Journal of Lie Theory
Volume **19** (2009) 395–407

A New Example of a Group-Valued Moment Map

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Communicated by B. Ørsted

Abstract. The purpose of this paper is to construct a new example of group-valued moment maps. As the main tool for construction of such an example we use quasi-symplectic implosion, which was introduced by J. Hurtubise, L. Jeffrey and R. Sjamaar. More precisely we show that there are certain strata of $D\mathbf{Sp}(n)_{\mathrm{impl}}$, the universal imploded space, where it is singular but whose closure is a smooth quasi-Hamiltonian $\mathbf{Sp}(n) \times T$ space.
Mathematics Subject Classification 2000: 53D20.
Key Words and Phrases: quasi-Hamiltonian space, moment map, quaternionic projective space.

1. Introduction

The notion of a group-valued moment map, which was introduced by Alekseev, Malkin and Meinrenken [1], is a natural generalization of classical Hamiltonian spaces. In contrast to their classical counterpart, the moment map takes values in a Lie group instead of the dual of the Lie algebra. Quasi-Hamiltonian manifolds and their moment maps share many of the features of the Hamiltonian ones, such as reduction, cross-section and implosion.

The motivation of [1] for developing the theory of group-valued moment map came from one particularly important result. They show that the moduli space $M(\Sigma)$ of flat connections on a closed Riemann surface Σ of genus k is a quasi-Hamiltonian quotient of G^{2k}, which possesses a natural quasi-Hamiltonian G-structure. Therefore it is a symplectic manifold, a result earlier obtained by Atiyah and Bott. They go further generalizing it to the case $M(\Sigma, \mathcal{C})$, the moduli space of flat connection on a punctured Riemann surface with fixed conjugacy classes of holonomies associated to the boundary components.

By imitating symplectic implosion [3], J. Hurtubise, L. Jeffrey, and R. Sjamaar introduced the notion of group-valued implosion [6]. While an imploded cross-section is almost always singular, the quasi-symplectic quotients are not. For example, using result of [3], one can show the imploded cross-section of $D(G)$ is singular unless the commutator subgroup of G is a product of copies of $\mathbf{SU}(2)$.

*Research partially supported by NSF grant DMS 03-06495.

ISSN 0949–5932 / $2.50

Usually they are not even orbifolds unless the universal cover of $[G, G]$ is a product of copies of $\mathbf{SU}(2)$. However, like in case of singular quotients [8], using imploded cross-section theorem one can show that imploded spaces partition into symplectic manifolds.

It was observed in [6] that there are certain strata of $D(G)_{\mathrm{impl}}$ where it is singular, but whose closure is smooth. This observation led them to construct a new class of examples of quasi-Hamiltonian manifolds. In particular when G is A-type i.e. $G = \mathbf{SU}(n)$, there is a one dimensional face of the alcove whose corresponding stratum has a smooth closure diffeomorphic to S^{2n}. As a result, they showed that S^{2n} is a quasi-Hamiltonian $\mathbf{U}(n)$ - space. Motivated by this example, we study the implosion for type C groups, i.e. $G = \mathbf{Sp}(n)$ unitary quaternionic group. We show that there is a certain stratum of $D(G)_{\mathrm{impl}}$ which has a smooth closure diffeomorphic to $\mathbf{HP}^n$. Unlike in [6] for S^{2n}, which they obtained by gluing two copies of $\mathbf{C}^n$, an analogous cover given by strata in our case is not affine. This makes computations more difficult, without referring to Hamiltonian spaces. On the other hand, it also gives new examples of multiplicity-free quasi-Hamiltonian spaces with non-effective $G \times T$ action. It is interesting to observe that both of these classes of examples do not have a symplectic or complex structure.

The organization of this paper is as follows. In section 2 we recall the definition, basic properties and related examples of group-valued moment maps. In the section 3 we review the definition and basic properties of group-valued implosion. In this section we also give an example of "Spinning Sphere" constructed in [6], as motivating example of our own construction. In the section 4 we give a construction of quasi-Hamiltonian structure on $\mathbf{HP}^n$ using an implosion. In the section 5 we discuss the existence of other examples for imploded spaces of arbitrary Lie groups. Our notation is almost the same as in [6]. Any other additions, changes are given in a notation index given at the end of the paper.

Acknowledgments. It is a pleasure to express my gratitude to my advisor, Reyer Sjamaar, for suggesting this problem and for his guidance throughout the work. I also would like to thank Victor Guillemin, Eckhard Meinrenken and Anton Alekseev for useful discussions and valuable suggestions.

