpha & \alpha & \cdots & \alpha \\ \hline 1 \\ \cline{1-1} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

For ease of notation, we write $h_\lambda(s; \alpha) = \ell(s) + \alpha a(s)$. Let

$$H_*(\lambda; \alpha) = \prod_{s\in\lambda}(h_\lambda(s; \alpha) + 1), \quad H^*(\lambda; \alpha) = \prod_{s\in\lambda}(h_\lambda(s; \alpha) + \alpha) \tag{3.3}$$

denote the products of the lower and upper hook lengths for λ. Note that

$$H_*(\lambda; 1) = H^*(\lambda; 1) = H(\lambda) = \prod_{s\in\lambda}(\ell(s) + a(s) + 1),$$

the standard product of hook lengths.

For actions (*i*)-(*iii*) one has zonal polynomials $Z_\lambda = J_\lambda(\cdot\ ; \alpha)$ with $\alpha = 2, 1, 1/2$ respectively. As explained in Section 1, we follow the convention from [Sta89] and [Mac87] regarding normalization for these polynomials. This ensures:

Lemma 3.2. ([Sta89] Theorem 5.6) The coefficient of m_λ in $J_\lambda(\cdot\ ; \alpha)$ is $H_*(\lambda; \alpha)$.

3.3. A norm calculation.

To reconcile Lemma 3.1 with Lemma 3.2 we must compute $||\xi_\lambda||^2_{\mathcal{F}}$, the square of the norm for the highest weight vector ξ_λ in P_λ.

Suppose that we express polynomials $f \in \mathbb{C}[V]$ using coordinates w_j with respect to an orthonormal basis for V. It is well known that the Fock inner product (2.18) can be evaluated using the rule

$$\langle f_1, f_2\rangle_{\mathcal{F}} = \big(f_1(\partial)\overline{f}_2\big)(0),$$

where "$f(\partial)$" is the operator obtained by substituting $\partial_j = \partial/\partial w_j$ for w_j in the expression for f. (See Section 7.5 in [BR04] for a proof.)

For $z \in M_n(\mathbb{C})$ the (i,j)'th entry z_{ij} is a coordinate with respect to the orthonormal basis $\{E_{ij} \ : \ 1 \le i,j \le n\}$. Thus in case (ii) we have $\langle f_1, f_2\rangle_{\mathcal{F}} = (f_1(\partial)\overline{f}_2)(0)$ where ∂ denotes the $n\times n$ matrix $[\partial_{ij} = \partial/\partial z_{ij}]$. In case (i)

$$\left\{E_{ii}, \frac{1}{\sqrt{2}}(E_{ij}+E_{ji})\right\}$$

is an orthonormal basis for $V = Sym(n,\mathbb{C})$. The coordinate functions with respect to this basis are

$$w_{ii} = z_{ii} \ \text{ and } \ w_{ij} = \sqrt{2}z_{ij} \ \text{ for } i<j.$$

In this case, the inner product on $\mathbb{C}[V]$ can be expressed using z_{ij} variables as

$$\langle f_1, f_2\rangle_{\mathcal{F}} = f_1(\partial^S)\overline{f}_2(0)$$

where

$$\partial^S = \begin{pmatrix} \partial_{11} & \frac{1}{2}\partial_{12} & \cdots & \frac{1}{2}\partial_{1n} \\ \frac{1}{2}\partial_{21} & \partial_{22} & \cdots & \frac{1}{2}\partial_{2n} \\ \vdots & & \ddots & \vdots \\ \frac{1}{2}\partial_{n1} & \frac{1}{2}\partial_{n2} & \cdots & \partial_{nn} \end{pmatrix}$$

subject to the convention: $z_{ij} = z_{ji}$ and $\partial_{ij} = \partial_{ji}$. Similar considerations apply in case (iii) to yield

$$\langle f_1, f_2\rangle_{\mathcal{F}} = f_1(\partial^\wedge)\overline{f}_2(0)$$

where

$$\partial^\wedge = \begin{pmatrix} 0 & \frac{1}{2}\partial_{1,2} & \cdots & \frac{1}{2}\partial_{1,2n} \\ \frac{1}{2}\partial_{2,1} & 0 & \cdots & \frac{1}{2}\partial_{2,2n} \\ \vdots & & \ddots & \vdots \\ \frac{1}{2}\partial_{2n,1} & \frac{1}{2}\partial_{2n,2} & \cdots & 0 \end{pmatrix}$$

with $z_{ij} = -z_{ji}$ and $\partial_{ij} = -\partial_{ji}$.

