

Lie Quasi-States

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Abstract. Lie quasi-states on a real Lie algebra are functionals which are linear on any abelian subalgebra. We show that on the symplectic Lie algebra of rank at least 3 there is only one continuous non-linear Lie quasi-state (up to a scalar factor, modulo linear functionals). It is related to the asymptotic Maslov index of paths of symplectic matrices.

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1. Introduction and main results

1.1. Lie quasi-states.

Let $W \subset \mathfrak{g}$ be a vector subspace of a finite-dimensional Lie algebra \mathfrak{g} over \mathbb{R} . A function $\zeta : W \rightarrow \mathbb{R}$ will be called *quasi-linear* if:

$$[x_1, x_2] = 0 \implies \zeta(c_1x_1 + c_2x_2) = c_1\zeta(x_1) + c_2\zeta(x_2) \quad \forall c_1, c_2 \in \mathbb{R}.$$

A quasi-linear function on the whole Lie algebra \mathfrak{g} will be called a *Lie quasi-state*.

Continuous Lie quasi-states on \mathfrak{g} form a vector space $\widehat{\mathcal{Q}}(\mathfrak{g})$. Set $\mathcal{Q}(\mathfrak{g}) := \widehat{\mathcal{Q}}(\mathfrak{g})/\mathfrak{g}^*$, where \mathfrak{g}^* is the dual space to \mathfrak{g} . It can be viewed as the space of *non-linear* continuous Lie quasi-states on \mathfrak{g} .

In the present paper we focus on Lie quasi-states on the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$, that is on the Lie algebra of the group $Sp(2n, \mathbb{R})$ of $2n \times 2n$ symplectic matrices. Our main finding is that the notion of a continuous Lie quasi-state is rigid in the following sense.

Theorem 1.1. *Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$, $n \geq 3$. Then $\dim \mathcal{Q}(\mathfrak{g}) = 1$.*

As we shall see below, the generator of $\mathcal{Q}(\mathfrak{g})$ looks as follows: its value on a matrix $B \in \mathfrak{sp}(2n, \mathbb{R})$ equals, roughly speaking, to the asymptotic Maslov index of the path e^{tB} as $t \rightarrow \infty$.

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Let us discuss the assumptions of the theorem. In the case $n = 1$, $\dim \mathcal{Q}(\mathfrak{sp}(2, \mathbb{R})) = +\infty$. Indeed, any two commuting matrices in $\mathfrak{sp}(2, \mathbb{R})$ differ by a scalar factor, and hence any odd homogeneous function on $\mathfrak{sp}(2, \mathbb{R})$ is a Lie quasi-state. The case $n = 2$ is so far absolutely open.

The next result shows that the continuity assumption in Theorem 1.1 is essential.

Theorem 1.2. *The space of (not necessarily continuous) Lie quasi-states on $\mathfrak{sp}(2n, \mathbb{R})$ which are bounded on a neighborhood of zero is infinite-dimensional for all $n \geq 1$.*

At the same time any Lie quasi-state which is *differentiable* at 0 is automatically linear since it is homogeneous of degree 1.

1.2. Origins of Lie quasi-states.

The interest to the notion of Lie quasi-states is three-fold.

LIE QUASI-STATES AND QUASI-MORPHISMS ON LIE GROUPS: Recall that a *homogeneous quasi-morphism* on a group G is a function $\mu : G \rightarrow \mathbb{R}$ such that

- There exists $C > 0$ so that $|\mu(xy) - \mu(x) - \mu(y)| \leq C$ for all $x, y \in G$.
- $\mu(x^k) = k\mu(x)$ for all $k \in \mathbb{Z}$, $x \in G$.

It is known that restriction of any homogeneous quasi-morphism to an abelian subgroup is a genuine morphism, and that homogeneous quasi-morphisms are conjugation invariant (see e.g. [2] for introduction to quasi-morphisms). Therefore, given a homogeneous quasi-morphism μ on a Lie group G , its pull-back to the Lie algebra \mathfrak{g} by the exponential map, which we will call *the directional derivative of μ* ,

$$\zeta : \mathfrak{g} \rightarrow \mathbb{R}, \quad a \mapsto \mu(\exp a),$$

is an Ad_G -invariant Lie quasi-state. Clearly, ζ is continuous whenever μ is continuous.

In fact, if G is a simply connected real Lie group with a Hermitian simple Lie algebra \mathfrak{g} (see Section 4 for the definition; in particular, $G = \widetilde{Sp}(2n, \mathbb{R})$, the universal cover of $Sp(2n, \mathbb{R})$), is such a Lie group and its Lie algebra is $\mathfrak{sp}(2n, \mathbb{R})$), then the space of homogeneous quasi-morphisms on G is 1-dimensional [18], cf. [3], and these quasi-morphisms are continuous [18]. We shall show that in such a case the space of Ad_G -invariant Lie quasi-states on \mathfrak{g} is also one-dimensional – see Section 4.

LIE QUASI-STATES AND GLEASON'S THEOREM: Gleason's theorem [10] is one of the most famous and important results in the mathematical formalism of quantum mechanics (see e.g. [17, 5]). In the finite-dimensional setting the proof of Gleason's theorem yields the following result about Lie quasi-states.

Theorem 1.3 (Gleason). *Let V be a finite-dimensional vector space over \mathbb{R} (respectively, over \mathbb{C}), equipped with a real (respectively, Hermitian) inner product.*

Denote by $\mathcal{S}(V)$ the subspace of the self-adjoint operators (viewed as the subspace of the Lie algebra of all operators on V). Let $\zeta : \mathcal{S}(V) \rightarrow \mathbb{R}$ be a quasi-linear function which is bounded on a neighborhood of zero in $\mathcal{S}(V)$. Assume also that $\dim V \geq 3$.

Then ζ is linear and has the form $\zeta(A) = \text{tr}(HA)$ for some $H \in \mathcal{S}(V)$.

Corollary 1.4. Any Lie quasi-state ζ on the Lie algebra $\mathfrak{u}(n)$, $n \geq 3$, which is bounded on a neighborhood of zero, is linear and has the form $\zeta(A) = \text{tr}(HA)$ for some $H \in \mathfrak{u}(n)$.

Indeed, $\mathfrak{u}(n) = i\mathcal{S}(\mathbb{C}^n)$, where $\mathcal{S}(\mathbb{C}^n)$ is the space of Hermitian $n \times n$ -matrices.

The statement of Theorem 1.3 is slightly different from the original formulation in Gleason’s paper [10]: instead of boundedness of ζ near zero Gleason assumes that ζ is non-negative on the set of non-negative self-adjoint operators. To obtain this non-negativity condition from the boundedness near zero one just needs to add to ζ a linear function $A \mapsto \text{Tr}(cA)$ for a sufficiently large positive $c \in \mathbb{R}$. After such a modification, ζ becomes positive on the set of all orthogonal projectors, and hence (by the spectral theorem) non-negative on all non-negative self-adjoint operators.

Our proof of Theorem 1.1 uses Gleason’s theorem. Let us mention that the most difficult and non-trivial part of the proof of Gleason’s Theorem 1.3 is to show that the boundedness of ζ near zero implies its continuity – the latter yields (by basic representation theory) that ζ is linear. Since in Theorem 1.1 we assume that ζ is continuous, our proof of the theorem does not use the difficult part of Gleason’s proof (and there is no analogue of this part in our proof due to Theorem 1.2).

LIE QUASI-STATES IN SYMPLECTIC TOPOLOGY: As the third point of interest in Lie quasi-states, we note that such functionals on the infinite-dimensional Poisson-Lie algebra of Hamiltonian functions on a symplectic manifold appeared recently in symplectic topology and Hamiltonian dynamics before they were properly studied in the finite-dimensional setting. We refer the reader to [6, 7, 8] for various aspects of this development.

1.3. Maslov quasi-state on $\mathfrak{sp}(2n, \mathbb{R})$.

The Lie quasi-state ζ_M generating the 1-dimensional space $\mathcal{Q}(\mathfrak{sp}(2n, \mathbb{R}))$, $n \geq 3$, comes from the Maslov index of paths of symplectic matrices and can be defined as follows. Given $B \in \mathfrak{sp}(2n, \mathbb{R})$, write the (unique) polar decomposition of the matrix e^{tB} as $e^{tB} = P(t)U(t)$. Here $P(t)$ is symplectic (i.e. belongs to $Sp(2n, \mathbb{R})$), symmetric and positive and $U(t)$ is symplectic and complex-linear. The real operator $U(t)$ can be identified with a unitary operator on \mathbb{C}^n and we denote by

$$\det_{\mathbb{C}}U(t) \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

the determinant of this unitary operator. Note that the families $P(t), U(t)$ are continuous in t . Now set

$$\zeta_M(B) := \lim_{t \rightarrow +\infty} \frac{1}{t} \arg \det_{\mathbb{C}}U(t).$$

One can check that ζ_M is a continuous non-linear Lie quasi-state – in fact, it is a directional derivative of a unique (up to a non-zero constant factor) homogeneous quasi-morphism on $\widetilde{Sp}(2n, \mathbb{R})$ (cf. Section 4).

For $n = 1$ one can easily write an explicit formula for ζ_M . Namely,

$$sp(2, \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, a, b, c \in \mathbb{R} \right\}$$

and

$$\zeta_M(A) = \begin{cases} \sqrt{|a^2 + bc|}, & \text{if } a^2 + bc < 0, b < 0, c > 0, \\ -\sqrt{|a^2 + bc|}, & \text{if } a^2 + bc < 0, b > 0, c < 0, \\ 0, & \text{if } a^2 + bc \geq 0. \end{cases} \quad (1)$$

As we see, ζ_M is continuous but not differentiable.

