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Comparison of Lattice Filtrations and Moy-Prasad Filtrations for Classical Groups

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Abstract. Let $F_\circ$ be a non-Archimedean local field of characteristic not 2. Let G be a classical group over $F_\circ$ which is not a general linear group, i.e. a symplectic, orthogonal or unitary group over $F_\circ$ (possibly with a skew-field involved). Let x be a point in the building of G. In this article, we prove that the lattice filtration $(\mathfrak{g}_{x,r})_{r\in\mathbb{R}}$ of $\mathfrak{g} = \mathrm{Lie}(G)$ attached to x by Broussous and Stevens, coincides with the filtration defined by Moy and Prasad.
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Introduction

Let V be a finite dimensional vector space over a locally compact non-Archimedean field F of characteristic not 2 — residual characteristic 2 is permitted —, and $F_\circ$ be a subfield of F such that $[F : F_\circ] \leq 2$. Let σ be the generator of $\mathrm{Gal}(F/F_\circ)$ if $F \neq F_\circ$, and $\sigma = \mathrm{Id}_F$ if $F = F_\circ$. We fix a non-degenerate σ-skew ε-hermitian form h on V, where $\varepsilon \in \{\pm 1\}$. Let $G = \mathrm{U}(h)$ be the subgroup of $\mathrm{GL}(V)$ formed of those g satisfying $h(gx, gy) = h(x, y)$ for all $x,\, y \in V$. It is the group of $F_\circ$-rational points of an $F_\circ$-algebraic group $\boldsymbol{G}$ whose connected component $\boldsymbol{G}^\circ$ is reductive. To each point x of the building $\mathcal{I}$ of G, let $(\mathfrak{g}_{x,r})_{r\in\mathbb{R}}$ be the filtration of the Lie algebra $\mathfrak{g}$ of G attached to x by Broussous and Stevens in [BS][1]; let also $(\mathfrak{g}^{\mathrm{MP}}_{x,r})_{r\in\mathbb{R}}$ be the filtration of $\mathfrak{g}$ attached to x by Moy and Prasad in [MP][2]. In this article, we prove that the two filtrations coincide:

$$\mathfrak{g}_{x,r} = \mathfrak{g}^{\mathrm{MP}}_{x,r}, \quad r \in \mathbb{R}.$$

This result is used by Broussous and Stevens in [BS] — this is the reason for which we proved it. Note that for a general linear group (i.e. an inner $F_\circ$-form of GL_n),

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[1] In [BS], the base field $F_\circ$ is supposed to be of odd residual characteristic, but the definition of $(\mathfrak{g}_{x,r})_{r\in\mathbb{R}}$ naturally extends to the residual characteristic 2 (see 1.7 and Remark 1.6).

[2] In [MP] the group G is supposed to be simply connected, which is not the case here in general. But the definition of $(\mathfrak{g}^{\mathrm{MP}}_{x,r})_{r\in\mathbb{R}}$ naturally extends to a general connected reductive group (see 3.5).

the analogous result is proved in [BL] Appendix A.

Recall that the Moy-Prasad filtrations are defined by descent from a maximal unramified extension $F_\circ^{\mathrm{nr}}$ of $F_\circ$. So let $L_\circ/F_\circ$ be a finite sub-extension of $F_\circ^{\mathrm{nr}}/F_\circ$ such that the $L_\circ$-group $\boldsymbol{G}^\circ \times_{F_\circ} L_\circ$ is quasi-split (note that we can choose $L_\circ$ such that $\boldsymbol{G}^\circ \times_{F_\circ} L_\circ$ is residually split, i.e. quasi-split and with the same relative rank as $\boldsymbol{G}^\circ \times_{F_\circ} F_\circ^{\mathrm{nr}}$). Denote by $G_{L_\circ}$ the group of $L_\circ$-rational points of $\boldsymbol{G}$, and by L the $L_\circ$-algebra $L_\circ \otimes_{F_\circ} F$. There are two cases: L is a field; or $L \simeq (L_\circ)^2$. If L is a field, then the form h extends to a non-degenerate $(\mathrm{Id}\otimes\sigma)$-skew ε-hermitian form h_L on V_L, and $G_{L_\circ} = \mathrm{U}(h_L)$. If $L \simeq (L_\circ)^2$, then F is isomorphic to a quadratic sub-extension of $L_\circ/F_\circ$, and we can suppose $F \subset L_\circ$; then we have $G_{L_\circ} \simeq \mathrm{GL}(L_\circ \otimes_F V)$.

Let $\mathcal{I}_{L_\circ}$ be the building of $G_{L_\circ}$, and Γ be the Galois group of $L_\circ/F_\circ$. There exists a unique G-invariant affine map $\mathcal{I} \to \mathcal{I}_{L_\circ}$, which allows us to identify $\mathcal{I}$ with the convex subset $(\mathcal{I}_{L_\circ})^\Gamma$ of $\mathcal{I}_{L_\circ}$ formed of those points which are Γ-invariant. The point $x \in \mathcal{I}$ defines two filtrations $(\mathfrak{g}_{L_\circ,x,r})_{r\in\mathbb{R}}$ and $(\mathfrak{g}^{\mathrm{MP}}_{L_\circ,x,r})_{r\in\mathbb{R}}$ of the Lie algebra $\mathfrak{g}_{L_\circ} = L_\circ \otimes_{F_\circ} \mathfrak{g}$ of $G_{L_\circ}$, where $(\mathfrak{g}_{L_\circ,x,r})_{r\in\mathbb{R}}$ is the filtration attached in [BS] to $x \in (\mathcal{I}_{L_\circ})^\Gamma$ if L is a field, and in [BL] if $L \simeq (L_\circ)^2$. By definition, we have the descent property:

$$(\mathfrak{g}^{\mathrm{MP}}_{L_\circ,x,r})^\Gamma = \mathfrak{g}^{\mathrm{MP}}_{x,r}, \quad r \in \mathbb{R}.$$

We prove we also have the descent property:

$$(\mathfrak{g}_{L_\circ,x,r})^\Gamma = \mathfrak{g}_{x,r}, \quad r \in \mathbb{R}.$$

This reduces the question to the quasi-split case. Now assume that the reductive $F_\circ$-group $\boldsymbol{G}^\circ$ is quasi-split. In that case, we can describe explicitly the intersection of $\mathfrak{g}_{x,r}$ (resp. $\mathfrak{g}^{\mathrm{MP}}_{x,r}$) with each root subspace of $\mathfrak{g}$ with respect to a maximal split torus S of G, and with the Lie algebra of the centralizer of S in G. This shows that both filtrations coincide.

Following a suggestion of Gopal Prasad, we also extend the result to a more general "unitary group" of type $\mathrm{U}(h)$, that is with a skew-field — in fact a quaternionic algebra — involved (cf. [BT4]). Such a group becomes, over a finite unramified extension $L_\circ$ of $F_\circ$, a unitary group of the previous type (i.e. of type $\mathrm{U}(h_L)$ with no skew-field involved) or a split general linear group. So taking into account [BL] Appendix A, the extended result is implied by the descent property for the Broussous-Stevens filtrations (for a general unitary group); this descent property is proved (briefly) as in the case with no skew-field involved.

Ultimately, we have that Broussous-Stevens filtrations and Moy-Prasad filtrations coincide for *almost all* (cf. Remark 1.1) $F_\circ$-forms of GL_n, Sp_{2n}, O_n.

In chapter 1, we introduce all objects and notation we need — unitary groups are defined in 1.1 in full generality, but from 1.2 untill the end of chapter 3, we assume that there is no skew-field involved —, and we state the result (Theorem 1.8). In chapter 2 we reduce the proof of the result to the quasi-split case. In chapter 3 we prove the result in the quasi-split case. In chapter 4 we extend the result to a general unitary group.

1. The objects

1.1. Unitary groups. Let $F_\circ$ be a locally compact non-Archimedean commutative field, and F be a Galois extension of $F_\circ$ of degree at most 2. We assume that the characteristic of $F_\circ$ is not 2. Let σ be the generator of $\mathrm{Gal}(F/F_\circ)$ if $[F:F_\circ]=2$, and the identity of F if $F=F_\circ$. Let D be a central division F-algebra of finite dimension d^2, endowed with an *involution* extending σ, still denoted by σ — i.e. σ is an anti-automorphism if D, such that $\sigma^2=\mathrm{Id}_D$ and $\sigma|_F$ is the generator of $\mathrm{Gal}(F/F_\circ)$. We know that $d=1$ or 2, i.e. $D=F$ or D is a quaternionic algebra over F. Let $D_\circ$ and D° be the sub-$F_\circ$-algebras of D defined by

$$D_\circ=\{\lambda\in D:\lambda^\sigma=\lambda\},\quad D^\circ=\{\lambda\in D:\lambda+\lambda^\sigma=0\}.$$

Since $\mathrm{char}(F)\neq 2$, we have the decomposition $D=D_\circ\oplus D^\circ$ (with $D^\circ=\{0\}$ if $\sigma=\mathrm{Id}$). The notations are coherent: if $D=F$, then $D_\circ$ coincides with $F_\circ$. Put $F^\circ=F\cap D^\circ$.

Let $\varepsilon\in\{\pm1\}$. We fix a finite dimensional *right* D-vector space V, and a σ-skew ε-hermitian form h on V, that is a $\mathbb{Z}$-bilinear map $V\times V\to F$ such that, for all $x,\,y\in V$ and all $\lambda,\,\mu\in D$, we have

$$h(x\lambda,y\mu)=\lambda^\sigma h(x,y)\mu,$$
$$h(y,x)=\varepsilon h(x,y)^\sigma.$$

The form h is supposed to be *non-degenerate.* Put

$$D_{\sigma,\varepsilon}=\{\lambda-\varepsilon\lambda^\sigma:\lambda\in D\}.$$

It is a subset of $\{\lambda\in D:\lambda^\sigma=-\varepsilon\lambda\}$, and since for $\lambda\in D$ such that $\lambda^\sigma=-\varepsilon\lambda$, we have $\lambda=\frac12\lambda-\varepsilon(\frac12\lambda)^\sigma$, the two sets coincide. So we have

$$D_{\sigma,\varepsilon}=D^\circ\ \text{ if }\varepsilon=1,$$
$$D_{\sigma,\varepsilon}=D_\circ\ \text{ if }\varepsilon=-1.$$

Denote by $\overline{D}_{\sigma,\varepsilon}$ the $F_\circ$-vector space $D/D_{\sigma,\varepsilon}$, and by $\lambda\mapsto\bar\lambda$ the canonical projection $D\to\overline{D}_{\sigma,\varepsilon}$. Let ξ be an element of F such that $\xi+\xi^\sigma=1$ (since the characteristic of $F_\circ$ is not 2, we can take $\xi=\frac12$). Let $q=q_h:V\to\overline{D}_{\sigma,\varepsilon}$ be the pseudo-quadratic form associated with h (cf. [BT4] 1.2), defined by

$$q(x)=\xi h(x,x)+D_{\sigma,\varepsilon}.$$

It is well defined: if ξ' is another element of F such that $\xi'+\xi'^\sigma=1$, then $\xi'-\xi\in F^\circ$, and $(\xi'-\xi)h(x,x)\in D_{\sigma,\varepsilon}$ for all $x\in V$. Note we also have

$$q(x)=\{\mu\in F:\mu+\varepsilon\mu^\sigma=h(x,x)\}+D_{\varepsilon,\sigma},\quad x\in V.$$

For all $x,\,y\in V$ and all $\lambda\in D$, we have

$$q(x\lambda)=\tfrac12\overline{\lambda^\sigma h(x,x)\lambda},$$

$$q(x+y) = q(x) + q(y) + \overline{h(x,y)}.$$

If $(\sigma, \varepsilon) = (\mathrm{Id}, 1)$, then $q : V \to D$ is a quadratic form in the usual sense and h is the bilinear form associated with q. If $(\sigma, \varepsilon) = (\mathrm{Id}, -1)$, which is equivalent to $\overline{D}_{\sigma,\varepsilon} = \{0\}$, then $q = 0$. If $(\sigma, \varepsilon) \neq (\mathrm{Id}, -1)$, then h is determined by q: it is the unique σ-skew ε-hermitian form on V verifying $\overline{h(x,y)} = q(x+y) - q(x) - q(y)$ for all x, $y \in V$.

Put $\widetilde{G} = \mathrm{GL}(V)$ $(= \mathrm{Aut}_D(V))$ and let $G = \mathrm{U}(h)$ be the subgroup of $\widetilde{G}$ formed of those g satisfying $h(gx, gy) = h(x, y)$ for all x, $y \in V$. Then G is the group of $F_\circ$-rational points of a linear algebraic group $\boldsymbol{G}$ defined over $F_\circ$, whose neutral component $\boldsymbol{G}^\circ$ is reductive.

Remark 1.1. The algebraic group $\mathbf{G}$ is an $F_\circ$-form of one of the (split) classical groups GL_n, Sp_{2n}, O_n. Moreover, by varying the data F, D, σ, ε, V, h, we obtain all $F_\circ$-forms of those classical groups, except the inner forms of GL_n, and certain forms corresponding to a Dynkin diagram of symmetric group $\mathcal{S}_3$ (e.g. O_8 with Dynkin diagram of type D_4). For the inner forms of GL_n, the comparison of lattice filtrations and Moy-Prasad filtrations is already done in [BL].

From now on, untill the end of chapter 3, we assume $D = F$ and we consider V as a left F-vector space.

1.2. Derived groups. Put $\widetilde{G}' = \mathrm{SL}(V)$ and let $G' = \mathrm{SU}(h)$ be the subgroup $G \cap \widetilde{G}'$ of $\widetilde{G}'$. Then G' is the group of $F_\circ$-rational points of a linear algebraic group $\boldsymbol{G}'$ defined over $F_\circ$. Put

$$F^1 = \{\lambda \in F^\times : \lambda^\sigma \lambda = 1\}.$$

Identifying $F^\times$ with the centre $F^\times \mathrm{Id}_V \subset \mathrm{GL}(V)$ of $\widetilde{G}$, we have the inclusion

$$F^1 G' \subset G \cap F^\times \widetilde{G}.$$

Moreover, G' is a cocompact subgroup of G (see 1.4 and the following remark).