Now let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with $\lambda_n > 0$. We have

$$||\xi_\lambda||^2_{\mathcal{F}} = \langle \xi_\lambda,\ \xi_1^{\lambda_1-\lambda_2}\cdots\xi_{n-1}^{\lambda_{n-1}-\lambda_n}\xi_n^{\lambda_n}\rangle_{\mathcal{F}} = \langle \xi_n(\widetilde{\partial})\xi_\lambda, \xi_{\lambda-(1^n)}\rangle_{\mathcal{F}} \tag{3.4}$$

where $(1^n) = (1,\ldots,1)$ and

$$\widetilde{\partial} = \partial^S,\ \partial \text{ or } \partial^\wedge \qquad \text{for actions } (i)\text{-}(iii) \text{ respectively.}$$

Lemma 3.3. $\xi_n(\widetilde{\partial})\xi_\lambda = a_\lambda \xi_{\lambda-(1^n)}$ where

$$a_\lambda = \begin{cases} (1/2^n)\prod_{i=1}^n(2\lambda_i+n-i) = (1/2^n)\prod_{s=(i,1)}(h_\lambda(s;2)+2) & \text{for action } (i) \\ \prod_{i=1}^n(\lambda_i+n-i) = \prod_{s=(i,1)}(h_\lambda(s;1)+1) & \text{for action } (ii) \\ \prod_{i=1}^n(\lambda_i/2+n-i) = \prod_{s=(i,1)}(h_\lambda(s;1/2)+1/2) & \text{for action } (iii) \end{cases}.$$

Proof. These are known equations. They can be derived by applying a Capelli identity in each case, as outlined below.

Case (i). The *symmetric Capelli identity* asserts that the $GL(n,\mathbb{C})$-invariant operator $\xi_n(z)\xi_n(\widetilde{\partial}) = det(z)det(\partial^S)$ on $\mathbb{C}[Sym(n,\mathbb{C})]$ can be rewritten as

$$det(z)det(\partial^S) = \frac{1}{2^n} det(\widetilde{E}_{ij} + \delta_{ij}(n-j)).$$

See [Tur47] and Section 11.2 in [HU91]. Here

$$\widetilde{E}_{ij} = \sum_k (z_{ik}\partial_{jk} + z_{ki}\partial_{kj})$$

gives the derived action of $E_{ij} \in gl(n,\mathbb{C})$ on $\mathbb{C}[V]$. To apply this expression correctly one must regard z_{ij}, z_{ji} as independent variables and expand $det(\widetilde{E}_{ij} + \delta_{ij}(n-j))$ in column order. One can argue that only the diagonal terms in this expansion can yield a non-zero result when applied to ξ_λ. So in fact

$$\xi_n(z)\xi_n(\widetilde{\partial})(\xi_\lambda) = \frac{1}{2^n}\prod_{i=1}^{n}(2z_{ii}\partial_{ii} + n - i)\xi_\lambda.$$

The result for action (i) now follows since

$$z_{ii}\partial_{ii}(\xi_\lambda) = (\lambda_i - \lambda_{i+1})\xi_\lambda + \cdots + (\lambda_{n-1} - \lambda_n)\xi_\lambda + \lambda_n\xi_\lambda = \lambda_i\xi_\lambda.$$

Case (ii). For action (ii) the value of a_λ is given by formula (11.1.15) in [HU91]. This is an application of the *classical* Capelli identity, analogous to the symmetric case discussed above.

Case (iii). A *skew* Capelli identity is given in general form in [HU91] and explicitly in [KW02]. The result is that $2^n\xi_n(z)\xi_n(\partial^\wedge) = Pf(z)Pz(\partial)$ acts on P_λ by the scalar

$$\prod_{i=1}^{n}(\lambda_i + 2(n-i)).$$

See (11.3.6) in [HU91] and (3.11) in [KW02]. Thus $\xi_n(z)\xi_n(\partial^\wedge)$ acts on P_λ by the scalar

$$a_\lambda = \frac{1}{2^n}\prod_{i=1}^{n}(\lambda_i + 2(n-i)) = \prod_{i=1}^{n}(\frac{\lambda_i}{2} + n - i). \qquad \blacksquare$$

Applying Lemma 3.3 to Equation 3.4 and proceeding inductively one obtains:

Lemma 3.4.

$$||\xi_\lambda||^2_{\mathcal{F}} = \begin{cases} H^*(\lambda;2)/2^{|\lambda|} & \text{for action } (i) \\ H(\lambda) & \text{for action } (ii) \\ H^*(\lambda;1/2) & \text{for action } (iii) \end{cases}.$$

3.4. The constant c_λ.

In view of Lemmas 3.1 and 3.2 we have $p_\lambda(x_t) = c_\lambda Z_\lambda(t^2)$ where

$$c_\lambda = \frac{1}{H_*(\lambda;\alpha)||\xi_\lambda||^2_{\mathcal{F}}}$$

with $\alpha = 2, 1, 1/2$ for actions (i)-(iii) respectively. Substituting the value for $||\xi_\lambda||^2_{\mathcal{F}}$ from Lemma 3.4 yields

$$c_\lambda = \begin{cases} 2^{|\lambda|}/(H_*(\lambda;2)H^*(\lambda;2)) & \text{for action } (i) \\ 1/H(\lambda)^2 & \text{for action } (ii) \\ 1/(H_*(\lambda;1/2)H^*(\lambda;1/2)) & \text{for action } (iii) \end{cases}.$$

This establishes (1.6), completing the proof of Theorem 1.1.

