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Factoring Tilting Modules for Algebraic Groups

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Abstract. Let G be a semisimple, simply-connected algebraic group over an algebraically closed field of characteristic $p > 0$. We observe that the tensor product of the Steinberg module with a minuscule module is always indecomposable tilting. Although quite easy to prove, this fact does not seem to have been observed before. It has the following consequence: If $p \geqslant 2h-2$ and a given tilting module has highest weight p-adically close to the rth Steinberg weight, then the tilting module is isomorphic to a tensor product of two simple modules, usually in many ways.
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Let G be a semisimple, simply-connected algebraic group over an algebraically closed field k of characteristic $p > 0$. For convenience we assume the underlying root system is indecomposable. Tensor products are over k unless otherwise specified. Fix a maximal torus T in G and write $X(T)$ for the character group of T. Note that $X(T) \simeq \mathbb{Z}^n$ for some n. By "G-module" we mean "rational G-module". Fix a Borel subgroup B containing T and let the negative roots be determined by B. Let

$$X(T)^+ = \{\lambda \in X(T) : (\alpha^\vee, \lambda) \geqslant 0,\ \text{all simple roots } \alpha\}$$

be the set of dominant weights and

$$X_r(T) = \{\lambda \in X(T)^+ : (\alpha^\vee, \lambda) < p^r,\ \text{all simple roots } \alpha\}.$$

The set $X_1(T)$ is known as the restricted region and its elements are often called restricted weights. For any $\lambda \in X(T)^+$ let

$\Delta(\lambda) =$ the Weyl module of highest weight λ;
$\nabla(\lambda) =$ the dual Weyl module of highest weight λ;
$L(\lambda) =$ the simple G-module of highest weight λ.

The main properties of these families of modules are summarized in [8], to which the reader should also refer for any unexplained notation or terminology.

Let $\mathcal{F}(\Delta)$ be the full subcategory of the category of G-modules whose objects have an ascending filtration with successive sub-quotients isomorphic to

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various Weyl modules; $\mathcal{F}(\nabla)$ is defined similarly with ∇ in place of Δ. Recall that the objects of $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ are called tilting modules and the category of tilting modules is closed under tensor products, direct sums, and direct summands. For each $\lambda \in X^+$, there is a unique (up to isomorphism) indecomposable tilting module of highest weight λ, denoted by $T(\lambda)$. Every tilting module is isomorphic to a direct sum of various $T(\lambda)$. Since $\Delta(\lambda)$ is isomorphic to the contravariant dual of $\nabla(\lambda)$ it follows immediately that whenever $\Delta(\lambda)$ is simple as a G-module, then

$$L(\lambda) \simeq \Delta(\lambda) \simeq \nabla(\lambda) \simeq T(\lambda). \tag{1}$$

Conversely, any simple tilting module must be a simple Weyl module.

A dominant weight is called *minuscule* if the weights of $\Delta(\lambda)$ form a single orbit under the action of the Weyl group W. This forces $\Delta(\lambda)$ to be simple, so (1) holds for any minuscule weight λ. When λ is minuscule we shall refer to any of the isomorphic modules in (1) as a minuscule module. Note that the zero weight is minuscule and the trivial module is a minuscule module by our definition. Minuscule weights are classified in [1, ch. VIII, prop. 7]. For the reader's convenience we list them in Table 1. In the table, $\varepsilon_1, \ldots, \varepsilon_n$ are the fundamental

Type	Highest Weight	Dimension	Name
A_n	ε_j $(1 \leqslant j \leqslant n)$	$\binom{n+1}{j}$	exterior powers of natural
B_n	ε_n	2^n	spin
C_n	ε_1	$2n$	natural
D_n	$\varepsilon_1, \varepsilon_{n-1}, \varepsilon_n$	$2n$, 2^{n-1}, 2^{n-1}	natural, $\frac{1}{2}$-spin, $\frac{1}{2}$-spin
E_6	$\varepsilon_1, \varepsilon_6$	27, 27	minimal
E_7	ε_7	56	minimal
E_8	none		
F_4	none		
G_2	none		

Table 1: Minuscule modules

weights, defined by the requirement $(\alpha_i^\vee, \varepsilon_j) = \delta_{i,j}$ for all i, j (with respect to the usual ordering of the simple roots). Note that all minuscule weights belong to the restricted region $X_1(T)$ for any p.

Let $\rho \in X(T)$ be half the sum of the positive roots. Write

$$\mathrm{St}_r := \Delta((p^r - 1)\rho)$$

for the rth Steinberg module; this is a simple tilting module for every $r > 0$. We write St for St_1.

Lemma. *If* λ *is minuscule then* $\mathrm{St} \otimes L(\lambda) \simeq T((p-1)\rho + \lambda)$.

