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LU-Decomposition of a Noncommutative Linear System and Jacobi Polynomials

Alfredo O. Brega and Leandro R. Cagliero*

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Abstract. In this paper we obtain the LU-decomposition of a non commutative linear system of equations that, in the rank one case, characterizes the image of the Lepowsky homomorphism $U(\mathfrak{g})^K \to U(\mathfrak{k})^M \otimes U(\mathfrak{a})$. Although this system can not be expressed as a single matrix equation with coefficients in $U(\mathfrak{k})$, it turns out that obtaining a triangular system equivalent to it, can be reduced to obtaining the LU-decomposition of a matrix $\widetilde{M_0}$ with entries in a polynomial algebra. We prove that both the L-part and U-part of $\widetilde{M_0}$ are expressed in terms of Jacobi polynomials. Moreover, each entry of the L-part of $\widetilde{M_0}$ and of its inverse is given by a single ultraspherical Jacobi polynomial. This fact yields a biorthogonality relation between the ultraspherical Jacobi polynomials.
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1. Introduction

The noncommutative linear system. Let k be a field of characteristic zero, let $\mathcal{A}$ be an associative, not necessarily commutative, k-algebra with unit and let $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule.

In this paper we perform a gaussian elimination process on a noncommutative homogeneous system of infinitely many linear equations and infinite unknowns of the following form,

$$\begin{array}{llllllll} Eb_0 & +Eb_1 & +Eb_2 & \dots & = & b_0E & -b_1E & +b_2E & \dots \\ E^2b_0 & +2E^2b_1 & +2^2E^2b_2 & \dots & = & b_0E^2 & -2b_1E^2 & +2^2b_2E^2 & \dots \\ E^3b_0 & +3E^3b_1 & +3^2E^3b_2 & \dots & = & b_0E^3 & -3b_1E^3 & +3^2b_2E^3 & \dots \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \end{array}, \tag{1.1}$$

where E is a given element of $\mathcal{A}$ and $b_0, b_1, b_2, \dots$ belong to $\mathcal{M}$ and are the unknowns.

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It is clear that $(b_0, b_1, \dots, b_d, 0, \dots)$ is a solution of the system (1.1) if and only if the polynomial $b = b_0 + b_1 t + b_2 t^2 + \cdots + b_d t^d$, which belongs to $\mathcal{M}[t] := \mathcal{M} \otimes \mathrm{k}[t]$, satisfies the equations

$$E^n b(n) = b(-n) E^n \quad \text{for all } n \in \mathbb{N}. \tag{1.2}$$

This system can not be expressed as a single matrix equation $AX = 0$ with A a matrix with coefficients in $\mathcal{A}$. In fact, noncommutative systems of (homogeneous) linear equations can seldom be expressed as a single matrix equation with coefficients in $\mathcal{A}$. Even when this is possible, only in exceptional cases a gaussian elimination process can be performed in a satisfactory way. Some papers dealing with this subject are [Co73], [CS98], [Or31] or [GGRW05].

On the other hand, by using the left and right regular actions $L, R : \mathcal{A} \to \mathrm{End}_{\mathrm{k}}(\mathcal{M})$ of $\mathcal{A}$ on $\mathcal{M}$, it is indeed possible to express every noncommutative system of linear equations as a single matrix equation but with coefficients in $\mathrm{End}_{\mathrm{k}}(\mathcal{M})$. In our case, the system (1.1) can be expressed as a single matrix equation $M_0 X = 0$, where $X = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix}$ and

$$M_0 = \begin{pmatrix} L_E - R_E & L_E + R_E & L_E - R_E & L_E + R_E & \dots \\ L_E^2 - R_E^2 & 2L_E^2 + 2R_E^2 & 4L_E^2 - 4R_E^2 & 8L_E^2 + 8R_E^2 & \dots \\ L_E^3 - R_E^3 & 3L_E^3 + 3R_E^2 & 9L_E^3 - 9R_E^2 & 27L_E^3 + 27R_E^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix has the special feature that its entries are homogeneous polynomials in the commuting variables L_E and R_E. This allows us to replace the matrix M_0 by the following matrix with entries in $\mathrm{k}[x]$,

$$\widetilde{M_0} = \begin{pmatrix} x-1 & x+1 & x-1 & x+1 & \dots \\ x^2-1 & 2x^2+2 & 4x^2-4 & 8x^2+8 & \dots \\ x^3-1 & 3x^3+3 & 9x^3-9 & 27x^3+27 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The main result of this paper is the LU-decomposition of the above matrix. It turns out that each entry of the L-factor is, up to a constant, a single ultraspherical Jacobi polynomial and each entry of the U-factor is also expressed in terms of a single Jacobi polynomial. Moreover, each entry of the inverse of the L-factor is also, up to a constant, a single ultraspherical Jacobi polynomial. This fact yields a biorthogonality relation between the ultraspherical polynomials that, up to our knoledge, is not known. It has recently appeared in the literature some other matrix identities (in particular LU-decompositions) involving Jacobi polynomials that translate sophisticated polynomials identities into very simple matrix identities (see for instance [KO07]).

Relevance of the system (1.1)**.** The interest on the system (1.1) comes from the fact that their solution set is closely related to invariant spaces under group actions. This is clear in the particular case in which E is the identity of $\mathcal{A}$, since the solutions of (1.1) are the even polynomials. It is not difficult to see that in other more general situations there exists a group G_E associated to E, acting on $\mathcal{M} \otimes_{\mathrm{k}} \mathrm{k}[t]$, such that

$$\{b \in \mathcal{M} \otimes_{\mathrm{k}} \mathrm{k}[t] : E^n b(n) = b(-n)E^n \text{ for all } n \in \mathbb{N}\} = \big(\mathcal{M} \otimes_{\mathrm{k}} \mathrm{k}[t]\big)^{G_E}.$$

A second and more relevant example of the relationship between the system (1.1) and invariants of groups appears in a problem from representation theory of Lie groups. Let G_o be a connected, noncompact, real semisimple Lie group with finite center, let $\mathfrak{g}$ be the complexification of the Lie algebra of G_o and let K_o denote a maximal compact subgroup of G_o. By the fundamental work of Harish-Chandra it is known that many deep questions concerning the infinite dimensional representation theory of G_o reduce to questions about the structure of the algebra $U(\mathfrak{g})^{K_o}$, the K_o-invariants in the universal enveloping algebra of $\mathfrak{g}$. In [Ti94], Tirao proved that the elements of the image of $U(\mathfrak{g})^{K_o}$ by the Lepowsky homomorphism (see [Le73]) satisfy a system of linear equations completely analogous to system (1.1). Moreover, for rank one classical Lie groups it is proved in [BCT08] that this image coincides with the solution set of the system introduced by Tirao. In order to prove this result we have strongly used the LU-decomposition obtained in this paper. These connections with groups invariants are not treated in this paper.

Main results. Let $P_n^{\alpha,\beta}$ denote the Jacobi polynomial of degree n associated to the numbers α and β. Usually α and β are complex numbers but, since we are working in the context of an arbitrary field of characteristic zero, they are assumed to be rational numbers. The polynomials corresponding to parameters $\alpha = \beta$, suitable normalized, are known as the *ultraspherical* or *Gegenbauer's* polynomials. Let

$$p_n^{\alpha,\beta}(x) = (x-1)^n P_n^{\alpha,\beta}\left(\tfrac{x+1}{x-1}\right).$$

It is clear that $p_n^{\alpha,\beta}(x)$ is again a polynomial and it can be expressed in terms of the hypergeometric function of Gauss (see (4.21.2) in [Sz59] or definition 2.5.1 in [AAR99]).

