Riesz Potentials and Fractional Maximal Function for the Dunkl Transform

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Abstract. In this article we investigate the $L^p \to L^q$ boundedness properties of the Riesz potentials $I^\kappa_\alpha$ and the related fractional maximal function $M^\kappa,\alpha$ associated to the Dunkl transform.

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1. Introduction

Let $G$ be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system $R_+$, normalized so that $\langle v, v \rangle = 2$ for all $v \in R_+$, where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product. For a nonzero vector $v \in \mathbb{R}^d$, let $\sigma_v$ denote the reflection with respect to the hyperplane perpendicular to $v$. i.e.

$$x \sigma_v = x - 2 \frac{\langle x, v \rangle}{\|v\|^2} v, \quad x \in \mathbb{R}^d.$$ 

Then $G$ is a subgroup of the orthogonal group generated by the reflections $\{\sigma_v, v \in R_+\}$. Let $\kappa$ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on $R_+$ with the property that $\kappa_u = \kappa_v$ whenever $\sigma_u$ is conjugate to $\sigma_v$ in $G$, then $v \mapsto \kappa_v$ is a $G$-invariant function. The weight function $h_\kappa$ is defined by

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v}, \quad x \in \mathbb{R}^d.$$ 

This is a $G$-invariant positive homogeneous function of degree $\gamma_\kappa = \sum_{v \in R_+} \kappa_v$.

For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$ the Dunkl transform is defined (see [4]) by

$$\widehat{f}(y) = c_h \int_{\mathbb{R}^d} f(x) E(x, -iy) h_\kappa^2(x) dx, \quad y \in \mathbb{R}^d,$$
where \( c_h \) is the following constant
\[
c_h^{-1} = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2}} h_n^2(x) dx,
\]
and where \( E(x, -iy) \) denotes the Dunkl kernel (for more details see the next section). The generalized translation operator is defined on \( L^2(\mathbb{R}^d, h_n^2) \) by the equation
\[
\widehat{\tau_y f}(x) = E(y, -ix) \widehat{f}(x), \quad x \in \mathbb{R}^d.
\]
It plays the role of the ordinary translation \( \tau_y f(\cdot) = f(\cdot - y) \) in \( \mathbb{R}^d \), since the Euclidean Fourier transform satisfies \( \widehat{\tau_y f}(x) = e^{-i\langle x, y \rangle} \widehat{f}(x) \).

For \( 0 < \alpha < 2\gamma_\kappa + d \), the Riesz potential \( I_\alpha^n f \) is defined on \( \mathcal{S}(\mathbb{R}^d) \) (the class of Schwartz functions) by (see [10])
\[
I_\alpha^n f(x) = (d_\kappa^n)^{-1} \int_{\mathbb{R}^d} \frac{\tau_y f(x)}{\|y\|^{2\gamma_\kappa + d - \alpha}} h_n^2(y) dy,
\]
where
\[
d_\kappa^n = 2^{-\gamma_\kappa - d/2 + \alpha} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(\gamma_\kappa + \frac{d - \alpha}{2})}.
\]
It is easy to see that the Riesz potentials operate on the Schwartz class \( \mathcal{S}(\mathbb{R}^d) \), as integral operators, and it is natural to inquire about their action on the spaces \( L^p(\mathbb{R}^d, h_n^2) \).

The main problem can be formulated as follows. Given \( \alpha \in [0, 2\gamma_\kappa + d] \) for what pair \((p, q)\) is it possible to extend (1) to a bounded operator from \( L^p(\mathbb{R}^d, h_n^2) \) to \( L^q(\mathbb{R}^d, h_n^2) \)? That is when do we have the inequality
\[
\|I_\alpha^n f\|_{L^q(\mathbb{R}^d, h_n^2)} \leq C \|f\|_{L^p(\mathbb{R}^d, h_n^2)}.
\]
A necessary condition is given in [10]. This condition says that (2) holds only if
\[
\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\gamma_\kappa + d}.
\]
Thangavelu and Xu proved also in [10] that the condition (3) is sufficient to ensure the boundedness of \( I_\alpha^n \) (save for \( p = 1 \) where a weak-type estimate holds) if one assumes that the reflection group \( G \) is \( \mathbb{Z}^d_2 \) or if \( f \) are radial functions and \( p \leq 2 \) (see [10], Theorem 4.4).