Remark 1.2. If $\dim_F(V) = 1$, then we have $G = F^1$ and $G' = \{1\}$; thus $\boldsymbol{G}^\circ(F) = G$ if $\sigma \neq \mathrm{Id}$, and $\boldsymbol{G}^\circ(F) = \{1\}$ if $\sigma = \mathrm{Id}$. Now suppose $\dim_F(V) = 2$ and $\sigma = \mathrm{Id}$. If $\varepsilon = 1$, then $G \simeq F_\circ^\times \ltimes \langle s \rangle$ with $\lambda^s = \lambda^{-1}$ for all $\lambda \in F_\circ^\times$, and $G' = \boldsymbol{G}^\circ(F) \simeq F_\circ^\times$; if $\varepsilon = -1$, then we have $G = G' = \boldsymbol{G}^\circ(F) = \mathrm{SL}(V)$.

In the small dimension cases of the Remark 1.2, the lattice filtration of the Lie algebra of G attached to a point x of the building of G coincides with the filtration defined by Moy and Prasad in [MP]: it is a straightforward consequence of the definitions if G is a torus (see 3.5), and it is a consequence of [BL] if $G \simeq \mathrm{SL}(2, F_\circ)$. So from now on, we assume that

$$\dim_{F_\circ}(V) \geq 3.$$

Then $\boldsymbol{G}'$ is connected ([BT4] 1.5) and semisimple (see [PR] 2.3). In particular, $\boldsymbol{G}'$ is a subgroup of $\boldsymbol{G}^\circ$.

1.3. Root systems. Recall that a subspace W of V is called totally isotropic if $h(W,W)=\{0\}$. We fix a Witt decomposition

$$V=V_-\oplus V_0\oplus V_+,$$

where V_- and V_+ are two isotropic subspaces of V of maximal dimension such that $V_-\cap V_+=\{0\}$, and $V_0=(V_-+V_+)^\perp$; here, for a subspace W of V, $W^\perp$ denotes the subspace $\{x\in V : h(x,W)=0\}$ of V. Put $n=\dim_F(V)$, $r=\dim_F(V_-)$ and $n_0=n-2r$ so we have $\dim_F(V_0)=n_0$. Note that $r=0$ if and only if the form h is *anisotropic*. Put $I=\{\pm 1,\dots,\pm r\}$ and let $(e_{-i})_{i=1,\dots,r}$ and $(e_i)_{i=1,\dots,r}$ be some basis of V_- and V_+ such that for all i, $j\in I$ and all $x\in V_0$, we have

$$h(e_i,e_j)=0 \;\text{ if } i\neq -j,$$

$$h(e_i,e_{-i})=\varepsilon(i),$$

$$h(e_i,x)=0,$$

$$q(x)\neq 0 \text{ if } x\neq 0;$$

where $\varepsilon(i)=1$ if $i>0$, and $\varepsilon(i)=\varepsilon$ if $i<0$. Denote by h_0 the restriction of h to $V_0\times V_0$. If $V_0\neq 0$, h_0 is a non-degenerate anisotropic σ-skew ε-hermitian form h_0 on V_0. Hence the form h is given by (for λ_i, $\mu_i\in F$ and x, $y\in V_0$):

$$h\left(\textstyle\sum_{i\in I}\lambda_i e_i+x,\sum_{i\in I}\mu_i e_i+y\right)=\textstyle\sum_{i\in I}\varepsilon(i)\lambda_i^\sigma\mu_{-i}+h_0(x,y).$$

Let S be the subgroup of G formed of those g satisfying $ge_i\in F_\circ e_i$ for all $i\in I$, and $gx=x$ for all $x\in V_0$. We have $S\subset G'$, and S is the group of $F_\circ$-rational points of a maximal $F_\circ$-split torus $\boldsymbol{S}$ in $\boldsymbol{G}^\circ$ (hence in $\boldsymbol{G}'$). For $i\in I$, let a_i be the algebraic character of $\boldsymbol{S}$ given by $se_i=a_i(s)^{-1}e_i$. We have $a_{-i}=-a_i$ in the group $\mathrm{X}^*(\boldsymbol{S})$ of algebraic characters of $\boldsymbol{S}$, denoted additively. The a_i for $i>0$ form a basis of $\mathrm{X}^*(\boldsymbol{S})$. For i, $j\in I$, $j\neq\pm i$, put $a_{i,j}=a_i+a_j$. Let $\Phi=\Phi(\boldsymbol{S},\boldsymbol{G})$ be the (relative) root system of $\boldsymbol{G}$. We have the following cases ([BT1] 10.1), where i, $j\in I$, $j\neq\pm i$:

(B): $\Phi=\{a_i,a_{i,j}\}$ when $V_0\neq\{0\}$ and $(\sigma,\varepsilon)=(\mathrm{Id},1)$;

(BC): $\Phi=\{a_i,2a_i,a_{i,j}\}$ when $V_0\neq\{0\}$ and $\sigma\neq\mathrm{Id}$;

(C): $\Phi=\{2a_i,a_{i,j}\}$ when $V_0=\{0\}$ and $(\sigma,\varepsilon)\neq(\mathrm{Id},1)$;

(D): $\Phi=\{a_{i,j}\}$ when $V_0=\{0\}$ and $(\sigma,\varepsilon)=(\mathrm{Id},1)$.

The case (C) can be divided in two sub-cases: $V_0=\{0\}$ and $(\sigma,\varepsilon)=(\mathrm{Id},-1)$ (the symplectic case); $V_0=\{0\}$ and $\sigma\neq\mathrm{id}$ (a quasi-split unitary case).

1.4. The groups $Z=Z_G(S)$ and $N=N_G(Z)$. For $i\in I$, put $V_i=Fe_i$. The centralizer $\boldsymbol{Z}$ of $\boldsymbol{S}$ in $\boldsymbol{G}$ is defined over $F_\circ$. Its group of $F_\circ$-rational points is the subgroup Z of G formed of those g satisfying $gV_i=V_i$ for all $i\in I\cup\{0\}$. We have $Z=\widetilde{Z}\cap G$ where $\widetilde{Z}$ is the Levi subgroup of $\widetilde{G}$ formed of those g satisfying $gV_i=V_i$ for all $i\in I\cup\{0\}$. The centralizer $\boldsymbol{Z}'$ of $\boldsymbol{S}$ in $\boldsymbol{G}'$ is also defined over F_0

(it is a Levi $F_\circ$-subgroup of a parabolic $F_\circ$-subgroup of $\boldsymbol{G}'$), and coincides with $\boldsymbol{Z} \cap \boldsymbol{G}'$. Its group of $F_\circ$-rational points is $Z' = Z \cap G'$. Let us describe Z and Z'. For $z \in Z$, put $\mu(z) = \prod_{i=1}^r a_i(z)$.

Suppose first $V_0 = \{0\}$. Then the decomposition

$$V = V_{-r} \oplus \cdots \oplus V_{-1} \oplus V_1 \oplus \cdots \oplus V_r$$

allows us to represent each element $g \in \widetilde{G}$ by a matrix $(g_{i,j})_{i,j \in I}$. An element $z \in Z$ is represented by a diagonal matrix

$$\mathrm{diag}(z_{-r}, \ldots, z_{-1}, z_1, \ldots, z_r) \in \mathrm{GL}(2r, F)$$

such that $z_{-i}^\sigma z_i = 1$ for $i = 1, \ldots, r$. Moreover, we have $z \in Z'$ if and only if $\mu(z) \in F_\circ$. So the map $z \mapsto (a_1(z), \ldots, a_r(z))$ identifies Z with $(F^\times)^r$, and the map $z \mapsto (a_1(z), \ldots, a_{r-1}(z), \mu(z))$ identifies Z' with $(F^\times)^{r-1} \times F_\circ^\times$.

Now suppose $V_0 \neq \{0\}$. Then the decomposition

$$V = V_{-r} \oplus \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_r$$

allows us to represent each element $g \in \widetilde{G}$ by a matrix $(g_{i,j})_{i,j \in I \cup \{0\}}$. An element $z \in Z$ is represented by a block diagonal matrix

$$\mathrm{diag}(z_{-r}, \ldots, z_{-1}, z_0, z_1, \ldots, z_r) \in \mathrm{GL}(2r + n_0, F)$$

such that $z_{-i}^\sigma z_i = 1$ for $i = 1, \ldots, r$, and $z_0 \in \mathrm{U}(h_0)$. Moreover, we have $z \in Z'$ if and only if $\mu(z)^{-1}\mu(z)^\sigma \det(z_0) = 1$. So the map $z \mapsto (z_0^{-1}, a_1(z), \ldots, a_r(z))$ identifies Z with $\mathrm{U}(h_0) \times (F^r)^\times$, and the map $z \mapsto (\mu(z)^{-1}\mu(z)^\sigma z_0, a_1(z), \ldots, a_r(z))$ identifies Z' with $\mathrm{SU}(h_0) \times (F^\times)^r$.

From the above description, the group Z' is a cocompact subgroup of Z. Since $G = ZG'$, we obtain that G' is a cocompact subgroup of G. If $\dim_F(V_0) \leq 1$, then the connected component $\boldsymbol{Z}$ of $\boldsymbol{Z}^\circ$ is a torus (hence a maximal torus of $\boldsymbol{G}^\circ$), and the groups $\boldsymbol{G}^\circ$ and $\boldsymbol{G}'$ are quasi-split over $F_\circ$. Conversely, if $\boldsymbol{G}^\circ$ is quasi-split over $F_\circ$, then $\dim_F(V_0) \leq 1$ ([BT4] 3.5). If $\sigma = \mathrm{Id}$, we have $\boldsymbol{Z}' = \boldsymbol{S} = \boldsymbol{Z}^\circ$ and $\boldsymbol{G}' = \boldsymbol{G}^\circ$.

The normalizer N of Z is the group of $F_\circ$-rational points of the $F_\circ$-subgroup $\boldsymbol{N}$ of $\boldsymbol{G}$ which stabilizes V_0 and permutes the lines V_i, $i \in I$. It is the semidirect product $\mathfrak{N} \ltimes Z$ where $\mathfrak{N}$ is the subgroup of N which fixes (pointwise) V_0 and permutes the e_i, $i \in I$.

1.5. Root subgroups. For i, $j \in I$, $j \neq \pm i$ and $u \in F$, let $u_{i,j}(u) \in G$ be the linear transformation of V defined by

$$x \mapsto x \text{ for all } x \in V_0,$$

$$e_i \mapsto e_i + \varepsilon(-j)u^\sigma e_{-j},$$

$$e_j \mapsto e_j - \varepsilon(i)ue_{-i},$$

$$e_k \mapsto e_k \quad \text{for all } k \in I \smallsetminus \{i, j\}.$$

The set $U_{a_{i,j}} = \{u_{i,j}(u) : u \in F\}$ is the group of $F_\circ$-rational points of the $F_\circ$-subgroup $\boldsymbol{U}_{a_{i,j}}$ of $\boldsymbol{G}$ associated with the (relative) root $a_{i,j}$.

Suppose $V_0 \neq \{0\}$ (case (B) or (BC)). Recall that for $x \in V_0$ and $\mu \in F$, we have $\mu \in q(x)$ if and only if $\mu + \varepsilon\mu^\sigma = h(x,x)$; in particular if $\sigma = \mathrm{Id}$ (case (B)), we have $q(x) = \frac{1}{2}h(x,x) \in F$. For $x \in V_0$ and $\mu \in q(x)$, let $u_i(x,\mu) \in G$ be the linear transformation of V defined by

$$y \mapsto y - \varepsilon(i)h(x,y)e_{-i} \text{ for all } y \in V_0,$$

$$e_i \mapsto e_i + x - \varepsilon(i)\mu e_{-i},$$

$$e_k \mapsto e_k \quad \text{for all } k \in I \smallsetminus \{i\}.$$

The set $U_{a_i} = \{u_i(x,\mu) : x \in V_0,\ \mu \in q(x)\}$ is the group of $F_\circ$-rational points of the

$F_\circ$-subgroup $\boldsymbol{U}_{a_i}$ of $\boldsymbol{G}$ associated with the root a_i. Moreover if $\sigma \neq \mathrm{Id}$ (case (BC)), the set $U_{2a_i} = \{u_i(0,v) : v \in F_{\sigma,\varepsilon}\}$ is the group of $F_\circ$-rational points of the $F_\circ$-subgroup $\boldsymbol{U}_{2a_i}$ of $\boldsymbol{G}$ associated with the root $2a_i$.

Suppose $V_0 = \{0\}$ and $(\sigma,\varepsilon) \neq (\mathrm{Id},1)$ (case (C)). For $i \in I$ and $v \in F_{\sigma,\varepsilon}$, let $u_i(0,v) \in G$ be the linear transformation of V defined by

$$e_i \mapsto e_i - \varepsilon(i)ve_{-i},$$

$$e_k \mapsto e_k \quad \text{for all } k \in I \smallsetminus \{i\}.$$

The set $U_{2a_i} = \{u_i(0,v) : v \in F_{\sigma,\varepsilon}\}$ is the group of $F_\circ$-rational points of the $F_\circ$-subgroup $\boldsymbol{U}_{2a_i}$ of $\boldsymbol{G}$ associated with the root $2a_i$.