Our results in terms of $\widetilde{M}_0$ are the following:

(1) The LU-decomposition of $\widetilde{M}_0$ is

$$\widetilde{M}_0 = \widetilde{L}_0 \widetilde{U}_0 \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 2 &&& \\ &&& 6 && \\ &&&& 24 & \\ &&&&& \ddots \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & \dots \\ & 1 & -3 & 7 & -15 & \dots \\ && 1 & -6 & 25 & \dots \\ &&& 1 & -10 & \dots \\ &&&& 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.3}$$

where $\widetilde{L}_0$ and $\widetilde{U}_0$ are, respectively, the lower and upper triangular matrices given by

$$\begin{aligned} (\widetilde{L}_0)_{ij} &= (-1)^{i-j}\, p_{i-j}^{-i,-i}(x), && i \ge j \ge 1; \\ (\widetilde{U}_0)_{ij} &= (-1)^{i-j}\, \frac{j}{i}\, (x-1)^{2i-j}\, p_{j-i}^{-j,-1}(x), && 1 \le i \le j, \end{aligned}$$

and the explicit numerical matrices occuring in (1.3) are respectively formed by the factorial numbers and (up to the minus signs) the Stirling numbers of the second kind.

(2) The inverse of the matrix $\widetilde{L}_0$ is the lower triangular matrix given by

$$(\widetilde{L}_0^{-1})_{ij} = (-1)^{i-j}\,\frac{j}{i}\,p^{j,j}_{i-j}(x), \quad i \geq j \geq 1.$$

This yields the following "discrete orthogonality" relationship that involves once many of the ultraspherical Jacobi polynomials with integer parameters,

$$\begin{pmatrix} \frac{1}{1}P_0^{1,1} & & & & \cdots \\ \frac{1}{2}P_1^{1,1} & \frac{2}{2}P_0^{2,2} & & & \cdots \\ \frac{1}{3}P_2^{1,1} & \frac{2}{3}P_1^{2,2} & \frac{3}{3}P_0^{3,3} & & \cdots \\ \frac{1}{4}P_3^{1,1} & \frac{2}{4}P_2^{2,2} & \frac{3}{4}P_1^{3,3} & \frac{1}{1}P_0^{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_0^{-1,-1} & & & & \cdots \\ P_1^{-2,-2} & P_0^{-2,-2} & & & \cdots \\ P_2^{-3,-3} & P_1^{-3,-3} & P_0^{-3,-3} & & \cdots \\ P_3^{-4,-4} & P_2^{-4,-4} & P_1^{-4,-4} & P_0^{-4,-4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & & & \cdots \\ & 1 & & & \cdots \\ & & 1 & & \cdots \\ & & & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Observe that this identity holds either for the polynomials $P_n^{\alpha,\alpha}$ or $p_n^{\alpha,\alpha}$.

(3) We also obtain the following infinite factorization of the matrices $\widetilde{L}_0$ and $\widetilde{L}_0^{-1}$ which, in particular, involve the cyclotomic polynomials.

$$\widetilde{L}_0 = \begin{pmatrix} p_0^{-1,-1} & & & & \cdots \\ p_1^{-2,-2} & p_0^{-2,-2} & & & \cdots \\ p_2^{-3,-3} & p_1^{-3,-3} & p_0^{-3,-3} & & \cdots \\ p_3^{-4,-4} & p_2^{-4,-4} & p_1^{-4,-4} & p_0^{-4,-4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & \cdots \\ 1{+}x & 1 & & & \cdots \\ 1{+}x{+}x^2 & 1{+}x & 1 & & \cdots \\ 1{+}x{+}x^2{+}x^3 & 1{+}x{+}x^2 & 1{+}x & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & \cdots \\ & 1 & & & \cdots \\ & 1{+}x & 1 & & \cdots \\ & 1{+}x{+}x^2 & 1{+}x & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & \cdots \\ & 1 & & & \cdots \\ & & 1 & & \cdots \\ & & 1{+}x & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdots$$

and

$$\widetilde{L}_0^{-1} = \begin{pmatrix} \frac{1}{1}p_0^{1,1} & & & & \cdots \\ \frac{1}{2}p_1^{1,1} & \frac{2}{2}p_0^{2,2} & & & \cdots \\ \frac{1}{3}p_2^{1,1} & \frac{2}{3}p_1^{2,2} & \frac{3}{3}p_0^{3,3} & & \cdots \\ \frac{1}{4}p_3^{1,1} & \frac{2}{4}p_2^{2,2} & \frac{3}{4}p_1^{3,3} & \frac{1}{1}p_0^{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \cdots \begin{pmatrix} 1 & & & & \cdots \\ & 1 & & & \cdots \\ & & 1 & & \cdots \\ & & -x{-}1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & \cdots \\ & 1 & & & \cdots \\ & -x{-}1 & 1 & & \cdots \\ & x & -x{-}1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & \cdots \\ -x{-}1 & 1 & & & \cdots \\ x & -x{-}1 & 1 & & \cdots \\ & x & -x{-}1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In these two identities the polynomials involved are the $p_n^{\alpha,\alpha}$ instead of the $P_n^{\alpha,\alpha}$.

(4) We also consider the following more general system,

$$E^n b(H+n) = b(H-n)E^n \quad \text{for all } n \in \mathbb{N}, \tag{1.4}$$

where H is a given element in $\mathcal{A}$ and $b \in \mathcal{M}[t]$. In the last part of the paper we give the LU-decomposition of this system as it is used in [BCT08].

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2. The matrix representation of the system

Let k be a field of characteristic zero, let $\mathcal{A}$ be an associative, not necessarily commutative, k-algebra with unit and let $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. Given an element $r \in \mathcal{A}$, let $L_r, R_r \in \mathrm{End}_{\mathrm{k}}(\mathcal{M})$ denote, respectively, the left and right actions by r, and let $\mathrm{ad}_r = L_r - R_r$ be the adjoint action of r in $\mathcal{M}$.

Recall that $\mathcal{M}[t] = \mathcal{M} \otimes_{\mathrm{k}} \mathrm{k}[t]$, and that given an element $r \in \mathcal{A}$ one has the evaluation map $\mathcal{M}[t] \to \mathcal{M}$ defined using the right action of $\mathcal{A}$ on $\mathcal{M}$ by $a \otimes b(t) \mapsto ab(r)$. If $r \in \mathcal{A}$ and $b \in \mathcal{M}[t]$ then the evaluation of b in r is given by $b(r) = b_0 + b_1 r + \cdots + b_n r^n$.