2. Quasi-Hamiltonian Manifolds

Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. Given a G-manifold M, there is an induced infinitesimal Lie algebra action

$$\xi_M(x) = \frac{d}{dt}|_{t=0} \exp(-t\xi).x \quad \text{for} \quad \xi \in \mathfrak{g}. \tag{2.1}$$

A Hamiltonian G-manifold is a symplectic G-manifold (M, ω) with an equivariant map, called moment map, $\Phi : M \to \mathfrak{g}^*$ satisfying the relation

$$\iota(\xi_M)\omega = d\langle \Phi, \xi \rangle. \tag{2.2}$$

Let $\theta_L, \theta_R \in \Omega^1(G, \mathfrak{g})$ be the Maurer-Cartan forms defined by $\theta_{L,g}(L(g)_*\xi) = \xi$ and $\theta_{R,g}(R(g)_*\xi) = \xi$ for $\xi \in \mathfrak{g}$, where $L(g)$ denotes the left multiplication and

$R(g)$ the right multiplication by g. Let $(\cdot\,,\cdot)$ be some choice of invariant inner product on $\mathfrak{g}$. Then there is a closed bi-invariant three-form on G

$$\chi = \frac{1}{12}(\theta_L, [\theta_L, \theta_L]) = \frac{1}{12}(\theta_R, [\theta_R, \theta_R])\,.$$

Definition 2.1. [1] A *quasi-Hamiltonian G-manifold* is a smooth G-manifold M equipped with a G-invariant two-form ω and a G-equivariant map $\Phi : M \to G$, called the *group-valued moment map*, such that the following properties hold:

(i) $d\omega = -\Phi^*\chi$

(ii) $ker\omega_x = \{\xi_M | \xi \in \mathrm{Ker}\,(Ad_{\Phi(x)} + 1)\}$ for all $x \in M$

(iii) $\iota(\xi_M)\omega = \frac{1}{2}\Phi^*(\theta_L + \theta_R, \xi)$

Basic examples of quasi-Hamiltonian manifolds are provided by conjugacy classes and the "double" $D(G)$. One can think of them as analogs of coadjoint orbits and cotangent bundle respectively.

Double D(G)

It is remarked in [1] that $D(G)$ plays the similar role in the category of quasi-Hamiltonian spaces as T^*G does in Hamiltonian spaces. As the space $D(G)$ is defined

$$D(G) := G \times G. \tag{2.3}$$

A $G \times G$ action on $D(G)$ is given by

$$(g_1, g_2).(u, v) = (g_1 u g_2^{-1}, Ad_{g_2} v). \tag{2.4}$$

Define a moment map $\Phi : D(G) \longrightarrow G \times G$ by $\Phi = \Phi_1 \times \Phi_2$ where

$$\Phi_1(u, v) = Ad_u v^{-1}, \quad \Phi_2(u, v) = v \tag{2.5}$$

and the two-form

$$\omega = -\frac{1}{2}(Ad_v u^*\theta_L, u^*\theta_L) - \frac{1}{2}(u^*\theta_L, v^*(\theta_L + \theta_R). \tag{2.6}$$

Note that there is a light change of coordinates from [1], namely $v \mapsto v^{-1}$. The following statement is proved [1, Proposition 3.2].

Proposition 2.2. *The $(D(G), \Phi, \omega)$ is a quasi-Hamiltonian $G \times G$-manifold.*

3. Group-valued imploded cross-section.

Let G be a simply connected compact Lie group with maximal torus T. Recall that the symplectic implosion is an "abelianization functor", which transforms a Hamiltonian G-manifold into a Hamiltonian T-space preserving some of properties of the manifold, but in the expense of producing singularities [3]. However, the singularities are not arbitrary in the sense that it "stratifies" into symplectic submanifolds in such a way that T action preserves the stratification.

Now let (M, ω, Φ) be a quasi-Hamiltonian G-space. In [1], it is shown that like in Hamiltonian case one can prove a convexity theorem. But one has to consider the image of moment map in an alcove instead of a Weyl chamber. Here the assumption of simply connectedness of the group is crucial, since otherwise the description of the space of conjugacy classes is quite complicated.

Let $\mathcal{C}$ be a Weyl chamber with its dual $\mathcal{C}^\vee$ in $\mathfrak{t}$. Let $\mathcal{A}$ be the unique (open) alcove contained in $\mathcal{C}^\vee$ such that $0 \in \bar{\mathcal{A}}$. Using the exponential map one can identify $\bar{\mathcal{A}}$ with space of conjugacy classes $T/W \cong G/AdG$, where W is the corresponding Weyl group. Let denote by G_g the centralizer of g in G. For points $m_1, m_2 \in \Phi^{-1}(\exp \bar{\mathcal{A}})$ define $m_1 \sim m_2$ if $m_2 = gm_1$ for some $g \in [G_{\Phi(m_1)}, G_{\Phi(m_1)}]$. One can check that $\sim$ is indeed an equivalence relation.

Definition 3.1. The imploded cross-section of M is the quotient space $M_{\mathrm{impl}} = \Phi^{-1}(\exp \bar{\mathcal{A}})/\sim$, equipped with the quotient topology. The imploded moment map Φ_{impl} is the continuous map $M_{\mathrm{impl}} \to T$ induced by Φ.

The space M_{impl} has many nice properties that smooth manifolds posses. It is Hausdorff, locally compact and second countable. The action of T preserves $\Phi^{-1}(\exp \bar{\mathcal{A}})$ and descends to a continuous action on M_{impl}.

