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Riesz Potentials and Fractional Maximal Function for the Dunkl Transform

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Abstract. In this article we investigate the $L^p \to L^q$ boundedness properties of the Riesz potentials I_α^κ and the related fractional maximal function $M_{\kappa,\alpha}$ associated to the Dunkl transform.
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1. Introduction

Let G be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system R_+, normalized so that $\langle v, v\rangle = 2$ for all $v \in R_+$, where $\langle .,. \rangle$ denotes the usual Euclidean inner product. For a nonzero vector $v \in \mathbb{R}^d$, let σ_v denote the reflection with respect to the hyperplane perpendicular to v i.e.

$$x\sigma_v = x - 2\frac{\langle x, v\rangle}{\|v\|^2}v, \quad x \in \mathbb{R}^d.$$

Then G is a subgroup of the orthogonal group generated by the reflections $\{\sigma_v,\ v \in R_+\}$. Let κ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on R_+ with the property that $\kappa_u = \kappa_v$ whenever σ_u is conjugate to σ_v in G, then $v \mapsto \kappa_v$ is a G-invariant function. The weight function h_κ is defined by

$$h_\kappa(x) = \prod_{v\in R_+} |\langle x, v\rangle|^{\kappa_v}, \qquad x \in \mathbb{R}^d.$$

This is a G-invariant positive homogeneous function of degree $\gamma_\kappa = \sum_{v\in R_+} \kappa_v$.

For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$ the Dunkl transform is defined (see [4]) by

$$\widehat{f}(y) = c_h \int_{\mathbb{R}^d} f(x)E(x, -iy)h_\kappa^2(x)dx, \qquad y \in \mathbb{R}^d,$$

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where c_h is the following constant

$$c_h^{-1} = \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} h_\kappa^2(x)dx,$$

and where $E(x, -iy)$ denotes the Dunkl kernel (for more details see the next section). The generalized translation operator is defined on $L^2(\mathbb{R}^d, h_\kappa^2)$ by the equation

$$\widehat{\tau_y f}(x) = E(y, -ix)\widehat{f}(x), \qquad x \in \mathbb{R}^d.$$

It plays the role of the ordinary translation $\tau_y f(.) = f(. - y)$ in $\mathbb{R}^d$, since the Euclidean Fourier transform satisfies $\widehat{\tau_y f}(x) = e^{-i\langle x,y\rangle}\widehat{f}(x)$.

For $0 < \alpha < 2\gamma_\kappa + d$, the Riesz potential $I_\alpha^\kappa f$ is defined on $\mathcal{S}(\mathbb{R}^d)$ (the class of Schwartz functions) by (see [10])

$$I_\alpha^\kappa f(x) = (d_\kappa^\alpha)^{-1} \int_{\mathbb{R}^d} \frac{\tau_y f(x)}{\|y\|^{2\gamma_\kappa+d-\alpha}} h_\kappa^2(y)dy, \tag{1}$$

where

$$d_\kappa^\alpha = 2^{-\gamma_\kappa - d/2+\alpha} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\gamma_\kappa + \frac{d-\alpha}{2})}.$$

It is easy to see that the Riesz potentials operate on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, as integral operators, and it is natural to inquire about their action on the spaces $L^p(\mathbb{R}^d, h_\kappa^2)$.

The main problem can be formulated as follows. Given $\alpha \in]0, 2\gamma_\kappa + d[$ for what pair (p, q) is it possible to extend (1) to a bounded operator from $L^p(\mathbb{R}^d, h_\kappa^2)$ to $L^q(\mathbb{R}^d, h_\kappa^2)$? That is when do we have the inequality

$$\|I_\alpha^\kappa f\|_{L^q(\mathbb{R}^d,h_\kappa^2)} \leq C \ \|f\|_{L^p(\mathbb{R}^d,h_\kappa^2)}. \tag{2}$$

A necessary condition is given in [10]. This condition says that (2) holds only if

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\gamma_\kappa + d}. \tag{3}$$

Thangavelu and Xu proved also in [10] that the condition (3) is sufficient to ensure the boundedness of I_α^κ (save for $p = 1$ where a weak-type estimate holds) if one assumes that the reflection group G is $\mathbb{Z}_2^d$ or if f are radial functions and $p \leq 2$ (see [10], Theorem 4.4).