Given two arbitrary elements E and H in $\mathcal{A}$ we are interested in the set of polynomials $b \in \mathcal{M}[t]$ that satisfy

$$E^n b(H+n) = b(H-n)E^n \quad \text{for all } n \in \mathbb{N}.$$

It is clear that $b = b_0 + b_1\, t + b_2\, t^2 + b_3\, t^3 + \dots$ satisfies the above system if and only if the vector $(b_0, b_1, b_2, \dots)$ is a solution of the following linear system,

$$\begin{array}{llll} Eb_0 & +Eb_1(H+1) & +Eb_2(H+1)^2 \dots = & b_0E + b_1(H-1)E + b_2(H-1)^2E \dots \\ E^2b_0 & +E^2b_1(H+2) & +E^2b_2(H+2)^2 \dots = & b_0E^2 + b_1(H-2)E^2 + b_2(H-2)^2E^2 \dots \\ E^3b_0 & +E^3b_1(H+3) & +E^3b_2(H+3)^2 \dots = & b_0E^3 + b_1(H-3)E^3 + b_2(H-3)^2E^3 \dots \\ \vdots & \vdots & \vdots & = \vdots \quad \vdots \quad \vdots \end{array}$$

or equivalently, if and only if $M\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix} = 0$, where

$$M = \begin{pmatrix} L_E & L_ER_{H+1} & L_ER^2_{H+1} & L_ER^3_{H+1} & \dots \\ L^2_E & L^2_ER_{H+2} & L^2_ER^2_{H+2} & L^2_ER^3_{H+2} & \dots \\ L^3_E & L^3_ER_{H+3} & L^3_ER^2_{H+3} & L^3_ER^3_{H+3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} R_E & R_ER_{H-1} & R_ER^2_{H-1} & R_ER^3_{H-1} & \dots \\ R^2_E & R^2_ER_{H-2} & R^2_ER^2_{H-2} & R^2_ER^3_{H-2} & \dots \\ R^3_E & R^3_ER_{H-3} & R^3_ER^2_{H-3} & R^3_ER^3_{H-3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is also clear that $M = M_0 P^t_{R_H}$ where

$$P_x = \left(\tbinom{i-1}{j-1} x^{i-j}\right) = \begin{pmatrix} 1 & & & & \dots \\ x & 1 & & & \dots \\ x^2 & 2x & 1 & & \dots \\ x^3 & 3x^2 & 3x & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$M_0 = \begin{pmatrix} L_E-R_E & L_E+R_E & L_E-R_E & L_E+R_E & \dots \\ L^2_E-R^2_E & 2L^2_E+2R^2_E & 4L^2_E-4R^2_E & 8L^2_E+8R^2_E & \dots \\ L^3_E-R^3_E & 3L^3_E+3R^3_E & 9L^3_E-9R^3_E & 27L^3_E+27R^3_E & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

3. The LU-decomposition

In this section we obtain the LU-decompositionof M_0.

Infinite matrices. We begin this section by recalling some general considerations on infinite matrices. Given an associative k-algebra $\mathcal{A}$, let $\mathcal{M}_\infty(\mathcal{A})$ denote the set of all infinite matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $a_{ij} \in \mathcal{A}$. The *n-minor* of a matrix $A \in \mathcal{M}_\infty(\mathcal{A})$ is the matrix corresponding to the upper-left corner of A of size $n \times n$. We say that a sequence of matrices B_k *converges* to B if for every i, j there exists k_0 such that $(B_k)_{ij} = B_{ij}$ for all $k \geq k_0$. A matrix $A \in \mathcal{M}_\infty(\mathcal{A})$ is said to be *row-finite* (resp. *column-finite*) if every row (resp. column) of A contains only a finite number of nonzero elements. Lower triangular and upper triangular matrices are, respectively, examples of row-finite and column-finite matrices.

It is clear that $\mathcal{M}_\infty(\mathcal{A})$ is not a ring since the multiplication of two matrices does not always exist. Nevertheless, if $A, B \in \mathcal{M}_\infty(\mathcal{A})$ and either A is row-finite or B is column-finite then AB do exist. In this context it is not difficult to prove the following proposition.

Proposition 3.1. *A lower or upper triangular matrix $A \in \mathcal{M}_\infty(\mathcal{A})$ is invertible if and only if all n-minors of A are invertible. Also, an LU-factorization of a matrix $A \in \mathcal{M}_\infty(\mathcal{A})$ exists if and only if it exists for all n-minors of A. Moreover, in this case the LU-factorization of A is unique.*

We now introduce some infinite matrices that will be used frequently in what follows.

- The Vandermonde matrix: $V_{ij} = i^{j-1}$ for $i, j \geq 1$.
- The diagonal matrix formed by the powers of $q \in \mathcal{A}$: $(D_q)_{ij} = \delta_{ij}\, q^i$.
- The diagonal matrix formed by the factorial numbers: $F_{ij} = \delta_{ij}\, (i-1)!$.
- The lower triangular matrix formed by the Pascal numbers: $P_{ij} = \binom{i-1}{j-1}$.
- The lower triangular matrix formed by the Stirling numbers of the second kind: $S_{ij} = S(i,j)$ for $i \geq j \geq 1$, where $S(i,j) = \frac{1}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} (j-k)^i$.

These matrices are the following,

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 4 & 8 & \dots \\ 1 & 3 & 9 & 27 & \dots \\ 1 & 4 & 16 & 64 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad D_q = \begin{pmatrix} q & & & & \\ & q^2 & & & \\ & & q^3 & & \\ & & & q^4 & \\ & & & & \ddots \end{pmatrix}, \qquad F = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 6 & \\ & & & & \ddots \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & & & & \dots \\ 1 & 1 & & & \dots \\ 1 & 2 & 1 & & \dots \\ 1 & 3 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & & & & & \dots \\ 1 & 1 & & & & \dots \\ 1 & 3 & 1 & & & \dots \\ 1 & 7 & 6 & 1 & & \dots \\ 1 & 15 & 25 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Observe that $P^{-1} = D_{-1}PD_{-1}$ and that the LDU-factorization of the matrix V is $V = PFS^t$.

Another property that will be used about the matrix S is the following. Recall the Pochhammer symbol defined by

$$(t)_0 = 1 \quad \text{and} \quad (t)_j = t(t+1)\dots(t+j-1),\ j \geq 1.$$

Then the matrix $D_{-1}s(S)^tD_{-1}$ transforms the coordinates of a given $b \in \mathcal{M}[t]$ with respect to the canonical basis $\{t^j\}$ into the coordinates of b with respect to the Pochhammer basis $\{(t)_j\}$. More precisely, if

$$\begin{aligned} b(t) &= b_0 + b_1t + b_2t^2 + b_3t^3 + \dots \\ &= a_0(t)_0 + a_1(t)_1 + a_2(t)_2 + a_3(t)_3 + \dots, \end{aligned}$$

then the coefficients a_j and b_j are related by $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \end{pmatrix} = D_{-1}s(S)^tD_{-1}\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \cdot \end{pmatrix}$. In particular, from this relation it follows that

$$\sum_{k=1}^{j}(-t)^k(S^{-1})_{jk} = (-1)^j(t)_j. \tag{3.1}$$

A brief review about Jacobi and ultraspherical polynomials. The *Jacobi* polynomials $P_n^{\alpha,\beta}$ are defined for nonnegative integers n and arbitrary (rational) numbers α and β as follows (see Chapter IV in [Sz59]),

$$P_n^{\alpha,\beta}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(n+\alpha+\beta+1)_k(\alpha+k+1)_{n-k}\left(\frac{x-1}{2}\right)^k.$$

The Jacobi polynomials can also be represented as

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right),$$

where ${}_2F_1$ is the hypergeometric function of Gauss. If $\alpha = \beta$ the normalized Jacobi polynomials

$$\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)}\,P_n^{\alpha,\alpha}(x)$$

are called *ultraspherical* polynomials or *Gegenbauer*'s polynomials.