1.4. Perspective and open questions.

Theorem 1.1 raises the following general problem: given a Lie algebra \mathfrak{g} , describe the space $\mathcal{Q}(\mathfrak{g})$ of continuous non-linear Lie quasi-states on \mathfrak{g} .

Besides the Lie algebras mentioned above, there are a few other (finite-dimensional) cases where the answer is known: for instance, the Heisenberg algebra (in this case $\mathcal{Q}(\mathfrak{g}) = 0$ – see Section 2) and the algebra $\mathfrak{so}(3, \mathbb{R})$ (in this case any two commuting elements must be proportional to each other so any continuous \mathbb{R} -homogeneous function is a continuous Lie quasi-state). A partial result on Lie quasi-states on $\mathfrak{gl}(n, \mathbb{R})$, needed for the proof of Theorem 1.1 can be found in Section 2. Otherwise, as far as the classical Lie algebras are concerned, the answer is unknown already for $\mathfrak{sl}(3, \mathbb{R})$.

Further, in view of Theorem 1.2 it would be interesting to relax the continuity assumption and to describe the space of non-linear Lie quasi-states on \mathfrak{g} that are *bounded on a neighborhood of 0*. We do not know the complete answer even for the case of $\mathfrak{sp}(2n, \mathbb{R})$. A possible interesting modification of the question above would be to explore Lie quasi-states on \mathfrak{g} that are *positive on a certain cone* in \mathfrak{g} . This is motivated by the original version of Gleason's theorem and by the theory of symplectic quasi-states which have this sort of property (see [6]).

Another set of questions arises from the relation between Lie quasi-states and homogeneous quasi-morphisms.

First, note that homogeneous quasi-morphisms appear as a part of a certain remarkable cohomological theory on groups, called *bounded cohomology* – see e.g. [2, 12, 16]. It would be interesting to find a helpful cohomology theory for Lie algebras incorporating Lie quasi-states.

Second, note that the directional derivative of a functional $\mu : G \rightarrow \mathbb{R}$ on a Lie group G is a Lie quasi-state on the Lie algebra \mathfrak{g} of G provided

$$\mu(xy) = \mu(x) + \mu(y) \text{ for all commuting } x, y \in G \quad (2)$$

(and, in particular, $\mu(x^k) = k\mu(x)$ for any $k \in \mathbb{Z}$). It would be interesting to find whether a continuous μ satisfying (2) always has to be a quasi-morphism and, more generally, to describe the quotient of the space of all such μ on a given G by the space of continuous homogeneous quasi-morphisms on G .

Third, given the Lie algebra \mathfrak{g} of a (simply connected) Lie group G , one can consider the following subsets of $\widehat{\mathcal{Q}}(\mathfrak{g})$:

$\widehat{\mathcal{Q}}_{qm}(\mathfrak{g}) := \{\text{the space of continuous Lie quasi-states on } \mathfrak{g} \text{ coming from continuous homogeneous quasi-morphisms on } G\};$

$\widehat{\mathcal{Q}}_{Ad}(\mathfrak{g}) := \{\text{the set of continuous Lie quasi-states on } \mathfrak{g} \text{ which are invariant under the adjoint action of } G\}.$

Clearly,

$$\widehat{\mathcal{Q}}_{qm}(\mathfrak{g}) \subset \widehat{\mathcal{Q}}_{Ad}(\mathfrak{g}) \subset \widehat{\mathcal{Q}}(\mathfrak{g}).$$

By Theorem 1.1 these spaces coincide for $\mathfrak{sp}(2n, \mathbb{R})$. It would be interesting to explore these inclusions for other algebras \mathfrak{g} . For instance, assume that \mathfrak{g} is a compact simple Lie algebra. In this case $\widehat{\mathcal{Q}}_{qm}(\mathfrak{g}) = 0$ since any continuous homogeneous quasi-morphism on a compact group has to be zero. Further, we show in Section 4 below that $\widehat{\mathcal{Q}}_{Ad}(\mathfrak{g}) = 0$. At the same time note that $\widehat{\mathcal{Q}}(\mathfrak{g})$ might sometimes be infinite-dimensional, for instance, if $\mathfrak{g} = \mathfrak{so}(3, \mathbb{R})$.

Let us mention finally that the study of homogeneous quasi-morphisms on groups is closely related to geometrical structures and dynamics on spaces where these groups act – see e.g. [9]. It would be interesting to understand geometric and/or dynamical meaning of non-linear Lie quasi-states, for instance, those constructed in Section 3 below.

2. Proof of the main theorem

Let ζ be a continuous Lie quasi-state on $\mathfrak{sp}(2n, \mathbb{R})$, $n \geq 3$. We want to show that ζ is a sum of $c\zeta_M$, $c \in \mathbb{R}$, and a linear functional on $\mathfrak{sp}(2n, \mathbb{R})$.

2.1. Preliminaries.

For each $k \in \mathbb{N}$ denote by $M_k(\mathbb{R})$ (respectively, $M_k(\mathbb{C})$) the spaces of real (respectively, complex) $k \times k$ -matrices.

Let $\omega = \sum_{k=1}^n dp_k \wedge dq_k$ be the standard linear symplectic form on the vector space \mathbb{R}^{2n} with the coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ on it.

For each $A \in M_{2n}(\mathbb{R})$ there exists a unique $A^\omega \in M_{2n}(\mathbb{R})$ such that $\omega(Ax, y) = \omega(x, Ay)$ for any $x, y \in \mathbb{R}^{2n}$. We say that A is ω -symmetric, if $A = A^\omega$, and skew-symplectic, if $A = -A^\omega$. With this terminology, $\mathfrak{sp}(2n, \mathbb{R})$ is the algebra of skew-symplectic matrices $A \in M_{2n}(\mathbb{R})$.

Given two vectors $\xi, \eta \in \mathbb{R}^{2n}$, define the following operators on \mathbb{R}^{2n} :

$$T_{\xi, \eta}(x) = \omega(\xi, x)\eta,$$

$$Y_{\xi, \eta}(x) := T_{\xi, \xi}(x) + T_{\eta, \eta}(x) = \omega(\xi, x)\xi + \omega(\eta, x)\eta,$$

$$Z_{\xi, \eta}(x) := T_{\eta, \xi}(x) + T_{\xi, \eta}(x) = \omega(\eta, x)\xi + \omega(\xi, x)\eta.$$

One readily checks that

$$T_{\xi, \xi}, Y_{\xi, \eta}, Z_{\xi, \eta} \in \mathfrak{sp}(2n, \mathbb{R}).$$

Note also that

$$Y_{\xi, \eta} = Y_{\eta, \xi}, \quad Z_{\xi, \eta} = Z_{\eta, \xi}$$

and $T_{\xi,\eta}, Z_{\xi,\eta}$ depend linearly on ξ and η . Finally, an easy computation shows that

$$\zeta_M(Y_{\xi,\eta}) = -|\omega(\xi, \eta)|, \quad \zeta_M(Z_{\xi,\eta}) = 0.$$

Consider $\mathbb{C}^{2n} = \mathbb{R}^{2n} \oplus i\mathbb{R}^{2n}$ as the complexification of \mathbb{R}^{2n} . We write elements of \mathbb{C}^{2n} as $v = a + ib$, $a, b \in \mathbb{R}^{2n}$.

Denote by (\cdot, \cdot) the standard Euclidean inner product on \mathbb{R}^{2n} and by $\langle \cdot, \cdot \rangle$ the standard Hermitian inner product on \mathbb{C}^{2n} so that

$$\langle a + ib, c + id \rangle = (a, c) + (b, d) + i(b, c) - i(a, d).$$

We say that a function $Q : \mathbb{C}^{2n} \rightarrow \mathbb{R}$ is a *real Hermitian quadratic form* if Q is a real quadratic form satisfying $Q(\lambda v) = |\lambda|^2 Q(v)$ for any $\lambda \in \mathbb{C}$, or, equivalently, if $Q(v) = \mathcal{H}(v, v)$ for some Hermitian form $\mathcal{H} : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \rightarrow \mathbb{C}$. By definition, a Hermitian form \mathcal{H} is given by $\mathcal{H}(v, w) = \langle Hv, w \rangle$ for some Hermitian $2n \times 2n$ -matrix H , which, in turn, can be always written as $H = A + iB$, where $A, B \in M_{2n}(\mathbb{R})$, $A = A^T$, $B = -B^T$. The corresponding real Hermitian quadratic form Q can then be written as

$$Q(a + ib) = (Aa, a) + (Ab, b) + 2(Ba, b). \tag{3}$$

On the other hand, since ω is non-degenerate, any real bilinear form on \mathbb{R}^{2n} can be uniquely represented as $\omega(C\cdot, \cdot)$ for some $C \in M_{2n}(\mathbb{R})$, and moreover, such a bilinear form is symmetric (respectively, anti-symmetric) if and only if $C = -C^\omega$ (respectively, $C = C^\omega$). Together with (3) this yields that any real Hermitian quadratic form Q on \mathbb{C}^{2n} can be written as

$$Q(a + ib) = \omega(Ca, a) + \omega(Cb, b) + \omega(Da, b), \tag{4}$$

where

$$C, D \in M_{2n}(\mathbb{R}), \quad C = -C^\omega, \quad D = D^\omega.$$

2.2. Reduction to the computation of ζ on $Y_{\xi,\eta}, Z_{\xi,\eta}$.

First, we will show the following

Proposition 2.1. *Any continuous Lie quasi-state ζ on $\mathfrak{sp}(2n, \mathbb{R})$ is completely determined by its values on elements of the form $Y_{\xi,\eta}, Z_{\xi,\eta}$.*

Proof. Since semi-simple (i.e. diagonalizable over \mathbb{C}) elements are dense in $\mathfrak{sp}(2n, \mathbb{R})$ and ζ is continuous, it is enough to show that for any semi-simple $A \in \mathfrak{sp}(2n, \mathbb{R})$ the computation of $\zeta(A)$ can be reduced to the computation of ζ on some $Y_{\xi,\eta}, Z_{\xi,\eta}$.