Our aim in this paper is to show that it is possible to remove this restrictive hypothesis and prove that (3) is a sufficient condition for all reflection groups. More precisely we have the following Theorem.

**Theorem 1.1.** Let \( \alpha \) be a real number such that \( 0 < \alpha < 2\gamma_\kappa + d \) and let \((p, q)\) be a pair of real numbers such that \( 1 \leq p < q < \infty \) and \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\gamma_\kappa + d} \). Then:

(i) If \( p > 1 \), then the mapping \( f \rightarrow I_\alpha^n f \) can be extended to a bounded operator from \( L^p(\mathbb{R}^d, h_n^2) \) to \( L^q(\mathbb{R}^d, h_n^2) \) and
\[
\|I_\alpha^n f\|_{\kappa, q} \leq A_{p, \alpha} \|f\|_{\kappa, p}, \quad f \in L^p(\mathbb{R}^d, h_n^2),
\]
where $A_{p,\alpha} > 0$ depends only on $p$ and $\alpha$.

(ii) If $p = 1$, $f \to T_\kappa f$ can be extended to a mapping of weak-type $(1, q)$ and

$$
\int_{\{x : |T_\kappa f(x)| > \lambda\}} h_\kappa^2(x) dx \leq A_\alpha \left( \frac{\|f\|_{\kappa,1}}{\lambda} \right)^q, \quad f \in L^1(\mathbb{R}^d, h_\kappa^2),
$$

where $A_\alpha > 0$ depends only on $\alpha$.

The notation $\|\cdot\|_{\kappa,p}$ is used here to denote the norm of $L^p(\mathbb{R}^d, h_\kappa^2)$.

The boundedness of Riesz potentials can be used to establish the boundedness properties of the fractional maximal operator associated to Dunkl transform.

For $0 < \alpha < 2\gamma_\kappa + d$ and $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $1 \leq p < \infty$, we define the fractional maximal $M_{\kappa,\alpha} f$ function by

$$
M_{\kappa,\alpha} f(x) = \sup_{r > 0} \frac{1}{m_\kappa r^{d+2\gamma_\kappa-\alpha}} \int_{\mathbb{R}^d} |f(y)| \tau_x \chi_B_r(y) h_\kappa^2(y) dy, \quad x \in \mathbb{R}^d,
$$

where

$$
m_\kappa = (c_\kappa 2^{\gamma_\kappa+\frac{d}{2}} \Gamma(\gamma_\kappa + \frac{d}{2} + 1))^{\frac{\alpha}{2\gamma_\kappa}}^{-1},
$$

and where $\chi_B_r$ denotes the characteristic function of the ball $B_r$ of radius $r$ centered at 0. We have the following corollary of Theorem 1.1.

**Corollary 1.2.** Let $\alpha$ be a real number such that $0 < \alpha < 2\gamma_\kappa + d$ and let $(p, q)$ be a pair of real numbers such that $1 \leq p < q < \infty$ and satisfying (3). Then:

(i) The maximal operator $M_{\kappa,\alpha}$ is bounded from $L^p(\mathbb{R}^d, h_\kappa^2)$ to $L^q(\mathbb{R}^d, h_\kappa^2)$ for $p > 1$.

(ii) $M_{\kappa,\alpha}$ is of weak type $(1, q)$, that is, for $f \in L^1(\mathbb{R}^d, h_\kappa^2)$

$$
\int_{\{x : M_{\kappa,\alpha} f(x) > \lambda\}} h_\kappa^2(x) dx \leq C_\alpha \left( \frac{\|f\|_{\kappa,1}}{\lambda} \right)^q, \quad \lambda > 0,
$$

where $c_\alpha > 0$ depends only on $\alpha$.

### 2. Background

Introduced by C. F. Dunkl in [2], the Dunkl operators $T_j$, $1 \leq j \leq d$, on $\mathbb{R}^d$ are the first-order differential-difference operators given by

$$
T_j f(x) = \partial_j f(x) + \sum_{v \in \mathbb{R}_+} k_v \frac{f(x) - f(x\sigma_v)}{\langle x, v \rangle} \langle v, e_j \rangle, \quad 1 \leq j \leq d,
$$

where $\partial_j$ denotes the usual partial derivatives and $e_1, \ldots, e_d$ the standard basis of $\mathbb{R}^d$. A fundamental property of these differential-difference operators is their commutativity, that is, $T_k T_l = T_l T_k$, $1 \leq k, l \leq d$.