It is well known that

$$P_n^{\alpha,\beta}(x) = (-1)^n P_n^{\beta,\alpha}(-x). \tag{3.2}$$

In particular, the ultraspherical Jacobi polynomials are even or odd according as n is even or odd. The ultraspherical Jacobi polynomials also satisfy the following identities

$$(n+2\alpha)P_n^{\alpha,\alpha}(x) = 2(n+\alpha)P_n^{\alpha-1,\alpha-1}(x) + x(n+\alpha)P_{n-1}^{\alpha,\alpha}(x) \tag{3.3}$$
$$x(n+2\alpha)P_n^{\alpha,\alpha}(x) = 2(n+1)P_{n+1}^{\alpha-1,\alpha-1}(x) + (n+\alpha)P_{n-1}^{\alpha,\alpha}(x), \tag{3.4}$$

which can be deduced from (4.7.14) and (4.7.28) in [Sz59]. The difference between these two identities gives,

$$\begin{aligned}&(1-x)(n+2\alpha)P_n^{\alpha,\alpha}(x)\\ &= -2(n+1)P_{n+1}^{\alpha-1,\alpha-1}(x) + 2(n+\alpha)P_n^{\alpha-1,\alpha-1}(x) - (1-x)(n+\alpha)P_{n-1}^{\alpha,\alpha}(x).\end{aligned} \tag{3.5}$$

On the other hand, multiplying by x equation (3.3) and subtracting from it equation (3.4) yields the following identity,

$$2(n+1)P_{n+1}^{\alpha-1,\alpha-1}(x) = 2x(n+\alpha)P_n^{\alpha-1,\alpha-1}(x) + (x^2-1)(n+\alpha)P_{n-1}^{\alpha,\alpha}(x). \tag{3.6}$$

Let us consider

$$p_n^{\alpha,\beta}(x) = (x-1)^n P_n^{\alpha,\beta}\left(\tfrac{x+1}{x-1}\right),$$

it is clear that $p_n^{\alpha,\beta}(x)$ is again a polynomial. These polynomials can be expressed in terms of the hypergeometric function of Gauss as follows (see (4.22.1) in [Sz59] and (2.3.14) in [AAR99]),

$$\begin{aligned}p_n^{\alpha,\beta}(x) &= \frac{(n+\alpha+\beta+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,-n-\alpha\\ -2n-\alpha-\beta\end{matrix};1-x\right)\\ &= \frac{(\beta+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,-n-\alpha\\ \beta+1\end{matrix};x\right).\end{aligned} \tag{3.7}$$

The LU-decomposition. The entries of M_0 are homogeneous polynomials in the variables L_E and R_E. Since an homogeneous polynomial $q(x_1,x_2)$ in two variables x_1 and x_2 is completely determined by the polynomial $\widetilde{q}(x) = q(x,1)$, we shall simplify the notation by substituting the variables

$$L_E \text{ by } x \qquad \text{and} \qquad R_E \text{ by } 1. \tag{3.8}$$

Under this transformations, the matrix M_0 becomes,

$$\widetilde{M_0} = \begin{pmatrix} x-1 & x+1 & x-1 & x+1 & \dots \\ x^2-1 & 2x^2+2 & 4x^2-4 & 8x^2+8 & \dots \\ x^3-1 & 3x^3+3 & 9x^3-9 & 27x^3+27 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In particular, observe that the entries of this matrix are given as follows,

$$(\widetilde{M}_0)_{ij} = i^{j-1}(x^i + (-1)^j), \quad i, j \geq 1. \tag{3.9}$$

Let us define now the lower triangular matrix $\widetilde{L}_0$ and the upper triangular matrix $\widetilde{U}_0$ as follows,

$$\begin{aligned} (\widetilde{L}_0)_{ij} &= (-1)^{i-j}\, p_{i-j}^{-i,-i}(x), & i \geq j \geq 1; \\ (\widetilde{U}_0)_{ij} &= (-1)^{i-j}\, \frac{j}{i}\, (x-1)^{2i-j}\, p_{j-i}^{-j,-1}(x), & 1 \leq i \leq j. \end{aligned}$$

In the following theorem we obtain the LU-decomposition of $\widetilde{M}_0$ as well as the inverse of $\widetilde{L}_0$.

Theorem 3.2. *The LU-decomposition of $\widetilde{M}_0$ is given by*

$$\widetilde{M}_0 = \widetilde{L}_0 \widetilde{U}_0 F D_{-1} S^t D_{-1}, \tag{3.10}$$

and the inverse of $\widetilde{L}_0$ is the following lower triangular matrix

$$(\widetilde{L}_0^{-1})_{ij} = (-1)^{i-j}\, \frac{j}{i}\, p_{i-j}^{j,j}(x), \quad i \geq j \geq 1. \tag{3.11}$$

Before proving this theorem we point out that it yields the following "discrete orthogonality" relationship that involves once many of the ultraspherical Jacobi polynomials with integer parameters,

$$\begin{pmatrix} \frac{1}{1}P_0^{1,1} & & & & \cdots \\ \frac{1}{2}P_1^{1,1} & \frac{2}{2}P_0^{2,2} & & & \cdots \\ \frac{1}{3}P_2^{1,1} & \frac{2}{3}P_1^{2,2} & \frac{3}{3}P_0^{3,3} & & \cdots \\ \frac{1}{4}P_3^{1,1} & \frac{2}{4}P_2^{2,2} & \frac{3}{4}P_1^{3,3} & \frac{1}{1}P_0^{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_0^{-1,-1} & & & & \cdots \\ P_1^{-2,-2} & P_0^{-2,-2} & & & \cdots \\ P_2^{-3,-3} & P_1^{-3,-3} & P_0^{-3,-3} & & \cdots \\ P_3^{-4,-4} & P_2^{-4,-4} & P_1^{-4,-4} & P_0^{-4,-4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & & & \cdots \\ & 1 & & & \cdots \\ & & 1 & & \cdots \\ & & & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Observe that this identity holds either for the polynomials $P_n^{\alpha,\alpha}$ or $p_n^{\alpha,\alpha}$.

Proof. [Proof of Theorem 3.2] We begin by proving (3.11). To do this we must prove that

$$\sum_{k=j}^{i} (-1)^{i-j}\, \frac{k}{i}\, p_{i-k}^{k,k}(x)\, p_{k-j}^{-k,-k}(x) = \delta_{i,j},$$

for $i \geq j \geq 1$. This amounts to proving that the following identity holds,

$$\sum_{k=0}^{n} \frac{k+j}{n+j}\, P_{n-k}^{k+j,k+j}(x)\, P_k^{-k-j,-k-j}(x) = \delta_{n,0}, \quad \text{for } n \geq 0 \text{ and } j \geq 1.$$

This is evident for $n = 0$. Hence, consider $n \geq 1$. If we use Rodrigues' formula (see (4.3.1) in [Sz59]) twice on the left hand side we obtain,

$$\begin{aligned}
&\frac{(-1)^n}{2^n n!(n+j)} \sum_{k=0}^{n} \binom{n}{k} (k+j) \left(\frac{d}{dx}\right)^{n-k} (1-x^2)^{n+j} \left(\frac{d}{dx}\right)^{k} (1-x^2)^{-j} \\
&= \frac{(-1)^n}{2^n n!(n+j)} \sum_{k=0}^{n} \binom{n}{k} k \left(\frac{d}{dx}\right)^{n-k} (1-x^2)^{n+j} \left(\frac{d}{dx}\right)^{k} (1-x^2)^{-j} \\
&\quad + \frac{(-1)^n j}{2^n n!(n+j)} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{dx}\right)^{n-k} (1-x^2)^{n+j} \left(\frac{d}{dx}\right)^{k} (1-x^2)^{-j} \\
&= \frac{(-1)^n\, 2jn}{2^n n!(n+j)} \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{d}{dx}\right)^{n-1-k} (1-x^2)^{n+j} \left(\frac{d}{dx}\right)^{k} (x(1-x^2)^{-j-1}) \\
&\quad + \frac{(-1)^n j}{2^n n!(n+j)} \left(\frac{d}{dx}\right)^{n} (1-x^2)^{n} \\
&= \frac{(-1)^n\, 2jn}{2^n n!(n+j)} \left(\frac{d}{dx}\right)^{n-1} (x(1-x^2)^{n-1}) + \frac{(-1)^n j}{2^n n!(n+j)} \left(\frac{d}{dx}\right)^{n} (1-x^2)^{n} \\
&= 0,
\end{aligned}$$

which completes the proof of (3.11).