Our aim in this paper is to show that it is possible to remove this restrictive hypothesis and prove that (3) is a sufficient condition for all reflection groups. More precisely we have the following Theorem.

Theorem 1.1. *Let α be a real number such that $0 < \alpha < 2\gamma_\kappa + d$ and let (p, q) be a pair of real numbers such that $1 \leq p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2\gamma_\kappa + d}$. Then:*
(i) If $p > 1$, then the mapping $f \to I_\alpha^\kappa f$ can be extended to a bounded operator from $L^p(\mathbb{R}^d, h_\kappa^2)$ to $L^q(\mathbb{R}^d, h_\kappa^2)$ and

$$\|I_\alpha^\kappa f\|_{\kappa,q} \leq A_{p,\alpha}\|f\|_{\kappa,p}, \qquad f \in L^p(\mathbb{R}^d, h_\kappa^2),$$

where $A_{p,\alpha} > 0$ *depends only on* p *and* α.
(ii) If $p = 1$, $f \to I_\alpha^\kappa f$ *can be extended to a mapping of weak-type* $(1, q)$ *and*

$$\int_{\{x:\ |I_\alpha^\kappa f(x)|>\lambda\}} h_\kappa^2(x)dx \le A_\alpha \left(\frac{\|f\|_{\kappa,1}}{\lambda}\right)^q, \qquad f \in L^1(\mathbb{R}^d, h_\kappa^2),$$

where $A_\alpha > 0$ *depends only on* α.

The notation $\|.\|_{\kappa,p}$ is used here to denote the norm of $L^p(\mathbb{R}^d, h_\kappa^2)$.

The boundedness of Riesz potentials can be used to establish the boundedness properties of the fractional maximal operator associated to Dunkl transform.

For $0 < \alpha < 2\gamma_\kappa + d$ and $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $1 \le p < \infty$, we define the fractional maximal $M_{\kappa,\alpha} f$ function by

$$M_{\kappa,\alpha} f(x) = \sup_{r>0} \frac{1}{m_\kappa r^{d+2\gamma_\kappa-\alpha}} \int_{\mathbb{R}^d} |f(y)| \tau_x \chi_{B_r}(y) h_\kappa^2(y)dy, \qquad x \in \mathbb{R}^d,$$

where

$$m_\kappa = (c_h 2^{\gamma_\kappa+\frac{d}{2}} \Gamma(\gamma_\kappa + \frac{d}{2} + 1))^{\frac{\alpha}{d+2\gamma_\kappa}-1},$$

and where χ_{B_r} denotes the characteristic function of the ball B_r of radius r centered at 0. We have the following corollary of Theorem 1.1.

Corollary 1.2. *Let* α *be a real number such that* $0 < \alpha < 2\gamma_k + d$ *and let* (p, q) *be a pair of real numbers such that* $1 \le p < q < \infty$ *and satisfying (3). Then:*
(i) The maximal operator $M_{\kappa,\alpha}$ *is bounded from* $L^p(\mathbb{R}^d, h_\kappa^2)$ *to* $L^q(\mathbb{R}^d, h_\kappa^2)$ *for* $p > 1$.
(ii) $M_{\kappa,\alpha}$ *is of weak type* $(1, q)$, *that is, for* $f \in L^1(\mathbb{R}^d, h_\kappa^2)$

$$\int_{\{x:\ M_{\kappa,\alpha} f(x)>\lambda\}} h_\kappa^2(x)dx \le C_\alpha \left(\frac{\|f\|_{\kappa,1}}{\lambda}\right)^q, \qquad \lambda > 0,$$

where $c_\alpha > 0$ *depends only on* α.