Closely related to them is the so-called intertwining operator $V_\kappa$ which is the unique linear isomorphism of $\bigoplus_{n \geq 0} \mathcal{P}_n$ determined by (see [4])

$$
V_\kappa(\mathcal{P}_n) = \mathcal{P}_n, \quad V_\kappa(1) = 1, \quad T_j V_\kappa = V_\kappa \partial_j, \quad \text{for } j = 1, \ldots, d,
$$
with $P_n$ the subspace of homogeneous polynomials of degree $n$ in $d$ variables. Even if the positivity of the intertwining operator has been established in [7] by M. Rosler, an explicit formula of $V_\kappa$ is not known in general. However, the operator $V_\kappa$ possesses the integral representation

$$V_\kappa f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y),$$

where $\mu_x$ is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$ (see [7], [12]).

The function $E(x, y) = V_\kappa^x [e^{x, y}]$, where the superscript means that $V_\kappa$ is applied to the $x$ variable, plays an important role in the development of the Dunkl transform which is defined on $L^1(\mathbb{R}^d, h^2_\kappa)$ by

$$\hat{f}(y) = c_h \int_{\mathbb{R}^d} f(x) E(x, -iy) h^2_\kappa(x) dx, \quad y \in \mathbb{R}^d.$$

If $\kappa = 0$, then $V_\kappa = id$ and the Dunkl transform coincides with the usual Fourier transform. If $d = 1$ and $G = \mathbb{Z}_2$ then the Dunkl transform is related closely to the Hankel transform in the real line (see [13]). In fact, in this case,

$$E(x, -iy) = \Gamma(\kappa + 1) \left(\frac{|xy|}{2}\right)^{-\kappa + \frac{1}{2}} J_{\kappa - \frac{1}{2}}(|xy|) - i \text{sign}(xy) J_{\kappa + \frac{1}{2}}(|xy|),$$

where $J_\alpha$ denotes the usual Bessel function

$$J_\alpha(t) = \left(\frac{t}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \alpha + 1)} \left(\frac{t}{2}\right)^{2n}.$$

Some of the properties of the kernel $E(x, y)$ and the Dunkl transform are collected below (see [4], [5]).

**Proposition 2.1.**

(i) $E(x, y) = E(y, x), \quad x, y \in \mathbb{R}^d$.

(ii) $|E(x, y)| \leq e^{\|x\|\|y\|}, \quad x, y \in \mathbb{C}^d$.

(iii) For $f \in L^1(\mathbb{R}^d, h^2_\kappa)$, $\hat{f}$ is in $C_0(\mathbb{R}^d)$.

(iv) The Dunkl transform is a topological automorphism of $S(\mathbb{R}^d)$.

(v) (Inversion formula) When both $f$ and $\hat{f}$ are in $L^1(\mathbb{R}^d, h^2_\kappa)$ we have

$$f(x) = \int_{\mathbb{R}^d} E(ix, y) \hat{f}(y) h^2_\kappa(y) dy.$$

(vi) (Plancherel Theorem) The Dunkl transform extends to an isometry of $L^2(\mathbb{R}^d, h^2_\kappa)$.

The Dunkl transform allows us to define a generalized translation operator on $L^2(\mathbb{R}^d, h^2_\kappa)$ by setting

$$\tau_y f(x) = E(y, -ix) \hat{f}(x), \quad x \in \mathbb{R}^d.$$

In the analysis of this generalized translation a particular role is played by the space (cf. [6], [7], [9] and [11])

$$A_\kappa(\mathbb{R}^d) = \{ f \in L^1(\mathbb{R}^d, h^2_\kappa) : \hat{f} \in L^1(\mathbb{R}^d, h^2_\kappa) \}.$$
Assume that

(i) For every case of $G f$ the case of $\phi$ where (ii) $\tau$ is a subspace of $L^1$. The operator $\tau_y$ satisfies the following properties:

**Proposition 2.2.** Assume that $f \in A_\kappa(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d, h_\kappa^2)$ is bounded. Then:

(i) $\int_{\mathbb{R}^d} \tau_y f(x) g(x) h_\kappa^2(x) dx = \int_{\mathbb{R}^d} f(x) \tau_{-y} g(x) h_\kappa^2(x) dx$.

(ii) $\tau_y f(x) = \tau_{-x} f(-y)$.

