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Nilpotent Lie Algebras without $\mathbb{Q}$-structures

Hatem Hamrouni and Salah Souissi

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Abstract. We give a list of examples of non isomorphic n-dimensional nilpotent Lie algebras without $\mathbb{Q}$-structures, for all $n \geq 7$.
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A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is said to be a $\mathbb{Q}$-algebra or a Lie algebra with a $\mathbb{Q}$-structure, if it admits rational structure constants with respect to some basis of $\mathfrak{g}$. As is well known in the case of nilpotent Lie algebras, the existence of $\mathbb{Q}$-structures is equivalent to that of a discrete uniform subgroups of the associated connected, simply connected Lie group ([5]). Every nilpotent Lie algebra of dimension less or equal to six is a $\mathbb{Q}$-algebra. In 1949, A. I. Malcev [5] gave for each integer $n \geq 16$ an example of a nilpotent Lie algebra of dimension n which is not a $\mathbb{Q}$-algebra. Later, in 1962, C. Y. Chao [2] showed a procedure for constructing nilpotent Lie algebras of dimension $n \geq 10$ with no rational structures. For the case of seventh dimension, there are uncountably many nilpotent Lie algebras (up to isomorphism). Since the set of Lie algebras, considered up to isomorphism, which admit $\mathbb{Q}$-structures, is countable, then there are seven dimensional nilpotent Lie algebras without rational structure ([6, p. 46-47]). The purpose of this short note is to produce a list of examples of non isomorphic seven dimensional nilpotent Lie algebras without rational structures. As a straight applications, for any $n \geq 8$, we construct n-dimensional nilpotent Lie algebras which do not admit any $\mathbb{Q}$-structure.

We denote by $\mathscr{D}$ the subset of $\mathrm{Mat}(6, \mathbb{R})$ consisting of all skew-symmetric matrices $\alpha = (\alpha_{ij})_{1 \leq i,j \leq 6}$ satisfying

$$\begin{aligned} -\alpha_{23}\alpha_{15} + \alpha_{13}\alpha_{24} &= 0 \\ \alpha_{12}\alpha_{34} - \alpha_{24}\alpha_{16} + \alpha_{14}\alpha_{25} &= 0 \\ \alpha_{ij} &= 0 \qquad (i + j > 7) \\ \alpha_{ij} &\neq 0 \qquad (i \neq j, i + j \leq 7) \end{aligned}$$

For $\alpha \in \mathscr{D}$, let us denote by $\mathfrak{g}_\alpha$ the seven dimensional real Lie algebra spanned by the elements $e_1, \ldots, e_7$ with Lie brackets given by

$$[e_i, e_j] = \alpha_{ij} e_{i+j} \qquad (1 \leq i < j \leq 7,\ i + j \leq 7)$$

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and the non-defined brackets being equal to zero or obtained by antisymmetry. It is clear that $\mathfrak{g}_\alpha$ is a 6-step nilpotent Lie algebra ([1]).

Theorem 0.1. *A necessary condition under which $\mathfrak{g}_\alpha$ admits a rational structure is*

$$\alpha_{14}\alpha_{25}\alpha_{16}^{-1}\alpha_{24}^{-1} \in \mathbb{Q}. \tag{1}$$

Proof. Let $\alpha \in \mathscr{D}$ and we assume that $\mathfrak{g}_\alpha$ has a rational structure. It is clear that the algebras in the descending central series of $\mathfrak{g}_\alpha$ are

$$\mathscr{C}^k(\mathfrak{g}_\alpha) = \mathbb{R}\text{-span}\,\{e_{k+1}, \ldots, e_7\} \qquad (k = 2, \ldots, 6)$$

By Corollary 5.2.2 of [3], $\mathscr{C}^k(\mathfrak{g}_\alpha)$ $(k = 2, \ldots, 6)$ are rational. On the other hand, let $\mathfrak{h}$ be the centralizer of $\mathscr{C}^5(\mathfrak{g})$ in $\mathfrak{g}$. It is easy to verify that $\mathfrak{h} = \mathbb{R}\text{-span}\,\{e_2, \ldots, e_7\}$. By Proposition 5 of [4], $\mathfrak{h}$ is also rational. Consequently, there exists a strong Malcev basis $(\epsilon_1, \ldots, \epsilon_7)$ of $\mathfrak{g}_\alpha$ with rational structure constants, passing through $\mathscr{C}^6(\mathfrak{g}_\alpha), \ldots, \mathscr{C}^2(\mathfrak{g}_\alpha)$ and $\mathfrak{h}$ ([3, Proposition 5.3.2]).