Recall that a Darboux basis $e_1, \dots, e_n, f_1, \dots, f_n$ on \mathbb{R}^{2n} is a basis which satisfies

$$\omega(e_i, e_k) = \omega(f_i, f_k) = 0, \quad \omega(e_i, f_k) = \delta_{ik},$$

for all $i, k = 1, \dots, n$. Here $\delta_{ik} = 0$ for $i \neq k$ and $\delta_{ii} = 1$.

By Williamson's results on the normal forms of skew-symplectic matrices with respect to the adjoint action of the symplectic group (see [19], cf. [1],

Appendix 6, and [14]), any semi-simple $A \in \mathfrak{sp}(2n, \mathbb{R})$ can be represented in an appropriate basis as a block-diagonal matrix, where the blocks correspond to symplectic subspaces of \mathbb{R}^{2n} spanned by vectors $e_k, \dots, e_{k+l}, f_k, \dots, f_{k+l}$ from a Darboux basis. Each such block, written in the basis $e_k, \dots, e_{k+l}, f_k, \dots, f_{k+l}$, belongs to one of the following three types:

- (i) $\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}$, corresponding to a 2-dimensional symplectic subspace of \mathbb{R}^{2n} spanned by some e_k, f_k .
- (ii) $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$, corresponding to a 2-dimensional symplectic subspace of \mathbb{R}^{2n} spanned by some e_k, f_k .
- (iii) $\begin{pmatrix} -a & b & 0 & 0 \\ -b & -a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix}$, corresponding to a 4-dimensional symplectic subspace of \mathbb{R}^{2n} spanned by some $e_k, e_{k+1}, f_k, f_{k+1}$.

Note that

- I. A block-diagonal $2n \times 2n$ -matrix having only one block which is the 2×2 -block (i) can be represented as aZ_{e_k, f_k} .
- II. A block-diagonal $2n \times 2n$ -matrix having only one block which is the 2×2 -block (ii) can be represented as bY_{e_k, f_k} .
- III. A block-diagonal $2n \times 2n$ -matrix A having only one block which is the 4×4 -block (iii) can be represented as

$$A = aZ_{e_k, f_k} + aZ_{e_{k+1}, f_{k+1}} - bY_{\frac{e_{k+1}-f_k}{\sqrt{2}}, \frac{e_k+f_{k+1}}{\sqrt{2}}} + bY_{\frac{e_k-f_{k+1}}{\sqrt{2}}, \frac{e_{k+1}+f_k}{\sqrt{2}}},$$

where

$$\left[aZ_{e_k, f_k} + aZ_{e_{k+1}, f_{k+1}}, -bY_{\frac{e_{k+1}-f_k}{\sqrt{2}}, \frac{e_k+f_{k+1}}{\sqrt{2}}} + bY_{\frac{e_k-f_{k+1}}{\sqrt{2}}, \frac{e_{k+1}+f_k}{\sqrt{2}}} \right] = 0, \tag{5}$$

$$[Z_{e_k, f_k}, Z_{e_{k+1}, f_{k+1}}] = 0, \tag{6}$$

$$\left[Y_{\frac{e_{k+1}-f_k}{\sqrt{2}}, \frac{e_k+f_{k+1}}{\sqrt{2}}}, Y_{\frac{e_k-f_{k+1}}{\sqrt{2}}, \frac{e_{k+1}+f_k}{\sqrt{2}}} \right] = 0. \tag{7}$$

Thus every semi-simple $A \in \mathfrak{sp}(2n, \mathbb{R})$ can be represented as a sum of *pairwise commuting* block-diagonal matrices, each of which has one of the forms I, II, III, and thus the computation of $\zeta(A)$ reduces to the computation of ζ on the block-diagonal matrices I, II, III. The latter in turn reduces to the computation of ζ on some $Y_{\xi, \eta}, Z_{\xi, \eta}$ – for I and II this is trivial and for III this follows from the commutation relations (5), (6), (7). ■

Given $\xi, \eta \in \mathbb{R}^{2n}$, denote

$$F(\xi, \eta) := \zeta(Y_{\xi, \eta}), \quad (8)$$

$$G(\xi, \eta) := \zeta(Z_{\xi, \eta}). \quad (9)$$

Let $\mathbb{C}^{2n} = \mathbb{R}^{2n} + i\mathbb{R}^{2n}$ be the complexification of \mathbb{R}^{2n} . Put

$$\mathcal{W} := \{x + iy \in \mathbb{C}^{2n}, \omega(x, y) > 0\}.$$

Consider the function $F(\xi, \eta)$ as a function on \mathbb{C}^{2n} : $F(\xi, \eta) := F(\xi + i\eta)$. We will prove the following key proposition:

Proposition 2.2.

- (i) *There exists a real Hermitian quadratic form on \mathbb{C}^{2n} which coincides with F on \mathcal{W} .*
- (ii) *The function G is a symmetric bilinear form on \mathbb{R}^{2n} .*

The proof of the proposition uses Gleason's theorem, and hence the assumption $n \geq 3$. Postponing the proof, let us show how the proposition implies the main theorem.

Deducing Theorem 1.1 from Proposition 2.2: By (4) we can write the real Hermitian quadratic form F as

$$F(\xi + i\eta) = \omega(C\xi, \xi) + \omega(C\eta, \eta) + \omega(D\xi, \eta) \quad \forall \xi + i\eta \in \mathcal{W} \quad (10)$$

for some $C, D \in M_{2n}(\mathbb{R})$, $C = -C^\omega$, $D = D^\omega$. Put

$$\mathcal{W}^- := \{x + iy \in \mathbb{C}^{2n}, \omega(x, y) < 0\}.$$

Observe that $Y_{\xi, \eta} = Y_{\xi, -\eta}$ and hence, since ζ is \mathbb{R} -homogeneous,

$$F(\xi + i\eta) = F(\xi - i\eta) \quad \forall \xi, \eta \in \mathbb{R}^{2n}.$$

Since $\xi + i\eta \in \mathcal{W}$ whenever $\xi - i\eta \in \mathcal{W}^-$ we get that

$$F(\xi + i\eta) = \omega(C\xi, \xi) + \omega(C\eta, \eta) - \omega(D\xi, \eta) \quad \forall \xi + i\eta \in \mathcal{W}^-.$$

Next, we claim that $D = c\mathbb{1}$ for some $c \in \mathbb{R}$. Indeed, otherwise there exist ξ, η so that $\omega(D\xi, \eta) \neq 0$ and $\omega(\xi, \eta) = 0$. Hence F is discontinuous at $\xi + i\eta$, so we get a contradiction.

Since $\mathcal{W} \cup \mathcal{W}^-$ is dense in \mathbb{C}^{2n} and F is continuous, we conclude that

$$F(\xi + i\eta) = \omega(C\xi, \xi) + \omega(C\eta, \eta) + c|\omega(\xi, \eta)| \quad \forall \xi + i\eta \in \mathbb{C}^{2n}. \quad (11)$$

Noting that $Y_{\xi, \xi} = Z_{\xi, \xi}$ (and hence $F(\xi + i\xi) = \zeta(Y_{\xi, \xi}) = \zeta(Z_{\xi, \xi}) = G(\xi, \xi)$) we get

$$F(\xi + i\xi) = G(\xi, \xi) = 2\omega(C\xi, \xi),$$

and therefore

$$G(\xi, \eta) = 2\omega(C\xi, \eta) . \tag{12}$$

Define a linear functional $\alpha : \mathfrak{sp}(2n, \mathbb{R}) \rightarrow \mathbb{R}$ by $\alpha(A) := -tr(CA)$. We claim that ζ equals to $\alpha - c\zeta_M$ on each $Y_{\xi, \eta}, Z_{\xi, \eta}$ – as it was explained in the beginning of this section, this claim would imply the theorem.

In order to prove the claim, we observe that

$$\begin{aligned} \alpha(Y_{\xi, \eta}) &= -tr(CY_{\xi, \eta}) = -tr(CT_{\xi, \xi}) - tr(CT_{\eta, \eta}) = \\ &= -trT_{\xi, C\xi} - trT_{\eta, C\eta} = \omega(C\xi, \xi) + \omega(C\eta, \eta), \end{aligned}$$

$$\begin{aligned} \alpha(Z_{\xi, \eta}) &= -tr(CZ_{\xi, \eta}) = -tr(CT_{\xi, \eta}) - tr(CT_{\eta, \xi}) = \\ &= -trT_{\xi, C\eta} - trT_{\eta, C\xi} = \omega(C\eta, \xi) + \omega(C\xi, \eta) = 2\omega(C\xi, \eta). \end{aligned}$$

Thus, by (11) and (12)

$$\zeta(Y_{\xi, \eta}) = \alpha(Y_{\xi, \eta}) + c|\omega(\xi, \eta)|,$$

$$\zeta(Z_{\xi, \eta}) = \alpha(Z_{\xi, \eta}) .$$

The claim follows, since

$$\zeta_M(Y_{\xi, \eta}) = -|\omega(\xi, \eta)|, \quad \zeta_M(Z_{\xi, \eta}) = 0.$$

This finishes the proof of the theorem modulo Proposition 2.2. ■

Outline of the proof of Proposition 2.2(i): We reduce the problem to the case when F is smooth on \mathcal{W} (see Section 2). Further, for any ω -compatible almost complex structure J on \mathbb{R}^{2n} the space $L_J := \xi + iJ\xi$ is Lagrangian with respect to the complexification $\omega^{\mathbb{C}}$ of ω . We observe that the transformation $Y_{\xi, J\xi}$ lies in the J -unitary subalgebra of $\mathfrak{sp}(2n, \mathbb{R})$, and hence Gleason’s theorem (complex version) yields that the restriction of F to each L_J is a Hermitian quadratic form, say, q_J . Note that $q_J = q_I$ on $L_J \cap L_I$ for every two compatible almost complex structures J, I . This yields a restriction on the differential $\partial q_J / \partial J$ which can be translated into a system of first order PDE’s. The analysis of the system (see Section 2) eventually yields that F on \mathcal{W} is quadratic (see Section 2).