2. Background

Introduced by C. F. Dunkl in [2], the Dunkl operators T_j, $1 \le j \le d$, on $\mathbb{R}^d$ are the first-order differential-difference operators given by

$$T_j f(x) = \partial_j f(x) + \sum_{v \in R_+} k_v \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} \langle v, e_j\rangle, \qquad 1 \le j \le d,$$

where ∂_j denotes the usual partial derivatives and $e_1, ..., e_d$ the standard basis of $\mathbb{R}^d$. A fundamental property of these differential-difference operators is their commutativity, that is, $T_k T_l = T_l T_k$, $1 \le k, l \le d$.

Closely related to them is the so-called intertwining operator V_κ which is the unique linear isomorphism of $\bigoplus_{n\ge0} \mathcal{P}_n$ determined by (see [4])

$$V_\kappa(\mathcal{P}_n) = \mathcal{P}_n, \quad V_\kappa(1) = 1, \quad T_j V_\kappa = V_\kappa \partial_j, \ for\ j = 1, ..., d,$$

with $\mathcal{P}_n$ the subspace of homogeneous polynomials of degree n in d variables. Even if the positivity of the intertwining operator has been established in [7] by M. Rösler, an explicit formula of V_κ is not known in general. However, the operator V_κ possesses the integral representation

$$V_\kappa f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y),$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$ (see [7], [12]).

The function $E(x,y) = V_\kappa^x[e^{\langle x,y\rangle}]$, where the superscript means that V_κ is applied to the x variable, plays an important role in the development of the Dunkl transform which is defined on $L^1(\mathbb{R}^d, h_\kappa^2)$ by

$$\widehat{f}(y) = c_h \int_{\mathbb{R}^d} f(x) E(x, -iy) h_\kappa^2(x) dx, \qquad y \in \mathbb{R}^d.$$

If $\kappa = 0$, then $V_\kappa = id$ and the Dunkl transform coincides with the usual Fourier transform. If $d = 1$ and $G = \mathbb{Z}_2$ then the Dunkl transform is related closely to the Hankel transform in the real line (see [13]). In fact, in this case,

$$E(x, -iy) = \Gamma(\kappa + \frac{1}{2})(\frac{|xy|}{2})^{-\kappa+\frac{1}{2}}[J_{\kappa-\frac{1}{2}}(|xy|) - i \operatorname{sign}(xy) J_{\kappa-\frac{1}{2}}(|xy|)],$$

where J_α denotes the usual Bessel function

$$J_\alpha(t) = (\frac{t}{2})^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} (\frac{t}{2})^{2n}.$$

Some of the properties of the kernel $E(x,y)$ and the Dunkl transform are collected below (see [4], [5]).

Proposition 2.1. *(i)* $E(x,y) = E(y,x)$, $\quad x,\ y \in \mathbb{R}^d$.
(ii) $|E(x,y)| \le e^{\|x\|\|y\|}$, $\quad x, y \in \mathbb{C}^d$.
(iii) For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, $\widehat{f}$ *is in* $\mathcal{C}_0(\mathbb{R}^d)$.
(iv) The Dunkl transform is a topological automorphism of $\mathcal{S}(\mathbb{R}^d)$.
(v) (Inversion formula) When both f *and* $\widehat{f}$ *are in* $L^1(\mathbb{R}^d, h_\kappa^2)$ *we have*

$$f(x) = \int_{\mathbb{R}^d} E(ix, y) \widehat{f}(y) h_\kappa^2(y) dy.$$

(vi) (Plancherel Theorem) The Dunkl transform extends to an isometry of $L^2(\mathbb{R}^d, h_\kappa^2)$.