Proof. In [4, Proposition 5.5] it is proved (by an application of Brauer's formula) that if $(\alpha_0^\vee, \lambda) \leqslant p$, where α_0 is the highest short root, then the character of $T((p-1)\rho+\lambda)$ is equal to the character of St multiplied by the character of the orbit of λ under the action of W. Now the tensor product $\mathrm{St} \otimes L(\lambda)$ in question is the tensor product of two tilting modules, hence is itself tilting. By highest weight considerations a copy of $T((p-1)\rho + \lambda)$ must occur as a direct summand. Thus we are done once we have verified that $(\alpha_0^\vee, \lambda) \leqslant p$. But this is easy to check, by comparing the classification of minuscule weights in Table 1 with a list of highest short roots (see [7, §12, Table 2]). ∎

We now want to generalize the above result. Say that a weight λ is r-minuscule if λ can be written in the form $\lambda = \sum_{j=0}^{r-1} p^j \lambda^j$, where each λ^j is minuscule. For such λ we obviously have

$$L(\lambda) \simeq L(\lambda^0) \otimes L(\lambda^1)^{[1]} \otimes \cdots \otimes L(\lambda^{r-1})^{[r-1]}$$

by Steinberg's tensor product theorem.

Let h be the Coxeter number of the underlying root system. Recall (Donkin [3, p. 47, Example 1]) that if $p \geqslant 2h-2$ and $\lambda \in X_r(T)$ then $T((p^r-1)\rho+\lambda)$ is isomorphic to the projective cover of $L((p^r-1)\rho + w_0\lambda)$ in the category of G_r-modules. Here G_r is the rth Frobenius kernel of G and w_0 is the longest element of the Weyl group. Donkin has conjectured that this holds for any p; see [3, (2.2)]. He proved in [3, (2.1)] (see also [8, II.E.9]) that $T(\tau) \otimes T(\mu)^{[r]}$ is tilting, for any $\tau \in (p^r-1)\rho + X_r(T)$, $\mu \in X(T)^+$, and morover if $p \geqslant 2h-2$ (or if the conjecture holds for $p < 2h-2$) then

$$T(\tau) \otimes T(\mu)^{[r]} \simeq T(\tau + p^r\mu). \tag{2}$$

This statement is known as the tensor product theorem for tilting modules.

Proposition. *Assume that Donkin's conjecture holds for G if $p < 2h-2$. If λ is r-minuscule and $\mu \in X(T)^+$ then*

$$T(\mu)^{[r]} \otimes \mathrm{St}_r \otimes L(\lambda) \simeq T(p^r\mu + (p^r-1)\rho + \lambda).$$

Proof. By Steinberg's tensor product theorem it follows that

$$\mathrm{St}_r \otimes L(\lambda) \simeq \textstyle\bigotimes_{j=1}^{r-1} \left(\mathrm{St} \otimes L(\lambda^j)\right)^{[j]}$$

where $\lambda = \sum_j \lambda^j p^j$ (with $\lambda^j \in X_1(T)$ for all j) is the p-adic expansion of λ. By the lemma we get

$$\mathrm{St}_r \otimes L(\lambda) \simeq \textstyle\bigotimes_{j=1}^{r-1} \left(T((p-1)\rho + \lambda^j)\right)^{[j]}$$

and by the tensor product theorem for tilting modules (see (2)) applied inductively it follows that

$$\mathrm{St}_r \otimes L(\lambda) \simeq T((p^r-1)\rho + \lambda).$$

Now tensor both sides by $T(\mu)^{[r]}$ and apply the tensor product theorem for tilting modules again to obtain the result. ∎

In general one would like to understand the indecomposable direct summands of modules of the form $L \otimes M$ where L is simple and M is either simple or tilting. The proposition provides many examples where such tensor products are in fact indecomposable tilting modules.

Corollary. *Assume that Donkin's conjecture holds for G if $p < 2h-2$. If λ is r-minuscule and $\mu \in X(T)^+$ then:*

(a) $T(p^r\mu + (p^r-1)\rho) \otimes L(\lambda) \simeq T(p^r\mu + (p^r-1)\rho + \lambda)$.

(b) *If* $T(\mu)$ *is simple then* $\mathrm{St}_r \otimes L(p^r\mu + \lambda) \simeq T(p^r\mu + (p^r-1)\rho + \lambda)$.

Proof. By the tensor product theorem for tilting modules we have $T(\mu)^{[r]} \otimes \mathrm{St}_r \simeq T(p^r\mu + (p^r-1)\rho)$. This proves (a).

If $T(\mu) \simeq L(\mu)$ then $T(\mu)^{r]} \otimes L(\lambda) \simeq L(\mu)^{r]} \otimes L(\lambda) \simeq L(p^r\mu + \lambda)$, by Steinberg's tensor product theorem. This proves (b). ∎

Remarks. 1. In case G is of Type A_1 or A_2 it is known that Donkin's conjecture holds for all p.