Let v_F be the unique valuation on F extending the normalized valuation on $F_\circ$, i.e. such that $v_F(F_\circ^\times) = \mathbb{Z}$. Recall that $(Z,(U_a)_{a\in\Phi})$ is a *generating root datum* in G ([BT1] 6.1.1, 6.1.3.c and 10.1.6). Let $\varphi = (\varphi_a)_{a\in\Phi}$ be the valuation of $(Z,(U_a)_{a\in\Phi})$ given by:

$$\varphi_{a_{i,j}}(u_{i,j}(u)) = v_F(u) \quad \text{for } i,\ j \in I,\ i \neq \pm j,\ u \in F;$$

$$\varphi_{a_i}(u_i(x,\mu)) = \tfrac{1}{2}v_F(\mu) \text{ for } i \in I,\ x \in V_0,\ \mu \in q(x) \text{ (case (B) or (BC))};$$

$$\varphi_{2a_i}(u_i(0,v)) = v_F(v) \text{ for } i \in I,\ v \in F_{\sigma,\varepsilon} \text{ (case (BC) or (C))}.$$

1.6. Building, norms and lattice-functions. Let $\mathcal{I} = \mathcal{I}(\boldsymbol{G},F_\circ)$ be the (non-enlarged) Bruhat-Tits building of G, i.e. the building of the valuated root datum $(Z,(U_a)_{a\in\Phi},\varphi)$. Since $\boldsymbol{G}'$ is semisimple and G' is cocompact in G, the connected centre of $\boldsymbol{G}^\circ$ is an anisotropic F-torus; thus we have $\mathcal{I} = \mathcal{I}(\boldsymbol{G}^\circ,F_\circ) = \mathcal{I}(\boldsymbol{G}',F_\circ)$ and $\mathcal{I}$ coincides with the enlarged building of $\boldsymbol{G}^\circ(F)$. Let $\mathcal{A}$ be the apartment of $\mathcal{I}$ attached to the maximal $F_\circ$-split torus $\boldsymbol{S}$ of $\boldsymbol{G}$. It is an affine space with underlying space $A = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{X}^*(\boldsymbol{S}),\mathbb{R})$. We identify $\mathcal{A}$ with A by taking $\varphi \in \mathcal{A}$ as the origin (cf. [BT1], §10). Thus $\mathrm{X}^*(\boldsymbol{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ becomes identified with the dual space $\mathrm{Hom}_{\mathbb{R}}(\mathcal{A},\mathbb{R})$.

Let $\tilde{\mathcal{I}}^1$ be the enlarged building of $\widetilde{G}$, and $\mathcal{N}^1 = \mathrm{Norm}^1_F(V)$ be the set of (F-)norms on V ([BT3] 1.1); recall that since F is complete, each norm on V

splits with respect to an F-basis of V. The group $\widetilde{G}$ acts naturally on $\mathcal{N}^1$ by $g \cdot \alpha(v) = \alpha(g^{-1}v)$ for $g \in \widetilde{G}$, $\alpha \in \mathcal{N}^1$ and $v \in V$. Moreover, $\mathcal{N}^1$ is endowed with an *affine structure* ([BT3] 1.27). From [BT3] 2.11, there exists a bijective $\widetilde{G}$-equivariant affine map $j : \tilde{\mathcal{I}}^1 \to \mathcal{N}^1$; moreover, up to translation by a real number, j is the unique $\widetilde{G}$-equivariant affine map from $\tilde{\mathcal{I}}^1$ to $\mathcal{N}^1$. Let $\mathcal{L}^1 = \mathrm{Latt}^1_{\mathfrak{o}_F}(V)$ be the set of ($\mathfrak{o}_F$-)lattice-functions in V ([BL] 2.1), where $\mathfrak{o}_F$ denotes the ring of integers of F. The group $\widetilde{G}$ acts on $\mathcal{L}^1$ via its action on V. For $\alpha \in \mathcal{N}^1$, let Λ_α be the $\mathfrak{o}_F$-lattice-function in V defined by

$$\Lambda_\alpha(r) = \{v \in V : \alpha(v) \geq r\}, \quad r \in \mathbb{R},$$

and for $\Lambda \in \mathcal{L}^1$, let α_Λ be the norm on V defined by

$$\alpha_\Lambda(v) = \sup\{r \in \mathbb{R} : v \in \Lambda(r)\}, \quad v \in V.$$

The maps $\mathcal{N}^1 \to \mathcal{L}^1$, $\alpha \mapsto \Lambda_\alpha$ and $\mathcal{L}^1 \to \mathcal{N}^1$, $\Lambda \mapsto \alpha_\Lambda$ are bijective, $\widetilde{G}$-equivariant and mutually inverse ([BL] 2.4); via these maps, we transfer to $\mathcal{L}^1$ the affine structure on $\mathcal{N}^1$. For $p \in \mathcal{I}^1$, denote by Λ_p the $\mathfrak{o}_F$-lattice-function $\Lambda_{j(p)}$ in V. By construction, the map $\tilde{\mathcal{I}}^1 \to \mathcal{L}^1$, $p \mapsto \Lambda_p$ is bijective, $\widetilde{G}$-equivariant and affine, and up to translation by a real number, it is the unique $\widetilde{G}$-equivariant affine map from $\tilde{\mathcal{I}}^1$ to $\mathcal{L}^1$.

Remark 1.3. Let $\tilde{\mathcal{I}}$ be the (non-enlarged) building of $\widetilde{G}$, and let $\mathcal{V}^1$ be the $\mathbb{R}$-vector space $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{X}^*(\widetilde{G}), \mathbb{R})$, where $\mathrm{X}^*(\widetilde{G})$ denotes the free $\mathbb{Z}$-module of rank 1 generated by the character $\det : \widetilde{G} \to F^\times$. We have the decomposition $\tilde{\mathcal{I}}^1 = \tilde{\mathcal{I}} \times \mathcal{V}^1$, and the action of $\widetilde{G}$ on $\tilde{\mathcal{I}}^1$ is given by the map

$$\widetilde{G} \times (\tilde{\mathcal{I}} \times \mathcal{V}^1),\ (g, (\bar{p}, v)) \mapsto g \cdot (x, v) = (g \cdot \bar{p}, v + \theta(v))$$

where $\theta(g) \in \mathcal{V}^1$ is defined by $\langle \det, \theta(g) \rangle = -v_F(\det(g))$. We also have some natural actions of $\mathbb{R}$ on $\mathcal{N}^1$ and on $\mathcal{L}^1$, given by the maps

$$\mathbb{R} \times \mathcal{N}^1 \to \mathcal{N}^1,\ (r, \alpha) \mapsto \alpha + r, \quad \mathbb{R} \times \mathcal{L}^1 \to \mathcal{L}^1,\ (r, \Lambda) \mapsto r \cdot \Lambda,$$

where $(\alpha + r)(v) = \alpha(v) + r$ for all $v \in V$, and $(r \cdot \Lambda)(r') = \Lambda(r' - r)$ for all $r' \in \mathbb{R}$. Let $\mathcal{N}$ (resp. $\mathcal{L}$) be the quotient of $\mathcal{N}^1$ (resp. $\mathcal{L}^1$) by the action of $\mathbb{R}$. The actions of $\widetilde{G}$ on $\mathcal{N}^1$ and $\mathcal{L}^1$ induce some actions on $\mathcal{N}$ and $\mathcal{L}$, and the affine structures on $\mathcal{N}^1$ and $\mathcal{L}^1$ induce some affine structures on $\mathcal{N}$ and $\mathcal{L}$. The maps $j : \tilde{\mathcal{I}}^1 \to \mathcal{N}^1$ and $\mathcal{N}^1 \to \mathcal{L}^1$, $\alpha \mapsto \Lambda_\alpha$ induce some maps $\tilde{\mathcal{I}} \to \mathcal{N}$ and $\mathcal{N} \to \mathcal{L}$ which are bijective, $\widetilde{G}$-equivariant and affine. So we obtain a canonical bijective $\widetilde{G}$-equivariant affine map $\tilde{\mathcal{I}} \to \mathcal{N}$ (resp. $\tilde{\mathcal{I}} \to \mathcal{L}$): it is the unique $\widetilde{G}$-equivariant affine map from $\tilde{\mathcal{I}}$ to $\mathcal{N}$ (resp. from $\tilde{\mathcal{I}}$ to $\mathcal{L}$).

The valuation v_F is an $F_\circ$-norm on F, and we define an $F_\circ$-norm $\overline{v}_F$ on the $F_\circ$-space $\overline{F}_{\sigma,\varepsilon} = F/F_{\sigma,\varepsilon}$:

$$\overline{v}_F = \sup\{v_F(\lambda + \mu - \varepsilon\mu^\sigma) : \mu \in F\}, \quad \lambda \in F.$$

Since F is complete, $F_{\sigma,\varepsilon}$ is closed in F and $\overline{v}_F$ is well-defined. Let us recall the definition 2.1 of [BT4]:

Definition 1.4. Let $\alpha \in \mathcal{N}^1$. We write $\alpha \leq h$ ("α minore h" in French) if

$$\alpha(x) + \alpha(y) \leq v_F(h(x,y)) \quad \text{for all } x,\, y \in V.$$

We write $\alpha \leq (h,q)$ ("α minore (h,q)" in French) if $\alpha \leq h$ and

$$\alpha(x) \leq \tfrac{1}{2}\overline{v}_F(q(x)) \quad \text{for all } x \in V.$$

We say that α is an MM-norm ("norme maximinorante" in French) for h (resp. for (h,q)) if $\alpha \leq h$ (resp. $\alpha \leq (h,q)$) and α is maximal for this property.

Let $\mathcal{N}^1_h$ be the subset of $\mathcal{N}^1$ formed of the MM-norms for h, and $\mathcal{N}^1_{h,q}$ be the subset of $\mathcal{N}^1_h$ formed of the MM-norms for (h,q).

Definition 1.5. We say that we are in the tame case if one of the following two conditions is satisfied:

$(\sigma,\varepsilon) = (\mathrm{id}, -1)$, i.e. $q = 0$;

the extension $F/F_\circ$ is tamely ramified.

If we are in the tame case, then we have $\mathcal{N}^1_{h,q} = \mathcal{N}^1_h$ ([BT4] 2.2).

Let $v_{V_0} = v_{V_0,h_0}$ be the $F_\circ$-norm on V_0 defined by

$$v_{V_0}(x) = \tfrac{1}{2}\overline{v}_F(q(x)), \quad x \in V_0.$$

Thus we have

$$v_{V_0}(x) = \tfrac{1}{2}\sup\{v_F(\lambda) : \lambda + \varepsilon\lambda^\sigma = h(x,x)\}, \quad x \in V_0.$$

Remark 1.6. Suppose $\varepsilon = 1$, and let ξ be an element of F such that $\xi + \xi^\sigma = 1$ and $v_F(\xi) \geq v_F(\xi')$ for all $\xi' \in F$ such that $\xi' + \xi'^\sigma = 1$. Put $l = \frac{1}{2}v_F(\xi)$. We have $l \leq 0$ with equality if and only if the extension $F/F_\circ$ is quadratic unramified or the residual characteristic of $F_\circ$ is not 2 (i.e. the extension $F/F_\circ$ is tamely ramified if $\sigma \neq \mathrm{Id}$, and the residual characteristic of $F_\circ$ is not 2 if $\sigma = \mathrm{Id}$). If $\sigma = \mathrm{Id}$, we have $\xi = \frac{1}{2}$. If $\sigma \neq \mathrm{Id}$, we can take $\xi = \frac{1}{2}$ if and only if and $l = 0$. Since $\varepsilon = 1$, for all $x \in V_0$, we have $h(x,x) \in F_\circ$ and

$$q(x) = \{\xi h(x,x) + \mu - \mu^\sigma : \mu \in F\} = \{\xi' h(x,x) : \xi' \in F,\, \xi' + \xi'^\sigma = 1\}.$$

Hence we obtain

$$v_{V_\circ}(x) = \tfrac{1}{2}v_F(h(x,x)) + l, \quad x \in V_0.$$

In particular if the residual characteristic of $F_\circ$ is not 2, then the $F_\circ$-norm v_{V_0} on V_0 is the one used by Broussous and Stevens [BS].

For $p \in \mathcal{A}$ $(= A)$, let α_p be the MM-norm for (h,q) on V defined by ([BT4] 2.9):

$$\alpha_p(x_0 + \textstyle\sum_{i\in I}\lambda_i e_i) = \inf(v_{V_0}(x_0), \inf_{i\in I}(v_F(\lambda_i) - a_i(p))),$$

where $\lambda_i \in F$ and $x_0 \in V_0$. The map $\mathcal{A} \to \mathcal{N}^1_{h,q}$, $p \mapsto \alpha_p$ is injective (loc. cit.) and N-equivariant ([BT4] 2.11), and it extends in a unique way to a G-equivariant map $\mathcal{I} \to \mathcal{N}^1_{h,q}$, $p \mapsto \alpha_p$ which is bijective and affine ([BT4] 2.12); moreover, this is the unique G-equivariant affine map from $\mathcal{I}$ to $\mathcal{N}^1_{h,q}$ (loc. cit.). Via j, the building $\mathcal{I}$ identifies with a G-stable convex subset of $\tilde{\mathcal{I}}^1$. Note that the apartment $\mathcal{A}$ of $\mathcal{I}$ is the intersection of $\mathcal{I}$ with an apartment $\tilde{\mathcal{A}}^1$ of $\tilde{\mathcal{I}}^1$ (cf. [BT4] 2.14). Put

$$\mathcal{L}^1_h = \{\Lambda_\alpha : \alpha \in \mathcal{N}^1_h\}.$$

and

$$\mathcal{L}^1_{h,q} = \{\Lambda_\alpha : \alpha \in \mathcal{N}^1_{h,q}\}.$$

By construction, $\mathcal{L}^1_{h,q}$ is a G-stable convex subset of $\mathcal{L}^1$, and the map $\tilde{\mathcal{I}}^1 \to \mathcal{L}^1$, $p \mapsto \Lambda_p$ induces by restriction a bijective, G-equivariant and affine map $\mathcal{I} \to \mathcal{L}^1_{h,q}$, which is the unique G-equivariant affine map from $\mathcal{I}$ to $\mathcal{L}^1_{h,q}$.

Let $\alpha \mapsto \overline{\alpha}$ be the involution on $\mathcal{N}^1$ defined by ([BT4] 2.5)

$$\overline{\alpha}(x) = \inf_{y\in V}(v_F(h(x,y) - \alpha(y)), \quad x \in V.$$

A norm α on V is a MM-norm for h if and only if $\overline{\alpha} = \alpha$ (loc. cit.). In other terms, a norm α on V is a MM-norm for h if and only if the lattice-function Λ_α in V is self-dual in the sense of [BS] ch. 3 (the proof of Corollary 3.4 applies in the same manner). So $\mathcal{L}^1_h$ is the set of self-dual lattice-functions in V, and it coincides with $\mathcal{L}^1_{h,q}$ if we are in the tame case.