4. Generalized binomial coefficients

Let $G_{\mathbb{C}} : V$ be a multiplicity free action with associated decomposition $\mathbb{C}[V] = \oplus_{\lambda\in\Lambda} P_\lambda$ The generalized binomial coefficient $\begin{bmatrix}\lambda\\ \mu\end{bmatrix}$ gives the eigenvalue for the $G_{\mathbb{C}}$-invariant operator $p_\mu(z,\partial)$ on P_λ. As explained in [BR98, BR04], these values admit alternate interpretations and have a rich combinatorial theory. Key to this theory is a Pieri type formula due to Z. Yan [Yan92].

4.1. Yan's Pieri formula. Recall that U denotes a maximal compact subgroup in $G_{\mathbb{C}}$ and $\langle\cdot,\cdot\rangle$ is a U-invariant hermitian inner product on V. We regard our canonical invariants p_λ as living in $\mathcal{P}(V_{\mathbb{R}})^U$. Let $|\lambda|$ denote the degree of homogeneity for P_λ. For actions (i)-(iii) this coincides with the number of parts in partition λ. Clearly $\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = 0$ unless $|\mu| \leq |\lambda|$. As the polynomial

$$\gamma(z) = \frac{1}{2}\langle z, z\rangle$$

also belongs to $\mathcal{P}(V_{\mathbb{R}})^U$, the product $\gamma^k p_\mu$ can be expressed as a linear combination of invariants p_λ with $|\lambda| = |\mu| + k$. In fact

$$\frac{\gamma^k p_\mu}{k!} = \sum_{|\lambda|=|\mu|+k} \begin{bmatrix}\lambda\\ \mu\end{bmatrix} p_\lambda. \tag{4.1}$$

Proofs for (4.1) can be found in [Yan92] and [BR98, BR04].

Using Equation 4.1 iteratively, one obtains a contraction formula for the generalized binomial coefficients $\begin{bmatrix}\lambda\\ \mu\end{bmatrix}$ with $|\lambda| = |\mu| + k$. Namely

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = \frac{1}{k!}\sum \begin{bmatrix}\varepsilon_1\\ \mu\end{bmatrix}\begin{bmatrix}\varepsilon_2\\ \varepsilon_1\end{bmatrix}\cdots\begin{bmatrix}\varepsilon_{k-1}\\ \varepsilon_{k-2}\end{bmatrix}\begin{bmatrix}\lambda\\ \varepsilon_{k-1}\end{bmatrix} \tag{4.2}$$

where the sum is over all $(\varepsilon_1,\ldots,\varepsilon_{k-1})$ with $|\varepsilon_j| = |\mu| + j$.

4.2. Stanley's Pieri formula. Let J_λ denote the Jack polynomial $J_\lambda(\cdot;\alpha)$ with some fixed parameter $\alpha > 0$. In [Sta89], R. Stanley expresses the product $J_{(k)}J_\mu$ in terms of J_λ's with $|\lambda| = |\mu| + k$. For $k = 1$ the result is Equation 4.4 below.

Suppose that λ, μ are two diagrams which differ by a single box $s_\circ = (i_\circ, j_\circ)$. For any box $s = (i,j)$ in λ let

$$h^{\uparrow}_\mu(\lambda,s) = \begin{cases} h_\lambda(s;\alpha)+\alpha & \text{if } j = j_\circ \\ h_\lambda(s;\alpha)+1 & \text{if } j \neq j_\circ \end{cases}, \quad h^{\downarrow}_\mu(\lambda,s) = \begin{cases} h_\lambda(s;\alpha)+1 & \text{if } j = j_\circ \\ h_\lambda(s;\alpha)+\alpha & \text{if } j \neq j_\circ \end{cases}$$

For $h^{\uparrow}_{\mu}(\lambda, s)$, this means: Use the upper hook length for s in λ when s and $s_\circ$ lie in the same column, otherwise use the lower hook length. (We have momentarily suppressed the dependence on α.)