A formula of $\tau_y f$ is known, at the moment, only in two cases. One is in the case of $G = \mathbb{Z}_2$ and $h_\kappa(x) = \{x\}^\kappa$ on $\mathbb{R}$ (see [8]):

$$
\tau_y f(x) = \frac{1}{2} \int_{-1}^{1} f\left(\sqrt{x^2 + y^2 - 2xyt}\right) \left(1 + \frac{x - y}{\sqrt{x^2 + y^2 - 2xyt}}\right) \phi_\kappa(t) dt
+ \frac{1}{2} \int_{-1}^{1} f\left(-\sqrt{x^2 + y^2 - 2xyt}\right) \left(1 - \frac{x - y}{\sqrt{x^2 + y^2 - 2xyt}}\right) \phi_\kappa(t) dt,
$$

where $\phi_\kappa(t) = b_\kappa(1 + t)(1 - t^2)^{\kappa-1}$, from which also follows a formula of $\tau_y f$ in the case of $G = \mathbb{Z}_2^d$. This formula implies easily the $L^p$-boundedness of $\tau_y$ in this case.

Another case where a formula of $\tau_y f$ is known is when $f$ are radial functions, $f(x) = f_0(||x||)$, and $G$ being any reflection group (see [6])

$$
\tau_y f(x) = V_\kappa[f_0(\sqrt{||x||^2 + ||y||^2 - 2||x||||y||\langle x', \cdot \rangle})](y), \quad x' = \frac{x}{||x||}, \quad y' = \frac{y}{||y||}
$$

from which it follows that $\tau_y f(x) \geq 0$ for all $y \in \mathbb{R}^d$ if $f(x) = f_0(||x||) \geq 0$.

Several essential properties of $\tau_y f$ ($f$ being radial) follow from this formula. This is collected in the following proposition (see [9]).

**Proposition 2.3.** (i) For every $f \in L^1_{rad}(\mathbb{R}^d, h_\kappa^2)$ (the subspace of radial functions in $L^p(\mathbb{R}^d, h_\kappa^2)$) we have:

$$
\int_{\mathbb{R}^d} \tau_y f(x) h_\kappa^2(x) dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(x) dx.
$$

(ii) For $1 \leq p \leq 2$, $\tau_y : L^p_{rad}(\mathbb{R}^d, h_\kappa^2) \rightarrow L^p_{rad}(\mathbb{R}^d, h_\kappa^2)$ is a bounded operator.

Apart from these two cases, we lack precise information, in particular about the boundedness of generalized translations (see [1]). Based on this fact we tried to develop an elementary approach of the Riesz potentials which works with a minimal knowledge about $\tau_y$ and gives the answer to the problem (1)-(2) in full generality.

### 3. Riesz Potentials

For the proof of Theorem 1.1, we need the following version of a classical Schur’s lemma.
Lemma 3.1. Assume that $k$ is a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ that satisfies the mixed-norm conditions:

$$C_1 = \sup_{x \in \mathbb{R}^d} \int |k(x, y)|h_\kappa^2(y)dy < \infty, \quad C_2 = \sup_{y \in \mathbb{R}^d} \int |k(x, y)|h_\kappa^2(x)dx < \infty.$$ 

Then the integral operator induced by the kernel $k(x, y)$ (i.e. the operator defined by $T_k f(x) = \int k(x, y)f(y)h_\kappa^2(y)dy$) defines a bounded mapping of $L^p(\mathbb{R}^d, h_\kappa^2)$ into itself for every $1 \leq p \leq \infty$, with

$$\|T_k\|_{L^p(\mathbb{R}^d, h_\kappa^2) \to L^p(\mathbb{R}^d, h_\kappa^2)} \leq C_1^{1-\frac{1}{p}} C_2^\frac{1}{p}.$$ 

**Proof of Theorem 1.1** We begin with this simple formula used by Thangavelu and Xu in [10] and in the same context

$$\|y\|^{-a} = \frac{1}{\Gamma\left(\frac{a}{2}\right)} \int_0^\infty s^{\frac{a}{2}} e^{-s\|y\|^2} ds, \quad y \in \mathbb{R}^d, \quad a > 0.$$ 