In order to prove (3.10) we consider the matrix $\widetilde{M_1} = \widetilde{M_0} D_{-1} (S^{-1})^t D_{-1} F^{-1}$. We claim that

$$(\widetilde{M_1})_{ij} = \frac{(i+1)_{j-1}}{(j-1)!} x^i - \frac{(-i+1)_{j-1}}{(j-1)!}.$$

Indeed,

$$\begin{aligned}
(\widetilde{M_1})_{ij} &= \left(\widetilde{M_0} D_{-1} (S^{-1})^t D_{-1} F^{-1}\right)_{ij} \\
&= \sum_{k=1}^{j} \frac{(-1)^{k+j}}{(j-1)!} (\widetilde{M_0})_{ik} (S^{-1})_{jk} \\
&= \sum_{k=1}^{j} \frac{(-1)^{k+j}}{(j-1)!} i^{k-1} (x^i + (-1)^k) (S^{-1})_{jk} \\
&= \frac{(-1)^{j+1} x^i}{(j-1)!} \sum_{k=1}^{j} (-i)^{k-1} (S^{-1})_{jk} + \frac{(-1)^j}{(j-1)!} \sum_{k=1}^{j} i^{k-1} (S^{-1})_{jk} \\
&= \frac{(i+1)_{j-1}}{(j-1)!} x^i - \frac{(-i+1)_{j-1}}{(j-1)!},
\end{aligned}$$

where the last equality follows from identity (3.1).

Now we must prove that $\widetilde{M_1} = \widetilde{L_0}\widetilde{U_0}$, or equivalently that

$$\widetilde{L_0}^{-1} \widetilde{M_1} = \widetilde{U_0}.$$

Recall that $(\widetilde{U_0})_{ij} = (-1)^{i-j}\, \frac{j}{i}\, (x-1)^{2i-j}\, p_{j-i}^{-j,-1}(x)$, for $i \leq j$. From (4.22.2) in [Sz59] we have

$$(\widetilde{U_0})_{ij} = \begin{cases} (-1)^{i-j-1}\, \frac{j}{j-i}\, (x-1)^{2i-j}\, x p_{j-i-1}^{-j,1}(x), & \text{if } i < j; \\ (x-1)^i, & \text{if } i = j; \end{cases}$$

and moreover, from (3.7) we obtain

$$(\widetilde{U}_0)_{ij} = \begin{cases} (-1)^{i+1} jx\ {}_2F_1\left({}^{-i+j+1,-i+1}_{\quad 2} ; x\right), & \text{if } i<j; \\ (x-1)^i, & \text{if } i=j. \end{cases}$$

It is easy to see that

$$(-1)^{i+1} ix\ {}_2F_1\left({}^{1,-i+1}_{\ \ 2} ; x\right) - (-1)^{i+1} = (x-1)^i.$$

Hence it is clear that the proof of (3.10) will be completed once we prove that, for all $i,j \geq 1$, the following two identities hold:

$$\sum_{k=1}^{i} (\widetilde{L}_0^{-1})_{ik} \frac{(k+1)_{j-1}}{(j-1)!} x^k = (-1)^{i+1} jx\ {}_2F_1\left({}^{-i+j+1,-i+1}_{\quad 2} ; x\right)$$

$$\sum_{k=1}^{i} (\widetilde{L}_0^{-1})_{ik} \frac{(-k+1)_{j-1}}{(j-1)!} = \begin{cases} (-1)^{i+1} jx\ {}_2F_1\left({}^{-i+j+1,-i+1}_{\quad 2} ; x\right), & \text{if } i>j; \\ (-1)^{i+1}, & \text{if } i=j; \\ 0, & \text{if } i<j. \end{cases}$$

We shall prove these identities by showing that the coefficients of x^{r+1} on both sides coincide for $r \geq -1$.

First of all we observe that the coefficient of x^{r+1} in the polynomial $(-1)^{i+1} jx\ {}_2F_1\left({}^{-i+j+1,-i+1}_{\quad 2} ; x\right)$ is zero for $r=-1$ and it is equal to

$$(-1)^{i+1} j \frac{(-i+j+1)_r(-i+1)_r}{(2)_r r!} = (-1)^{i+1} j \frac{(-i+j+1)_r(-i+1)_r}{(r+1)! r!}, \tag{3.12}$$

for $r \geq 0$. In particular, it is zero for $r \geq i$.

We now consider the first identity. Its left hand side is equal to

$$\begin{aligned} &\sum_{k=1}^{i} (\widetilde{L}_0^{-1})_{ik} \frac{(k+1)_{j-1}}{(j-1)!} x^k \\ &\qquad = \sum_{k=1}^{i} (-1)^{i-k}\ \frac{k}{i}\ p_{i-k}^{k,k}(x) \frac{(k+1)_{j-1}}{(j-1)!} x^k \\ &\qquad = \sum_{k=1}^{i} (-1)^{i-k}\ \frac{k}{i}\ \frac{(k+1)_{i-k}}{(i-k)!}\ {}_2F_1\left({}^{-i+k,-i}_{\ \ k+1} ; x\right) \frac{(k+1)_{j-1}}{(j-1)!} x^k \\ &\qquad = \sum_{k=1}^{i} \sum_{s=0}^{\infty} (-1)^{i-k}\ \frac{k}{i}\ \frac{(k+1)_{i-k}}{(i-k)!}\ \frac{(-i+k)_s(-i)_s}{(k+1)_s s!} \frac{(k+1)_{j-1}}{(j-1)!} x^{k+s}, \end{aligned}$$

hence the coefficient of x^{r+1} is

$$\sum_{k=1}^{\min(i,r+1)} (-1)^{i-k}\ \frac{k}{i}\ \frac{(k+1)_{i-k}}{(i-k)!}\ \frac{(-i+k)_{r+1-k}(-i)_{r+1-k}}{(k+1)_{r+1-k}(r+1-k)!} \frac{(k+1)_{j-1}}{(j-1)!},$$