Now set

$$H^{\uparrow}_{\mu}(\lambda;\alpha) = \prod_{s\in\lambda} h^{\uparrow}_{\mu}(\lambda, s), \quad H^{\downarrow}_{\mu}(\lambda;\alpha) = \prod_{s\in\sigma} h^{\downarrow}_{\mu}(\lambda, s).$$

Note that $h^{\uparrow}_{\mu}(\lambda, s)h^{\downarrow}_{\mu}(\lambda, s) = (h_\lambda(s;\alpha)+\alpha)(h_\lambda(s;\alpha)+1)$ and hence

$$H^{\uparrow}_{\mu}(\lambda;\alpha)H^{\downarrow}_{\mu}(\lambda;\alpha) = H^*(\lambda;\alpha)H_*(\lambda;\alpha). \tag{4.3}$$

Stanley's Pieri formula with $k=1$ and fixed parameter $\alpha > 0$ now reads:

$$J_{(1)}J_\mu = \alpha \sum_{\substack{\lambda\supset\mu \\ |\lambda|=|\mu|+1}} \frac{H^{\uparrow}_{\lambda}(\mu;\alpha)}{H^{\uparrow}_{\mu}(\lambda;\alpha)} J_\lambda. \tag{4.4}$$

The polynomial $J_{(1)}$ in (4.4) is just

$$\begin{aligned} J_{(1)}(t_1,\ldots,t_n) &= m_{(1)}(t_1,\ldots,t_n) = t_1+\cdots+t_n \\ &= e_1(t_1,\ldots,t_n), \end{aligned}$$

the first elementary symmetric polynomial, independent of α.

4.3. The generalized binomial coefficients for actions (*i*)-(*iii*).

Now let $G_{\mathbb{C}} : V$ be one of actions (*i*)-(*iii*). On the cross section $\mathcal{X} = \{x_t \,:\, t \geq 0\}$ we have

$$\gamma(x_t) = \frac{\langle x_t, x_t\rangle}{2} = \frac{t_1^2+\cdots+t_n^2}{2} = \frac{1}{2}e_1(t^2).$$

Using Theorem 1.1 together with Pieri formulas (4.1) and (4.4) we can now write

$$\begin{aligned} \sum_{|\lambda|=|\mu|+1} \begin{bmatrix}\lambda \\ \mu\end{bmatrix} p_\lambda(x_t) &= \gamma(x_t)p_\mu(x_t) \\ &= \frac{1}{2}e_1(t^2)c_\mu Z_\mu(t^2) \\ &= \frac{\alpha}{2} \sum_{\substack{\lambda\supset\mu \\ |\lambda|=|\mu|+1}} c_\mu \frac{H^{\uparrow}_{\lambda}(\mu;\alpha)}{H^{\uparrow}_{\mu}(\lambda;\alpha)} Z_\lambda(t^2) \\ &= \frac{\alpha}{2} \sum_{\substack{\lambda\supset\mu \\ |\lambda|=|\mu|+1}} \frac{c_\mu}{c_\lambda}\frac{H^{\uparrow}_{\lambda}(\mu;\alpha)}{H^{\uparrow}_{\mu}(\lambda;\alpha)} p_\lambda(x_t), \end{aligned}$$

with $\alpha = 2, 1$ or $1/2$ for actions (*i*)-(*iii*) respectively. Thus

$$\begin{bmatrix}\lambda \\ \mu\end{bmatrix} = \frac{\alpha}{2}\frac{c_\mu}{c_\lambda}\frac{H^{\uparrow}_{\lambda}(\mu;\alpha)}{H^{\uparrow}_{\mu}(\lambda;\alpha)}. \tag{4.5}$$

Substituting the values for the normalization constants c_μ, c_λ from (1.6) into (4.5) and applying (4.3) yields the following.

Lemma 4.1. For $|\lambda| = |\mu| + 1$ one has $\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = 0$ unless $\lambda \supset \mu$ in which case

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = \begin{cases} \frac{1}{2} H^{\downarrow}_{\mu}(\lambda; 2)/H^{\downarrow}_{\lambda}(\mu; 2) & \text{for action } (i) \\ \frac{1}{2} H(\lambda)/H(\mu) & \text{for action } (ii)\text{, and} \\ \frac{1}{4} H^{\downarrow}_{\mu}(\lambda; 1/2)/H^{\downarrow}_{\lambda}(\mu; 1/2) & \text{for action } (iii). \end{cases}$$

Suppose that λ is obtained from μ by adding a single box $s_\circ = (i_\circ, j_\circ)$. For each box $s = (i, j)$ in μ one has $h^{\downarrow}_{\mu}(\lambda, s) = h^{\downarrow}_{\lambda}(\mu, s)$ unless s is in the same row or column as $s_\circ$. So the hook quotients in Lemma 4.1 are:

$$\frac{H^{\downarrow}_{\mu}(\lambda; \alpha)}{H^{\downarrow}_{\lambda}(\mu; \alpha)} = \prod_{j<j_\circ} \frac{h_\mu((i_\circ, j); \alpha) + \alpha}{h_\mu((i_\circ, j); \alpha)} \times \prod_{i<i_\circ} \frac{h_\mu((i, j_\circ); \alpha) + 1}{h_\mu((i, j_\circ); \alpha)}.$$

The generalized binomial coefficients $\begin{bmatrix}\lambda\\ \mu\end{bmatrix}$ with $|\lambda| \geq |\mu|$ are determined by Lemma 4.1 together with the contraction formula 4.2.