Applying this formula with $a = 2\gamma_\kappa + d - \alpha$ and changing the order of integrals in (1), we obtain

$$I_\alpha^\gamma f(x) = \frac{2^{\gamma_\kappa + \frac{d}{2} - \alpha}}{\Gamma\left(\frac{a}{2}\right)} \int_0^\infty \tau_y f(x) e^{-s\|y\|^2} h_\kappa^2(y)dy ds, \quad f \in \mathcal{S}(\mathbb{R}^d).$$ 

By Proposition 2.2, we have

$$\int_{\mathbb{R}^d} \tau_y f(x) e^{-s\|y\|^2} h_\kappa^2(y)dy = \int_{\mathbb{R}^d} \tau_{-x} f(-y) e^{-s\|y\|^2} h_\kappa^2(y)dy$$

$$= \int_{\mathbb{R}^d} \tau_{-x} f(y) e^{-s\|y\|^2} h_\kappa^2(y)dy$$

$$= \int_{\mathbb{R}^d} f(y) \tau_x (e^{-s\|y\|^2})(y) h_\kappa^2(y)dy.$$ 

We have thus the identity

$$I_\alpha^\gamma f(x) = \frac{2^{\gamma_\kappa + \frac{d}{2} - \alpha}}{\Gamma\left(\frac{a}{2}\right)} \int_0^\infty \int_{\mathbb{R}^d} f(y) \tau_x (e^{-s\|y\|^2})(y) h_\kappa^2(y)dy ds, \quad f \in \mathcal{S}(\mathbb{R}^d). \quad (4)$$

With the aid of this identity it will not be difficult to extend the mapping $f \to I_\alpha^\gamma f$ to all functions $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $p \geq 1$. Indeed we first notice that for each $x \in \mathbb{R}^d$, the function $y \to \tau_x (e^{-s\|y\|^2})(y)$ is positive and satisfies (see [8])

$$\tau_x (e^{-s\|y\|^2})(y) = e^{-s(\|x\|^2 + \|y\|^2)} E(2sx, y), \quad s > 0.$$ 

Using Proposition 2.1, we deduce that

$$\tau_x (e^{-s\|y\|^2})(y) \leq e^{-s(\|x\|^2 - \|y\|^2)} \leq 1. \quad (5)$$
On the other hand, applying Proposition 2.3, we obtain
\[
\int_{\mathbb{R}^d} \tau_x(e^{-s||y||^2})(y)h^2_\alpha(y)dy = \int_{\mathbb{R}^d} e^{-s||y||^2}h^2_\alpha(y)dy = \frac{1}{c_h(2\alpha)^{2\gamma_\alpha+\frac{d}{2}}}, \quad s > 0.
\] (6)

Let \((p, q)\) be a pair of real numbers satisfying (3) and let \(f \in L^p(\mathbb{R}^d, h^2_\alpha)\) normalized so that \(\|f\|_{\kappa, p} = 1\). We shall prove that the integral
\[
\frac{2\gamma_\alpha+\frac{d}{2}-\alpha}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty s^{\gamma_\alpha+\frac{d}{2}-\alpha} \int_{\mathbb{R}^d} f(y)\tau_x(e^{-s||y||^2})(y)h^2_\alpha(y)dy \frac{ds}{s}
\] (7)
converges absolutely for almost every \(x\). Towards this let us decompose (7) as a sum of two terms \(S_1f(x) + S_2f(x)\) where
\[
S_1f(x) = \frac{2\gamma_\alpha+\frac{d}{2}-\alpha}{\Gamma\left(\frac{d}{2}\right)} \int_0^\sigma s^{\gamma_\alpha+\frac{d}{2}-\alpha} \int_{\mathbb{R}^d} f(y)\tau_x(e^{-s||y||^2})(y)h^2_\alpha(y)dy \frac{ds}{s},
\]
\[
S_2f(x) = \frac{2\gamma_\alpha+\frac{d}{2}-\alpha}{\Gamma\left(\frac{d}{2}\right)} \int_\sigma^\infty s^{\gamma_\alpha+\frac{d}{2}-\alpha} \int_{\mathbb{R}^d} f(y)\tau_x(e^{-s||y||^2})(y)h^2_\alpha(y)dy \frac{ds}{s}.
\]

At this instance \(\sigma > 0\) is a fixed positive constant which need not to be specified (it suffices to take \(\sigma = 1\) for example).