The Dunkl transform allows us to define a generalized translation operator on $L^2(\mathbb{R}^d, h_\kappa^2)$ by setting

$$\widehat{\tau_y f}(x) = E(y, -ix) \widehat{f}(x), \qquad x \in \mathbb{R}^d.$$

In the analysis of this generalized translation a particular role is played by the space (cf. [6], [7] , [9] and [11])

$$A_\kappa(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d, h_\kappa^2) : \quad \widehat{f} \in L^1(\mathbb{R}^d, h_\kappa^2)\}.$$

Note that $A_\kappa(\mathbb{R}^d)$ is contained in the intersection of $L^1(\mathbb{R}^d, h_\kappa^2)$ and L^∞ and hence is a subspace of $L^2(\mathbb{R}^d, h_\kappa^2)$. The operator τ_y satisfies the following properties:

Proposition 2.2. *Assume that $f \in A_\kappa(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d, h_\kappa^2)$ is bounded. Then:*
(i) $\displaystyle\int_{\mathbb{R}^d} \tau_y f(x) g(x) h_\kappa^2(x)dx = \int_{\mathbb{R}^d} f(x)\tau_{-y} g(x) h_\kappa^2(x)dx.$
(ii) $\tau_y f(x) = \tau_{-x} f(-y)$.

A formula of $\tau_y f$ is known, at the moment, only in two cases. One is in the case of $G = \mathbb{Z}_2$ and $h_\kappa(x) = |x|^\kappa$ on $\mathbb{R}$ (see [8]):

$$\begin{aligned}\tau_y f(x) = \quad & \tfrac{1}{2}\int_{-1}^{1} f(\sqrt{x^2+y^2-2xyt})(1 + \tfrac{x-y}{\sqrt{x^2+y^2-2xyt}})\phi_\kappa(t)dt \\ & +\tfrac{1}{2}\int_{-1}^{1} f(-\sqrt{x^2+y^2-2xyt})(1 - \tfrac{x-y}{\sqrt{x^2+y^2-2xyt}})\phi_\kappa(t)dt,\end{aligned}$$

where $\phi_\kappa(t) = b_\kappa(1+t)(1-t^2)^{\kappa-1}$, from which also follows a formula of $\tau_y f$ in the case of $G = \mathbb{Z}_2^d$. This formula implies easily the L^p-boundedness of τ_y in this case.

Another case where a formula of $\tau_y f$ is known is when f are radial functions, $f(x) = f_0(\|x\|)$, and G being any reflection group (see [6])

$$\tau_y f(x) = V_\kappa[f_0(\sqrt{\|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\langle x', .\rangle})](y'), \quad x' = \frac{x}{|x|}, \quad y' = \frac{y}{|y|}$$

from which it follows that $\tau_y f(x) \geq 0$ for all $y \in \mathbb{R}^d$ if $f(x) = f_0(\|x\|) \geq 0$. Several essential properties of $\tau_y f$ (f being radial) follow from this formula. This is collected in the following proposition (see [9]).

Proposition 2.3. *(i) For every $f \in L^1_{rad}(\mathbb{R}^d, h_\kappa^2)$ (the subspace of radial functions in $L^p(\mathbb{R}^d, h_\kappa^2)$) we have:.*

$$\int_{\mathbb{R}^d} \tau_y f(x) h_\kappa^2(x)dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(x)dx.$$

(ii) For $1 \leq p \leq 2$, $\tau_y :\ L^p_{rad}(\mathbb{R}^d, h_\kappa^2) \to L^p_{rad}(\mathbb{R}^d, h_\kappa^2)$ is a bounded operator.

Apart from these two cases, we lack precise information, in particular about the boundedness of generalized translations (see [1]). Based on this fact we tried to develop an elementary approach of the Riesz potentials which works with a minimal knowledge about τ_y and gives the answer to the problem (1)-(2) in full generality.