1.7. Square lattice-functions. Denote by $\tilde{\mathfrak{g}} = \mathrm{Lie}(\widetilde{G})$ the Lie algebra of $\widetilde{G}$, and by $\mathfrak{g} = \mathrm{Lie}(G)$ that of G. So we have $\tilde{\mathfrak{g}} = \mathrm{End}_F(V)$. For $g \in \tilde{\mathfrak{g}}$, denote by g^{σ_h} the adjoint of g with respect to h, i.e. the unique element of $\tilde{\mathfrak{g}}$ such that $h(gx,y) = h(x, g^{\sigma_h}y)$ for all $x, y \in V$. The map $\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$, $g \mapsto g^{\sigma_h}$ is an involution, and we have $\mathfrak{g} = \{g \in \tilde{\mathfrak{g}} : g + g^{\sigma_h} = 0\}$. For $\Lambda \in \mathcal{L}^1$, denote by $\mathrm{End}(\Lambda)$ the ($\mathfrak{o}_F$-)lattice-function in $\tilde{\mathfrak{g}}$ defined by

$$\mathrm{End}(\Lambda)(r) = \{g \in \tilde{\mathfrak{g}} : g\Lambda(s) \subset \Lambda(s+r),\, s \in \mathbb{R}\}, \quad r \in \mathbb{R}.$$

The lattice-functions in $\tilde{\mathfrak{g}}$ arising in this way are called *square lattice-functions*. Let $\mathcal{L}^2 = \mathrm{Latt}^2_{\mathfrak{o}_F}(\tilde{\mathfrak{g}})$ be the set of square lattice-functions in $\tilde{\mathfrak{g}}$. For $p \in \tilde{\mathcal{I}}$, we put

$$\tilde{\mathfrak{g}}_{p,r} = \mathrm{End}(\Lambda_p)(r), \quad r \in \mathbb{R}.$$

The group $\widetilde{G}$ acts on $\mathcal{L}^2$ via its action on $\tilde{\mathfrak{g}}$, and the map
$\tilde{\mathcal{I}}^1 \to \mathcal{L}^2$, $p \mapsto \tilde{\mathfrak{g}}_{p,\cdot}$ is surjective and $\widetilde{G}$-equivariant ([BL] §4).

Remark 1.7. The map $\tilde{\mathcal{I}}^1 \to \mathcal{L}^1$, $p \mapsto \Lambda_p$ depends on the choice of $j : \tilde{\mathcal{I}}^1 \to \mathcal{N}^1$, but the map $\tilde{\mathcal{I}}^1 \to \mathcal{L}^2$, $p \mapsto \tilde{\mathfrak{g}}_{p,\cdot}$ does not depend on it. In fact, for $\Lambda, \Lambda' \in \mathcal{L}^1$, we have $\mathrm{End}(\Lambda') = \mathrm{End}(\Lambda)$ if and only if there exists $r \in \mathbb{R}$ such that $\Lambda' = r \cdot \Lambda$ ([BT3] 1.13). In particular, the map $\tilde{\mathcal{I}}^1 \to \mathcal{L}^2$, $p \mapsto \tilde{\mathfrak{g}}_{p,\cdot}$ factorizes through the non-enlarged building $\tilde{\mathcal{I}}$ (cf. Remark 1.3). We obtain a bijective and $\widetilde{G}$-equivariant map $\tilde{\mathcal{I}} \to \mathcal{L}^2$.

The involution σ_h on $\tilde{\mathfrak{g}}$ induces also an involution on $\mathcal{L}^2$, still denoted by σ_h: for $\Lambda \in \mathcal{L}^1$, we put

$$\mathrm{End}(\Lambda)^{\sigma_h}(r) = \mathrm{End}(\Lambda)(r)^{\sigma_h}.$$

Let $\mathcal{L}_h^2$ be the subset of $\mathcal{L}^2$ formed of those lattice-functions which are σ_h-invariant. For $\alpha \in \mathcal{N}^1$, we have ([BT4] 2.5)

$$\mathrm{End}(\Lambda_\alpha)^{\sigma_h} = \mathrm{End}(\Lambda_{\overline{\alpha}}).$$

This implies (loc. cit., Cor. 2) that the map $\mathcal{L}^1 \to \mathcal{L}^2$, $\Lambda \mapsto \mathrm{End}(\Lambda)$ induces a bijection from $\mathcal{L}_h^1$ to $\mathcal{L}_h^2$. Thus if we are in the tame case, then the map $\tilde{\mathcal{I}}^1 \to \mathcal{L}^2$, $p \mapsto \tilde{\mathfrak{g}}_{p,\cdot}$ induces a G-equivariant bijection from $\mathcal{I}$ to $\mathcal{L}_h^2$.

Let p be a point of $\mathcal{I}$. Let $\mathfrak{g}_{p,\cdot}$ be the $\mathfrak{o}_{F_\circ}$-lattice-function in $\mathfrak{g}$ defined by

$$\mathfrak{g}_{p,r} = \tilde{\mathfrak{g}}_{p,r} \cap \mathfrak{g} = \{g \in \mathfrak{g} : g\Lambda_p(s) \subset \Lambda_p(s+r),\, s \in \mathbb{R}\}, \quad r \in \mathbb{R},$$

and let $(\mathfrak{g}_{p,r}^{\mathrm{MP}})_{r\in\mathbb{R}}$ be the filtration of $\mathfrak{g}$ attached to p by Moy and Prasad ([MP], see 3.5). The following theorem is the main result of this paper.

Theorem 1.8. *For all $p \in \mathcal{I}$, we have*

$$\mathfrak{g}_{p,r} = \mathfrak{g}_{p,r}^{\mathrm{MP}}, \quad r \in \mathbb{R}.$$

2. Reduction to the quasi-split case

2.1. Extension of the base field. Let $L_\circ$ be a finite extension of $F_\circ$. Put $L = L_\circ \otimes_{F_\circ} F$. It is a commutative $L_\circ$-algebra, endowed with an involution $\mathrm{Id} \otimes \sigma$, still denoted by σ. The field $L_\circ$ is the set of fixed points of σ in L. Since F is a separable extension of $F_\circ$ of degree ≤ 2, there are two cases: L is field, in which case it is an extension of degree $[F : F_\circ]$ of $L_\circ$; or L is a cyclic $L_\circ$-algebra with group $\Sigma = \{1, \sigma\}$, i.e. a product $L_1 \times L_2$ of two extension L_1 and L_2 of $F_\circ$ isomorphic to $L_\circ$, such that $\sigma L_1 = L_2$. Denote by $L_{\sigma,\varepsilon}$ the $L_\circ$-vector space $L_\circ \otimes_{F_\circ} F_{\sigma,\varepsilon}$. So $\overline{F}_{\sigma,\varepsilon}$ identifies with an $F_\circ$-subspace of $\overline{L}_{\sigma,\varepsilon} = L/L_{\sigma,\varepsilon}$. Moreover, we have

$$L_{\sigma,\varepsilon} = \{\lambda - \varepsilon\lambda^\sigma : \lambda \in L\} = \{\lambda \in L : \lambda^\sigma = -\varepsilon\lambda\}.$$

Denote by V_L the L-vector-space $L_\circ \otimes_{F_\circ} V = L \otimes_F V$. Even if L is not a field, by replacing $F_\circ$ with $L_\circ$ and F with L, we define the notion of σ-skew ε-hermitian form on V_L. The σ-skew ε-hermitian form h on V extends to a σ-skew ε-hermitian form h_L on V_L, which is non-degenerate since h is non-degenerate. Let $q_L = q_{h_L} : V_L \to \overline{L}_{\sigma,\varepsilon}$ be the pseudo-quadratic form associated with h_L as in 1.1. Put $\widetilde{G}_L = \mathrm{GL}(V_L)$ and let $G_{L_\circ} = \mathrm{U}(h_L)$ the subgroup of $\widetilde{G}_L$ formed of those g satisfying $h_L(gx, gy) = h_L(x, y)$ for all $x, y \in V_L$. Then $G_{L_\circ}$ is the group of $L_\circ$-rational points of the $L_\circ$-algebraic group $\boldsymbol{G}_{L_\circ} = \boldsymbol{G} \times_{F_\circ} L_\circ$. Put $\mathfrak{g}_{L_\circ} = \mathrm{Lie}(G_{L_\circ})$; so we have $\mathfrak{g}_{L_\circ} = L_\circ \otimes_{F_\circ} \mathfrak{g}$.

Let us consider the first case: L is a field. We can replace $F_\circ$ with $L_\circ$ and F with L in all the constructions of chapter 1. In particular for $p \in \mathcal{I}_{L_\circ} = \mathcal{I}(\boldsymbol{G}_{L_\circ}, L_\circ)$, denote by $\mathfrak{g}_{L_\circ,p,\cdot}$ the square $\mathfrak{o}_{L_\circ}$-lattice-function in $\mathfrak{g}_{L_\circ}$ defined in 1.7, and by $(\mathfrak{g}^{\mathrm{MP}}_{L_\circ,p,\cdot})_{r\in\mathbb{R}}$ the filtration of $\mathfrak{g}_{L_\circ}$ attached to p by Moy and Prasad.

2.2. The case $L_\circ\otimes_{F_\circ}F\simeq(L_\circ)^2$. Now let us consider the second case: $L\simeq(L_\circ)^2$. Then $[F:F_\circ]=2$ and, up to $F_\circ$-isomorphism, F is contained in $L_\circ$. So we can (and do) assume $F\subset L_\circ$ and $L=(L_\circ)^2$. The embedding $F\to L$, $\lambda=1\otimes\lambda$ identifies F with the subset $\{(\lambda,\lambda^\sigma):\lambda\in F\}$, and for $\lambda,\mu\in F$, we have $(\lambda,\mu)^\sigma=(\mu,\lambda)$. Let $\xi_1=(1,0)$ and $\xi_2=(0,1)$ be the two minimal idempotents of L.

If X is an F-vector-space, the L-vector space $X_L=L_\circ\otimes_{F_\circ}X=L\otimes_F X$ is a product of two copies of $X_{L_\circ}=L_\circ\otimes_F X$: putting $X_{L_\circ,i}=\xi_i X_L$, we have

$$L_\circ\otimes_{F_\circ}X=X_{L_\circ,1}\times X_{L_\circ,2}.$$

In particular we have

$$V_L=V_{L_\circ,1}\times V_{L_\circ,2}.$$

For $i=1,\,2$, we have $gV_{L_\circ,i}\subset V_{L_\circ,i}$ for all $g\in\mathrm{End}_L(V_L)$. Thus we have

$$\mathrm{End}_L(V_L)=\mathrm{End}_{L_\circ}(V_{L_\circ,1})\times\mathrm{End}_{L_\circ}(V_{L_\circ,2})$$

with $\mathrm{End}_{L_\circ}(V_{L_\circ,i})=\xi_i\mathrm{End}_L(V_L)$, identifying $L_\circ$ with $\xi_i L$. Since h is non-degenerate, the map $x\mapsto h(x,\cdot)$ defines a σ-isomorphism from V to the dual space $V^*=\mathrm{Hom}_F(V,F)$, and by extension of scalars, the map $x\mapsto h_L(x,\cdot)$ defines a σ-isomorphism from V_L to the dual space $V_L^*=\mathrm{Hom}_L(V_L,L)=(V^*)_L$. Since $\xi_1^\sigma=\xi_2$ and $\xi_1\xi_2=0$, for $i=1,\,2$ and $x,\,y\in V_{L_\circ,i}$, we have

$$h_L(x,y)=h_L(\xi_i x,\xi_i y)=\xi_i\xi_i^\sigma h_L(x,y)=0.$$

Hence $V_{L_\circ,1}$ and $V_{L_\circ,2}$ are two maximal totally isotropic subspaces of V_L. Thus we obtain that the map $x\mapsto h_L(x,\cdot)$ induces an isomorphism of $L_\circ$-vector spaces

$$\varphi:V_{L_\circ,1}\to V^*_{L_\circ,2}=\mathrm{Hom}_{L_\circ}(V_{L_\circ,2},L_\circ).$$

Now let $g=(g_1,g_2)\in\widetilde{G}_L=\mathrm{GL}(V_{L_\circ,1})\times\mathrm{GL}(V_{L_\circ,2})$. By definition, we have $g\in\mathrm{U}(h_L)$ if and only if $\varphi(g_1x_1)(g_2x_2)=\varphi(x_1)(x_2)$ for all $(x_1,x_2)\in V_{L_\circ,1}\times V_{L_\circ,2}$, i.e. if and only if

$$g_2={}^{\mathrm{t}}\varphi^{-1}\circ{}^{\mathrm{t}}g_1^{-1}\circ{}^{\mathrm{t}}\varphi.$$

So by restriction the map $\widetilde{G}_L\to\mathrm{GL}(V_{L_\circ,1})$, $(g_1,g_2)\mapsto g_1$ gives an isomorphism of groups $\iota:\mathrm{U}(h_L)\to\mathrm{GL}(V_{L_\circ,1})$ which is defined over $L_\circ$, i.e. which comes from an isomorphism of algebraic groups defined over $L_\circ$. Moreover, ι restricts to an isomorphism of groups $\iota':\mathrm{SU}(h_L)\to\mathrm{SL}(V_{L_\circ,1})$ which is also defined over $L_\circ$.

Let $\mathcal{I}_{L_\circ}=\mathcal{I}(\mathbf{G}_{L_\circ},L_\circ)$ be the (non-enlarged) building of $G_{L_\circ}=\mathrm{U}(h_L)$, and $\mathcal{L}_{L_\circ,1}$ be the quotient of $\mathcal{L}^1_{L_\circ,1}=\mathrm{Latt}^1_{\mathfrak{o}_{L_\circ}}(V_{L_\circ,1})$ by the action of $\mathbb{R}$ (cf. Remark 1.3). The group $\mathrm{GL}(V_{L_\circ,1})$ acts on $\mathcal{I}_{L_\circ}$ via ι, and from the Remark 1.3, there exists a *unique* $\mathrm{GL}(V_{L_\circ,1})$-equivariant affine map $\mathcal{I}_{L_\circ}\to\mathcal{L}_{L_\circ,1}$, still denoted by ι. Let p be a point of $\mathcal{I}_{L_\circ}$. Let $\mathfrak{g}_{L_\circ,p,\cdot}$ be the $\mathfrak{o}_{L_\circ}$-lattice-function in $\mathfrak{g}_{L_\circ}$ defined by

$$\mathfrak{g}_{L_\circ,p,r}=\mathrm{Lie}(\iota)^{-1}(\mathrm{End}_{L_\circ}(V_{L_\circ,1})_{\iota(p),r}),\quad r\in\mathbb{R}.$$

Let also $(\mathfrak{g}^{\mathrm{MP}}_{L_\circ,p,r})_{r\in\mathbb{R}}$ be the filtration of $\mathfrak{g}_{L_\circ}$ attached to p by Moy and Prasad.