Let us estimate \(\|S_1f\|_\infty\). Let \(x \in \mathbb{R}^d\), we have
\[
|S_1f(x)| \leq \frac{2\gamma_\alpha+\frac{d}{2}-\alpha}{\Gamma\left(\frac{d}{2}\right)} \int_0^\sigma s^{\gamma_\alpha+\frac{d}{2}-\alpha} \sup_x \int_{\mathbb{R}^d} |f(y)|\tau_x(e^{-s||y||^2})(y)h^2_\alpha(y)dy \frac{ds}{s}.
\]

However, it follows from (5) that
\[
\sup_x \int_{\mathbb{R}^d} |f(y)|\tau_x(e^{-s||y||^2})(y)dy \leq \|f\|_{\kappa, 1},
\]
and from (6)
\[
\sup_x \int_{\mathbb{R}^d} |f(y)|\tau_x(e^{-s||y||^2})(y)dy \leq \left(\sup_x \int_{\mathbb{R}^d} \tau_x(e^{-s||y||^2})(y)h^2_\alpha(y)dy\right)\|f\|_\infty
\]
\[
\leq \frac{1}{c_h(2\alpha)^{2\gamma_\alpha+\frac{d}{2}}} \|f\|_\infty.
\]

Using complex interpolation we deduce then that
\[
\sup_x \int_{\mathbb{R}^d} |f(y)|\tau_x(e^{-s||y||^2})(y)dy \leq \left(\frac{1}{c_h(2\alpha)^{\gamma_\alpha+\frac{d}{2}}} \right)^{1-p} \|f\|_{\kappa, p}
\]
\[
= \frac{2^{-\left(1-\frac{p}{2}\right)(\gamma_\alpha+\frac{d}{2})} - \frac{\left(1-\frac{p}{2}\right)(\gamma_\alpha+\frac{d}{2})}{c_h^{1-\frac{p}{2}}},
\]
and then
\[
|S_1f(x)| \leq \frac{2\pi^{\frac{\gamma_\alpha+\frac{d}{2}}{2}}}{c_h^{\frac{1}{2} \pi \Gamma\left(\frac{d}{2}\right)}} \frac{1}{\Gamma\left(\frac{d}{2}\right)} \int_0^\sigma s^{\frac{1}{2} \pi \Gamma\left(\frac{d}{2}\right)-\frac{\gamma_\alpha+\frac{d}{2}}{2}} \frac{ds}{s}
\]
\[
= A \sigma^\frac{1}{2}(2\gamma_\alpha+\frac{d}{2})^{-\alpha}, \quad x \in \mathbb{R}^d,
\]
Lemma 3.1 which gives that

\[ \int f(y) \tau_x(e^{-\|y\|^2})(y) h_{\kappa}^2(y)dy \leq \frac{2^{-\alpha}}{2\Gamma(\frac{\alpha}{2})c_{\|x\|}} \int_{\lambda}^{\infty} s^{\frac{\alpha}{2} - 1} \frac{ds}{s} = \frac{2^{1-\alpha}}{2\Gamma(\frac{\alpha}{2})c_{\|x\|}} \sigma^{-\frac{\alpha}{2}} = B\sigma^{-\frac{\alpha}{2}}. \]  

(9)

The \( L^p \) norm of \( \int f(y) \tau_x(e^{-\|y\|^2})(y) h_{\kappa}^2(y)dy \) can be easily estimated by using Lemma 3.1 which gives

\[ \|S_2 f\|_{\kappa,p} \leq \frac{2^{-\alpha}}{\Gamma(\frac{\alpha}{2})c_{\kappa,p}} \int_{\lambda}^{\infty} s^{\frac{\alpha}{2} - 1} \frac{ds}{s} = \frac{2^{1-\alpha}}{2\Gamma(\frac{\alpha}{2})c_{\kappa,p}} \sigma^{-\frac{\alpha}{2}} = B\sigma^{-\frac{\alpha}{2}}. \]  

(10)

Putting together (8) and (9) we deduce that (7) converges absolutely for almost every \( x \in \mathbb{R}^d \).