3. Riesz Potentials

For the proof of Theorem 1.1, we need the following version of a classical Schur's lemma.

Lemma 3.1. *Assume that k is a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ that satisfies the mixed-norm conditions:*

$$C_1 = \sup_{x\in\mathbb{R}^d} \int |k(x,y)|h_\kappa^2(y)dy < \infty, \quad C_2 = \sup_{y\in\mathbb{R}^d} \int |k(x,y)|h_\kappa^2(x)dx < \infty.$$

Then the integral operator induced by the kernel $k(x,y)$ (i.e. the operator defined by $T_k f(x) = \int k(x,y)f(y)h_\kappa^2(y)dy$) defines a bounded mapping of $L^p(\mathbb{R}^d, h_\kappa^2)$ into itself for every $1 \le p \le \infty$, with

$$\|T_k\|_{L^p(\mathbb{R}^d,h_\kappa^2)\to L^p(\mathbb{R}^d,h_\kappa^2)} \le C_1^{1-\frac{1}{p}} C_2^{\frac{1}{p}}.$$

Proof of Theorem 1.1 We begin with this simple formula used by Thangavelu and Xu in [10] and in the same context

$$\|y\|^{-a} = \frac{1}{\Gamma(\frac{a}{2})} \int_0^\infty s^{\frac{a}{2}} e^{-s\|y\|^2} \frac{ds}{s}, \qquad y \in \mathbb{R}^d, \quad a > 0.$$

Applying this formula with $a = 2\gamma_\kappa + d - \alpha$ and changing the order of integrals in (1), we obtain

$$I_\alpha^\kappa f(x) = \frac{2^{\gamma_\kappa + \frac{d}{2} - \alpha}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty s^{\gamma_\kappa + \frac{d-\alpha}{2}} \Big(\int_{\mathbb{R}^d} \tau_y f(x) e^{-s\|y\|^2} h_\kappa^2(y)dy\Big) \frac{ds}{s}, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

By Proposition 2.2, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \tau_y f(x) e^{-s\|y\|^2} h_\kappa^2(y)dy &= \int_{\mathbb{R}^d} \tau_{-x} f(-y) e^{-s\|y\|^2} h_\kappa^2(y)dy \\
&= \int_{\mathbb{R}^d} \tau_{-x} f(y) e^{-s\|y\|^2} h_\kappa^2(y)dy \\
&= \int_{\mathbb{R}^d} f(y) \tau_x(e^{-s\|y\|^2})(y) h_\kappa^2(y)dy.
\end{aligned}$$

We have thus the identity

$$I_\alpha^\kappa f(x) = \frac{2^{\gamma_\kappa + \frac{d}{2} - \alpha}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty s^{\gamma_\kappa + \frac{d-\alpha}{2}} \int_{\mathbb{R}^d} f(y) \tau_x(e^{-s\|y\|^2})(y) h_\kappa^2(y)dy \frac{ds}{s}, \ f \in \mathcal{S}(\mathbb{R}^d). \quad (4)$$

With the aid of this identity it will not be difficult to extend the mapping $f \to I_\alpha^\kappa f$ to all functions $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $p \ge 1$. Indeed we first notice that for each $x \in \mathbb{R}^d$, the function $y \to \tau_x(e^{-s\|y\|^2})(y)$ is positive and satisfies (see [8])

$$\tau_x(e^{-s\|y\|^2}) = e^{-s(\|x\|^2 + \|y\|^2)} E(2sx, y), \qquad s > 0.$$

Using Proposition 2.1, we deduce that

$$\tau_x(e^{-s\|y\|^2})(y) \le e^{-s(\|x\| - \|y\|)^2} \le 1. \quad (5)$$

On the other hand, applying Proposition 2.3, we obtain

$$\int_{\mathbb{R}^d} \tau_x(e^{-s\|y\|^2})(y)h_\kappa^2(y)dy = \int_{\mathbb{R}^d} e^{-s\|y\|^2} h_\kappa^2(y)dy = \frac{1}{c_h(2s)^{2\gamma_\kappa+\frac{d}{2}}}, \quad s > 0. \tag{6}$$