Remark 2.1. Suppose $L_\circ/F_\circ$ is a Galois extension with Galois group Γ. Then we can make the action of Γ on $G_{L_\circ}$ explicit ([BT4] 1.13, Rem.). Let Γ' be the subgroup $\mathrm{Gal}(L_\circ/F)$ of Γ. The group Γ acts naturally on $V_L = L_\circ \otimes_{F_\circ} V$ and on $\mathrm{End}_L(V_L) = L_\circ \otimes_{F_\circ} \mathrm{End}_F(V)$. For $i = 1, 2$, the subspace $V_{L_\circ,i}$ of V_L is Γ'-stable, whence we have a natural action of Γ' on $\mathrm{End}_{L_\circ}(V_{L_\circ,i})$. If $\gamma \in \Gamma \smallsetminus \Gamma'$, i.e. if the restriction of γ to F is σ, the automorphism $\gamma \otimes \mathrm{Id}$ of $\mathrm{End}_L(V_L)$, denoted by $g \mapsto g^\gamma$, induces two γ-isomorphisms $\mathrm{End}_{L_\circ}(V_{L_\circ,1}) \to \mathrm{End}_{L_\circ}(V_{L_\circ,2})$ and $\mathrm{End}_{L_\circ}(V_{L_\circ,2}) \to \mathrm{End}_{L_\circ}(V_{L_\circ,1})$, still denoted by $g \mapsto g^\gamma$. Then for $\gamma \in \Gamma$ and $(g_1, g_2) \in \mathrm{End}_{L_\circ}(V_{L_\circ,1}) \times \mathrm{End}_{L_\circ}(V_{L_\circ,2})$, we have

$$(g_1, g_2)^\gamma = (g_1^\gamma, g_2^\gamma) \ \text{ if } \gamma \in \Gamma',$$

$$(g_1, g_2)^\gamma = (g_2^\gamma, g_1^\gamma) \ \text{ if } \gamma \in \Gamma \smallsetminus \Gamma'.$$

In particular, the map $\iota : \mathrm{U}(h_L) \to \mathrm{GL}(V_{L_\circ,1})$ is Γ'-equivariant, and for $\gamma \in \Gamma \smallsetminus \Gamma'$ and $g = (g_1, g_2) \in \mathrm{U}(h_L)$, we have

$$g^\gamma = (g_2^\gamma, g_1^\gamma) = (({}^{\mathrm{t}}\varphi^{-1} \circ {}^{\mathrm{t}}g_1{}^{-1} \circ {}^{\mathrm{t}}\varphi)^\gamma, g_1^\gamma).$$

2.3. Unramified descent: buildings. Let us turn to the general case: L is a field or $L \simeq (L_\circ)^2$. Suppose moreover that the extension $L_\circ/F_\circ$ is unramified, and let Γ be the group $\mathrm{Gal}(L_\circ/F_\circ)$. We know ([BT4] 4.1) that there exists a unique G-equivariant and affine map $\mathcal{I} \to \mathcal{I}_{L_\circ}$, whose image is the subset $(\mathcal{I}_{L_\circ})^\Gamma$ formed of those points which are fixed by Γ.

We can describe the canonical bijection $\mathcal{I} \to (\mathcal{I}_{L_\circ})^\Gamma$ in terms of norms (resp. of lattice-functions). Let $v_{L_\circ}$ be the normalized valuation on $L_\circ$. The $L_\circ$-algebra L is endowed with the $L_\circ$-algebra norm v_L defined by $v_L = v_{L_\circ} \otimes v_F$. The ring of integers $\mathfrak{o}_L = \mathfrak{o}_{L_\circ} \otimes_{\mathfrak{o}_{F_\circ}} \mathfrak{o}_F$ of L coincides with the set of $\lambda \in L$ such that $v_L(\lambda) = 0$. Let $\mathcal{N}_L^1 = \mathrm{Norm}_L^1(V_L)$ be the set of L-norms on V_L, and $\mathcal{L}_L^1 = \mathrm{Latt}_{\mathfrak{o}_L}^1(V_L)$ be the set of $\mathfrak{o}_L$-lattice-functions in V_L. We define, as in 1.6, the subsets $\mathcal{N}^1_{L,(h_L,q_L)} \subset \mathcal{N}^1_{L,h_L}$ of $\mathcal{N}_L^1$ (Definition 1.4 is valid even if L is not a field), and the subsets $\mathcal{L}^1_{L,(h_L,q_L)} \subset \mathcal{L}^1_{L,h_L}$ of $\mathcal{L}_L^1$. For $\alpha \in \mathcal{N}^1$, denote by α_L the L-norm $v_{L_\circ} \otimes \alpha$ on V_L, and, for $\Lambda \in \mathcal{L}^1$, denote by Λ_L the $\mathfrak{o}_L$-lattice-function in V_L defined by $\Lambda_L(r) = \mathfrak{o}_{L_\circ} \otimes_{\mathfrak{o}_{F_\circ}} \Lambda(r)$, $r \in \mathbb{R}$. For $\alpha \in \mathcal{N}_L^1$, denote by Λ_α the $\mathfrak{o}_L$-lattice-function in V_L defined (as in 1.6) by $\Lambda_\alpha(r) = \{v \in V_L : \alpha(v) \geq r\}$, $r \in \mathbb{R}$. Hence we have $\Lambda_{\alpha_L} = (\Lambda_\alpha)_L$, for all $\alpha \in \mathcal{N}^1$. On the other hand, the group Γ acts naturally on $\mathcal{N}_L^1$ (resp. on $\mathcal{L}_L^1$), stabilizing the subset $\mathcal{N}^1_{L,(h_L,q_L)}$ (resp. $\mathcal{L}^1_{L,(h_L,q_L)}$), and the map $\mathcal{N}_L^1 \to \mathcal{L}_L^1$, $\alpha \mapsto \Lambda_\alpha$ is Γ-equivariant.

The map $\alpha \mapsto \alpha_L$ from $\mathcal{N}^1$ to $\mathcal{N}_L^1$ is injective, $\widetilde{G}$-equivariant and affine, and it induces a bijection onto the convex subset $(\mathcal{N}_L^1)^\Gamma$ of $\mathcal{N}_L^1$ formed by those norms which are Γ-invariant. If $\alpha \in \mathcal{N}$, from [BT4] 4.2, we have $\alpha \leq (h, q)$ if and only if $\alpha_L \leq (h_L, q_L)$. Hence the map $\alpha \mapsto \alpha_L$ induces a G-equivariant affine bijection from $\mathcal{N}^1_{h,q}$ to the convex subset $\mathcal{N}^{1,\natural}_{L,(h_L,q_L)}$ of $(\mathcal{N}_L^1)^\Gamma$ formed by those norms β such that $\beta \leq (h_L, q_L)$ and which are maximal for this property. A priori we have the inclusion $(\mathcal{N}^1_{L,(h_L,q_L)})^\Gamma \subset \mathcal{N}^{1,\natural}_{L,(h_L,q_L)}$, but we know this inclusion is an equality ([BT4] 4.7 and 4.9).

First case: L is a field. From 1.6, there exists a unique $G_{L_\circ}$-equivariant affine map $\mathcal{I}_{L_\circ} \to \mathcal{N}^1_{L,(h_L,q_L)}$, which is Γ-equivariant by unicity. It induces a G-equivariant affine map $\mathcal{I} \to (\mathcal{N}^1_{L,(h_L,q_L)})^\Gamma$, which (by unicity again) coincides with the canonical bijection $\mathcal{I} \to \mathcal{N}^1_{h,q}$ composed with the G-equivariant affine bijection

$$\mathcal{N}^1_{h,q} \to (\mathcal{N}^1_{L,(h_L,q_L)})^\Gamma,\ \alpha \mapsto \alpha_L.$$

So via the canonical bijections $\mathcal{I} \to \mathcal{N}^1_{h,q}$ and $\mathcal{I}_{L_\circ} \to \mathcal{N}^1_{L,(h_L,q_L)}$, the canonical bijection $\mathcal{I} \to (\mathcal{I}_{L_\circ})^\Gamma$ is given by $\alpha \mapsto \alpha_L$.

Second case: $L \simeq (L_\circ)^2$. We take the hypotheses and notation of 2.2. Let Γ' be the subgroup $\mathrm{Gal}(L/F)$ of Γ. The $L_\circ$-norm v_L on $L = L_\circ \times L_\circ$ is given by

$$v_L(\lambda,\mu) = \inf\{v_{L_\circ}(\lambda), v_{L_\circ}(\mu)\}, \quad \lambda,\, \mu \in L_\circ.$$

For $i = 1,\, 2$, put $\mathcal{N}^1_{L_\circ,i} = \mathrm{Latt}^1_{\mathfrak{o}_{L_\circ}}(V_{L_\circ,i})$. For $\alpha \in \mathcal{N}^1_{L_\circ,1}$, denote by $\overline{\alpha}$ the $L_\circ$-norm on $V_{L_\circ,2}$ defined by

$$\overline{\alpha}(x_2) = \inf_{x_1 \in V_{L_\circ,1}}(v_{L_\circ}(h_L(x_1,x_2) - \alpha(x_1)), \quad x_2 \in V_{L_\circ,2}$$

and by $\alpha \oplus \overline{\alpha}$ the L-norm on $V_L = V \times V$ defined by

$$(\alpha \oplus \overline{\alpha})(x_1,x_2) = \inf(\alpha(x_1), \overline{\alpha}(x_2)), \qquad x_1 \in V_{L_\circ,1},\, x_2 \in V_{L_\circ,2}.$$

From the lemma of §4.8 in [BT4], the map $\alpha \mapsto \alpha \oplus \overline{\alpha}$ is a bijection from $\mathcal{N}^1_{L_\circ,1}$ to the subset $\mathcal{N}^1_{L,h_L} = \mathcal{N}^1_{L,(h_L,q_L)}$ of $\mathcal{N}^1_L$. Via this bijection, we obtain an action of Γ on $\mathcal{N}^1_{L_\circ,1}$ which extends the natural action of Γ' (cf. Remark 2.1). For $\alpha \in \mathcal{N}^1$, let $\alpha_{L_\circ,1}$ be the $L_\circ$-norm $v_{L_\circ} \otimes \alpha$ on $V_{L_\circ,1}$. Let us identify $\widetilde{G}$ with a subgroup of $\mathrm{GL}(V_{L_\circ,1})$ via the map $g \mapsto \xi_1(1 \otimes g)$. Then the map $\alpha \mapsto \alpha_{L_\circ,1}$ from $\mathcal{N}^1$ to $\mathcal{N}^1_{L_\circ,1}$ is injective, $\widetilde{G}$-equivariant and affine, and it induces a bijection onto the convex subset $(\mathcal{N}^1_{L_\circ,1})^{\Gamma'}$ of $\mathcal{N}^1_{L_\circ,1}$ formed by those norms which are Γ'-invariant. Moreover, if $\alpha \in \mathcal{N}^1$, the Γ'-invariant $L_\circ$-norm $\alpha_{L_\circ}$ on $V_{L_\circ,1}$ is Γ-invariant if and only if $\overline{\alpha} = \alpha$, i.e. if and only if $\alpha \in \mathcal{N}^1_h$. So the map $\alpha \mapsto \alpha_{L_\circ,1}$ induces a G-equivariant affine bijection from $\mathcal{N}^1_h$ to the convex subset $(\mathcal{N}^1_{L_\circ,1})^\Gamma$ of $\mathcal{N}^1_{L_\circ,1}$ formed by those norms which are Γ-invariant. Let $\mathcal{N}_{L_\circ,1}$ be the quotient of $\mathcal{N}^1_{L_\circ,1}$ by the action of $\mathbb{R}$. The action of Γ on $\mathcal{N}^1_{L_\circ,1}$ induces an action on $\mathcal{N}_{L_\circ,1}$, and we denote by $(\mathcal{N}_{L_\circ,1})^\Gamma$ the convex subset of $\mathcal{N}_{L_\circ,1}$ formed by those elements which are Γ-invariant. For $\alpha \in \mathcal{N}^1_{L_\circ,1}$ and $c \in \mathbb{R}$, we have

$$(\alpha + c, \overline{\alpha + c}) = (\alpha + c, \overline{\alpha} - c);$$

so if α is Γ-invariant, then $\alpha + c$ is Γ-invariant if and only if $c = 0$. Thus the canonical projection $\mathcal{N}^1_{L_\circ,1} \to \mathcal{N}_{L_\circ,1},\ \alpha \mapsto \alpha'$ induces an injective map $(\mathcal{N}^1_{L_\circ,1})^\Gamma \to (\mathcal{N}_{L_\circ,1})^\Gamma$, which is also surjective: for $\alpha \in \mathcal{N}^1_{L_\circ,1}$ such that α' is Γ-invariant, since Γ induces on the class $\{\alpha + c : c \in \mathbb{R}\}$ a finite group of affine automorphisms, there exists a $c \in \mathbb{R}$ such that $\alpha + c$ is Γ-invariant. Thus we have a G-equivariant affine bijection

$$\mathcal{N}^1_h \to (\mathcal{N}_{L_\circ,1})^\Gamma,\ \alpha \mapsto (\alpha_{L_\circ,1})'.$$

So via the canonical bijections $\mathcal{I} \to \mathcal{N}^1_h$ and $\mathcal{I}_{L\circ} \to \mathcal{N}_{L_\circ,1}$ (cf. Remark 1.3), the canonical bijection $\mathcal{I} \to (\mathcal{I}_{L_\circ})^\Gamma$ is given by $\alpha \mapsto (\alpha_{L_\circ,1})'$.