Let now \( \lambda > 0 \) and let us estimate

\[ \int_{\{x: |S_1 f(x)| > \lambda\}} h_{\kappa}^2(x)dx \leq \int_{\{x: |S_1 f(x)| > \frac{\lambda}{2}\}} h_{\kappa}^2(x)dx + \int_{\{x: |S_2 f(x)| > \frac{\lambda}{2}\}} h_{\kappa}^2(x)dx, \]

we choose \( \sigma \) to satisfy

\[ A\sigma^{\frac{1}{2}} = \frac{\lambda}{2}, \]  

(10)

so that (thanks to (8))

\[ \int_{\{x: |S_1 f(x)| > \frac{\lambda}{2}\}} h_{\kappa}^2(x)dx = 0. \]

We get

\[ \int_{\{x: |S_2 f(x)| > \lambda\}} h_{\kappa}^2(x)dx \leq \int_{\{x: |S_2 f(x)| > \frac{\lambda}{2}\}} h_{\kappa}^2(x)dx \leq \left( \frac{2}{\lambda} \right)^p \|S_2 f\|_{\kappa,p}^p, \]

and then by (9)

\[ \int_{\{x: |S_2 f(x)| > \lambda\}} h_{\kappa}^2(x)dx \leq \frac{2^p B^p \sigma^{-\frac{2p}{\lambda^p}}}{\lambda^p}. \]

Using (10) we deduce that

\[ \int_{\{x: |S_2 f(x)| > \lambda\}} h_{\kappa}^2(x)dx \leq \frac{2^p B^p (2A)^{\frac{\alpha^2}{2\kappa,2x-\alpha}}}{\lambda^{\frac{2^p(2\kappa,2x-\alpha)}{\kappa,2x-\alpha}}} = C_{\kappa,\alpha} \left( \frac{\|f\|_{\kappa,p}}{\lambda} \right)^q. \]
since $q = \frac{p(2\gamma_\kappa + d)}{2\gamma_\kappa + d - \alpha p}$ and $\|f\|_{\kappa,p} = 1$.

The previous considerations show that the mapping $f \to I_\alpha f$ is of weak type $(p, q)$:

$$\int_{\{x: |I_\alpha f(x)| > \lambda\}} h_\kappa^2(x)dx \leq C_{p,\alpha} \left(\frac{\|f\|_{\kappa,p}}{\lambda}\right)^q,$$

and $1 \leq p < q < \infty$ with $1 = \frac{q}{p} - \frac{\alpha}{2\gamma_\kappa + d}$. The special case for $p = 1$ gives then part $(ii)$ of Theorem 1.1, and part $(i)$ follows by an obvious use of real interpolation properties of the spaces $L^p(\mathbb{R}^d, h_\kappa^2)$ and Marcinkiewicz interpolation theorem.

**Proof of Corollary 1.2** This is an immediate consequence of Theorem 1.1 and the following pointwise inequality

$$M_{\kappa,\alpha} f(x) = \sup_{r > 0} \frac{1}{m_{\kappa r} + 2\gamma_\kappa - \alpha} \int_{\mathbb{R}^d} |f(y)| \tau_x \chi_{B_r}(y) h_\kappa^2(y) dy \leq C_{\kappa,d,\alpha} I_\alpha^\kappa(|f|)(x),$$

where $C_{\kappa,d,\alpha}$ depends only on $\kappa$, $d$ and $\alpha$.

Finally one should observe that (3) is also necessary for the boundedness of the maximal fractional operator $M_{\kappa,\alpha}$ from the spaces $L^p(\mathbb{R}^d, h_\kappa^2)$ when $p > 1$ and for the weak-type estimate of $M_{\kappa,\alpha}$ when $p = 1$. The proof of this fact is straightforward. It suffices to consider the dilation operator $\delta_r f(x) = f(rx)$, $r > 0$, and to observe that $\delta_r^{-1} M_{\kappa,\alpha} \delta_r = r^{-\alpha} M_{\kappa,\alpha}$. It follows then that $\|\delta_r f\|_{\kappa,p} = r^{\frac{2\gamma_\kappa + d}{p}} \|f\|_{\kappa,p}$. By dilation, the estimate $\|M_{\kappa,\alpha} f\|_{\kappa,q} \leq C\|f\|_{\kappa,p}$ (where we assume $p > 1$) implies then that

$$\|M_{\kappa,\alpha} f\|_{\kappa,q} \leq C r^{\alpha + \frac{2\gamma_\kappa + d}{q} - \frac{2\gamma_\kappa + d}{p}} \|f\|_{\kappa,p}, \quad r > 0.$$ 

Letting $r \to \infty$ and $r \to 0$ we get

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\gamma_\kappa + d}.$$ 

The same argument applies in the weak case.

**References**