which is zero if $r \geq i$. On the other hand, if $r < i$, this coefficient is

$$\begin{aligned}
&\frac{1}{i(j-1)!}\sum_{k=1}^{r+1}(-1)^{i-k}\,\frac{(k+1)_{i-k}}{(i-k)!}\,\frac{(-i+k)_{r+1-k}(-i)_{r+1-k}}{(k+1)_{r+1-k}(r+1-k)!}(k)_j\\
&=\frac{(i-1)!}{(j-1)!(r+1)!}\sum_{k=1}^{r+1}(-1)^{i-k}\,\frac{(-i+k)_{r+1-k}(-i)_{r+1-k}}{(i-k)!(r+1-k)!}\frac{(r+1)_j(k)_{r+1-k}}{(k+j)_{r+1-k}}\\
&=\frac{(-1)^{r+1+i}(r+1)_j(i-1)!}{(j-1)!(r+1)!(i-r-1)!}\sum_{k=1}^{r+1}\frac{(-i)_{r+1-k}}{(r+1-k)!}\frac{(-r)_{r+1-k}}{(-r-j)_{r+1-k}}\\
&=\frac{(-1)^{r+1+i}(r+1)_j(i-1)!}{(j-1)!(r+1)!(i-r-1)!}\sum_{s=0}^{r}\frac{(-i)_s}{s!}\frac{(-r)_s}{(-r-j)_s}\\
&=\frac{(-1)^{r+1+i}(r+1)_j(i-1)!}{(j-1)!(r+1)!(i-r-1)!}{}_2F_1\left({-r,-i \atop -r-j};1\right)\\
&=\frac{(-1)^{r+1+i}(r+1)_j(i-1)!}{(j-1)!(r+1)!(i-r-1)!}\frac{(i-r-j)_r}{(-r-j)_r},
\end{aligned}$$

where the last equality follows from the Chu-Vandermonde identity (see Corollary 2.2.3 in [AAR99]). It is now straightforward to see that this number is equal to the coefficient of x^{r+1} on the right hand side, which is given in (3.12). This completes the proof of the first identity.

We now prove the second identity. Its left hand side is equal to

$$\begin{aligned}
&\sum_{k=1}^{i}(\widetilde{L}_0^{-1})_{ik}\frac{(-k+1)_{j-1}}{(j-1)!}\\
&\qquad=\sum_{k=1}^{i}\sum_{s=0}^{\infty}(-1)^{i-k}\,\frac{k}{i}\,\frac{(k+1)_{i-k}}{(i-k)!}\,\frac{(-i+k)_s(-i)_s}{(k+1)_s s!}\frac{(-k+1)_{j-1}}{(j-1)!}x^s.
\end{aligned}$$

Then the coefficient of x^0 in this sum is

$$\sum_{k=j}^{i}(-1)^{i-k}\,\frac{k}{i}\,\frac{(k+1)_{i-k}}{(i-k)!}\,\frac{(-k+1)_{j-1}}{(j-1)!}.$$

It is easy to see that this coefficient is zero except for $i = j$. In this last case it is equal to $(-1)^{i-1}$. For $r \geq 0$ the coefficient of x^{r+1} is

$$\frac{(-i+1)_r}{(r+1)!(j-1)!}\sum_{k=j}^{i-r-1}(-1)^{i-k}\,\frac{(k+1)_{i-k}}{(i-k)!}\,\frac{(-i+k)_{r+1}(-k)_j}{(k+1)_{r+1}}$$

and it is clear that this sum is equal to zero for $j \geq i - r$, and in particular for

$j \geq i$. If $j \leq i-r-1$ this sum is equal to

$$
\begin{aligned}
&\frac{(-i+1)_r}{(r+1)!(j-1)!} \sum_{k=j}^{i-r-1} (-1)^{i-r-k+1} \frac{(r+k+2)_{i-r-1-k}(-k)_j}{(i-r-1-k)!} \\
&= \frac{(-1)^j(i-r-j)_j(-i+1)_r}{(r+1)!(j-1)!} \sum_{k=j}^{i-r-1} \frac{(-i)_{i-r-1-k}}{(i-r-1-k)!} \frac{(-i+j+r+1)_{i-r-1-k}}{(-i+r+1)_{i-r-1-k}} \\
&= \frac{(-1)^j(i-r-j)_j(-i+1)_r}{(r+1)!(j-1)!} \sum_{s=0}^{i-r-1-j} \frac{(-i)_s}{s!} \frac{(-i+j+r+1)_s}{(-i+r+1)_s} \\
&= \frac{(-1)^j(i-r-j)_j(-i+1)_r}{(r+1)!(j-1)!} \, {}_2F_1\left({-i+j+r+1, -i \atop -i+r+1} ; 1 \right) \\
&= \frac{(-1)^j(i-r-j)_j(-i+1)_r}{(r+1)!(j-1)!} \frac{(r+1)_{i-r-1-j}}{(-i+r+1)_{i-r-1-j}},
\end{aligned}
$$

where the last equality follows, again, from the Chu-Vandermonde identity. It is easy to see that this number is equal to the coefficient of x^{r+1} on the right hand side, which is given in (3.12). This completes the proof of the theorem. ■

4. Further identities

In this section we obtain an infinite factorization of the matrices $\widetilde{L}_0$ and $\widetilde{L}_0^{-1}$ that relates the Jacobi and cyclotomic polynomials.

Definition 4.1. Given a lower triangular matrix $T_0 \in \mathcal{M}_\infty(\mathcal{A})$ the *left iterated matrix* T_0^{L} corresponding to T_0 is the infinite (from right to left) product

$$T_0^{\mathrm{L}} = \ldots s^3(T_0)\, s^2(T_0)\, s(T_0)\, T_0.$$

It is clear that this product converges to a lower triangular matrix. Similarly, the *right iterated matrix* T_0^{R} corresponding to T_0 is the lower triangular matrix given by the infinite (from left to right) product

$$T_0^{\mathrm{R}} = T_0\, s(T_0)\, s^2(T_0)\, s^3(T_0)\ldots.$$

It is clear that if T_0 is invertible then T_0^{L} is also invertible and

$$\left(T_0^{\mathrm{L}}\right)^{-1} = \left(T_0^{-1}\right)^{\mathrm{R}}.$$

Moreover, T_0^{L} and T_0^{R} are respectively characterized by the identities

$$T_0^{\mathrm{L}} = s(T_0^{\mathrm{L}})\, T_0 \qquad \text{and} \qquad T_0^{\mathrm{R}} = T_0\, s(T_0^{\mathrm{R}}). \tag{4.1}$$

As an example, notice that the classical recurrence relations

$$\binom{i}{j} = \binom{i-1}{j-1} + \binom{i-1}{j} \qquad \text{and} \qquad S(i,j) = S(i-1,j-1) + jS(i-1,j)$$

that define, respectively, the Pascal and Stirling numbers correspond to the matrix identities $P = s(P)\,T_{0,P}$ and $S = s(S)\,T_{0,S}$ where

$$T_{0,P} = \begin{pmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ & 1 & 1 & & \cdots \\ & & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \text{and} \qquad T_{0,S} = \begin{pmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ & 2 & 1 & & \cdots \\ & & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus P and S are the left iterated matrices $P = T_{0,P}^{\mathrm{L}}$ and $S = T_{0,S}^{\mathrm{L}}$. Moreover, P and S are also right iterated matrices. Since $P = \begin{pmatrix} 1 & & & \cdots \\ 1 & 1 & & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} s(P)$ and $S = P\,s(S)$ it follows that, $P = \begin{pmatrix} 1 & & & \cdots \\ 1 & 1 & & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{\mathrm{R}}$ and $S = P^{\mathrm{R}}$.