We have decomposition of M_{impl} into orbit spaces

$$M_{\mathrm{impl}} = \coprod_{\sigma \leq \mathcal{A}} \Phi^{-1}(\exp \sigma)/[G_\sigma, G_\sigma], \tag{3.1}$$

where σ ranges over the faces of the alcove $\mathcal{A}$ and G_σ is the centralizer of $\exp \sigma$. Let us denote the piece $\Phi^{-1}(\exp \sigma)/[G_\sigma, G_\sigma]$ by X_σ. It is proved in [6] that each X_σ stratifies into symplectic manifolds. Let $\{X_i | i \in I\}$ be the collection of all strata of all pieces X_σ. Then the imploded cross-section M_{impl} is the disjoint union

$$M_{\mathrm{impl}} = \coprod_{i \in I} X_i \tag{3.2}$$

such that each piece X_i is a symplectic manifold.

Theorem 3.2. *[6, Theorem 3.17] The decomposition* (3.2) *of the imploded cross-section is a locally finite partition into locally closed subspaces, each of which is a symplectic manifold. There is a unique open stratum, which is dense in M_{impl} and symplectomorphic to the principal cross section of M. The action of the maximal torus T on M_{impl} preserves the decomposition and the imploded moment map $\Phi_{\mathrm{impl}} : M_{\mathrm{impl}} \to T$ restricts to a moment map for the T-action on each stratum.*

Therefore we call M_{impl} a *stratified quasi-Hamiltonian T-space.*

Imploded cross-section of the double.

In the example of quasi-Hamiltonian manifolds we have seen that $D(G) := G \times G$ possesses quasi-Hamiltonian $G \times G$-structure.

Let M be an arbitrary quasi-Hamiltonian G-space. By fusion operation [1], which one thinks of as product of quasi-Hamiltonian spaces, one obtains a quasi-Hamiltonian $G \times G$-manifold $M \circledast D(G)$. Now define $j : M \to M \circledast D(G)$ by $j(m) = (m, 1, \Phi(m))$. Then one of the main results of [6, Theorem 4.6] states:

Theorem 3.3 (universality of imploded double). *Let M be a quasi-Hamiltonian G-manifold. The map j induces a homeomorphism*

$$j_{\mathrm{impl}} : M_{\mathrm{impl}} \xrightarrow{\cong} (M \circledast D(G)_{\mathrm{impl}})//G$$

which maps strata to strata and whose restriction to each stratum is an isomorphism of quasi-Hamiltonian T-manifolds.

This result reduces the study of imploded spaces to the implosion of the double of the corresponding Lie group.

A smoothness criterion and quasi-Hamiltonian structure on S^{2n}

The implosion of double is a singular space, however the singularities on certain strata are removable. In order to show that one has to use the explicit correspondence between $D(G)$ and T^*G.

Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using bi-invariant inner product on $\mathfrak{g}$. Trivializing T^*G in a left-invariant manner, define $G \times G$-equivariant map $\mathcal{H} = id \times \exp : T^*G \to D(G)$. Let (T^*G, ω_0, Ψ_0) be a Hamiltonian $G \times G$ manifold, where ω_0 is the canonical symplectic form on cotangent bundle and a moment map $\Psi_0(g, \lambda) = (-Ad_g\lambda, \lambda)$. Let O be the set of all $\xi \in \mathfrak{t}$ with $(2\pi i)^{-1}\alpha(\xi) < 1$ for all positive roots α and $U = (AdG)O$.

Lemma 3.4. *[6, Proposition 4.15] The triple $(T^*G, \mathcal{H}^*\omega, \mathcal{H}^*\Psi)$ is the exponentiation of (T^*G, ω_0, Ψ_0). In particular, $G \times U$ is a quasi-Hamiltonian $G \times G$-manifold.*

Now using local diffeomorphism given by $\mathcal{H}$ and of [3, Proposition 6.15] we have

Theorem 3.5 (Smoothness criterion). *[6, Theorem 4.20] Let σ be a face of $\mathcal{A}$ satisfying $[G_\sigma, G_\sigma] \cong \mathbf{SU}(2)^k$ (resp. $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma] \cong \mathfrak{su}(2)^k$) for some $k \geq 0$ and possessing a vertex ξ such that* $\exp\xi$ *is central. Then $D(G)_{\mathrm{impl}}$ is a smooth quasi-Hamiltonian $G \times T$-manifold(resp. orbifold) in a neighborhood of the stratum corresponding to σ.*

A partial converse of this result is also true. Suppose that σ contains a vertex ξ such that $\exp\xi$ is central and $D(G)_{\mathrm{impl}}$ is smooth in a neighborhood of the corresponding stratum. Then $[G_\sigma, G_\sigma] \cong \mathbf{SU}(2)^k$. On the other hand, there are certain strata where $D(G)_{\mathrm{impl}}$ is singular, but their closure is a smooth quasi-Hamiltonian manifold.

Let G be $\mathbf{SU}(n)$. Consider an edge σ_{01} of an alcove with centralizer $G_{01} = \mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(n-1))$. By the argument above we know that for $n > 3$ the corresponding stratum X_{01} in X consists of genuine singularities. Nevertheless the following result asserts that it is a smooth quasi-Hamiltonian manifold and in fact diffeomorphic to S^{2n}.