Outline of the proof of Proposition 2.2(ii): The proof of (ii) is a bit trickier. First, we show that for $n \geq 3$ any continuous Lie quasi-state on $\mathfrak{gl}(n, \mathbb{R})$, when restricted to rank 1 operators $B_{\xi, \eta}x = (x, \eta)\xi$, is given by $\zeta(B_{\xi, \eta}) = trNB_{\xi, \eta}$ for some fixed matrix N . Interestingly enough, the proof of this statement is very similar to the proof of Proposition 2.2(i) outlined above, but *over the field* \mathbb{R} , see Section 2.

This result readily yields that for a fixed ξ , the restriction of $G(\xi, \cdot)$ to every Lagrangian subspace of \mathbb{R}^{2n} is linear, see Section 2.

Finally, we make a detour to the Heisenberg Lie algebra: we show that on this algebra every (not necessarily continuous) quasi-state is linear, see Section 2.

As an immediate consequence we get that $G(\xi, \eta)$ is linear in the variable η . Since G is symmetric in ξ and η , this completes the proof.

The rest of Section 2 contains a proof of Proposition 2.2. The plan described above serves as a rough guideline only, the specific details are often formulated in a different language and appear in a different order.

2.3. Smoothing Lie quasi-states. Eventually, we wish to reduce Proposition 2.2 to the case when the functions F and G are smooth on $\mathcal{W} \cup \mathcal{W}^-$. For that purpose we show that a continuous Lie quasi-state on any Lie algebra can be suitably approximated by Lie quasi-states which are smooth along the orbits of the adjoint action of the Lie group. We thank Semyon Alesker for explaining this to us.

Let \mathfrak{g} be the Lie algebra of a (connected) Lie group G . Fix a norm $\|\cdot\|$ on \mathfrak{g} , set

$$S := \{x \in \mathfrak{g}, \|x\| = 1\}$$

and define a metric d on $\widehat{\mathcal{Q}}(\mathfrak{g})$ by

$$d(\zeta_1, \zeta_2) := \sup_{x \in S} |\zeta_1(x) - \zeta_2(x)|.$$

Limits with respect to d will be called d -limits.

We will say that a function $\zeta : \mathfrak{g} \rightarrow \mathbb{R}$ is *orbit-smooth* if the restriction of ζ on any orbit of the adjoint action of G on \mathfrak{g} is smooth, or, in other words, if the function $g \mapsto \zeta(gx)$ is a smooth function on G for every $x \in \mathfrak{g}$. (Here gx denotes the adjoint action of g on x).

Proposition 2.3. *Every $\zeta \in \widehat{\mathcal{Q}}(\mathfrak{g})$ is a d -limit of orbit-smooth continuous Lie quasi-states.*

Proof. Let μ be a right-invariant smooth measure on G . We will measure diameters of sets in G with respect to some fixed distance function defined by a Riemannian metric on G .

Let φ_i , $i \in \mathbb{N}$, be a delta-like sequence of C^∞ -smooth functions on G . In particular, assume that

- $\varphi_i \geq 0$ for all i ;
- $\text{supp } \varphi_i \subset U_i$, $i \in \mathbb{N}$, for some open neighborhoods U_i of $\mathbb{1} \in G$ such that $\text{diam } U_i \rightarrow 0$ as $i \rightarrow +\infty$;
- $\int_G \varphi_i d\mu = 1$ for all i .

Given $i \in \mathbb{N}$, put

$$\zeta_i(x) := \int_G \zeta(hx) \varphi_i(h) d\mu(h).$$

Note that since ζ is continuous, $\zeta_i : \mathfrak{g} \rightarrow \mathbb{R}$ is continuous as well. Moreover, for any $x_0 \in \mathfrak{g}$ and $g \in G$ we have

$$\zeta_i(gx_0) = \int_G \zeta(hgx_0) \varphi_i(h) d\mu(h) = \int_G \zeta(h'x_0) \varphi_i(h'g^{-1}) d\mu(h'g^{-1}),$$

where $h' = hg$. Since μ is right-invariant, the latter integral is equal to

$$\int_G \zeta(h'x_0)\varphi_i(h'g^{-1})d\mu(h').$$

Obviously, this integral depends smoothly on g . Thus for any $i \in \mathbb{N}$ the functional ζ_i is orbit-smooth.

Let us estimate $d(\zeta_i, \zeta)$. Note that

$$\begin{aligned} |\zeta_i(x) - \zeta(x)| &= \int_G |\zeta(hx) - \zeta(x)|\varphi_i(h)d\mu(h) = \\ &= \int_{U_i} |\zeta(hx) - \zeta(x)|\varphi_i(h)d\mu(h) \leq \max_{h \in U_i} |\zeta(hx) - \zeta(x)|. \end{aligned}$$

Since $\text{diam } U_i \rightarrow 0$ and ζ is uniformly continuous on S , there exist $\delta_1(i)$, $0 < \delta_1(i) < 1$, and $\delta_2(i) > 0$, such that $\lim_{i \rightarrow +\infty} \delta_1(i) = 0$, $\lim_{i \rightarrow +\infty} \delta_2(i) = 0$, and such that for any $x \in S$ and any $h \in U_i$

$$1 - \delta_1(i) \leq \|hx\| \leq 1 + \delta_1(i),$$

$$\left| \zeta\left(\frac{hx}{\|hx\|}\right) - \zeta(x) \right| \leq \delta_2(i).$$

Then, by the homogeneity of ζ , we get for any $x \in S$ and any $h \in U_i$

$$\begin{aligned} |\zeta(hx) - \zeta(x)| &= \left| \|hx\| \zeta\left(\frac{hx}{\|hx\|}\right) - \zeta(x) \right| \leq \\ &\leq \left| \|hx\| \zeta\left(\frac{hx}{\|hx\|}\right) - \|hx\| \zeta(x) \right| + \left| \|hx\| \zeta(x) - \zeta(x) \right| \leq \\ &\leq \|hx\| \delta_2(i) + |\zeta(x)| \delta_1(i) \leq \left(1 + \delta_1(i)\right) \delta_2(i) + \max_S |\zeta| \cdot \delta_1(i). \end{aligned}$$

Thus

$$d(\zeta_i, \zeta) = \max_{x \in S} |\zeta(hx) - \zeta(x)| \leq \left(1 + \delta_1(i)\right) \delta_2(i) + \max_S |\zeta| \cdot \delta_1(i),$$

and therefore $\lim_{i \rightarrow +\infty} d(\zeta_i, \zeta) = 0$. ■

2.4. Functions whose restrictions on Lagrangian subspaces are quadratic forms.

Recall that

$$\mathcal{W} = \{x + iy \in \mathbb{C}^{2n}, \omega(x, y) > 0\}.$$

Denote by \mathcal{J} the space of all $J \in M_{2n}(\mathbb{R})$ such that $J^2 = -\mathbb{1}$ and $(\cdot, \cdot)_J := \omega(\cdot, J\cdot)$ is a J -invariant inner product on \mathbb{R}^{2n} . (Such a J is called a *complex structure on \mathbb{R}^{2n} compatible with ω*).

Given $J \in \mathcal{J}$, define a complex vector subspace L_J of \mathbb{C}^{2n} by

$$L_J := \{x + iJx \in \mathbb{C}^{2n}, x \in \mathbb{R}^{2n}\}.$$

Consider a complex-valued symplectic form $\omega^{\mathbb{C}}$ on $\mathbb{C}^{2n} = \mathbb{R}^{2n} \oplus i\mathbb{R}^{2n}$ which is the complexification of ω :

$$\omega^{\mathbb{C}}(a + ib, c + id) = \omega(a, c) - \omega(b, d) + i(\omega(b, c) + \omega(a, d)).$$

Note that each L_J , $J \in \mathcal{J}$, is Lagrangian with respect to $\omega^{\mathbb{C}}$ and, more generally, an $\omega^{\mathbb{C}}$ -Lagrangian complex vector subspace $L \subset \mathbb{C}^{2n}$ has the form $L = L_J$ for some $J \in \mathcal{J}$ if and only if $L \setminus 0 \subset \mathcal{W}$. The set of the subspaces L_J , $J \in \mathcal{J}$, is open in the set of $\omega^{\mathbb{C}}$ -Lagrangian complex vector subspaces of \mathbb{C}^{2n} .

Proposition 2.4. *Let $F : \mathcal{W} \rightarrow \mathbb{R}$ be a C^3 -smooth function. Assume that for any $\omega^{\mathbb{C}}$ -Lagrangian complex subspace $L_J \subset \mathcal{W}$, $J \in \mathcal{J}$, the restriction of F to $L_J \cap \mathcal{W}$ is a real Hermitian quadratic form. Then F is the restriction to \mathcal{W} of a real Hermitian quadratic form on \mathbb{C}^{2n} .*

The proof will be based on the following proposition. Denote the space of symmetric complex $n \times n$ -matrices by $\mathcal{S}_n(\mathbb{C})$. Let (z, w) be complex linear coordinates on the vector space $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$, where $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$. For an open connected neighborhood \mathcal{V} of 0 in $\mathcal{S}_n(\mathbb{C})$ put

$$\mathcal{C}_{\mathcal{V}} := \{(z, Az) \in \mathbb{C}^{2n} : z \neq 0, A \in \mathcal{V}\}.$$

One readily checks that the set $\mathcal{C}_{\mathcal{V}}$ is open and invariant under multiplication by non-zero scalars from \mathbb{C} .