We shall now express the matrices $\widetilde{L}_0$ and $\widetilde{L}_0^{-1}$ respectively as a right and left iterated matrices. Let T_0 be the lower triangular matrix given by

$$T_0 = \begin{pmatrix} 1 & & & & & \cdots \\ -x-1 & 1 & & & & \cdots \\ x & -x-1 & 1 & & & \cdots \\ 0 & x & -x-1 & 1 & & \cdots \\ 0 & 0 & x & -x-1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is clear that its inverse is the following lower triangular matrix whose entries are the cyclotomic polynomials

$$T_0^{-1} = \begin{pmatrix} 1 & & & & \cdots \\ 1+x & 1 & & & \cdots \\ 1+x+x^2 & 1+x & 1 & & \cdots \\ 1+x+x^2+x^3 & 1+x+x^2 & 1+x & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 4.2. *The matrices $\widetilde{L}_0$ and $\widetilde{L}_0^{-1}$ satisfy the following identities*

$$\widetilde{L}_0^{-1} = (T_0)^L \qquad \textit{and} \qquad \widetilde{L}_0 = \left(T_0^{-1}\right)^R.$$

In other words,

$$\begin{pmatrix} \frac{1}{1}p_0^{1,1} & & & & \cdots \\ \frac{1}{2}p_1^{1,1} & \frac{2}{2}p_0^{2,2} & & & \cdots \\ \frac{1}{3}p_2^{1,1} & \frac{2}{3}p_1^{2,2} & \frac{3}{3}p_0^{3,3} & & \cdots \\ \frac{1}{4}p_3^{1,1} & \frac{2}{4}p_2^{2,2} & \frac{3}{4}p_1^{3,3} & \frac{1}{1}p_0^{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$= \cdots \begin{pmatrix} 1 & & & & \cdots \\ & 1 & & & \cdots \\ & & 1 & & \cdots \\ & & -x-1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & \cdots \\ & 1 & & & \cdots \\ & -x-1 & 1 & & \cdots \\ & x & -x-1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & \cdots \\ -x-1 & 1 & & & \cdots \\ x & -x-1 & 1 & & \cdots \\ & x & -x-1 & 1 & \cdots \\ \vdots & \vdots & & & \ddots \end{pmatrix}$$

and

$$\begin{pmatrix} p_0^{-1,-1} & & & & \cdots \\ p_1^{-2,-2} & p_0^{-2,-2} & & & \cdots \\ p_2^{-3,-3} & p_1^{-3,-3} & p_0^{-3,-3} & & \cdots \\ p_3^{-4,-4} & p_2^{-4,-4} & p_1^{-4,-4} & p_0^{-4,-4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & \cdots \\ 1{+}x & 1 & & & \cdots \\ 1{+}x{+}x^2 & 1{+}x & 1 & & \cdots \\ 1{+}x{+}x^2{+}x^3 & 1{+}x{+}x^2 & 1{+}x & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & \cdots \\ & 1 & & & \cdots \\ & 1{+}x & 1 & & \cdots \\ & 1{+}x{+}x^2 & 1{+}x & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & \cdots \\ & 1 & & & \cdots \\ & & 1 & & \cdots \\ & & 1{+}x & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdots$$

We notice that in these two identities the polynomials involved are $p_n^{\alpha,\alpha}$ instead of $P_n^{\alpha,\alpha}$.

Proof. From equation 4.1 we only have to prove that

$$(-1)^{i-j}\,\frac{j}{i}\,p_{i-j}^{j,j}(x) = \sum_{k=j}^{i}(-1)^{i-k}\,\frac{k-1}{i-1}\,p_{i-k}^{k-1,k-1}(x)\,(T_0)_{kj}$$

for $i \geq 2$, or equivalently

$$\frac{j}{i}\,p_{i-j}^{j,j}(x) = \frac{j-1}{i-1}\,p_{i-j}^{j-1,j-1}(x) + \frac{j(x+1)}{i-1}\,p_{i-j-1}^{j,j}(x) + \frac{(j+1)x}{i-1}\,p_{i-j-2}^{j+1,j+1}(x).$$

This holds if and only if

$$\begin{aligned} &\frac{j}{i}\,P_{i-j}^{j,j}\left(\tfrac{x+1}{x-1}\right) \\ &\quad = \tfrac{j-1}{i-1}\,P_{i-j}^{j-1,j-1}\left(\tfrac{x+1}{x-1}\right) + \tfrac{j}{i-1}\tfrac{x+1}{x-1}\,P_{i-j-1}^{j,j}\left(\tfrac{x+1}{x-1}\right) + \tfrac{j+1}{i-1}\tfrac{x}{(x-1)^2}\,P_{i-j-2}^{j+1,j+1}\left(\tfrac{x+1}{x-1}\right), \end{aligned}$$

which, in turn, holds if and only if

$$\begin{aligned} &4(i-1)j\,P_{i-j}^{j,j}(x) \\ &\quad = 4i(j-1)\,P_{i-j}^{j-1,j-1}(x) + 4ijx\,P_{i-j-1}^{j,j}(x) + i(j+1)(x^2-1)\,P_{i-j-2}^{j+1,j+1}(x). \end{aligned}$$

This last identity is true since it is $(1+j)$ times equation (3.6) with $n = i-j-1$ and $\alpha = j+1$, plus $2(j-1)$ times equation (3.3) with $n = i-j$ and $\alpha = j$. ∎

5. A triangular system equivalent to the original system

Our original motivation to study the LU-decomposition of the matrix M_0 was that we needed to have a triangular system equivalent to the system (1.4) in order to derive some properties about the K_o-invariants in the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$ (see Section 1). This can now be achieved as follows.

Recall that a polynomial $b = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots \in \mathcal{M}[t]$ satisfies the system (1.4) if and only if the vector $(b_0, b_1, b_2, \dots)$ satisfies $M\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix} = 0$, where $M = M_0 P_{R_H}^t$ and

$$M_0 = \begin{pmatrix} L_E - R_E & L_E + R_E & L_E - R_E & L_E + R_E & \dots \\ L_E^2 - R_E^2 & 2L_E^2 + 2R_E^2 & 4L_E^2 - 4R_E^2 & 8L_E^2 + 8R_E^2 & \dots \\ L_E^3 - R_E^3 & 3L_E^3 + 3R_E^3 & 9L_E^3 - 9R_E^3 & 27L_E^3 + 27R_E^3 & \dots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

Since the LU-decomposition of

$$\widetilde{M}_0 = \begin{pmatrix} x-1 & x+1 & x-1 & x+1 & \dots \\ x^2-1 & 2x^2+2 & 4x^2-4 & 8x^2+8 & \dots \\ x^3-1 & 3x^3+3 & 9x^3-9 & 27x^3+27 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is $\widetilde{M}_0 = \widetilde{L}_0 \widetilde{U}_0 F D_{-1} S^t D_{-1}$ (see Theorem 3.2), we conclude that the coefficient matrix of a triangular system equivalent to the system (1.4) is

$$U_0 F D_{-1} S^t D_{-1} P_{R_H}^t \tag{5.1}$$

where U_0 is the upper triangular matrix given by

$$(U_0)_{ij} = (-1)^{i-j} \frac{j}{i} \operatorname{ad}_E^{2i-j} R_E^{j-i} \, p_{j-i}^{-j,-1}\left(\tfrac{L_E}{R_E}\right), \quad \text{for } 1 \le i \le j.$$