Theorem 3.6. *[6, Theorem 4.26] The closure of the stratum X_{01} of $X = D\mathbf{SU}(n)_{\mathrm{impl}}$ is a smooth quasi-Hamiltonian $\mathbf{U}(n)$-manifold diffeomorphic to S^{2n}. Furthermore antipodal map of S^{2n} corresponds to involution of X_{01} obtained by lifting symmetry of the alcove $\mathcal{A}$ that reverses the edge σ_{01}.*

The proof of this result relies on two main facts. First of all, a symmetry of the alcove of $\mathbf{SU}(n)$. Namely, the center $\mathrm{Z}(\mathbf{SU}(n))$ acts transitively on vertices which makes possible to shift any vertex of the alcove to the origin. Second one is that a closure of big open stratum around each vertex is an affine space, which readily gives an affine cover. As we will see in the next section this is not the case for $\mathbf{Sp}(n)$.

4. Imploded cross-section of $\mathbf{Sp}(n)$

In the previous section we give a construction of a quasi-Hamiltonian structure on a sphere using an imploded cross-section. In this section we show using a different approach that a quaternionic projective space has a quasi-Hamiltonian structure.

Preliminaries

Let $\mathbf{H}$ the set of quaternionic numbers. Then $\mathbf{H}^*$, the set of nonzero quaternions, acts on $\mathbf{H}^{n+1} \setminus \{0\}$ by the multiplication on the right. The quotient space of this action is known as quaternionic projective space, denoted by $\mathbf{HP}^n$.

Let $G = \mathbf{Sp}(n)$, the group of unitary $n \times n$ matrices over the quaternions, with a maximal torus $T = \{\mathtt{diag}(e^{2\pi i x_1}, ..., e^{2\pi i x_n})\}$. As an invariant inner product on $\mathfrak{sp}(n)$ we take

$$(\xi, \eta) = -(4\pi^2)^{-1}\mathrm{Re}(\mathtt{tr}(\xi \cdot \eta)) \quad \text{for} \quad \xi, \eta \in \mathfrak{sp}(n). \tag{4.1}$$

If we identify $\mathfrak{t}$ with $\mathbf{R}^n$ via the map $x \mapsto 2\pi i\, \mathtt{diag}(x_1, ..., x_n)$, then the simple roots have form

$$(2\pi i)^{-1}\alpha_k(x) = x_k - x_{k+1} \quad \text{for} \quad k = 1, ..., n-1 \quad \text{and} \quad (2\pi i)^{-1}\alpha_n(x) = 2x_n$$

with a maximal root $(2\pi i)^{-1}\tilde{\alpha}(x) = 2x_1$. The corresponding alcove is the n-simplex $0 < x_n < ... < x_1 < 1/2$. By slightly abusing our notation, we denote by σ_{01} the edge of the simplex with vertices σ_0, σ_1 which exponentiate to torus elements of the form $\mathtt{diag}(t_1, 1, ..., 1)$ with vertices $I = \mathtt{diag}(1, 1, ..., 1)$ and $\mathtt{diag}(-1, 1, ..., 1)$ correspondingly. Their centralizers are respectively $G_{01} = \mathbf{U}(1) \times \mathbf{Sp}(n-1)$, $G_0 = \mathbf{Sp}(n)$ and $G_1 = \mathbf{Sp}(1) \times \mathbf{Sp}(n-1)$. One can immediately see that $\exp(\sigma_1)$ is not central, which implies that there is no Weyl group element shifting it to the origin. Define a map $\mathcal{H} : G \times \mathcal{A} \to \coprod_{\sigma \leq \mathcal{A}} G/[G_\sigma, G_\sigma]$ by $\mathcal{H}(g, x) = g[G_{\exp x}, G_{\exp x}]$. By (3.1) the stratum corresponding to a general face is given by $X_\sigma = G/[G_\sigma, G_\sigma] \times \exp\sigma$ and therefore we have

$$X_0 = \mathcal{H}(I,0) \times \{I\} \cong \{\mathtt{pt}\}, \tag{4.2}$$

$$X_{01} = \mathbf{Sp}(n)/\mathbf{Sp}(n-1) \times \{\mathtt{diag}(e^{2\pi i x_1}, 1, ..., 1) | x_1 \in (0, \frac{1}{2})\} \cong S^{4n-1} \times (0,1), \tag{4.3}$$

$$X_1 = \mathbf{Sp}(n)/(\mathbf{Sp}(n-1) \times \mathbf{Sp}(1)) \times \mathtt{diag}(-1, 1, ..., 1) \cong \mathbf{HP}^{n-1}. \tag{4.4}$$

Consider the closure of the stratum corresponding to σ_{01}, which is $\bar{X}_{01} = X_0 \bigsqcup X_{01} \bigsqcup X_1$. Notice that we have a bijections $X_0 \bigsqcup X_{01} \cong \mathbf{H}^n$ and $X_1 \cong \mathbf{HP}^{n-1}$, and one would expect that $\bar{X}_{01}$ and $\mathbf{HP}^n$ are homeomorphic. However, the covering obtained from strata, $U_0 = X_0 \bigsqcup X_{01}$ and $U_1 = X_{01} \bigsqcup X_1$ is not an affine cover, since U_1 is a $\mathbf{H}$-line bundle over $\mathbf{HP}^{n-1}$. Hence, we have to construct directly a homeomorphism from $\bar{X}_{01}$ to $\mathbf{HP}^n$.