2. Given two simple modules L, M one may express each one as a twisted tensor product of restricted simple modules

$$L \simeq L_0 \otimes L_1^{[1]} \otimes L_2^{[2]} \otimes \cdots$$
$$M \simeq M_0 \otimes M_1^{[1]} \otimes M_2^{[2]} \otimes \cdots$$

by Steinberg's tensor product theorem. Interchanging L_j and M_j in arbitrary selected positions j results in two new simple modules L', M' such that $L \otimes M \simeq L' \otimes M'$. This is immediate by commutativity of tensor product. Applying this observation to the pair St_r, $L(p^r\mu + \lambda)$ in part (b) of the corollary produces many factorizations

$$T(p^r\mu + (p^r-1)\rho + \lambda) \simeq L(\lambda') \otimes L(\mu')$$

where λ', μ' are the highest weights of the rearranged tensor products.

3. There exist factorizations of indecomposable tilting modules not of the form in the proposition or corollary. For example, for $G = \mathrm{SL}_3$ in characteristic 3 one has from [6, 5.2] the factorization $T(3,0) \simeq L(2,0) \otimes L(1,0)$.

4. The proposition and corollary are most effective when p is small, in which case more weights are close to a Steinberg weight. For instance, for $G = \mathrm{SL}_2$ in characteristic 2 it follows from the corollary that every indecomposable tilting module is isomorphic to a tensor product of two simple modules. This was known previously; see [5, 2.5].

5. The above results can be extended to the reductive case by the usual arguments, although if $p = 2$ one has to deal with the possibility that the Steinberg module (as defined above) may fail to exist; e.g. consider $G = \mathrm{GL}_2$. Thus one may need to pass to a covering. We leave the details to the reader.

Another possibility in the reductive case is to replace ρ by any weight ρ' satisfying the condition $(\alpha^\vee, \rho') = 1$ for all simple roots α. The resulting modules $\Delta((p^r-1)\rho')$ satisfy the desired properties of Steinberg modules and may be used in place of the St_r in the above arguments; see the remark in [8, II.3.18]. In case $G = \mathrm{GL}_n$ one may wish to apply this remark with the weight $\rho' = \sum_{j=1}^{n}(n-j)\varepsilon_j$ replacing ρ.

6. The tensor product $\mathrm{St} \otimes L(\lambda)$ considered in the lemma (for minuscule λ) is projective as a kG^F-module, where F is the p-Frobenius endomorphism of G and G^F is the finite Chevalley group of F-fixed points in G. Moreover, if $U(\mu)$ denotes the projective cover of $L(\mu)$ in the category of kG^F-modules (for $\mu \in X_1(T)$) then $U((p-1)\rho + w_0\lambda)$ occurs once (see [9], [2]) as a direct summand

of $\mathrm{St} \otimes L(\lambda)$, viewed as a kG^F-module. It is natural to ask if $U((p-1)\rho + w_0\lambda) \simeq \mathrm{St} \otimes L(\lambda)$. This is not always the case; see [10] for a study of this question.

References

[1] Bourbaki, N., "Groupes et algèbres de Lie," Chapitres 7 et 8, Hermann, 1975.

[2] Chastkofsky, L., *Characters of projective indecomposable modules for finite Chevalley groups*, in: The Santa Cruz Conference on Finite Groups, Univ. California, Santa Cruz, Calif., 1979, Proc. Sympos. Pure Math. **37**, Amer. Math. Soc., Providence, R. I., 1980, 359–362.

[3] Donkin, S. On tilting modules for algebraic groups, *Math. Z.* **212** (1993), 39–60.

[4] —, *Tilting modules for algebraic groups and finite dimensional algebras*, in: Handbook of Tilting Theory, London Math. Soc. Lecture Note Ser., **332**, Cambridge Univ. Press, Cambridge 2007, 215–257.

[5] Doty, S. R., and A. E. Henke, *Decomposition of tensor products of modular irreducibles for* SL_2, Quart. J. Math. **56** (2005), 189–207.

[6] Doty, S. R., and S. Martin, *Decomposition of tensor products of modular irreducible representations for* SL_3, Preprint.

[7] Humphreys, J. E., "Introduction to Lie Algebras and Representation Theory," Second Printing, revised, Graduate Texts in Mathematics **9**, Springer-Verlag, New York-Berlin, 1978.

[8] Jantzen, J. C., "Representations of Algebraic Groups," 2nd ed., Mathematical Surveys and Monographs **107**, Amer. Math. Soc., Providence 2003.

[9] —, *Zur Reduktion modulo p der Charaktere von Deligne und Lusztig*, J. Algebra **70** (1981), 452–474.

[10] Tsushima, Y., *On certain projective modules for finite groups of Lie type*, Osaka J. Math. **27** (1990), 947–962.

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