Proposition 2.5. *Let \mathcal{V} be an open connected neighborhood of 0 in $\mathcal{S}_n(\mathbb{C})$, $n \geq 3$. Let $F : \mathcal{C}_{\mathcal{V}} \rightarrow \mathbb{R}$ be a C^3 -smooth function so that*

$$F(\lambda v) = |\lambda|^2 F(v) \quad \forall \lambda \in \mathbb{C} \setminus \{0\}, \quad \forall v \in \mathcal{C}_{\mathcal{V}}. \quad (13)$$

Assume that the restriction of F to any vector subspace

$$L_A := \{w = Az\} \subset \mathbb{C}^{2n}, \quad A \in \mathcal{V},$$

is a real Hermitian quadratic form. Then the function F is the restriction to $\mathcal{C}_{\mathcal{V}}$ of some real Hermitian quadratic form on \mathbb{C}^{2n} .

As one can easily see, Proposition 2.4 and Proposition 2.5 fail for $n = 1$. Nevertheless they do hold for $n = 2$ though in this case one needs to modify the proof of Proposition 2.5 slightly – see Remark 2.7 below.

Deducing Proposition 2.4 from Proposition 2.5:

Pick any $L := L_{J_0}$, $J_0 \in \mathcal{J}$. Using the linear Darboux theorem for complex symplectic forms choose complex coordinates $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$ on the vector space \mathbb{C}^{2n} so that $\omega^{\mathbb{C}} = dz \wedge dw$ and $L = \{w = 0\}$. Fix a sufficiently small open connected neighborhood \mathcal{V} of zero in $\mathcal{S}_n(\mathbb{C})$. Then $\mathcal{C}_{\mathcal{V}} \subset \mathcal{W}$ (since $L \setminus 0 \subset \mathcal{W}$) and F is C^3 -smooth on $\mathcal{C}_{\mathcal{V}}$ (since it is C^3 -smooth on \mathcal{W}). Moreover, any L_A , $A \in \mathcal{V}$, has the form $L_A = L_J$ for some $J \in \mathcal{J}$. Therefore, by Proposition 2.5, F coincides on $\mathcal{C}_{\mathcal{V}}$ with the restriction of some real Hermitian

quadratic form defined on \mathbb{C}^{2n} . Now letting J_0 vary inside \mathcal{J} we see that \mathcal{W} can be covered by open cones, invariant under the multiplication by non-zero complex scalars, on each of which F coincides with the restriction of a real Hermitian quadratic form. Since F is C^3 -smooth on \mathcal{W} and \mathcal{W} is path-connected, this yields that F coincides on the whole \mathcal{W} with the restriction of some real Hermitian quadratic form defined on \mathbb{C}^{2n} . ■

Proof of Proposition 2.5: Represent \mathbb{C}^{2n} as $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$, where z and w are coordinates along, respectively, the first and the second factors. Accordingly, we will write the vectors in \mathbb{C}^{2n} in the form $z \oplus w$. Given $A \in \mathcal{V}$, write

$$F(z, Az) = \langle \langle H(A)z, z \rangle \rangle,$$

where $H(A)$ is a Hermitian $n \times n$ -matrix and $\langle \langle \cdot, \cdot \rangle \rangle$ is the standard Hermitian inner product on \mathbb{C}^n . Since F is C^3 -smooth, the matrix $H(A)$ depends C^3 -smoothly on $A \in \mathcal{V}$.

We want to show that there exists a Hermitian $2n \times 2n$ -matrix \mathcal{H} such that for any $A \in \mathcal{V}$ and any z

$$F(z, Az) = \langle \mathcal{H}(z \oplus Az), z \oplus Az \rangle, \tag{14}$$

$\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product on \mathbb{C}^{2n} . Write \mathcal{H} as a matrix with four $n \times n$ -blocks:

$$\mathcal{H} = \begin{pmatrix} P & Q \\ \bar{Q}^T & R \end{pmatrix}.$$

Note that P and R are Hermitian $n \times n$ -matrices. Rewriting (14) we see that we need to show that

$$H(A) = P + QA + \bar{A}\bar{Q}^T + \bar{A}RA,$$

or, in terms of matrix coefficients,

$$H_{ij}(A) = P_{ij} + \sum_{\alpha} Q_{i\alpha}A_{\alpha j} + \sum_{\alpha} \bar{A}_{i\alpha}\bar{Q}_{j\alpha} + \sum_{\alpha,\beta} \bar{A}_{i\alpha}R_{\alpha\beta}A_{\beta j}, \quad \forall i, j = 1, \dots, n. \tag{15}$$

Remark 2.6. The coordinates on the space of symmetric matrices are A_{ij} with $i \leq j$. Nevertheless, we shall also use the coordinates A_{ij} with $i > j$ identifying them with A_{ji} : $A_{ij} = A_{ji}$.

For any $1 \leq s < t \leq n$ put

$$u_{ts}(z) = u_{st}(z) = z_s z_t, \quad u_{ss}(z) = -z_t^2, \quad u_{tt}(z) = -z_s^2.$$

Define a matrix $V^{s,t}(z) = (V_{ij}^{s,t}(z))_{i,j=1,\dots,n}$ by

$$V_{ij}^{s,t}(z) = \begin{cases} u_{ij}(z), & \text{if } i, j \in \{s, t\}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for any z

$$V^{s,t}(z)z = 0.$$

Thus for any s, t the expression

$$\langle H(A + \epsilon V^{s,t}(z))z, z \rangle = F(z, Az + \epsilon V^{s,t}(z)z) = F(z, Az)$$

does not depend on ϵ , where ϵ is a complex parameter. Differentiating by ϵ at $\epsilon = 0$ we get the following system of equations for every $i, j, s, t, s < t$, and any $A \in \mathcal{V}$:

$$\sum_{i,j} \left(\frac{\partial H_{ij}}{\partial A_{st}}(A)u_{st}(z)z_j\bar{z}_i + \frac{\partial H_{ij}}{\partial A_{ss}}(A)u_{ss}(z)z_j\bar{z}_i + \frac{\partial H_{ij}}{\partial A_{tt}}(A)u_{tt}(z)z_j\bar{z}_i + \right. \\ \left. + \frac{\partial H_{ij}}{\partial \bar{A}_{st}}(A)\bar{u}_{st}(z)z_j\bar{z}_i + \frac{\partial H_{ij}}{\partial \bar{A}_{ss}}(A)\bar{u}_{ss}(z)z_j\bar{z}_i + \frac{\partial H_{ij}}{\partial \bar{A}_{tt}}(A)\bar{u}_{tt}(z)z_j\bar{z}_i \right) = 0$$

for any z, \bar{z} . This is a polynomial in z, \bar{z} and, since it vanishes on an open set, all its coefficients have to vanish. These coefficients can be found by collecting similar terms in the last equation. A straightforward analysis of these terms yields that for any $i, j, s, t = 1, \dots, n, s < t$, the partial derivatives of H_{ij} at any $A \in \mathcal{V}$ satisfy the following equations:

$$\frac{\partial H_{it}}{\partial A_{st}} = \frac{\partial H_{is}}{\partial A_{ss}}, \tag{16}$$

$$\frac{\partial H_{is}}{\partial A_{st}} = \frac{\partial H_{it}}{\partial A_{tt}}, \tag{17}$$

$$\frac{\partial H_{tj}}{\partial \bar{A}_{st}} = \frac{\partial H_{sj}}{\partial \bar{A}_{ss}}, \tag{18}$$

$$\frac{\partial H_{sj}}{\partial \bar{A}_{st}} = \frac{\partial H_{tj}}{\partial \bar{A}_{tt}}. \tag{19}$$

Furthermore,

$$\frac{\partial H_{ij}}{\partial A_{\alpha\beta}} = 0 \text{ if } j \notin \{\alpha, \beta\}, \tag{20}$$

$$\frac{\partial H_{ij}}{\partial \bar{A}_{\alpha\beta}} = 0 \text{ if } i \notin \{\alpha, \beta\}. \tag{21}$$

Note that equations (16) and (17) can be summarized as

$$\frac{\partial H_{ij}}{\partial A_{lj}} = \frac{\partial H_{ir}}{\partial A_{lr}} \quad \forall i, j, l, r.$$

Differentiating this equation by A_{kj} and using (20) one gets that

$$\frac{\partial^2 H_{ij}}{\partial A_{kj} \partial A_{lj}} = \frac{\partial^2 H_{ir}}{\partial A_{kj} \partial A_{lr}} = 0,$$

provided $r \notin \{k, j\}$. But such an r always exists since $n \geq 3$. Thus

$$\frac{\partial^2 H_{ij}}{\partial A_{kj} \partial A_{lj}} = 0 \quad \forall i, j, k, l. \tag{22}$$

Similarly, using equations (18),(19) and (21) one gets that

$$\frac{\partial^2 H_{ij}}{\partial \bar{A}_{ik} \partial \bar{A}_{il}} = 0 \quad \forall i, j, k, l. \tag{23}$$

Observe now that for all i, j all the third derivatives of H_{ij} with respect to A_{st}, \bar{A}_{st} vanish. Indeed, in any third derivative either A -variables or \bar{A} -variables appear at least twice, and the result follows from the vanishing of the corresponding lower order derivatives, see formulas (20)–(23). Hence, each H_{ij} , as a function of the variables A_{st}, \bar{A}_{st} , is a (non-homogeneous) quadratic polynomial. The equations above on the first and second partial derivatives of H_{ij} allow to recover the coefficients of this quadratic polynomial and check that this polynomial indeed has the form (15). ■

Remark 2.7. In the case $n = 2$, a slightly more fine analysis of equations (16)–(21) yields the same result: H_{ij} 's are non-homogeneous quadratic polynomials of variables A_{st}, \bar{A}_{st} of the form (15).