Let (p, q) be a pair of real numbers satisfying (3) and let $f \in L^p(\mathbb{R}^d, h_\kappa^2)$ normalized so that $\|f\|_{\kappa,p} = 1$. We shall prove that the integral

$$\frac{2^{\gamma_\kappa+\frac{d}{2}-\alpha}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty s^{\gamma_\kappa+\frac{d-\alpha}{2}} \int_{\mathbb{R}^d} f(y)\tau_x(e^{-s\|y\|^2})(y)h_\kappa^2(y)dy\frac{ds}{s} \tag{7}$$

converges absolutely for almost every x. Towards this let us decompose (7) as a sum of two terms $S_1f(x) + S_2f(x)$ where

$$S_1f(x) = \frac{2^{\gamma_\kappa+\frac{d}{2}-\alpha}}{\Gamma(\frac{\alpha}{2})} \int_0^\sigma s^{\gamma_\kappa+\frac{d-\alpha}{2}} \int_{\mathbb{R}^d} f(y)\tau_x(e^{-s\|y\|^2})(y)h_\kappa^2(y)dy\frac{ds}{s},$$

$$S_2f(x) = \frac{2^{\gamma_\kappa+\frac{d}{2}-\alpha}}{\Gamma(\frac{\alpha}{2})} \int_\sigma^\infty s^{\gamma_\kappa+\frac{d-\alpha}{2}} \int_{\mathbb{R}^d} f(y)\tau_x(e^{-s\|y\|^2})(y)h_\kappa^2(y)dy\frac{ds}{s}.$$

At this instance $\sigma > 0$ is a fixed positive constant which need not to be specified (it suffices to take $\sigma = 1$ for example).

Let us estimate $\|S_1f\|_\infty$. Let $x \in \mathbb{R}^d$, we have

$$|S_1f(x)| \leq \frac{2^{\gamma_\kappa+\frac{d}{2}-\alpha}}{\Gamma(\frac{\alpha}{2})} \int_0^\sigma s^{\gamma_\kappa+\frac{d-\alpha}{2}} \sup_x \int_{\mathbb{R}^d} |f(y)|\tau_x(e^{-s\|y\|^2})(y)dy\frac{ds}{s}.$$

However, it follows from (5) that

$$\sup_x \int_{\mathbb{R}^d} |f(y)|\tau_x(e^{-s\|y\|^2})(y)dy \leq \|f\|_{\kappa,1},$$

and from (6)

$$\begin{aligned}\sup_x \int_{\mathbb{R}^d} |f(y)|\tau_x(e^{-s\|y\|^2})(y)dy &\leq \left(\sup_x \int_{\mathbb{R}^d} \tau_x(e^{-s\|y\|^2})(y)h_\kappa^2(y)dy\right)\|f\|_\infty \\ &\leq \frac{1}{c_h(2s)^{2\gamma_\kappa+\frac{d}{2}}}\|f\|_\infty.\end{aligned}$$

Using complex interpolation we deduce then that

$$\begin{aligned}\sup_x \int_{\mathbb{R}^d} |f(y)|\tau_x(e^{-s\|y\|^2})(y)dy &\leq \left(\frac{1}{c_h(2s)^{\gamma_\kappa+\frac{d}{2}}}\right)^{1-\frac{1}{p}}\|f\|_{\kappa,p} \\ &= \frac{2^{-(1-\frac{1}{p})(\gamma_\kappa+\frac{d}{2})}s^{-(1-\frac{1}{p})(\gamma_\kappa+\frac{d}{2})}}{c_h^{1-\frac{1}{p}}},\end{aligned}$$