Homeomorphism between $\bar{X}_{01}$ and $\mathbf{HP}^n$

Define a map

$$\mathcal{G} : X_{01} \to \mathbf{HP}^n \quad , \quad (\mathcal{H}(g,x), \exp x) \mapsto \left[\sqrt{(1-2x_1)}, \sqrt{2x_1} g.v\right], \tag{4.5}$$

where $v = (1, 0, ..., 0) \in \mathbf{H}^n$. One can easily check that it is well-defined, i.e. does not depend on the equivalence class of g in $\mathbf{Sp}(n)/\mathbf{Sp}(n-1)$. Moreover, $\mathcal{G}$ is a continuous, injective map on X_{01} (or $0 < x_1 < \frac{1}{2}$) and can be extended continuously to a bijective map on $\bar{X}_{01}$. Indeed, on X_0 (or $x_1 = 0$) we have $\mathcal{G}(\mathcal{H}(g,x), x) = [1, 0, ..., 0]$ and on X_1 (or $x_1 = \frac{1}{2}$) we have $\mathcal{G}(\mathcal{H}(g,x), x) = [0, g.v]$.

It is known (see [6]) that an imploded space is Hausdorff, locally compact and second countable. Thus, we can prove

Theorem 4.1. *The map $\mathcal{G} : \bar{X}_{01} \to \mathbf{HP}^n$ is a homeomorphism.*

This statement allows us to define a smooth structure on $\bar{X}_{01}$ as pull back of the smooth structure on $\mathbf{HP}^n$ via $\mathcal{G}$. Now we would like to give a description of the inverse to $\mathcal{G}$. For this we use the following decomposition $Y_0 \bigsqcup Y_{01} \bigsqcup Y_1$ of $\mathbf{HP}^n$, where

$$Y_0 = \{[1, 0, ..., 0]\},$$

$$Y_{01} = \{Z \in \mathbf{HP}^n | \, Z_1 \neq 0 \, \text{and} \, \sum_{l=2}^{n+1} |Z_l|^2 \neq 0\},$$

$$Y_1 = \{Z \in \mathbf{HP}^n | \, Z_1 = 0\}.$$

We define a map $\mathcal{F} : \mathbf{HP}^n \to \bar{X}_{01}$ stratawise as follows. First, we map

$$Y_0 \mapsto X_0.$$

Second, on Y_{01}

$$\begin{tikzcd} & X_{01} \arrow[d] \\ Y_{01} \arrow[r] \arrow[ur, dashed, "\mathcal{F}|_{Y_{01}}"] & S^{4n-1} \times \exp\sigma_{01} \end{tikzcd} \tag{4.6}$$

it is given through homeomorphisms $Y_{01} \to S^{4n-1} \times \exp\sigma_{01}$ and $X_{01} \to S^{4n-1} \times \exp\sigma_{01}$, defined respectively

$$[Z_1, Z_2, ..., Z_{n+1}] \mapsto \left(\frac{Z_2\bar{Z}_1}{|Z_1|\sqrt{\sum_{l=2}^{n+1}|Z_l|^2}}, ..., \frac{Z_{n+1}\bar{Z}_1}{|Z_1|\sqrt{\sum_{l=2}^{n+1}|Z_l|^2}} \right) \times \mathtt{diag}(e^{\lambda\pi i}, 1, ..., 1)$$

and

$$\Big(\mathcal{H}(g,x), \exp x\Big) \mapsto (g.v, \exp x)$$

where λ

$$\lambda = \frac{\sum_{l=2}^{n+1}|Z_l|^2}{\sum_{l=1}^{n+1}|Z_l|^2}. \tag{4.7}$$

Therefore $\mathcal{F}|_{Y_{01}}$ is of the form

$$[Z_1, Z_2, ..., Z_{n+1}] \mapsto \Big(\mathcal{H}(g,x), \exp x\Big)$$

where g is a matrix $(f_{p,q}(Z))_{p,q=1}^n$ whose first column is

$$f_{p,1}(Z) = \frac{Z_{p+1}\bar{Z}_1}{|Z_1|\sqrt{\sum_{l=2}^{n+1}|Z_l|^2}} \quad \text{for} \quad p = 1, ..., n \tag{4.8}$$

and $x = (\lambda/2, 0, ...0)$. Third, for the stratum Y_1 we have a commutative diagram

$$\begin{tikzcd} & X_1 \arrow[d] \\ Y_1 \arrow[r] \arrow[ur, dashed, "\mathcal{F}|_{Y_1}"] & \mathbf{HP}^{n-1} \times \mathtt{diag}(-1, 1, ..., 1) \end{tikzcd} \tag{4.9}$$

where $\mathcal{F}_{Y_1}$ is determined by homeomorphisms $Y_1 \to \mathbf{HP}^{n-1} \times \mathtt{diag}(-1, 1, ..., 1)$ and $X_1 \to \mathbf{HP}^{n-1} \times \mathtt{diag}(-1, 1, ..., 1)$ given by