A completely similar argument yields the following proposition, which we will need later and which is an analogue of Proposition 2.5 for functions on a real vector space.

Let (\cdot, \cdot) be the Euclidean inner product on \mathbb{R}^n . Set

$$\mathcal{U} := \{x \times y \in \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \mid (x, y) > 0\}.$$

Denote by $\mathcal{S}_n^+(\mathbb{R})$ the space of symmetric real positive-definite $n \times n$ -matrices.

Proposition 2.8. *Let $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $n \geq 2$, be a continuous function which is C^3 -smooth on \mathcal{U} . Assume that the restriction of F on any vector subspace*

$$L_A := \{y = Ax\} \subset \mathbb{R}^n \times \mathbb{R}^n, \quad A \in \mathcal{S}_n^+(\mathbb{R}),$$

is a quadratic form. Then there exists a quadratic form Q on \mathbb{R}^{2n} which coincides with F on \mathcal{U} .

Remark 2.9. Denote by \mathcal{L} the Lagrangian Grassmannian of the symplectic vector space \mathbb{R}^{2n} . For a connected open subset $U \subset \mathcal{L}$ consider the set $\mathcal{C} := \bigcup_{L \in U} L \setminus \{0\}$. Let $F : \mathcal{C} \rightarrow \mathbb{R}$ be a continuous function whose restriction to every $L \in U$ is a quadratic form. *Is it true that F is the restriction of some quadratic form defined on \mathbb{R}^{2n} to \mathcal{C} ?* The analogue over \mathbb{R} of Proposition 2.5 above yields the affirmative answer provided $n \geq 3$ and F is C^3 -smooth. We already mentioned that a small modification of our argument settles the $n = 2$ case, and there is a strong evidence that C^2 -smoothness suffices as well. It would be interesting to understand what is precisely the minimal regularity assumption on F for which the question admits the positive answer.

In case when $U = \mathcal{L}$, that is when F is defined on the whole \mathbb{R}^{2n} , the answer is affirmative: one uses the smoothing in the spirit of Section 2 in order

to reduce the problem to the case when F is not only continuous on \mathbb{R}^{2n} but also smooth on $\mathbb{R}^{2n} \setminus 0$. Interestingly enough, even when $U = \mathcal{L}$ there exist discontinuous F whose restrictions to every Lagrangian subspace are quadratic forms. These examples were constructed by Gleason in [11].

Let us mention that this circle of problems extends verbatim into the complex setting.

2.5. Proof of Proposition 2.2(i).

Note that the group $Sp(2n, \mathbb{R})$ acts transitively on the set of pairs $\xi \times \eta \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ such that $\omega(\xi, \eta) = 1$. Hence, the set $\{Y_{\xi, \eta} \mid \omega(\xi, \eta) = 1\}$ is an orbit of the adjoint action of $Sp(2n, \mathbb{R})$ on $\mathfrak{sp}(2n, \mathbb{R})$. Hence, by Proposition 2.3, we can assume without loss of generality that ζ is C^∞ -smooth on this orbit. This yields that F is C^∞ -smooth on the set $\{\xi + i\eta \in \mathbb{C}^{2n} \mid \omega(\xi, \eta) = 1\}$. One readily checks that

$$F(\lambda v) = |\lambda|^2 F(v) \quad \forall \lambda \in \mathbb{C}, \quad (24)$$

which yields that F is C^∞ -smooth on $\mathcal{W} = \{\xi + i\eta \in \mathbb{C}^{2n}, \omega(\xi, \eta) > 0\}$.

Lemma 2.10. *The restriction of F on any L_J , $J \in \mathcal{J}$, is a Hermitian quadratic form (for the definitions of L_J and \mathcal{J} see Section 2 above).*

Proof. Given $J \in \mathcal{J}$, the space of $A \in \mathfrak{sp}(2n, \mathbb{R})$ commuting with J is a Lie subalgebra $\mathfrak{u}(J)$ of $\mathfrak{sp}(2n, \mathbb{R})$ isomorphic to the real Lie algebra $\mathfrak{u}(n)$ of skew-Hermitian complex $n \times n$ -matrices. As one can easily check, $Y_{\xi, J\xi} \in \mathfrak{u}(J)$ for any $\xi \in \mathbb{R}^{2n}$ and any $J \in \mathcal{J}$. By Corollary 1.4, the restriction of ζ on $\mathfrak{u}(J) \cong \mathfrak{u}(n)$ is linear and we can write

$$F(\xi + iJ\xi) = \zeta(Y_{\xi, J\xi}) = \operatorname{tr}(HY_{\xi, J\xi})$$

for some $H \in M_{2n}(\mathbb{R})$. On the other hand,

$$\begin{aligned} \operatorname{tr}(HY_{\xi, J\xi}) &= \operatorname{tr}(HT_{\xi, \xi}) + \operatorname{tr}(HT_{J\xi, J\xi}) = \\ &= \operatorname{tr}T_{\xi, H\xi} + \operatorname{tr}T_{J\xi, HJ\xi} = \omega(\xi, H\xi) + \omega(J\xi, HJ\xi). \end{aligned}$$

Thus $F(\xi + iJ\xi) = \operatorname{tr}(HY_{\xi, J\xi})$ is a quadratic form in ξ . In view of (24), it means that the restriction of F on L_J is a real Hermitian quadratic form. \blacksquare

Combining Lemma 2.10 with Proposition 2.4 finishes the proof of Proposition 2.2(i). \blacksquare

2.6. Evaluating a continuous Lie quasi-state on $\mathfrak{gl}(n, \mathbb{R})$ on rank 1 matrices.

Here is another auxiliary proposition that we will need later.

Proposition 2.11. *Let ζ be a continuous Lie quasi-state on the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$, $n \geq 3$. Let $\mathcal{P}_1 \subset \mathfrak{gl}(n, \mathbb{R})$ be the set of matrices of rank 1. Then there exists a matrix $N \in M_n(\mathbb{R})$ so that $\zeta(A) = \operatorname{tr}NA$ for any $A \in \mathcal{P}_1$.*

Remark 2.12. If ζ is a continuous Lie quasi-state on $\mathfrak{gl}(n, \mathbb{R})$ and matrices $A, B, A + B \in \mathfrak{gl}(n, \mathbb{R})$ are diagonalizable over \mathbb{R} , then

$$\zeta(A + B) = \zeta(A) + \zeta(B).$$

Indeed, any matrix diagonalizable over \mathbb{R} is a sum of commuting rank-1 matrices.

Proof of Proposition 2.11.

By Proposition 2.3, we can assume without loss of generality that ζ is smooth on any orbit of the adjoint $GL_n(\mathbb{R})$ -action on $\mathfrak{gl}(n, \mathbb{R})$ and, in particular, on $\mathcal{P}'_1 := \{A \in \mathcal{P}_1 \mid trA = 1\}$ which is such an orbit.

As before we denote by (\cdot, \cdot) the Euclidean inner product on \mathbb{R}^n . Given $\xi, \eta \in \mathbb{R}^n$, define an operator $B_{\xi, \eta} \in \mathcal{P}_1$ on \mathbb{R}^n by

$$B_{\xi, \eta}(x) := (x, \eta)\xi.$$

One can easily check that if $(\xi, \eta) > 0$ then $B_{\frac{\xi}{\sqrt{(\xi, \eta)}}, \frac{\eta}{\sqrt{(\xi, \eta)}}} \in \mathcal{P}'_1$. Define a function $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ by

$$f(\xi, \eta) := \zeta(B_{\xi, \eta}).$$

Then

$$f(\lambda\xi, \eta) = f(\xi, \lambda\eta) = \lambda f(\xi, \eta) \quad \forall \lambda \in \mathbb{R}, \tag{25}$$

and therefore if $(\xi, \eta) > 0$, then

$$f(\xi, \eta) = \zeta(B_{\xi, \eta}) = (\xi, \eta)\zeta\left(B_{\frac{\xi}{\sqrt{(\xi, \eta)}}, \frac{\eta}{\sqrt{(\xi, \eta)}}}\right) = (\xi, \eta)f\left(\frac{\xi}{\sqrt{(\xi, \eta)}}, \frac{\eta}{\sqrt{(\xi, \eta)}}\right).$$

Since ζ is smooth on \mathcal{P}'_1 , we get that f is a smooth function on

$$\mathcal{U} := \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \mid (\xi, \eta) > 0\}.$$

As before, denote by $\mathcal{S}_n^+(\mathbb{R})$ the space of all symmetric real positive-definite $n \times n$ -matrices. For any $M \in \mathcal{S}_n^+(\mathbb{R})$ define an inner product $(\cdot, \cdot)_M$ on \mathbb{R}^n by

$$(x, y)_M := (Mx, y).$$

Denote by \mathcal{S}_M the space of all real $n \times n$ -matrices symmetric with respect to this inner product:

$$\mathcal{S}_M := \{A \in M_n(\mathbb{R}) \mid (Ax, y)_M = (x, Ay)_M \quad \forall x, y \in \mathbb{R}^n\}.$$

By Theorem 1.3, for any $M \in \mathcal{S}_n^+(\mathbb{R})$ there exists $T_M \in \mathcal{S}_M$ such that

$$\zeta(A) = trT_M A \quad \forall A \in \mathcal{S}_M.$$

Given $M \in \mathcal{S}_n^+(\mathbb{R})$, define

$$L_M := \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \mid \eta = M\xi\}.$$

One can easily check that for any $(\xi, \eta) \in L_M$ the operator $B_{\xi, \eta}$ lies in \mathcal{S}_M and hence

$$f(\xi, \eta) = \zeta(B_{\xi, \eta}) = \text{tr}(T_M B_{\xi, \eta}) = \text{tr} B_{T_M \xi, \eta} = (T_M \xi, \eta).$$

Thus the restriction of f on any L_M , $M \in \mathcal{S}_n^+(\mathbb{R})$, is a quadratic form. Applying Proposition 2.8 we get that there exists a quadratic form on \mathbb{R}^{2n} which coincides with f on \mathcal{U} . It follows from (25) that

$$f(\xi, \eta) = (N\xi, \eta) \tag{26}$$

for some matrix N .