Since the map $\mathcal{N}_L^1 \to \mathcal{L}_L^1$, $\alpha \mapsto \Lambda_\alpha$ is Γ-equivariant, the translation of the description above in terms of lattice-functions is straightforward and left to the reader.

2.4. Unramified descent: square lattice-functions. We continue with the hypotheses and notation of 2.3. Let p be a point in $\mathcal{I}$, identified with a point in $(\mathcal{I}_{L_\circ})^\Gamma$ by the canonical bijection $\mathcal{I} \to (\mathcal{I}_{L_\circ})^\Gamma$. By construction, the filtrations of $\mathfrak{g}$ and $\mathfrak{g}_{L_\circ}$ attached to p by Moy and Prasad, satisfy the descent property:

$$(\mathfrak{g}^{\mathrm{MP}}_{L_\circ,p,r})^\Gamma = \mathfrak{g}^{\mathrm{MP}}_{p,r}, \quad r \in \mathbb{R}.$$

Let us prove that the lattice-functions $\mathfrak{g}_{p,\cdot}$ in $\mathfrak{g}$ and $\mathfrak{g}_{L_\circ,p,\cdot}$ in $\mathfrak{g}_{L_\circ}$ satisfy the same descent property. Put $\tilde{\mathfrak{g}}_L = \mathrm{End}_L(V_L)$; so the $L_\circ$-algebras $\tilde{\mathfrak{g}}_L$ and $L_\circ \otimes_{F_\circ} \tilde{\mathfrak{g}}$ $(= L_\circ \otimes_{F_\circ} \mathrm{End}_F(V))$ are canonically isomorphic.

First case: L is a field. As in 1.7, the point $p \in (\mathcal{I}_{L_\circ})^\Gamma \subset \mathcal{I}_{L_\circ}$ defines a square $\mathfrak{o}_L$-lattice-function $\tilde{\mathfrak{g}}_{L,p,\cdot}$ in $\tilde{\mathfrak{g}}_L$. More precisely, from 1.6 there exists a unique $G_{L_\circ}$-equivariant affine map $\mathcal{I}_{L_\circ} \to \mathcal{L}^1_{L,(h_L,q_L)}, p' \mapsto \mathbf{\Lambda}_{p'}$, which is Γ-equivariant and, from 2.3, induces a G-equivariant affine map $\mathcal{I}^\Gamma_{L_\circ} \to (\mathcal{L}^1_{L,(h_L,q_L)})^\Gamma$. By definition, for $p' \in \mathcal{I}_{L_\circ}$, we have $\tilde{\mathfrak{g}}_{L,p',\cdot} = \mathrm{End}(\mathbf{\Lambda}_{p'})$. Since p is Γ-invariant, we have $\mathbf{\Lambda}_p = (\Lambda_p)_L$ (cf. 2.3). Thus we have

$$\tilde{\mathfrak{g}}_{L,p,r} = \mathfrak{o}_{L_\circ} \otimes_{\mathfrak{o}_{F_\circ}} \tilde{\mathfrak{g}}_{p,r}, \quad r \in \mathbb{R}.$$

This implies the descent property:

$$(\tilde{\mathfrak{g}}_{L,p,r})^\Gamma = \tilde{\mathfrak{g}}_{p,r}, \quad r \in \mathbb{R}.$$

Since $(\tilde{\mathfrak{g}}_L)^\Gamma = \tilde{\mathfrak{g}}$, we obtain the descent property:

$$(\mathfrak{g}_{L_\circ,p,r})^\Gamma = \mathfrak{g}_{p,r}, \quad r \in \mathbb{R}.$$

Second case: $L \simeq (L_\circ)^2$. We take the hypotheses and notation of 2.2. For $r \in \mathbb{R}$, the $\mathfrak{o}_{L_\circ}$-lattice $\mathfrak{g}_{L_\circ,p,r}$ in $\mathfrak{g}_{L_\circ}$ is the set of

$$(g_1, g_2) \in \mathrm{End}_{L_\circ}(V_{L_\circ,1}) \times \mathrm{End}_{L_\circ}(V_{L_\circ,2})$$

such that $g_1 \in \mathrm{End}_{L_\circ}(V_{L_\circ,1})_{\iota(p),r}$ and $g_2 + {}^{\mathrm{t}}\varphi^{-1} \circ {}^{\mathrm{t}}g_1 \circ {}^{\mathrm{t}}\varphi = 0$. Let Γ' be the subgroup $\mathrm{Gal}(L_\circ/F)$ of Γ. Fix a real number r. Since the map $\iota : \mathrm{U}(h_L) \to \mathrm{GL}(V_{L_\circ,1})$ is Γ'-equivariant, we have $\iota(p) \in (\mathcal{L}_{L_\circ,1})^{\Gamma'}$. On the other hand, the isomorphism $\varphi : V_{L_\circ,1} \to V^*_{L_\circ,2}$ is also Γ'-equivariant. From the first case above, we may and do assume that $L_\circ = F$. So φ is an isomorphism from $V_{F,1}$ to $V_{F,2}$, $\iota(p)$ is an element of $\mathcal{L}_{F,1} = \mathrm{Latt}^1_{\mathfrak{o}_F}(V_{F,1})/\mathbb{R}$, and $\mathfrak{g}_{F,p,r}$ is the $\mathfrak{o}_F$-lattice in $\mathfrak{g}_F = F \otimes_{F_\circ} \mathfrak{g} \subset \tilde{\mathfrak{g}}_F = \mathrm{End}_F(V_{F,1}) \times \mathrm{End}_F(V_{F,2})$ formed by those (g_1, g_2) satisfying $g_1 \in \mathrm{End}_F(V_{F,1})_{\iota(p),r}$ and $g_2 + {}^{\mathrm{t}}\varphi^{-1} \circ {}^{\mathrm{t}}g_1 \circ {}^{\mathrm{t}}\varphi = 0$. For $i = 1, 2$, the map $v \mapsto \xi_i(1 \otimes v)$ identifies V with $V_{F,i}$; hence we have the identifications $\mathrm{End}_F(V_{F,i}) = \tilde{\mathfrak{g}}$ and $\mathcal{L}_{F,1} = \mathcal{L}$. From Remark 2.1, the action of $\Gamma = \{1, \sigma\}$ on $\tilde{\mathfrak{g}}_F = \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ is given by $(g_1, g_2)^\sigma = (g_2, g_1)$, for all $g_1, g_2 \in \tilde{\mathfrak{g}}$. We obtain that the map $\mathfrak{g}_F \to \tilde{\mathfrak{g}}, (g_1, g_2) \mapsto g_1$ identifies $(\mathfrak{g}_{F,p,r})^\Gamma = \mathfrak{g}_{F,p,r} \cap \mathfrak{g}$ with the $\mathfrak{o}_F$-lattice in $\tilde{\mathfrak{g}}$ formed by those $g \in \tilde{\mathfrak{g}}_{\iota(p),r}$ satisfying $g + {}^{\mathrm{t}}\varphi^{-1} \circ {}^{\mathrm{t}}g \circ {}^{\mathrm{t}}\varphi = 0$. But, for all $x, y \in V$, we have

$$\varphi(x)(y) = h_L((x,0),(0,y)) = h_L((x,x),(y,y)) = h(x,y).$$

Hence for $g \in \tilde{\mathfrak{g}}$, we have
$$g^{\sigma_h} = {}^{\mathrm{t}}\varphi^{-1} \circ {}^{\mathrm{t}}g \circ {}^{\mathrm{t}}\varphi$$
and $g + {}^{\mathrm{t}}\varphi^{-1} \circ {}^{\mathrm{t}}g \circ {}^{\mathrm{t}}\varphi = 0$ if and only if $g \in \mathfrak{g}$. Thus we obtain the descent property:
$$(\mathfrak{g}_{F,p,r})^{\Gamma} = \tilde{\mathfrak{g}}_{\iota(p),r} \cap \mathfrak{g}.$$
The action of Γ on $\mathcal{I}_F$ induces an action on $\mathfrak{L}$: for $\Lambda \in \mathfrak{L}$, we put $\Lambda^{\sigma} = \iota(\iota^{-1}(\Lambda)^{\sigma})$. By construction, the map $\iota : \mathcal{I}_F \to \mathfrak{L}$ is Γ-invariant, and induces a bijection, denoted by $\iota^{\natural}$, from $\mathcal{I}$ to the convex subset $\mathfrak{L}^{\Gamma}$ of $\mathfrak{L}$ formed by those elements which are Γ-invariant. Moreover, $\iota^{\natural}$ is G-equivariant and affine. From 2.3, by restriction the canonical projection $\mathfrak{L}^1 \to \mathfrak{L}$ induces a bijection $\psi : \mathfrak{L}^1_h \to \mathfrak{L}^{\Gamma}$ which is G-equivariant and affine. This implies that the composition $\psi \circ \iota^{\natural} : \mathcal{I} \to \mathfrak{L}^1_h$ is bijective, G-equivariant and affine. Since we are in the tame case (the extension $F/F_\circ$ is unramified), we have $\mathfrak{L}^1_h = \mathfrak{L}^1_{h,q}$ and $\psi \circ \iota^{\natural}$ is the unique G-equivariant affine map from $\mathcal{I}$ to $\mathfrak{L}^1_h$. So we obtain that $\iota(p) \in \mathfrak{L}^{\Gamma}$, $\psi(\iota(p)) = \Lambda_p$ and $\tilde{\mathfrak{g}}_{\iota(p),r} = \tilde{\mathfrak{g}}_{p,r}$. Turning to the general unramified extension $L_\circ/F_\circ$ (i.e. removing the hypothesis $L_\circ = F$), we have proved the descent property:
$$(\mathfrak{g}_{L_\circ,p,r})^{\Gamma} = \mathfrak{g}_{p,r}, \quad r \in \mathbb{R}.$$

2.5. Reduction to the quasi-split case. Let $L_\circ$ be a finite unramified extension of $F_\circ$ such that the reductive $L_\circ$-group $\boldsymbol{G}^\circ \times_{F^\circ} L_\circ$ is quasi-split. Put $L = L_\circ \otimes_{F_\circ} F$ as before. Suppose that for all $p \in \mathcal{I}_{L_\circ}$, we have
$$\mathfrak{g}_{L_\circ,p,r} = \mathfrak{g}^{\mathrm{MP}}_{L_\circ,p,r}, \quad r \in \mathbb{R}.$$
Then the descent property proved in 2.4 implies that for all $p \in \mathcal{I} = (\mathcal{I}_{L_\circ})^{\Gamma}$, we have
$$\mathfrak{g}_{p,r} = \mathfrak{g}^{\mathrm{MP}}_{p,r}, \quad r \in \mathbb{R}.$$
If $L \simeq (L_\circ)^2$, we know from [BL] that for all $p \in \mathcal{I}_{L_\circ}$, the filtration $(\mathfrak{g}_{L_\circ,p,r})_{r\in\mathbb{R}}$ of $\mathfrak{g}$ coincide with $(\mathfrak{g}^{\mathrm{MP}}_{L_\circ,p,r})_{r\in\mathbb{R}}$. Thus we have reduced the question to the quasi-split case: we may suppose that $\boldsymbol{G}^\circ$ is quasi-split over $F_\circ$; note that we may also suppose that $\boldsymbol{G}^\circ$ is residually split over $F_\circ$, i.e. quasi-split and of the same relative rank as $\boldsymbol{G}^\circ \times_{F_\circ} L'_\circ$ for all finite unramified extensions $L'_\circ$ of $L_\circ$.

3. Proof in the quasi-split case

3.1. Root subgroups again. In this chapter, we assume that the group $\boldsymbol{G}$ is quasi-split over $F_\circ$. Recall that since $\boldsymbol{G}$ is quasi-split, we have $\dim_F(V_0) \leq 1$. From [BT1] 10.1.3, if $\sigma \neq \mathrm{Id}$, by a "change of coordinate" we may (and do) assume $\varepsilon = 1$ if $V_0 \neq \{0\}$, and $\varepsilon = -1$ if $V_0 = \{0\}$. Then we have $\varepsilon = -1$ if and only if we are in case (C). Let us fix an element $e_0 \in V_0$ such that $e_0 \neq 0$ if $V_0 \neq \{0\}$ (case (B) or (BC)). Moreover if $V_0 \neq \{0\}$, by another "change of coordinate" we may (and do) assume that $h(e_0, e_0) = 1$ (loc. cit.).

For $i, j \in I$, $j \neq \pm i$ and $u \in F$, the element $u_{i,j}(u) \in U_{a_i}$ (cf. 1.5) acts trivially on $\sum_{k\in I\cup\{0\},k\neq\pm i,\pm j} V_i$, stabilizes $X_{i,j} = V_{-i} + V_{-j} + V_j + V_i$, and induces on $X_{i,j}$ an automorphism whose matrix with respect to e_{-i}, e_{-j}, e_j, e_i is given by

$$\begin{pmatrix} 1 & 0 & -\varepsilon(i)u & 0 \\ 0 & 1 & 0 & \varepsilon(-j)u^\sigma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Suppose $V_0 \neq \{0\}$ (case (B) or (BC)). For $i \in I$ and $u, v \in F$ such that $v + v^\sigma = uu^\sigma$ (i.e. such that $v = \frac{1}{2}u^2$ if we are in case (B)), the element $u_i(u, v) = u_i(ue_0, v) \in U_{a_i}$ (cf. 1.5) acts trivially on $\sum_{k \in I, k \neq \pm i} V_i$, stabilizes $X_i = V_{-i} + V_0 + V_i$, and induces on X_i an automorphism whose matrix with respect to e_{-i}, e_0, e_i is given by

$$\begin{pmatrix} 1 & -u^\sigma & -v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose $V_0 = \{0\}$ and $(\sigma, \varepsilon) \neq (\mathrm{Id}, 1)$ (case (C)). For $i \in I$ and $v \in F_{\sigma,\varepsilon}$, the element $u_i(0, v) \in U_{2a_i}$ acts trivially on $\sum_{k \in I, k \neq \pm i} V_i$, stabilizes $X_i = V_{-i} + V_i$, and induces on X_i an automorphism whose matrix with respect to e_{-i}, e_i is given by

$$\begin{pmatrix} 1 & -\varepsilon(i)v \\ 0 & 1 \end{pmatrix}.$$

For $i, j \in I$, $j \neq \pm i$, the group-law on $U_{i,j}$ is given by $u_{i,j}(u)u_{i,j}(u') = u_{i,j}(u+u')$ for all $u, u' \in F$. For $i \in I$, to describe the group-law on U_{a_i} (case (B) or (BC)) and the group-law on U_{2a_i} (case (BC) or (C)), it is useful to introduce the $F_\circ$-spaces $H_2 = \{0\} \times F_{\sigma,\varepsilon}$ and $H = \{(u, v) \in F \times F : v + \varepsilon v^\sigma = uu^\sigma\} \supset H_2$. The space H is endowed with a multiplicative group-law

$$(u, v)(u', v') = (u + u', v + v' + u^\sigma u')$$

which makes H_2 a subgroup of H: for $v, v' \in F_{\sigma,\varepsilon}$, we have $v + v' \in F_{\sigma,\varepsilon}$ and $(0, v)(0, v') = (0, v + v')$. The group-laws on U_{a_i} and U_{2a_i} are obtained from the group-laws on H and H_2 by transport of structure via u_i. In case (BC), since H_2 is normal in H, we can define the quotient group $\overline{U}_{a_i} = U_{a_i}/U_{2a_i}$. It is the group of $F_\circ$-rational points of the (geometric) quotient group $\boldsymbol{U}_{a_i}/\boldsymbol{U}_{2a_i}$. In case (B), we put $\overline{U}_{a_i} = U_{a_i}$.