$$[0, Z_2, ..., Z_{n+1}] \mapsto ([Z_2, ..., Z_{n+1}], \mathtt{diag}(-1, 1, ..., 1))$$

and

$$\Big(\mathcal{H}(g,x), \exp x\Big) \mapsto (g.v, \exp x)$$

respectively. Therefore $\mathcal{F}|_{Y_1}$ is of the form

$$[0, Z_2, ..., Z_{n+1}] \mapsto \mathcal{H}(g,x) \times \mathtt{diag}(-1, 1, ..., 1) \quad \text{in} \quad X_1,$$

where g is a matrix $f_{p,q}(Z)$ whose first column is

$$f_{p,1}(Z) = \frac{Z_{p+1}}{\sqrt{\sum_{l=2}^{n+1}|Z_l|^2}} \quad \text{for} \quad p = 1, ..., n$$

and $x = (1/2, 0, ..., 0)$. So we have

Lemma 4.2. *The map $\mathcal{F}$ is the inverse of $\mathcal{G}$ and hence smooth.*

Now if we define an action of $\mathbf{Sp}(n) \times T$ on $\mathbf{HP}^n$ by

$$(g,t)[Z_1, ..., Z_{n+1}] = [Z_1 t_1, g.(Z_2, ..., Z_{n+1})]\ , \tag{4.10}$$

where $t = \mathtt{diag}(t_1, ..., t_n)$ then one can easily show

Lemma 4.3. *The map $\mathcal{G}$, hence $\mathcal{F}$, is a $\mathbf{Sp}(n) \times T$-equivariant.*

Quasi-Hamiltonian structure on $\mathbf{HP}^n$

First, let us recall the quasi-Hamiltonian structure on a stratum X_σ. Since the moment map Φ_2 defined as in (2.5) is transversal to all faces of the alcove, using quasi-symplectic reduction with quasi-Hamiltonian cross-section theorem one can show

Lemma 4.4. *[6, Lemma 4.5] For every $\sigma \le \mathcal{A}$, the subspace $X_\sigma = G/[G_\sigma, G_\sigma] \times \exp\sigma$ of $D(G)_{\mathrm{impl}}$ is a quasi-Hamiltonian $G \times T$-manifold. The moment map $X_\sigma \to G \times T$ is the restriction to X_σ of the continuous map $\Phi_{\mathrm{impl}} \to G \times T$ induced by $\Phi : D(G) \to G \times G$.*

Next we compute the corresponding 2-form ω_σ on X_σ. Let $(g, \exp x)$ be an arbitrary point on X_σ. A tangent vector at $(g, \exp x)$ is of the form

$$((L(g)_*\xi, (L(\exp x))_*\eta)$$

where $\xi \in \mathfrak{g}$ and $\eta \in \zeta + \mathfrak{z}(\mathfrak{g}_\sigma)$ for some $\zeta \in \mathfrak{z}(\mathfrak{g}_\sigma)^\perp = [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$ [6, Lemma A.3]. Then a simple calculation yields

$$\begin{aligned}(\omega_\sigma)_{(g,\exp x)}&((L(g)_*\xi_1, L(\exp x)_*\eta_1), (L(g)_*\xi_2, L(\exp x)_*\eta_2)) \\ &= -\frac{1}{2}((Ad_{\exp x} - Ad_{\exp(-x)})\xi_1, \xi_2) - (\xi_1, \eta_2) + (\xi_2, \eta_1).\end{aligned} \tag{4.11}$$

One can check that it does not depend on the equivalence class of ξ_i in $\mathfrak{g}/[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$. Consider the 2-form ω_{01} on an open stratum X_{01}. In what follows we compute the pull back of this 2-form via $\mathcal{F}$ and show that it extends smoothly to all of $\mathbf{HP}^n$. Since ω_{01} is $\mathbf{Sp}(n) \times T$-invariant, it suffices to consider vectors of the form $z_0 = [s, 1, 0, ..., 0]$, where

$$s = \frac{|Z_1|}{\sqrt{\sum_{l=2}^{n+1} |Z_l|^2}}..$$

The tangent space at z_0 will be

$$T_{z_0}\mathbf{HP}^n = \{(w_1, ..., w_{n+1}) \in \mathbf{H}^{n+1} | sw_1 + w_2 = 0\}, \tag{4.12}$$

where $w_l = w_{l1} + w_{l2}i + w_{l3}j + w_{l4}k$. Let us first find the corresponding tangent vectors at $\mathcal{F}(z_0)$, or more precisely corresponding pull-backs ξ_i, η_i to elements of Lie algebra as in (4.11). Let v and w be tangent vectors of form (4.12). Note, since

$$\mathcal{F}(z_0) = (\mathcal{H}(I, (\lambda/2, 0, ..., 0)), \mathtt{diag}(\exp(\lambda\pi i), 1, ..., 1)) =: (\mathcal{H}(g, x), x), \tag{4.13}$$