Observe now that $B_{-\xi, \eta} = -B_{\xi, \eta}$ and hence $f(-\xi, \eta) = -f(\xi, \eta)$. Furthermore, $\xi \times \eta \in \mathcal{U}$ whenever $(-\xi) \times \eta \in \mathcal{U}^-$, where

$$\mathcal{U}^- := \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \mid (\xi, \eta) < 0\}.$$

Thus equality (26) holds on \mathcal{U}^- as well. Since $\mathcal{U} \cup \mathcal{U}^-$ is dense in \mathbb{R}^{2n} , we conclude that $f(\xi, \eta) = (N\xi, \eta)$ for all ξ and η which means that $\zeta(B_{\xi, \eta}) = \text{tr} N B_{\xi, \eta}$. This completes the proof. \blacksquare

2.7. Lie quasi-states on a Heisenberg algebra.

Here we will prove an auxiliary result which is, in fact, equivalent to the claim that any (not necessarily continuous) Lie quasi-state on a Heisenberg Lie algebra has to be linear.

Proposition 2.13. *Assume that the restriction of a function $\phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $n \geq 2$ to any isotropic vector subspace of $(\mathbb{R}^{2n}, \omega)$ is linear (here ω is a linear symplectic form on \mathbb{R}^{2n}). Then ϕ is linear.*

Proof. Let $e_1, \dots, e_n, f_1, \dots, f_n$ be the Darboux basis of \mathbb{R}^{2n} corresponding to the coordinate system $p_1, \dots, p_n, q_1, \dots, q_n$.

Any $a \in \mathbb{R}^{2n}$ can be written as

$$a = a_1 + \dots + a_n, \quad a_i \in \text{Span}(e_i, f_i), \quad i = 1, \dots, n,$$

so that

$$\omega(a_i, a_j) = 0, \quad \forall i \neq j,$$

and therefore

$$\phi(a) = \phi(a_1) + \dots + \phi(a_n).$$

Thus it suffices to prove that the restriction of ϕ on each $\text{Span}(e_i, f_i)$, $i = 1, \dots, n$, is linear.

Restricting ϕ on 1-dimensional vector subspaces of \mathbb{R}^{2n} – which are all isotropic, of course – we get that ϕ is homogeneous.

Denote

$$k := \phi(e_1) + \phi(f_1) - \phi(e_1 + f_1).$$

Fix $i \in \mathbb{N}$, $2 \leq i \leq n$. We need to show that

$$\phi(a + b) = \phi(a) + \phi(b) \quad \forall a, b \in \text{Span}(e_i, f_i). \tag{27}$$

If $\omega(a, b) = 0$, this follows from the hypothesis of the proposition. Otherwise, after permuting if necessary a and b , assume that $\omega(a, b) = -C^2 < 0$. Then the vectors $Ce_1 + a$ and $Cf_1 + b$ are ω -orthogonal. Hence,

$$\begin{aligned} \phi(Ce_1) + \phi(a) + \phi(Cf_1) + \phi(b) &= \phi(Ce_1 + a) + \phi(Cf_1 + b) = \\ &= \phi(Ce_1 + a + Cf_1 + b) = \phi(Ce_1 + Cf_1) + \phi(a + b). \end{aligned}$$

Therefore, by the homogeneity of ϕ ,

$$\phi(a + b) - \phi(a) - \phi(b) = Ck, \quad \forall a, b \in \text{Span}(e_i, f_i).$$

Substituting $-a, -b$ instead of a, b in the last equation and using again the homogeneity of ϕ we get that $Ck = 0$, hence $k = 0$, which yields (27). Thus the restriction of ϕ on each $\text{Span}(e_i, f_i)$, $i = 2, \dots, n$, is linear. Switching $i \neq 1$ and 1 we see immediately that the restriction of ϕ on $\text{Span}(e_1, f_1)$ is linear as well, which finishes the proof. ■

2.8. Proof of Proposition 2.2(ii).

Now we will deal with the function G defined in (9). Let us fix $\xi \in \mathbb{R}^{2n}$ and show that $G(\xi, \cdot) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is a linear function – since G is obviously symmetric with respect to ξ, η , this would show that G is a symmetric bilinear form on \mathbb{R}^{2n} . By Proposition 2.13, it is enough to show that

$$\omega(\eta_1, \eta_2) = 0 \implies G(\xi, c_1\eta_1 + c_2\eta_2) = c_1G(\xi, \eta_1) + c_2G(\xi, \eta_2) \quad \forall c_1, c_2 \in \mathbb{R}. \quad (28)$$

Let us, indeed, assume that $\omega(\eta_1, \eta_2) = 0$. Choose a Lagrangian subspace L_2 containing η_1, η_2 .

If $\xi \in L_2$, then a direct check shows that $[Z_{\xi, \eta_1}, Z_{\xi, \eta_2}] = 0$ and hence (28) follows by the definition of a Lie quasi-state.

If $\xi \notin L_2$, then, as one can easily check, there exists a Lagrangian subspace L_1 transversal to L_2 and containing ξ . Define a Lie subalgebra

$$\mathcal{R}(L_1, L_2) := \{A \in \mathfrak{sp}(2n, \mathbb{R}) \mid AL_1 \subset L_1, AL_2 \subset L_2\}$$

of $\mathfrak{sp}(2n, \mathbb{R})$. An easy check using the linear Darboux theorem shows that the mapping $A \mapsto A|_{L_1}$ establishes a Lie algebras isomorphism between $\mathcal{R}(L_1, L_2)$ and $\mathfrak{gl}(L_1) \approx \mathfrak{gl}(n, \mathbb{R})$. Furthermore, any transformation $Z_{\xi, v}$ with $v \in L_1$ lies in $\mathcal{R}(L_1, L_2)$. Its image under the above isomorphism has rank 1 provided $v \neq 0$. Applying Proposition 2.11 to the elements $Z_{\xi, \eta_1}, Z_{\xi, \eta_2}$ and $Z_{\xi, \eta_1 + \eta_2} = Z_{\xi, \eta_1} + Z_{\xi, \eta_2}$ of $\mathcal{R}(L_1, L_2)$ yields (28).

This finishes the proof of Proposition 2.2 and hence of Theorem 1.1. ■

3. Discontinuous Lie quasi-states

In this section we prove Theorem 1.2. We start from the following general observation.

Proposition 3.1. *Assume $L \subset \mathfrak{g}$ is an abelian subalgebra of a Lie algebra \mathfrak{g} and $L_0 \subset L$ is a vector subspace so that*

$$[x, v] = 0, x \in \mathfrak{g}, v \in L \setminus L_0 \implies x \in L.$$

Let $\alpha : L \rightarrow \mathbb{R}$ be a linear functional on L such that $\alpha \not\equiv 0$ and $\alpha|_{L_0} \equiv 0$. Define a functional $\zeta : \mathfrak{g} \rightarrow \mathbb{R}$ as

$$\zeta(x) = \begin{cases} 0, & \text{if } x \notin L, \\ \alpha(x), & \text{if } x \in L. \end{cases}$$

Then ζ is a Lie quasi-state.

Proof. Assume that $[x, y] = 0$. We have to show that

$$\zeta(x + y) = \zeta(x) + \zeta(y). \quad (29)$$

Vectors $x, y, x + y$ pairwise commute. If at least one of them does not lie in L , two others must lie in $(\mathfrak{g} \setminus L) \cup L_0$. Thus ζ vanishes on each of these vectors and so (29) holds. If $x, y \in L$ equation (29) follows from the linearity of α . ■

In the case of $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$ one can construct L, L_0 as in Proposition 3.1 in the following way.

Lemma 3.2. *Let $A \in \mathfrak{sp}(2n, \mathbb{R})$ and let $p(t)$ be a real polynomial. Then $p(A) \in \mathfrak{sp}(2n, \mathbb{R})$ if and only if $p(t)$ includes only the odd powers of t (i.e. p is an odd function of t).*

This is an immediate consequence of the fact that $A \in \mathfrak{sp}(2n, \mathbb{R})$ if and only if $AJ = -JA^T$, where J is the standard complex structure on \mathbb{R}^{2n} .

Take any matrix $A \in \mathfrak{sp}(2n, \mathbb{R})$ whose Jordan form is a $2n \times 2n$ Jordan block with the eigenvalue 0 (the existence of such an A is well-known – see e.g. [19], cf. [1], [14]). Define L_0, L as follows:

$$L := \{a_1 A + a_3 A^3 + \dots + a_{2n-1} A^{2n-1}, a_1, a_3, \dots, a_{2n-1} \in \mathbb{R}\},$$

$$L_0 := \{a_3 A^3 + \dots + a_{2n-1} A^{2n-1}, a_3, \dots, a_{2n-1} \in \mathbb{R}\} \subset L.$$

Let us show that subspaces L_0 and L satisfy the hypothesis of Proposition 3.1.