Theorem 4.2. For $|\lambda| = |\mu| + k$ one has $\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = 0$ unless $\lambda \supset \mu$. When $\lambda \supset \mu$ the generalized binomial coefficient is determined as follows.

- *Action (i)*:
$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = \frac{1}{2^k k!} \sum \prod_{j=1}^{k} \frac{H^{\downarrow}_{\lambda^{(j-1)}}(\lambda^{(j)}; 2)}{H^{\downarrow}_{\lambda^{(j)}}(\lambda^{(j-1)}; 2)}$$
where the sum is over all standard tableaux $\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda$ of shape $\lambda - \mu$.

- *Action (ii)*:
$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = \frac{\mathcal{C}_{\lambda,\mu}}{2^k k!} \frac{H(\lambda)}{H(\mu)}$$
where $\mathcal{C}_{\lambda,\mu}$ is the number of standard tableaux of shape $\lambda - \mu$.

- *Action (iii)*:
$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = \frac{1}{4^k k!} \sum \prod_{j=1}^{k} \frac{H^{\downarrow}_{\lambda^{(j-1)}}(\lambda^{(j)}; 1/2)}{H^{\downarrow}_{\lambda^{(j)}}(\lambda^{(j-1)}; 1/2)}$$
where the sum is over all standard tableaux $\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda$ of shape $\lambda - \mu$.

For action (*ii*) factors $H(\lambda^{(j)})$ cancel for $j = 1, \ldots, k-1$ using any tableau $\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda$. This accounts for the simpler formula for the generalized binomial coefficients in this case. Such cancellation does not occur with the weighted hook products $H^{\downarrow}_{\lambda^{(j-1)}}(\lambda^{(j)}; \alpha)$ that appear in connection with actions (*i*) and (*iii*).

4.4. An example. To illustrate Theorem 4.2 we will calculate

$$\begin{bmatrix}(4,2,2)\\ (3,1,1)\end{bmatrix}$$

for actions (i)-(iii) and any $n \geq 3$. The diagram $\lambda = (4,2,2)$ can be obtained from $\mu = (3,1,1)$ by adding three boxes in three different ways. These are indicated by the tableaux T_i listed in the first column of Table 2. Each corresponds to a sequence of diagrams

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \lambda^{(3)} = \lambda,$$

where $\lambda^{(i)}$ is obtained from $\lambda^{(i-1)}$ by addition of the box labelled i.

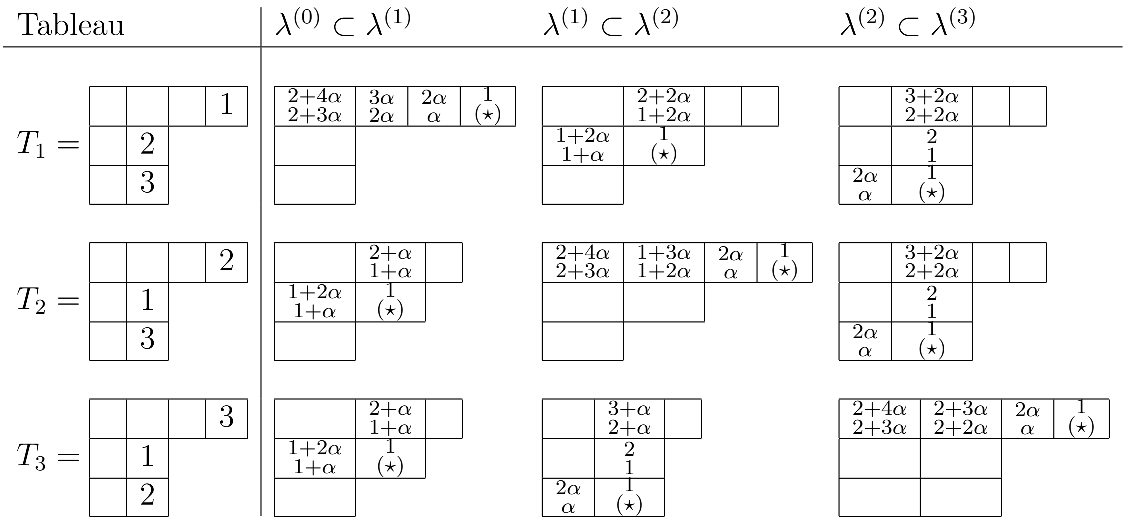

Table 2:

As $\mathcal{C}_{\lambda,\mu} = 3$ we obtain

$$\begin{bmatrix}(4,2,2)\\(3,1,1)\end{bmatrix} = \frac{3}{2^3 3!}\frac{H((4,2,2))}{H((3,1,1))} = \frac{3}{2^3 3!}\frac{6\cdot5\cdot2\cdot1\cdot3\cdot2\cdot2\cdot1}{5\cdot2\cdot1\cdot2\cdot1} = \frac{9}{4}$$

for action (ii). For actions (i) and (iii) Table 2 gives a weighted hook length for boxes in $\lambda^{(j)}$ and $\lambda^{(j-1)}$ that lie in the same row or column as the new box $(\star)$ added at each stage. These are lower hook lengths for boxes in the same column as $(\star)$ and upper hook lengths for the boxes in the same row. The products

$$\pi_i = \prod_{j=1}^{3} \frac{H^{\downarrow}_{\lambda^{(j-1)}}(\lambda^{(j)}, \alpha)}{H^{\downarrow}_{\lambda^{(j-1)}}(\lambda^{(j-1)}; \alpha)}$$

using tableau T_i are

$$\begin{aligned}
\pi_1 &= \frac{2+4\alpha}{2+3\alpha}\cdot\frac{3\alpha}{2\alpha}\cdot\frac{2\alpha}{\alpha} \times \frac{2+2\alpha}{1+2\alpha}\cdot\frac{1+2\alpha}{1+\alpha} \times \frac{3+2\alpha}{2+2\alpha}\cdot\frac{2}{1}\cdot\frac{2\alpha}{\alpha},\\
\pi_2 &= \frac{2+\alpha}{1+\alpha}\cdot\frac{1+2\alpha}{1+\alpha} \times \frac{2+4\alpha}{2+3\alpha}\cdot\frac{1+3\alpha}{1+2\alpha}\cdot\frac{2\alpha}{\alpha} \times \frac{3+2\alpha}{2+2\alpha}\cdot\frac{2}{1}\cdot\frac{2\alpha}{\alpha},\\
\pi_3 &= \frac{2+\alpha}{1+\alpha}\cdot\frac{1+2\alpha}{1+\alpha} \times \frac{3+\alpha}{2+\alpha}\cdot\frac{2}{1}\cdot\frac{2\alpha}{\alpha} \times \frac{2+4\alpha}{2+3\alpha}\cdot\frac{2+3\alpha}{2+2\alpha}\cdot\frac{2\alpha}{\alpha}.
\end{aligned}$$

For action (i) one sets $\alpha = 2$ and obtains

$$\begin{bmatrix}(4,2,2)\\(3,1,1)\end{bmatrix} = \frac{1}{2^3 3!}\big(\pi_1 + \pi_2 + \pi_3\big) = \frac{325}{144}.$$

For action (iii) we take $\alpha = 1/2$ to obtain

$$\begin{bmatrix}(4,2,2)\\(3,1,1)\end{bmatrix} = \frac{1}{4^3 3!}\big(\pi_1 + \pi_2 + \pi_3\big) = \frac{17}{63}.$$

5. Some related actions

In this section we extend our results for actions (ii) and (iii) to encompass two closely related actions:

$(ii)'$ $G_{\mathbb{C}} = GL(n,\mathbb{C}) \times GL(m,\mathbb{C})$ acts on the space $V = M_{n,m}(\mathbb{C})$ of $n \times m$ matrices via $g \cdot z = g_1^{-t} z g_2^{-1}$. Without loss of generality, we assume below that $n \le m$.

$(iii)'$ $G_{\mathbb{C}} = GL(2n+1,\mathbb{C})$ acts on $V = Skew(2n+1,\mathbb{C})$ via $g \cdot z = g^{-t} z g^{-1}$.

As is well known, actions $(ii)'$ and $(iii)'$ are multiplicity free and the decomposition of $\mathbb{C}[V]$ into $G_{\mathbb{C}}$-irreducible components parallels that for (ii) and (iii) [How89, How95, GW98, BR04]. That is, $\mathbb{C}[V] = \oplus_{\lambda \in \Lambda} P_\lambda$ where, as before,

- Λ is the set of partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ with at most n parts,
- the irreducible representation $G_{\mathbb{C}} : P_\lambda$ has highest weight

$$2\lambda = \begin{cases} (\lambda;\lambda) & \text{in case } (ii)' \\ (\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n) & \text{in case } (iii)' \end{cases}$$

- and ξ_λ, given by Equation 2.8, is a highest weight vector in P_λ.