As an example, we show how the last two equations of this triangular system look like for a polynomial b of degree m. In this case we must consider the $(m+1)$-minor of the matrix $U_0 F D_{-1} S^t D_{-1} P_{R_H}^t$, that is the matrix whose entries are $(U_0 F D_{-1} S^t D_{-1} P_{R_H}^t)_{ij}$ for $1 \le i \le j \le m+1$. The last two equations correspond to the 2×2 matrix system

$$\begin{pmatrix} (m-1)!\operatorname{ad}_E^m & m!\operatorname{ad}_E^m R_H - \frac{(m+1)!}{2}\operatorname{ad}_E^m + (m+1)!\operatorname{ad}_E^{m-1} L_E \\ 0 & m!\operatorname{ad}_E^{m+1} \end{pmatrix} \begin{pmatrix} b_{m-1} \\ b_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If we simplify the constants the corresponding equations are:

$$\begin{aligned} \operatorname{ad}_E^m(b_{m-1}) + m\operatorname{ad}_E^m(b_m H) - \tbinom{m+1}{2}\operatorname{ad}_E^m(b_m) + m(m+1)\operatorname{ad}_E^{m-1}(E b_m) &= 0, \\ \operatorname{ad}_E^{m+1}(b_m) &= 0. \end{aligned}$$

This triangular system presents the following inconvenient for our needs: the operator ad_E acts, in general, on elements of the form $b_j H^l$ (see the first equation above) and we would prefer to have ad_E acting directly on the coefficients of the polynomial b. In the enveloping algebra $U(\mathfrak{k})$, where this system comes from, the elements H and E satisfy $[H, E] = cE$ where $c \in \mathbb{C}$, and this fact allows us to express this triangular system in such a way that ad_E acts only on the coefficients of the polynomial b.

In order to do this we make use of Theorem 4.2. Assume from now on that $[H, E] = cE$ with $c \in \mathrm{k}$, and let us denote by Eq^0 the original system (1.4), that is, Eq^0_n is the equation

$$E^n b(H+n) = b(H-n)E^n$$

with $n \in \mathbb{N} \cup \{0\}$ (we added the trivial equation corresponding to $n = 0$ for technical reasons). From Theorem 4.2 we know that if we define recursively the system Eq^k by

$$\mathrm{Eq}^k_n = \begin{cases} \mathrm{Eq}^{k-1}_n, & \text{if } 1 \le n \le k; \\ \mathrm{Eq}^{k-1}_n - (L_E + R_E)(\mathrm{Eq}^{k-1}_{n-1}) \,+\, L_E(\mathrm{Eq}^{k-1}_{n-2}), & \text{if } k < n; \end{cases}$$

then the system

$$\mathrm{Eq}^\infty = \{\mathrm{Eq}^n_{n+1} : n \ge 0\}$$

is exactly the triangular system given by (5.1). In the next theorem we give an expression for the equations Eq^k_n where ad_E acts directly on the coefficients of the polynomial b. To do this we introduce the following notation. For $h \in \mathrm{k}$, $h \neq 0$, let $\partial_h : \mathcal{M}[t] \to \mathcal{M}[t]$ denote the h-discrete derivative

$$\partial_h b(t) = \frac{b(t) - b(t-h)}{h}, \qquad \text{for } b \in \mathcal{M}[t].$$

Also, let $\dot{E} : \mathcal{M}[t] \to \mathcal{M}[t]$ denote the map $\dot{E}(\sum b_j t^j) = \sum \mathrm{ad}_E(b_j) t^j$. Observe that $\dot{E}$ commutes with ∂_h for all $h \in \mathrm{k}$, and it is straightforward to prove that

$$\mathrm{ad}_E\big(b(t+H)\big) = \dot{E}(b)(t+H) - c\,\partial_c b(t+H)E. \tag{5.2}$$

Theorem 5.1. *For $k \ge 0$ and $n \ge k$ the equation Eq^k_n is given by*

$$\sum_{i=0}^{k} \sum_{j=0}^{n-k} (-1)^i \tbinom{k}{i} \tbinom{n-k}{j}\, \partial_1^k \partial_{c-1}^i \dot{E}^{n-(j+i)}(b)(H + n - cj - i)E^{j+i}$$
$$= \sum_{i=0}^{k} (-1)^i \tbinom{k}{i}\, \partial_1^k \partial_{c-1}^i \dot{E}^{k-i}(b)(H - n + 2k - i)E^{n-(k-i)}.$$

In particular, the n-th equation of the system Eq^∞ is

$$\sum_{i=0}^{n} (-1)^i \tbinom{n}{i}\, \partial_1^n \partial_{c-1}^i \dot{E}^{n+1-i}(b)(H + n + 1 - i)E^i$$
$$+ \sum_{i=0}^{n} (-1)^i \tbinom{n}{i}\, \partial_1^n \partial_{c-1}^i \dot{E}^{n-i}(b)(H + n + 1 - c - i)E^{i+1}$$
$$= \sum_{i=0}^{n} (-1)^i \tbinom{n}{i}\, \partial_1^n \partial_{c-1}^i \dot{E}^{n-i}(b)(H + n - 1 - i)E^{i+1}.$$

This theorem can be proved directly by induction. Another possibility is to make use of identity (5.2) to interchange, by induction, the orders of the actions of ad_E and R_H in the expression of the triangular system given by (5.1). In any case the proof is rather technical and long, and thus we prefer to omit it. As a corollary we have the following particular cases.

Corollary 5.2. *If $[H,E]=0$ then the n-th equation of the system* Eq^∞ *is*

$$\sum_{i=0}^{n}\Big[\tbinom{n}{i}\,\dot{E}^{n+1-i}\partial_1^{n+i}(b)(H+n+1)E^i+\tbinom{n}{i}\dot{E}^{n-i}\partial_1^{n+i}(b)(H+n+1)E^{i+1}\Big]$$
$$=\sum_{i=0}^{n}\tbinom{n}{i}\,\dot{E}^{n-i}\partial_1^{n+i}(b)(H+n-1)E^{i+1}.$$

If $[H,E]=E$ then the n-th equation of the system Eq^∞ *is*

$$\dot{E}^{n+1}\partial_1^{n}(b)(H+n+1)+\dot{E}^{n}\partial_1^{n+1}(b)(H+n)E=0.$$

Proof. If E and H commute then $c=0$ and hence $\partial_{c-1}b(t)=-\partial_1 b(t+1)$. Therefore $\partial_{c-1}^i b(t)=(-1)^i\partial_1^i b(t+i)$ and the result follows.

On the other hand, if $[H,E]=E$ then $c=1$ and hence $\partial_{c-1}=\partial_0=0$. Therefore all the terms of Eq_{n+1}^n, for $i>0$, are zero and thus Eq_{n+1}^n is

$$\dot{E}^{n+1}\partial_1^{n}(b)(H+n+1)+\dot{E}^{n}\partial_1^{n}(b)(H+n)E=\dot{E}^{n}\partial_1^{n}(b)(H+n-1)E,$$

which is equal to the equation stated in the corollary. ■

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Alfredo O. Brega
CIEM-FaMAF
Universidad Nacional de Córdoba
Ciudad Universitaria
5000 Córdoba, Argentina
brega@famaf.unc.edu.ar

Leandro R. Cagliero
CIEM-FaMAF
Universidad Nacional de Córdoba
Ciudad Universitaria
5000 Córdoba, Argentina
cagliero@famaf.unc.edu.ar

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