Lemma 3.3.

$$[x, v] = 0, v \in L \setminus L_0 \implies [x, A^{2m-1}] = 0 \quad \forall m = 1, \dots, n.$$

Proof. Without loss of generality,

$$v = A + a_3 A^3 + \dots + a_{2n-1} A^{2n-1}.$$

We prove the statement of the lemma using an inverse induction by m (from $m = n$ to $m = 1$).

For $m = n$ note that $v^{2n-1} = A^{2n-1}$ (we use here that $A^{2n} = 0$), and hence $[x, v^{2n-1}] = [x, A^{2n-1}] = 0$.

Assume now that $[x, A^{2j-1}] = 0$ for all $j = m + 1, \dots, n$. Put

$$B = A + \sum_{i=2}^m a_{2i-1} A^{2i-1} .$$

The inductive assumption together with $[x, v] = 0$ yields $[x, B] = 0$. But

$$B^{2m-1} = A^{2m-1} + \sum_{i \geq 2m+1} c_i A^i$$

for some coefficients c_i . Together with the inductive assumption this yields $[x, A^{2m-1}] = [x, B^{2m-1}] = 0$, as required. ■

Finally, we recall without proof a standard fact from the linear algebra.

Lemma 3.4. *Assume that a real square matrix Δ is a Jordan block with the eigenvalue zero. Then any matrix commuting with Δ is a polynomial of Δ .*

Proof of Theorem 1.2: Assume that $v \in L \setminus L_0$ and $[x, v] = 0$. Then, by Lemma 3.3, $[x, A] = 0$. Hence, by Lemma 3.4, x is a polynomial of A , which, by Lemma 3.2, belongs to L . This shows that subspaces L_0 and L satisfy the hypothesis of Proposition 3.1. Applying this proposition and varying the matrix A together with a functional α , we get a continuum of linearly independent discontinuous Lie quasi-states on $\mathfrak{sp}(2n, \mathbb{R})$. All of them are bounded in a neighborhood of zero. ■

4. *Ad*-invariant Lie quasi-states

In this section we will discuss Lie quasi-states invariant under the adjoint action of a Lie group.

Note that the adjoint actions on \mathfrak{g} of different connected Lie groups with the same Lie algebra \mathfrak{g} have the same orbits because such groups are locally isomorphic (see e.g. [13], p. 109) so that the local isomorphism intertwines their adjoint actions, and any connected Lie group is generated by a neighborhood of the identity. We say that a function on the Lie algebra \mathfrak{g} is *Ad*-invariant if it is constant on any orbit of the adjoint action on \mathfrak{g} of any Lie group with the Lie algebra \mathfrak{g} .

Note also that if G_1 is a (closed) Lie subgroup of a Lie group G_2 and $\mathfrak{g}_1 \subset \mathfrak{g}_2$ are the corresponding Lie algebras, then any *Ad*-invariant function on \mathfrak{g}_2 (that is, invariant with respect to the adjoint action of G_2) restricts to an *Ad*-invariant function on \mathfrak{g}_1 (that is, invariant with respect to the adjoint action of G_1).

Recall that a real Lie algebra is called *compact* if it is the Lie algebra of some compact real Lie group. A Lie group G will be called *Hermitian* (see e.g. [15]) if

- G is connected and non-compact;
- the Lie algebra of G is simple;
- the associated homogeneous space G/K , where K is the maximal compact subgroup of G , has a complex-manifold structure and G acts on it by holomorphic transformations.

There is a complete classification of the Lie algebras of Hermitian Lie groups (see e.g. [15]). In particular, $Sp(2n, \mathbb{R})$ is a Hermitian Lie group.

We will now classify Ad -invariant Lie quasi-states on compact Lie algebras and the Lie algebras of Hermitian Lie groups. Let us emphasize that a priori these quasi-states are not assumed to be continuous.

Theorem 4.1. *Any Ad -invariant Lie quasi-state on any compact Lie algebra is linear. If the compact Lie algebra has a trivial center (in particular, if it is simple), any Ad -invariant Lie quasi-state on it vanishes identically.*

Theorem 4.2. *Let G be a simply connected Hermitian Lie group and let \mathfrak{g} be its Lie algebra. Let μ be the unique (up to a multiplicative constant) non-trivial homogeneous quasi-morphism on G and let $\xi : \mathfrak{g} \rightarrow \mathbb{R}$ be its directional derivative:*

$$\xi(x) := \mu(\exp(x)) \quad \forall x \in \mathfrak{g} .$$

Then any Ad -invariant Lie quasi-state ζ on \mathfrak{g} is proportional to ξ .

We refer to [18] (cf. [3]) for the uniqueness (up to a multiplicative constant) of a non-trivial homogeneous quasi-morphism on G .

Proof of Theorem 4.1: Any compact Lie algebra can be represented as a direct sum of an abelian Lie algebra and a number of compact simple Lie algebras (see e.g. [13], p. 132). Any Lie quasi-state on an abelian Lie algebra is linear. This shows that the first claim of the theorem follows from the second one.

Let us show that any Ad -invariant Lie quasi-state ζ on a compact simple Lie algebra \mathfrak{g} vanishes identically – this would immediately imply the second claim of the theorem. Denote by G a compact connected Lie group whose Lie algebra is \mathfrak{g} . Any element of \mathfrak{g} lies in a Cartan subalgebra of \mathfrak{g} , that is the abelian subalgebra which is the Lie algebra of a maximal torus in G , and any two Cartan subalgebras are mapped into each other by the adjoint action of G (see e.g. [4], p.152). Thus it suffices to show that ζ vanishes on a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Since \mathfrak{h} is abelian, the restriction of ζ to \mathfrak{h} is linear. Since ζ is Ad -invariant, this linear function on \mathfrak{h} is invariant under the actions of Ad_g , $g \in G$, that preserve \mathfrak{h} , that is under the action of the Weyl group W of G on \mathfrak{h} . Since the action of W on the simple Lie algebra \mathfrak{h} has only trivial invariant subspaces (see e.g. [4], p. 172 and p.251), we have $\zeta|_{\mathfrak{g}} \equiv 0$, as required. ■

Proof of Theorem 4.2:

1) The structure of the Hermitian Lie group on G gives rise to the following features of the Cartan decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$, where \mathfrak{t} is the Lie algebra of a maximal compact subgroup K of G : The center \mathfrak{c} of \mathfrak{t} is one-dimensional and contains a preferred element, say J , so that ad_J preserves \mathfrak{p} and acts on \mathfrak{p} as a complex structure (see e.g. [15], Theorem 7.117 and p. 513). We shall normalize ξ and ζ by $\xi(J) = \zeta(J) = 1$.

2) Let us check that any Ad -invariant Lie quasi-state must vanish on \mathfrak{p} by using a trick by Ben Simon and Hartnick [3]: they noticed that since $(ad_J)^2 = -\mathbf{1}$, one has $\exp(\pi \cdot ad_J) = -\mathbf{1}$ on \mathfrak{p} (here π is the number $\pi = 3.14\dots$). It follows that $\zeta(x) = -\zeta(x)$ for all $x \in \mathfrak{p}$ which yields the claim.

3) Every element $x \in \mathfrak{g}$ can be written as $s + n$, where s is semi-simple, n is nilpotent and $[s, n] = 0$. By Jacobson-Morozov theorem ([15, p.620], pass to the complexification and use that n is a real nilpotent), there exists $y \in \mathfrak{g}$ with $[y, n] = n$. Therefore, setting $f = \exp(y)$ we get (passing to the series for the exponent) that $Ad_f(n) = e \cdot n$, where e is the number $e = 2.71\dots$. This yields

$$\zeta(n) = \zeta(e \cdot n) = e \cdot \zeta(n) .$$

Therefore $\zeta(n) = 0$ and so $\zeta(x) = \zeta(s)$. Thus it suffices to check that $\zeta = \xi$ on the semi-simple elements.

4) Every semi-simple element of \mathfrak{g} lies in some Cartan subalgebra. Every Cartan subalgebra of \mathfrak{g} is conjugate to a Cartan subalgebra of the form $\mathfrak{a} + \mathfrak{h}$ where $\mathfrak{a} \subset \mathfrak{p}$ and $\mathfrak{h} \subset \mathfrak{t}$ ([15, Proposition 6.59]). Thus every semi-simple element of \mathfrak{g} is conjugate to $a + h$ with $a \in \mathfrak{a}, h \in \mathfrak{h}$ and $[a, h] = 0$. Since ζ and ξ vanish on \mathfrak{a} (step 2), it suffices to show that $\zeta = \xi$ on \mathfrak{h} . In fact, we shall show that any Ad -invariant Lie quasi-state $\zeta : \mathfrak{g} \rightarrow \mathbb{R}$ vanishes on the algebra $\mathfrak{t}' := [\mathfrak{t}, \mathfrak{t}]$, which would yield the desired result in view of the decomposition $\mathfrak{t} = \mathfrak{t}' \oplus \mathfrak{c}$, where \mathfrak{c} is the center of \mathfrak{t} . Indeed, \mathfrak{t}' is a compact Lie algebra with a trivial center (see e.g. [15, p.513]) and the restriction of $\zeta|_{\mathfrak{t}'}$ is an Ad -invariant Lie quasi-state on \mathfrak{t}' . Therefore, by Theorem 4.1, ζ vanishes identically on \mathfrak{t}' , which finishes the proof.■

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