and then

$$\begin{aligned}|S_1f(x)| &\leq \frac{2^{\frac{1}{p}(\gamma_\kappa+\frac{d}{2})-\alpha}}{c_h^{1-\frac{1}{p}}\Gamma(\frac{\alpha}{2})}\int_0^\sigma s^{\frac{1}{p}(\gamma_\kappa+\frac{d}{2})-\frac{\alpha}{2}}\frac{ds}{s} \\ &= A\,\sigma^{\frac{1}{2}(\frac{2\gamma_\kappa+d}{p}-\alpha)}, \quad x \in \mathbb{R}^d,\end{aligned}$$

and so
$$\|S_1 f\|_\infty \le A\sigma^{\frac{1}{2}(\frac{2\gamma_\kappa+d}{p}-\alpha)}. \tag{8}$$
Let us estimate $\|S_2 f\|_{\kappa,p}$. We write
$$\|S_2 f\|_{\kappa,p} \le \frac{2^{\gamma_\kappa+\frac{d}{2}-\alpha}}{\Gamma(\frac{\alpha}{2})}\int_\sigma^\infty s^{\gamma_\kappa+\frac{d-\alpha}{2}}\|\int_{\mathbb{R}^d} f(y)\tau_x(e^{-s\|y\|^2})(y)h_\kappa^2(y)dy\|_{\kappa,p}\frac{ds}{s}.$$
The L^p norm of $\displaystyle\int_{\mathbb{R}^d} f(y)\tau_x(e^{-s\|y\|^2})(y)h_\kappa^2(y)dy$ can be easily estimated by using Lemma 3.1 which gives
$\|\displaystyle\int_{\mathbb{R}^d} f(y)\tau_x(e^{-s\|y\|^2})(y)h_\kappa^2(y)dy\|_{\kappa,p} \le$
$$\left(\sup_x \int_{\mathbb{R}^d} \tau_x(e^{-s\|y\|^2})(y)h_\kappa^2(y)dy\right)^{1-\frac{1}{p}}\left(\sup_y \int_{\mathbb{R}^d} \tau_x(e^{-s\|y\|^2})(y)h_\kappa^2(x)dx\right)^{\frac{1}{p}},$$
and then by (6)
$$\|S_2 f\|_{\kappa,p} \le \frac{2^{-\alpha}}{\Gamma(\frac{\alpha}{2})c_h}(\int_\sigma^\infty s^{-\frac{\alpha}{2}}\frac{ds}{s}) = \frac{2^{1-\alpha}}{2\Gamma(\frac{\alpha}{2})c_h}\sigma^{-\frac{\alpha}{2}} = B\sigma^{-\frac{\alpha}{2}}. \tag{9}$$
Putting together (8) and (9) we deduce that (7) converges absolutely for almost every $x \in \mathbb{R}^d$.

Let now $\lambda > 0$ and let us estimate
$$\int_{\{x:\ |I_\alpha^\kappa f(x)|>\lambda\}} h_\kappa^2(x)dx \le \int_{\{x:\ |S_1 f(x)|>\frac{\lambda}{2}\}} h_\kappa^2(x)dx + \int_{\{x:\ |S_2 f(x)|>\frac{\lambda}{2}\}} h_\kappa^2(x)dx,$$
we choose σ to satisfy
$$A\sigma^{\frac{1}{2}(\frac{2\gamma_\kappa+d}{p}-\alpha)} = \frac{\lambda}{2}, \tag{10}$$
so that (thanks to (8))
$$\int_{\{x:\ |S_1 f(x)|>\frac{\lambda}{2}\}} h_\kappa^2(x)dx = 0.$$
We get
$$\int_{\{x:\ |I_\alpha^\kappa f(x)|>\lambda\}} h_\kappa^2(x)dx \le \int_{\{x:\ |S_2 f(x)|>\frac{\lambda}{2}\}} h_\kappa^2(x)dx \le \left(\frac{2}{\lambda}\right)^p \|S_2 f\|_{\kappa,p}^p,$$
and then by (9)
$$\int_{\{x:\ |I_\alpha^\kappa f(x)|>\lambda\}} h_\kappa^2(x)dx \le \frac{2^p B^p \sigma^{-\frac{\alpha}{2}p}}{\lambda^p}.$$
Using (10) we deduce that
$$\int_{\{x:\ |I_\alpha^\kappa f(x)|>\lambda\}} h_\kappa^2(x)dx \le \frac{2^p B^p (2A)^{\frac{\alpha p^2}{2\gamma_\kappa+d-\alpha p}}}{\lambda^{\frac{p(2\gamma_\kappa+d)}{2\gamma_\kappa+d-\alpha p}}} = C_{p,\alpha}\left(\frac{\|f\|_{\kappa,p}}{\lambda}\right)^q$$