the first component of the image is already an element of Lie algebra, while the second one has to be translated by an appropriate element of Lie group (that is $\exp x$). Denote by $(A^v_{p,q})^n_{p,q=1}$ and $(B^v_{p,q})^n_{p,q=1}$ $((A^w_{p,q})^n_{p,q=1}$ and $(B^w_{p,q})^n_{p,q=1})$ the matrix representation of ξ_1 and η_1 (correspondingly ξ_2 and η_2). Then substituting these to the first term of (4.11) and using (4.13) expression for $\mathcal{F}(z_0)$ we have

$$((Ad_{\exp x} - Ad_{\exp(-x)})\xi_1, \xi_2) = (4\pi^2)^{-1}\mathrm{Re}\Big(\Big[\exp(\lambda\pi i)A^v_{11}\exp(-\lambda\pi i) - \exp(-\lambda\pi i)A^v_{11}\exp(\lambda\pi i)\Big]\bar{A}^w_{11} - \Big[\exp(\lambda\pi i) - \exp(-\lambda\pi i)\Big]\sum_{p=2}^{n} A^v_{1p}A^w_{p1} + \sum_{p=2}^{n} A^v_{1p}\Big[\exp(-\lambda\pi i) - \exp(\lambda\pi i)\Big]\bar{A}^w_{p1}\Big), \tag{4.14}$$

where the inner product is given by (4.1) and

$$A^v_{11} = -(s+s^{-1})(v_{12}i + v_{13}j + v_{14}k), \tag{4.15}$$

and by the skew-symmetry

$$A^v_{p1} = -A^v_{1p} = v_{(p+1)1} + v_{(p+1)2}i + v_{(p+1)3}j + v_{(p+1)4}k. \tag{4.16}$$

There are similar relations to (4.15) and (4.16) if we replace v by w. Thus we can rewrite (4.14) in the following form

$$((Ad_{\exp x} - Ad_{\exp(-x)})\xi_1, \xi_2) = \frac{\sin(2\pi\lambda)}{2\pi^2}(s+s^{-1})(v_{13}w_{14} - w_{13}v_{14}) - \frac{\sin(\pi\lambda)}{\pi^2}\sum_{p=3}^{n+1}(v_{p1}w_{p2} - w_{p1}v_{p2} - v_{p3}w_{p4} + w_{p3}v_{p4}). \tag{4.17}$$

Hence, the corresponding two-form will be

$$\frac{\sin(2\pi\lambda)}{2\pi^2}(s+s^{-1})dx_{13}dx_{14} - \frac{\sin(\pi\lambda)}{\pi^2}\sum_{p=3}^{n+1}(dx_{p1}dx_{p2} - dx_{p3}dx_{p4}), \tag{4.18}$$

where x's are just real coordinates for Z's, such that $Z_l = x_{l1} + x_{l2}i + x_{l3}j + x_{l4}k$. Note we write two-forms multiplicatively without usual wedge product. For the remaining part of (4.11) we have:

$$-(\xi_1, \eta_2) + (\xi_2, \eta_1) = (4\pi^2)^{-1}\mathrm{Re}(-A^v_{11}\bar{B}^w_{11} + A^w_{11}\bar{B}^v_{11}), \tag{4.19}$$

where

$$B^v_{11} = -\frac{2\pi i s}{s^2+1}v_{11}, \tag{4.20}$$

and the corresponding two-form will be

$$\frac{1}{2\pi}dx_{11}dx_{12}. \tag{4.21}$$

Combining (4.18) with (4.21) yields

$$\mathcal{F}^*\omega_{01} = \tfrac{1}{2\pi}dx_{11}dx_{12} - \tfrac{\sin(2\pi\lambda)}{2\pi^2}(s+s^{-1})dx_{13}dx_{14} + \tfrac{\sin(\pi\lambda)}{\pi^2}\textstyle\sum_{p=3}^{n+1}(dx_{p1}dx_{p2} - dx_{p3}dx_{p4}). \tag{4.22}$$

It is a smooth two-form defined on open dense subset Y_{01} of $\mathbf{HP}^n$. Moreover we can show

Lemma 4.5. *The two-form $\mathcal{F}^*\omega_{01}$ extends smoothly to all of* $\mathbf{HP}^n$.

Proof. It suffices to check two critical cases $Z_1 = 0$, a line at infinity, and $[1, 0, ..., 0]$, a point at infinity. As $|Z_1|$ approaches to 0, λ tends to 1 and therefore the third expression on the right hand side of (4.22) vanishes. Now since $\lambda = 1 - s^2$, we have $s \to 0$ and hence

$$\frac{\sin(2\pi\lambda)}{2\pi^2}(s + s^{-1}) \longrightarrow -\frac{1}{\pi}.$$

So in the neighborhood of $Z_1 = 0$ the two-form $\mathcal{F}^*\omega_{01}$ can be written as

$$\frac{1}{2\pi}dx_{11}dx_{12} + \frac{1}{\pi}dx_{13}dx_{14}.$$

In the similar fashion one can show that in the neighborhood of $[1, 0, ..., 0]$ it is given by:

$$\frac{1}{2\pi}dx_{11}dx_{12} - \frac{1}{\pi}dx_{13}dx_{14}.$$

This finishes the proof of this lemma. ∎

Notice that the obtained 2-form is given in dehomogenized coordinates

$$\left[\frac{|Z_1|}{\sqrt{\sum_{l=2}^{n+1}|Z_l|^2}}, \frac{Z_2\bar{Z}_1}{|Z_1|\sqrt{\sum_{l=2}^{n+1}|Z_l|^2}}, \ldots\ldots, \frac{Z_{n+1}\bar{Z}_1}{|Z_1|\sqrt{\sum_{l=2}^{n+1}|Z_l|^2}}\right]. \tag{4.23}$$