Since $\mathbb{R}\text{-span}\,\{e_k, \ldots, e_7\} = \mathbb{R}\text{-span}\,\{\epsilon_k, \ldots, \epsilon_7\}$ $(k = 1, \ldots, 7)$ then for each $i = 1, \ldots, 7$ we have

$$\epsilon_i = \sum_{j=i}^{7} a_{ij} e_j,$$

with $a_{ii} \neq 0$. On the other hand, for $i, j = 1, \ldots, 7$ such that $i \neq j$ and $i + j \leq 7$ there exist $\alpha'_{ij}, \beta, \ldots, \lambda \in \mathbb{Q}$ such that

$$[\epsilon_i, \epsilon_j] = \alpha'_{ij}\epsilon_{i+j} + \beta\epsilon_{i+j+1} + \ldots + \lambda\epsilon_7.$$

Next, we calculate

$$\begin{aligned}
[\epsilon_1, \epsilon_4] &= \left[\sum_{j=1}^{7} a_{1j}e_j, \sum_{j=4}^{7} a_{4j}e_j\right] \\
&= a_{11}a_{44}[e_1, e_4] + \ldots \\
&= a_{11}a_{44}\alpha_{14}e_5 + \ldots.
\end{aligned}$$

Then

$$[\epsilon_2, [\epsilon_1, \epsilon_4]] = a_{22}a_{11}a_{44}\alpha_{14}\alpha_{25}e_7.$$

On the other hand, we have

$$\begin{aligned}
[\epsilon_2, [\epsilon_1, \epsilon_4]] &= [\epsilon_2, \alpha'_{14}\epsilon_5] \\
&= \alpha'_{25}\alpha'_{14}\epsilon_7 \\
&= \alpha'_{25}\alpha'_{14}a_{77}e_7.
\end{aligned}$$

Consequently

$$a_{22}a_{11}a_{44}\alpha_{14}\alpha_{25} = \alpha'_{25}\alpha'_{14}a_{77} \tag{2}$$

Similarly, calculating $[\epsilon_1, [\epsilon_2, \epsilon_4]]$ we obtain

$$a_{22}a_{11}a_{44}\alpha_{24}\alpha_{16} = \alpha'_{24}\alpha'_{16}a_{77} \tag{3}$$

Combining (2) and (3) we deduce

$$\alpha_{14}\alpha_{25}\alpha_{16}^{-1}\alpha_{24}^{-1} = \alpha'_{14}\alpha'_{25}\alpha'^{-1}_{16}\alpha'^{-1}_{26}. \tag{4}$$

Since the structure constants α'_{ij} relative to the basis $(\epsilon_1, \dots, \epsilon_n)$ are all rational, then $\alpha_{14}\alpha_{25}\alpha_{16}^{-1}\alpha_{24}^{-1} \in \mathbb{Q}$. ■

For $a \in \mathbb{R} \setminus \mathbb{Q}$, let us put

$$\alpha(a) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & a & 0 \\ -1 & -1 & 0 & 1-a & 0 & 0 \\ -1 & -1 & a-1 & 0 & 0 & 0 \\ -1 & -a & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Corollary 0.2. *The Lie algebras* $\mathfrak{g}_{\alpha(a)}$ $(a \in \mathbb{R}\setminus\mathbb{Q})$ *are non isomorphic nilpotent Lie algebras without* $\mathbb{Q}$*-structures.*

Proof. We have $\alpha_{14}\alpha_{25}\alpha_{16}^{-1}\alpha_{24}^{-1} = a \notin \mathbb{Q}$. ■

With Theorem 5 of [4], this gives

Corollary 0.3. *Let* $n \geq 8$ *and* $a \in \mathbb{R} \setminus \mathbb{Q}$. *Then* $\mathfrak{g}_{\alpha(a)} \oplus \mathbb{R}^{n-7}$ *is a nilpotent Lie algebra of dimension* n *without* $\mathbb{Q}$*-structure.*

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H. Hamrouni and S. Souissi
Department of Mathematics
Faculty of Sciences at Sfax
Route Soukra, B.P. 1171. 3000 Sfax
Tunisia
hatemhhamrouni@voila.fr

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