since $q = \dfrac{p(2\gamma_\kappa + d)}{2\gamma_\kappa + d - \alpha p}$ and $\|f\|_{\kappa,p} = 1$.

The previous considerations show that the mapping $f \to I_\alpha^\kappa f$ is of weak type (p,q):

$$\int_{\{x:\ \ |I_\alpha^\kappa f(x)|>\lambda\}} h_\kappa^2(x)dx \leq C_{p,\alpha}\left(\frac{\|f\|_{\kappa,p}}{\lambda}\right)^q, \qquad f \in L^p(\mathbb{R}^d, h_\kappa^2), \qquad \lambda > 0,$$

and $1 \leq p < q < \infty$ with $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{2\gamma_\kappa + d}$. The special case for $p = 1$ gives then part (ii) of Theorem 1.1, and part (i) follows by an obvious use of real interpolation properties of the spaces $L^p(\mathbb{R}^d, h_\kappa^2)$ and Marcinkiewicz interpolation theorem. □

Proof of Corollary 1.2 This is an immediate consequence of Theorem 1.1 and the following pointwise inequality

$$M_{\kappa,\alpha}f(x) = \sup_{r>0} \frac{1}{m_\kappa r^{d+2\gamma_\kappa-\alpha}} \int_{\mathbb{R}^d} |f(y)|\tau_x\chi_{B_r}(y)h_\kappa^2(y)dy \leq C_{\kappa,d,\alpha} I_\alpha^\kappa(|f|)(x),$$

where $C_{\kappa,d,\alpha}$ depends only on κ, d and α. □

Finally one should observe that (3) is also necessary for the boundedness of the maximal fractional operator $M_{\kappa,\alpha}$ from the spaces $L^p(\mathbb{R}^d, h_\kappa^2)$ to the space $L^q(\mathbb{R}^d, h_\kappa^2)$ when $p > 1$ and for the weak-type estimate of $M_{\kappa,\alpha}$ when $p = 1$. The proof of this fact is straightforward. It suffices to consider the dilation operator $\delta_r f(x) = f(rx)$, $r > 0$, and to observe that $\delta_{r^{-1}} M_{\kappa,\alpha}\delta_r = r^{-\alpha}M_{\kappa,\alpha}$. It follows then that $\|\delta_r f\|_{\kappa,p} = r^{-\frac{2\gamma_\kappa+d}{p}}\|f\|_{\kappa,p}$. By dilation, the estimate $\|M_{\kappa,\alpha}f\|_{\kappa,q} \leq C\|f\|_{\kappa,p}$ (where we assume $p > 1$) implies then that

$$\|M_{\kappa,\alpha}f\|_{\kappa,q} \leq Cr^{\alpha+\frac{2\gamma_\kappa+d}{q}-\frac{2\gamma_\kappa+d}{p}}\|f\|_{\kappa,p}, \quad r > 0.$$

Letting $r \to \infty$ and $r \to 0$ we get $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{\alpha}{2\gamma_\kappa + d}$. The same argument applies in the weak case.

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