For $Q = q_1 + q_2 i + q_3 j + q_4 k$ we define $\mathrm{Im}_i(Q) = q_2$, then we have

$$\mathrm{Im}_i(d\bar{Z}_p dZ_p) = 2(dx_{p1}dx_{p2} - dx_{p3}dx_{p4}). \tag{4.24}$$

Now using (4.23) and (4.24) in homogenous coordinates the first two terms vanish, our 2-form will take the form

$$\begin{aligned}&\frac{\sin(\lambda\pi)}{\pi^2}\left(|Z_1|^2\sum_{l=2}^{n+1}|Z_l|^2\right)^{-1}\left[\sum_{p=3}^{n+1}|Z_p|^2\mathrm{Im}_i(dZ_1 d\bar{Z}_1) -\right.\\ &\mathrm{Im}_i(Z_1 d\bar{Z}_p dZ_p \bar{Z}_1) + \left(\sum_{p=3}^{n+1}|Z_p|^2\mathrm{Im}_i((Z_1 d\bar{Z}_1)) - \mathrm{Im}_i(Z_1\bar{Z}_p dZ_p\bar{Z}_1)\right)\times\\ &\left.\left(\frac{Z_1 d\bar{Z}_1 + dZ_1\bar{Z}_1}{|Z_1|^2} + \frac{\sum_{l=2}^{n+1}(Z_l d\bar{Z}_l + dZ_l\bar{Z}_l)}{\sum_{l=2}^{n+1}|Z_l|^2}\right)\right].\end{aligned} \tag{4.25}$$

The last thing we need to show that there is well-defined smooth moment map. Define a map $\Phi : \mathbf{HP}^n \to \mathbf{Sp}(n) \times T$ stratawise, such that the following diagram commutes

$$\begin{tikzcd} & X_\sigma \arrow[dl, "\mathcal{G}|_{X_\sigma}"'] \arrow[dr, "\Phi_\sigma"] & \\ Y_\sigma \arrow[rr, dashed, "\Phi"] & & \mathbf{Sp}(n) \times T, \end{tikzcd} \tag{4.26}$$

for each face σ in the closure. Then on each strata it has the form

$$[Z_1, ..., Z_{n+1}] \mapsto (AB^{-1}A^{-1}, B) \tag{4.27}$$

where $B = (B_{pq})_{p,q=1}^n$

$$B = \mathtt{diag}(\exp(\lambda\pi i), 1, ..., 1) \tag{4.28}$$

and $A = (A_{pq})_{p,q=1}^n$ is any representative of $\mathcal{H}(g, x)$. One can easily check that it does not depend on the representative of $\mathcal{H}(g, x)$. Evidently, Φ is uniquely determined and $\mathbf{Sp}(n) \times T$-equivariant. We have to show that it is smooth. From the construction one can see that B_{pq} are smooth. As for the first component of Φ, on Y_{01} we have

$$AB^{-1}A^{-1} = \mathrm{Id}_n + C,$$

where $C = (C_{pq})_{p,q=1}^n$:

$$C_{pq} = A_{p1}\bar{B}_{11}\bar{A}_{q1} - A_{p1}\bar{A}_{q1},$$

or more precisely

$$C_{pq} = \left(|Z_1|^2 \sum_{l=2}^{n+1} |Z_l|^2\right)^{-1} Z_{p+1}\Big[\bar{Z}_1 \exp(\pi i\lambda) Z_1 - |Z_1|^2\Big]\bar{Z}_{q+1}. \tag{4.29}$$

We can easily see that it is smooth for $Z_1 \neq 0$ and $\sum_{l=2}^{n+1} |Z_l|^2 \neq 0$. Hence it is smooth on Y_{01}. Using almost the same argument as in Lemma 4.5 we can show it is smooth in these two cases as well. Now summarizing these facts we have

Theorem 4.6. *The closure of the stratum X_{01} of $X = D\mathbf{Sp}(n)_{\mathrm{impl}}$ is a smooth quasi-Hamiltonian $\mathbf{Sp}(n) \times T$-manifold diffeomorphic to n-dimensional quaternionic projective space with the 2-form and the moment map determined by* (4.25) *and* (4.27) *respectively.*

5. Other Lie groups

In this paper and in [6] were discussed a closure of certain stratum of imploded spaces of type A and C Lie groups. It is natural to ask whether there are other examples of quasi-Hamiltonian spaces appearing in this context. Surprisingly, the answer to this question is negative [4]. To be precise, let G be any connected and simply-connected Lie group. We showed using the argument in [7] and computation of dimensions of strata that a stratum of DG_{impl} has a smooth closure only in above mentioned examples. That is, S^{2n} and $\mathbf{HP}^n$ are the only examples where stratum has a smooth closure.

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Received April 1, 2009
and in final form July 30, 2009