3.2. Basis of the Lie algebras. Recall that $\mathfrak{g}$ denotes the Lie algebra of G, and $\tilde{\mathfrak{g}}$ the Lie algebra of $\widetilde{G}$. For $a \in \Phi$, let $\mathfrak{u}_a$ be the subspace of $\mathfrak{g}$ defined by $\mathfrak{u}_a = \mathrm{Lie}(U_a)$ if $2a \notin \Phi$, and by $\mathfrak{u}_a = \mathrm{Lie}(\overline{U}_a)$ if $2a \in \Phi$. For $i, j \in I$, or $i, j \in I \cup \{0\}$ if $V_0 \neq \{0\}$, let $\widetilde{E}_{i,j} \in \tilde{\mathfrak{g}}$ be the standard elementary matrix. For $i, j \in I$, $j \neq \pm i$, $u \in F$, $v \in F_{\sigma,\varepsilon}$, we put:

$$E_{a_{i,j}}(u) = -\varepsilon(i)u\widetilde{E}_{-i,j} + \varepsilon(-j)u^\sigma \widetilde{E}_{-j,i};$$

$$E_{a_i}(u) = -u^\sigma \widetilde{E}_{-i,0} + u\widetilde{E}_{0,i} \text{ (case (B) or (BC))};$$

$$E_{2a_i}(v) = -\varepsilon(i)v\widetilde{E}_{-i,i} \text{ (case (BC) or (C))}.$$

Let $a \in \Phi$. If $\frac{1}{2}a \notin \Phi$, the map $u \mapsto E_a(u)$ is an isomorphism from F to $\mathfrak{u}_a$. If $\frac{1}{2}a \in \Phi$ (case (BC)), the map $v \mapsto E_a(v)$ is an isomorphism from F° to $\mathfrak{u}_a$.

Denote by $\mathfrak{z} = \mathrm{Lie}(Z)$ the Lie algebra of Z, and by $\mathfrak{z}'$ the Lie algebra of Z'. For $i \in I$, $u \in F$, $v \in F^\circ$, we put:

$$E_i(u) = u^\sigma \widetilde{E}_{-i,-i} - u\widetilde{E}_{i,i};$$

$$E_0(v) = v\widetilde{E}_{0,0} \text{ (case (B) or (BC))}.$$

For $i \in I$, the map $u \mapsto \widetilde{E}_i$ is an isomorphism from F to a subspace $\mathfrak{z}_i$ of $\mathfrak{z}$. If $V_0 \neq \{0\}$ (case (B) or (BC)), the map $v \mapsto E_0(v)$ is an isomorphism from F° to a subspace $\mathfrak{z}_0$ of $\mathfrak{z}$. Note that in case (B), we have $\mathfrak{z}_0 = \{0\}$. If $V_0 = \{0\}$ (case (C) or (D)), we put $\mathfrak{z}_0 = \{0\}$. So $\mathfrak{z}_0 \neq \{0\}$ if and only if we are in case (BC).

We have the decompositions

$$\mathfrak{g} = \mathfrak{z} \oplus \bigoplus_{a \in \Phi} \mathfrak{u}_a, \qquad \mathfrak{z} = \bigoplus_{i \in I \cup \{0\}} \mathfrak{z}_i.$$

3.3. Lattice-functions and square lattice-functions. Recall that $\mathcal{A}$ is the apartment of the building $\mathcal{I}$ of G attached to the maximal $F_\circ$-split torus $\boldsymbol{S}$ of $\boldsymbol{G}$. Put

$$l = \tfrac{1}{2}\sup\{r \in \mathbb{R} : v_F(\lambda) = r,\ \lambda \in F,\ \lambda + \lambda^\sigma = 1\}.$$

Recall we have $l \leq 0$, and $l = 0$ if and only if the extension $F/F_\circ$ is quadratic unramified or the residual characteristic of $F_\circ$ is not 2. If $V_0 \neq 0$, from Remark 1.6, we have

$$v_{V_0}(\lambda e_0) = v_F(\lambda) + l.$$

Put $a_0 = -l$ if $V_0 \neq \{0\}$, and $a_0 = -\infty$ if $V_0 = \{0\}$. We consider a_0 as a constant function on $\mathcal{A}$.

Let p be a point in $\mathcal{A}$. The MM-norm α_p for (h,q) on V and the lattice-function Λ_p in V are given by

$$\alpha_p\left(\textstyle\sum_{i \in I \cup \{0\}} \lambda_i e_i\right) = \inf\{v_F(\lambda_i) - a_i(p) : i \in I \cup \{0\}\}, \quad \lambda_i \in F,$$

and

$$\Lambda_p(r) = \{x \in V : \alpha_p(x) \geq r\},\ r \in \mathbb{R}.$$

Put $\eta = 1$ if the extension $F/F_\circ$ is unramified, and $\eta = 2$ if it is ramified. So ηv_F is the normalized valuation on F, and we have

$$\Lambda_p(r) = \oplus_{i \in I \cup \{0\}} \mathfrak{p}_F^{\lceil \eta(r + a_i(p)) \rceil} e_i\ ,\ r \in \mathbb{R},$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x (if $V_0 = \{0\}$, then we have $\mathfrak{p}_F^{\lceil \eta(r + a_0(p)) \rceil} e_0 = \{0\}$). The square lattice-function $\tilde{\mathfrak{g}}_{p,\cdot}$ in $\tilde{\mathfrak{g}}$ attached to p is given by

$$\tilde{\mathfrak{g}}_{p,r} = \{g \in \tilde{\mathfrak{g}} : g\Lambda_p(s) \subset \Lambda_p(s + r),\ s \in \mathbb{R}\}, r \in \mathbb{R}.$$

Lemma 3.1. *Let* $(p,r)\in\mathcal{A}\times\mathbb{R}$. *We have*

$$\tilde{\mathfrak{g}}_{p,r}=\bigoplus_{i,\,j}\mathfrak{p}_F^{\lceil\eta(r+a_i(p)-a_j(p))\rceil}\widetilde{E}_{i,j}$$

where i, j *run over the elements of* I *in case (C) or (D), and over the elements of* $I\cup\{0\}$ *in case (B) or (BC).*

Proof. Let $i,\,j\in I$ and $u\in F$. By definition, the element $u\widetilde{E}_{i,j}$ belongs to $\tilde{\mathfrak{g}}_{p,r}$ if and only if $u\mathfrak{p}_F^{\lceil\eta(s+a_j(p)\rceil}\subset\mathfrak{p}_F^{\lceil\eta(s+r+a_i(p)\rceil}$ for all $s\in\mathbb{R}$, i.e. if and only if $\eta v_F(u)\geq\lceil r+a_i(p)-a_j(p)\rceil$. Hence we have $\tilde{\mathfrak{g}}_{p,r}\cap F\widetilde{E}_{i,j}=\mathfrak{p}_F^{\lceil\eta(r+a_i(p)-a_j(p))\rceil}\widetilde{E}_{i,j}$. In case (B) or (BC), the same proof applies for $i,\,j\in I\cup\{0\}$. ■

For $p\in\mathfrak{I}$, recall that the $\mathfrak{o}_{F_\circ}$-lattice-function $\mathfrak{g}_{p,\cdot}$ in $\mathfrak{g}$ is defined by

$$\mathfrak{g}_{p,r}=\tilde{\mathfrak{g}}_{p,r}\cap\mathfrak{g},\ r\in\mathbb{R}.$$

For $(p,r)\in\mathcal{A}\times\mathbb{R}$, we have the decomposition

$$\mathfrak{g}_{p,r}=\bigoplus_{i\in I\cup\{0\}}(\tilde{\mathfrak{g}}_{p,r}\cap\mathfrak{z}_i)\oplus\bigoplus_{a\in\Phi}(\tilde{\mathfrak{g}}_{p,r}\cap\mathfrak{u}_a).$$

Lemma 3.2. *Let* $(p,r)\in A\times\mathbb{R}$. *Then:*

for $i\in I$, *we have* $\tilde{\mathfrak{g}}_{p,r}\cap\mathfrak{z}_i=E_i(\mathfrak{p}_F^{\lceil\eta r\rceil})$;

in case (BC), we have $\tilde{\mathfrak{g}}_{p,r}\cap\mathfrak{z}_0=E_0(\mathfrak{p}_F^{\lceil\eta r\rceil}\cap F^\circ)$;

for $i,\,j\in I$, $i\neq\pm j$, *we have* $\tilde{\mathfrak{g}}_{p,r}\cap\mathfrak{u}_{a_{i,j}}=E_{a_{i,j}}(\mathfrak{p}_F^{\lceil\eta(r-a_{i,j}(p))\rceil})$;

in case (B) or (BC), for $i\in I$, *we have* $\tilde{\mathfrak{g}}_{p,r}\cap\mathfrak{u}_{a_i}=E_{a_i}(\mathfrak{p}_F^{\lceil\eta(r-l-a_i(p)\rceil})$;

in case (BC) or (C), for $i\in I$, *we have* $\tilde{\mathfrak{g}}_{p,r}\cap\mathfrak{u}_{2a_i}=E_{2a_i}(\mathfrak{p}_F^{\lceil\eta(r-2a_i(p)\rceil}\cap F_{\sigma,\varepsilon})$.

Proof. Let $i,\,j\in I$, $i\neq\pm j$, and $u\in F$. By definition, the element $E_{a_{i,j}}(u)$ belongs to $\tilde{\mathfrak{g}}_{p,r}$ if and only if $u\widetilde{E}_{-i,j}\in\mathfrak{p}_F^{\lceil\eta(r+a_{-i}(p)-a_j(p))\rceil}\widetilde{E}_{-i,j}$ and $u^\sigma\widetilde{E}_{-j,i}\in\mathfrak{p}_F^{\lceil\eta(r+a_{-j}(p)-a_i(p))\rceil}\widetilde{E}_{-j,i}$; i.e., since $a_{-i}=-a_i$ and $a_{-j}=-a_j$, if and only if $u\in\mathfrak{p}_F^{\lceil\eta(r-(a_i(p)+a_j(p))\rceil}$. Hence we have

$$\tilde{\mathfrak{g}}_{p,r}\cap\mathfrak{u}_{a_{i,j}}=E_{a_{i,j}}(\mathfrak{p}_F^{\lceil\eta(r-a_{i,j}(p))\rceil}).$$

The same proof shows that for $i\in I$, we have

$$\tilde{\mathfrak{g}}_{p,r}\cap\mathfrak{z}_i=E_i(\mathfrak{p}_F^{\lceil\eta r\rceil}).$$

Suppose we are in case (B) or (BC), and let $i\in I$ and $u\in F$. By definition, the element $E_{a_i}(u)$ belongs to $\tilde{\mathfrak{g}}_{p,r}$ if and only if $u^\sigma\widetilde{E}_{-i,0}\in\mathfrak{p}_F^{\lceil\eta(r+a_{-i}(p)-a_0(p))\rceil}\widetilde{E}_{-i,0}$

and $u\widetilde{E}_{0,i} \in \mathfrak{p}_F^{\lceil \eta(r+a_0(p)-a_i(p))\rceil}\widetilde{E}_{0,i}$; i.e., since $a_{-i} = -a_i$ and $a_0(p) = -l \geq 0$, if and only if $u \in \mathfrak{p}_F^{\lceil \eta(r+a_0(p)-a_i(p))\rceil}$. Hence we have

$$\tilde{\mathfrak{g}}_{p,r} \cap \mathfrak{u}_{a_i} = E_{a_i}(\mathfrak{p}_F^{\lceil \eta(r-l-a_i(p))\rceil}).$$

Suppose we are in case (BC) or (C), and let $v \in F_{\sigma,\varepsilon}$. By definition, the element $E_{2a_i}(v)$ belongs to $\tilde{\mathfrak{g}}_{p,r}$ if and only if $v\widetilde{E}_{-i,i} \in \mathfrak{p}_F^{\lceil \eta(r+a_{-i}(p)-a_i(p))\rceil}\widetilde{E}_{-i,i}$; i.e., since $a_{-i} = -a_i$, if and only if $v \in \mathfrak{p}_F^{\lceil \eta(r-2a_i(p))\rceil}$. Hence we have

$$\tilde{\mathfrak{g}}_{p,r} \cap \mathfrak{u}_{2a_i} = E_{2a_i}(\mathfrak{p}_F^{\lceil \eta(r-2a_i(p))\rceil} \cap F_{\sigma,\varepsilon}).$$

In case (BC), the same proof shows that we have

$$\tilde{\mathfrak{g}}_{p,r} \cap \mathfrak{z}_0 = E_0(\mathfrak{p}_F^{\lceil \eta r\rceil} \cap F^{\circ}).$$

The lemma is proved. ■

3.4. Filtrations of the root subgroups. For $a \in \Phi$ and $r \in \mathbb{R}$, let $U_{a,r} = U^{\varphi}_{a,r}$ be the compact subgroup of U_a defined by

$$U_{a,r} = \{g \in U_a : \varphi_a(g) \geq r\},$$

where $\varphi = (\varphi_a)_{a\in\Phi}$ is the valuation of the root datum $(Z, (U_a)_{a\in\Phi})$ defined in 1.5. For $a \in \Phi$ such that $2a \in \Phi$ (case (BC)), since $\varphi_{2a} = 2\varphi_a|_{U_{2a}}$, we have

$$U_{a,r} \cap U_{2a} = U_{2a,r}.$$

Put

$$\Gamma_a = \varphi_a(U_a \smallsetminus \{1\}) \subset \mathbb{R},\ \ a \in \Phi;$$

$$\overline{\Gamma}_a = \{\varphi_a(u) : u \in U_a \smallsetminus \{1\},\, \varphi_a(u) = \sup \varphi_a(uU_{2a})\},\ a \in \Phi,\ 2a \in \Phi.$$

So if we *are not* in case (BC), we have $\Gamma_a = v_F(F^{\times})$.