The usual maximal compact subgroup of $G_{\mathbb{C}}$ is

$U = U(n) \times U(m)$ or $U(2n+1)$ in cases $(ii)'$ and $(iii)'$ respectively.

One obtains a canonical invariant $p_\lambda \in \mathcal{P}(V_{\mathbb{R}})^U \cong \mathbb{C}[V \oplus V^*]^{G_{\mathbb{C}}}$ for each $\lambda \in \Lambda$.

Let

$$V_\circ = \begin{cases} M_n(\mathbb{C}) & \text{in case } (ii)' \\ Skew(2n,\mathbb{C}) & \text{in case } (iii)', \end{cases}$$

$$G_{\mathbb{C}}^\circ = \begin{cases} GL(n,\mathbb{C}) \times GL(n,\mathbb{C}) \\ \text{or } GL(2n,\mathbb{C}), \end{cases} \qquad U_\circ = \begin{cases} U(n) \times U(n) \\ \text{or } U(2n), \end{cases}$$

and consider the obvious embeddings

$$V_\circ \subset V, \quad G_{\mathbb{C}}^\circ \subset G_{\mathbb{C}}, \quad U_\circ \subset U.$$

It is clear that each U-orbit in V meets $V_\circ$. Hence the invariant $p_\lambda \in \mathcal{P}(V_{\mathbb{R}})^U$ is determined by its restriction to $V_\circ$. Moreover $p_\lambda|_{V_\circ}$ belongs to $\mathcal{P}((V_\circ)_{\mathbb{R}})^{U_\circ}$ because $U_\circ \subset U$ preserves $V_\circ$.

Proposition 5.1. The restrictions $p_\lambda|_{V_\circ}$ of the canonical invariants for actions $(ii)'$ and $(iii)'$ to $V_\circ$ coincide with the canonical invariants for actions (ii) and (iii).

In view of Theorems 1.1 and 4.2, Proposition 5.1 has an immediate corollary:

Corollary 5.2. The canonical invariants $p_\lambda \in \mathcal{P}(V_\mathbb{R})^U$ for actions $(ii)'$ and $(iii)'$ are determined by their restrictions to the cross-section $\mathcal{X} \subset V_\circ$ given by (1.5). These are precisely the symmetric functions $p_\lambda(x_t) = c_\lambda Z_\lambda(t^2)$ from cases (ii) and (iii) in Theorem 1.1. Moreover the generalized binomial coefficients for actions $(ii)'$ and $(iii)'$ coincide with those for (ii) and (iii), obtained above in Theorem 4.2.

Proof. [Proof of Proposition 5.1] We write the decomposition for $\mathbb{C}[V_\circ]$ under the action of $G_\mathbb{C}^\circ$ as

$$\mathbb{C}[V_\circ] = \bigoplus_{\lambda\in\Lambda} P_\lambda^\circ,$$

let $\xi_\lambda^\circ \in P_\lambda^\circ$ denote highest weight vector (2.8) and $p_\lambda^\circ \in \mathcal{P}(V_\mathbb{R})^{U_\circ}$ the associated canonical invariant. Let

$$r : \mathbb{C}[V] \to \mathbb{C}[V_\circ]$$

be the restriction map. We will prove that

$$r(P_\lambda) = P_\lambda^\circ. \tag{5.1}$$

- *Case (ii)'*: By induction we can assume here that $m = n+1$. The ($G_\mathbb{C} = Gl(n,\mathbb{C}) \times GL(n+1,\mathbb{C})$)-irreducible space P_λ decomposes under the action of $G_\mathbb{C}^\circ = GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ as

  $$P_\lambda = \bigoplus_\nu P_{\lambda,\nu}, \quad \text{where}$$

  the sum is over all partitions $\nu = (\nu_1, \ldots, \nu_n)$ that interlace $(\lambda_1, \ldots, \lambda_n, 0)$:

  $$\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \nu_{n-1} \geq \lambda_n \geq \nu_n \geq 0$$

  and the irreducible representation $G_\mathbb{C}^\circ : P_{\lambda,\nu}$ has highest weight $(\lambda;\nu)$. This is an immediate consequence of the branching rules for $GL(n,\mathbb{C}) \subset GL(n+1,\mathbb{C})$ ([GW98], Theorem 8.1.1). As r is $G_\mathbb{C}^\circ$-equivariant,

  $$K_{\lambda,\nu} = Ker(r|_{P_{\lambda,\nu}})$$

  is $G_\mathbb{C}^\circ$-invariant. Thus $K_{\lambda,\nu} = \{0\}$ or $K_{\lambda,\nu} = P_{\lambda,\nu}$. If $K_{\lambda,\nu} = \{0\}$ then the highest weight $(\lambda;\nu)$ occurs in $\mathbb{C}[V_\circ]$ on the subspace $r(P_{\lambda,\nu})$. But we know that only highest weights of the form $2\lambda = (\lambda;\lambda)$ occur in $\mathbb{C}[V_\circ]$. So $r(P_{\lambda,\nu}) = \{0\}$ unless $\nu = \lambda$. On the other hand $\xi_\lambda \in P_{\lambda,\lambda}$ has $r(\xi_\lambda) = \xi_\lambda^\circ$. So $r|_{P_{\lambda,\lambda}}$ is injective and $r(P_{\lambda,\lambda})$ is a copy of the irreducible representation for $G_\mathbb{C}^\circ$ with highest weight 2λ. Thus

  $$r(P_\lambda) = r(P_{\lambda,\lambda}) = P_\lambda^\circ.$$

- *Case (iii)'*: The ($G_{\mathbb{C}} = GL(2n+1, \mathbb{C})$)-irreducible space P_λ has highest weight $(\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n, 0)$. This decomposes under the action of $G^\circ_{\mathbb{C}} = GL(n, \mathbb{C})$ as
$$P_\lambda = \bigoplus_\nu P_{\lambda,\nu}$$
where, as in case $(ii)'$, the sum is over all partitions $\nu = (\nu_1, \ldots, \nu_n)$ with
$$\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \nu_{n-1} \geq \lambda_n \geq \nu_n \geq 0.$$
Here $P_{\lambda,\nu}$ has highest weight $(\lambda_1, \nu_1, \cdots, \lambda_n, \nu_n)$. From this one argues, as in the preceding case, that $r(P_{\lambda,\nu}) = \{0\}$ for $\nu \neq \lambda$ and that $r(P_{\lambda,\lambda}) = P^\circ_\lambda$.

Since $p_\lambda \in (P_\lambda \otimes \overline{P}_\lambda)^U$, Equation 5.1 now implies

$$p_\lambda|_{V_\circ} \in \Big[(r(P_\lambda) \otimes \overline{r(P_\lambda)})^{U_\circ} = (P^\circ_\lambda \otimes \overline{P^\circ_\lambda})^{U_\circ} = \mathbb{C}p^\circ_\lambda\Big].$$

Thus $p_\lambda|_{V_\circ}$ is a scalar multiple of p°_λ. To show that, in fact, $p_\lambda|_{V_\circ} = p^\circ_\lambda$, we note that, as in Lemma 3.1, the coefficients of $m_\lambda(t^2)$ in $p_\lambda(x_t)$ and $p^\circ_\lambda(x_t)$ are $1/||\xi_\lambda||^2_{\mathcal{F}}$ and $1/||\xi^\circ_\lambda||^2_{\mathcal{F}}$ respectively. (Here $||\cdot||_{\mathcal{F}}$ denotes the Fock norm on both $\mathbb{C}[V]$ and $\mathbb{C}[V_\circ]$.) But $||\xi_\lambda||_{\mathcal{F}} = ||\xi^\circ_\lambda||_{\mathcal{F}}$. Indeed, the derivation given for Lemma 3.4 in cases (*ii*) and (*iii*) also encompasses actions $(ii)'$ and $(iii)'$. In particular, the Capelli and skew-Capelli identities from [HU91] and [KW02] apply in these cases, just as in the proof of Lemma 3.3. ∎

6. Related work

We conclude with some remarks concerning [KS96] and related work. For each fixed partition $\mu \in \Lambda$, Knop and Sahi show that the function

$$e_\mu : \Lambda \to \mathbb{C}, \quad e_\mu(\lambda) = \begin{bmatrix} \lambda \\ \mu \end{bmatrix}$$

extends in a natural way to a polynomial function on $\mathbb{C}^n$. This is non-homogeneous of degree $|\mu|$ and is uniquely determined by conditions of vanishing and shifted symmetry. Working from this characterization, it is shown that the e_μ's satisfy a system of *difference* equations. The canonical invariant p_μ can, moreover, be obtained from the terms of highest degree in e_μ. The difference equations for e_μ imply that p_μ is an eigenfunction for certain *differential* operators. These are the differential operators of Debiard [Deb83] and Sekiguchi [Sek77], which can be used to define Jack polynomials [Mac87]. Thus, up to normalization, p_μ is a Jack polynomial and e_μ a *shifted* Jack polynomial. Combinatorial properties of shifted Jack functions were subsequently developed by Okounkov and Olshanski in [OO97]. See also [OO98a, OO98b] for the important special case of shifted Shur functions, with relevance for action (*ii*). More recent work of Knop [Kno00, Kno01, Kno03] provides far reaching extensions of the techniques outlined above, encompassing many further examples of multiplicity free actions.

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