Let ξ be an element of F such that $\xi + \xi^{\sigma} = 1$ and $v_F(\xi) = l$. Recall that if $\sigma \neq \mathrm{Id}$ and the residual characteristic of $F_{\circ}$ is not 2, we may take $\xi = \frac{1}{2}$. Suppose we are in case (BC), and let $i \in I$. For $u \in F$, we put

$$\overline{u}_i(u) = u_i(u, \xi u u^{\sigma}) \ (\mathrm{mod}\, U_{2a_i}) \in \overline{U}_{a_i}$$

and

$$\overline{\varphi}_{a_i}(\overline{u}_i(u)) = l + v_F(u).$$

Then we have

$$\overline{\Gamma}_{a_i} = \overline{\varphi}_{a_i}(\overline{U}_{a_i} \smallsetminus \{1\})$$

and

$$\Gamma_{a_i} = \overline{\Gamma}_{a_i} \coprod \tfrac{1}{2}\Gamma_{2a_i}$$

(disjoint union). More explicitly, we have $\overline{\Gamma}_{a_i} = l + \eta^{-1}\mathbb{Z}$, $\Gamma_{2a_i} = \frac{1}{2}v_F(F^{\circ} \smallsetminus \{0\})$ and $\Gamma_{a_i} = \frac{1}{2}v_F(F^{\times})$.

Let p be a point of $\mathcal{A}$. Denote by φ_p the valuation $\varphi + a(p)$ of the root datum $(Z, (U_a)_{a\in\Phi})$. For $a \in \Phi$ and $r \in \mathbb{R}$, let $U_{a,p,r} = U_{a,r}^{\varphi_p}$ be the compact subgroup of U_a defined by

$$U_{a,p,r} = \{g \in G : g \in U_a : \varphi_a(g) \geq r - a(p)\}.$$

For $a \in \Phi$ such that $2a \in \Phi$ (case (BC)) and $r \in \mathbb{R}$, let $\overline{U}_{a,p,r}$ be the compact subgroup of $\overline{U}_a$ defined by

$$\overline{U}_{a,p,r} = \{g \in G : g \in \overline{U}_a : \overline{\varphi}_a(g) \geq r - a(p)\}.$$

3.5. Moy-Prasad filtrations of $\mathfrak{g}$. Let p be a point in $\mathcal{A}$. For $a \in \Phi$ and $r \in \mathbb{R}$, the subgroup $U_{a,p,r}$ of U_a is the group of $\mathfrak{o}_{F_\circ}$-rational points of a smooth affine $\mathfrak{o}_{F_\circ}$-group-scheme $\mathfrak{U}_{a,p,r}$. Its Lie algebra $\mathrm{Lie}(\mathfrak{U}_{a,p,r})$ is an $\mathfrak{o}_{F_\circ}$-lattice in $\mathrm{Lie}(U_a)$. If, moreover, $2a \notin \Phi$, we put $\mathfrak{u}_a = \mathrm{Lie}(\mathfrak{U}_{a,p,r})$.

For $a \in \Phi$ such that $2a \in \Phi$ (case (BC)), the subgroup $\overline{U}_{a,p,r}$ of $\overline{U}_a$ is the group of $\mathfrak{o}_{F_\circ}$-rational points of a smooth affine $\mathfrak{o}_{F_\circ}$-group-scheme $\overline{\mathfrak{U}}_{a,p,r}$. Denote by $\mathfrak{u}_{a,p,r} = \mathrm{Lie}(\overline{\mathfrak{U}}_{a,p,r})$ its Lie algebra; it is an $\mathfrak{o}_{F_\circ}$-lattice in $\mathfrak{u}_a$ $(= \mathrm{Lie}(\overline{U}_a))$ and identifies with a sub-$\mathfrak{o}_{F_\circ}$-module of $\mathrm{Lie}(\mathfrak{U}_{a,p,r})$. Moreover we have the decomposition

$$\mathrm{Lie}(\mathfrak{U}_{a,p,r}) = \mathfrak{u}_{a,p,r} \oplus \mathfrak{u}_{2a,p,r}.$$

Let $\mathrm{X}^*(\boldsymbol{Z})$ be the group of algebraic characters of $\boldsymbol{Z}$. Since Z splits over F, each element of $\mathrm{X}^*(\boldsymbol{Z})$ is defined over F. For $r \in \mathbb{R}$, let $\mathfrak{z}_r$ be the $\mathfrak{o}_{F_\circ}$-lattice in $\mathfrak{z}$ defined by

$$\mathfrak{z}_r = \{x \in \mathfrak{z} : v_F(\mathrm{d}\chi(x)) \geq r,\ \forall \chi \in \mathrm{X}^*(\boldsymbol{Z})\}.$$

The filtration $(\mathfrak{g}_{p,r}^{\mathrm{MP}})_{r\in\mathbb{R}}$ of $\mathfrak{g}$ attached to p by Moy and Prasad [MP] is given by

$$\mathfrak{g}_{p,r}^{\mathrm{MP}} = \mathfrak{z}_r \oplus \bigoplus_{a\in\Phi} \mathfrak{u}_{a,p,r},\ r \in \mathbb{R}.$$

Remark 3.3. The Moy-Prasad filtration of $\mathfrak{g}$ attached to a point p in the building $\mathfrak{I}$ of G is usually defined by descent from a maximal unramified extension $F_\circ^{\mathrm{nr}}$ of $F_\circ$ (the point p being canonically identified with a $\mathrm{Gal}(F_\circ^{\mathrm{nr}}/F_\circ)$-invariant point in the building of $\boldsymbol{G}(F_\circ^{\mathrm{nr}})$). If the group $\boldsymbol{G}$ is residually split over $F_\circ$, the filtration $(\mathfrak{g}_{p,r}^{\mathrm{MP}})_{r\in\mathbb{R}}$ of $\mathfrak{g}$ defined above is clearly the Moy-Prasad filtration attached to p. If the group $\boldsymbol{G}$ is quasi-split but not residually split over $F_\circ$ (i.e. if F is a quadratic unramified extension of $F_\circ$), then $\boldsymbol{G}$ splits over F and $\boldsymbol{G}(F) \simeq \mathrm{GL}(V)$. In that case, chapter 2 implies that the filtration $(\mathfrak{g}_{p,r}^{\mathrm{MP}})_{r\in\mathbb{R}}$ of $\mathfrak{g}$ defined above is the Moy-Prasad filtration attached to p (more generally, the result follows [BT2] 5.1.20, Rem. 2). Note that we can also avoid the problem by assuming directly that $\boldsymbol{G}$ is residually split over $F_\circ$ (see 2.5).

Proposition 3.4. *Let p be a point in $\mathfrak{I}$. We have*

$$\mathfrak{g}_{p,r} = \mathfrak{g}_{p,r}^{\mathrm{MP}},\ r \in \mathbb{R}.$$

Proof. We may assume that $p \in \mathcal{A}$. Then the proposition follows from Lemma 3.2 and the description of the root subgroups $U_{a,p,r}$ given in 3.4. ∎

This proposition, together with the reduction step in chapter 2, implies Theorem 1.8.

4. The general case ($D \neq F$)

4.1. Reduction to the case $D = F$. In this chapter, we show briefly how to generalize the main result (Theorem 1.8) to a general classical group (cf. Remark 1.1). So notation and hypothesis are those of 1.1, but we assume $D \neq F$.

Let $L_\circ$ be a finite extension of $F_\circ$ of degree a multiple of $2d = 4$. Put $L = L_\circ \otimes_{F_\circ} F$ and $D_L = L_\circ \otimes_{F_\circ} D$ $(= L \otimes_F D)$. Then D_L is a semisimple $L_\circ$-algebra with centre L, endowed with an involution $\mathrm{Id} \otimes \sigma$, still denoted by σ. Denote by V_L the right D_L-module $L_\circ \otimes_{F_\circ} V = L \otimes_F V$. The form h on V extends to a non-degenerate σ-skew ε-hermitian form $h_L : V_L \times V_L \to D_L$. Put $\widetilde{G}_L = \mathrm{Aut}_{D_L}(V_L)$ and $G_{L_\circ} = \mathrm{U}(h_L)$. Then $G_{L_\circ}$ is the group of $L_\circ$-rational points of the $L_\circ$-algebraic group $\mathbf{G}_{L_\circ} = \mathbf{G} \times_{F_\circ} L_\circ$.

Let us consider the *first case*: L is a field. Then L is an extension of degree $[L_\circ : F_\circ]$ of F, and D_L is a *split* central simple L-algebra. Moreover, the restriction of σ $(= \mathrm{Id} \otimes \sigma)$ to L, say σ_1, is the generator of $\mathrm{Gal}(L/L_\circ)$. Let us choose a simple *right* D_L-module M. We have the canonical identifications $L = \mathrm{End}_{D_L}(M)$ and $D_L = \mathrm{End}_L(M)$. Let M^σ be the simple *left* D_L-module deduced from M via σ, i.e. the additive group M endowed with the action of D_L given by $(a, x) \mapsto xa^\sigma$ for $a \in D_L$, $x \in M$. The dual $M^\star = \mathrm{Hom}_L(M, L)$ is also a simple left D_L-module — canonically identified with $\mathrm{Hom}_{D_L}(M, D_L)$, cf. [BT3] 1.16. Thus we have $L = \mathrm{End}_{D_L}(M^*)$ and $D_L = \mathrm{End}_L(M^*)$. Moreover there exists an isomorphism of D_L-modules $s : M^\sigma \to M^*$ which is *admissible* in the following sense (cf. [BT4] 1.7, 1.8): let $\beta_s : M \times M \to D_L$ be the σ-skew form given by $\beta_s(x, y) = s(x)(y)$; the admissibility condition on s says there exists $\eta \in \{\pm 1\}$ such that $\beta_s(y, x) = \eta \beta_s(x, y)^\sigma$. In other words, β_s is a σ-skew η-hermitian form on M, which is non-degenerate by construction.

Put $V_1 = \mathrm{Hom}_{D_L}(M, V_L)$. It is a finite dimensional vector space over L, and the map

$$V_1 \otimes_L M \to V_L,\ v \otimes x \mapsto v(x)$$

is an isomorphism of (right) D_L-vector spaces, which induces a canonical identification of L-algebras (cf. [BT3] 1.16)

$$\mathrm{End}_L(V_1) = \mathrm{End}_{D_L}(V_L).$$

This gives an identification of $\widetilde{G}_1 = \mathrm{Aut}_L(V_1)$ with $\widetilde{G}_L$. Now put $\varepsilon_1 = \varepsilon\eta$. From [BT4] 1.10, Prop., there exists a unique σ-skew form $h_1 : V_1 \times V_1 \to L$ such that

$$h(u(x), v(y)) = \beta_s(xh_1(u, v), y) = h_1(u, v)^{\sigma_1} \beta_s(x, y);$$

it is non-degenerate and ε_1-hermitian. Let also $q_1 : V_1 \to L/L_{\sigma_1,\varepsilon_1}$ be the pseudo-quadratic form associated with h_1 as in 1.1. Put $G_1 = \mathrm{U}(h_1)$. It is the group of

$L_\circ$-rational points of an algebraic $L_\circ$-group $\mathbf{G}_1$ whose connected component $\mathbf{G}_1^\circ$ is reductive, and the identification $\widetilde{G}_L = \widetilde{G}_1$ induces an identification $G_{L_\circ} = G_1$.

Remark 4.1. The form h_1 constructed above depends on the choices of M and s. The simple right D_L-module M is unique up to isomorphism. Now suppose M is fixed, and let $s' : M^\sigma \to M^*$ be another admissible isomorphism of D_L-modules. Then $s' = \lambda s$ for an element λ in $L^\times$ such that $\lambda^{\sigma_1} = \mu\lambda$, $\mu \in \{\pm 1\}$, and the non-degenerate σ_1-skew form $h_1' : V_1 \times V_1 \to L$ defined by s_1' is given by $h_1' = \lambda h_1$; it is ε_1'-hermitian, $\varepsilon_1' = \varepsilon_1\mu$. So h_1 is well defined up to a "change of coordinates" in the sense of [BT1], 10.1.3.

Now let us consider the *second case*: $L \simeq (L_\circ)^2$. As in 2.2, we assume $F \subset L_\circ$ and $L = (L_\circ)^2$. Hence $L_\circ$ is an extension of even degree of F. Using the notation and identifications of 2.2, we have $D_L = D_{L_\circ,1} \times D_{L_\circ,2}$ where $D_{L_\circ,i} = \xi_i D_L$ is a central simple $L_\circ$-algebra (recall that ξ_1, ξ_2 are the two minimal idempotents of L). The involution σ ($= \mathrm{Id} \otimes \sigma$) on D_L induces an anti-isomorphism $D_{L_\circ,1} \to D_{L_\circ,2}$ (resp. $D_{L_\circ,2} \to D_{L_\circ,1}$), still denoted by σ. For $a \in D_{L_\circ,i}$, we have $(a^\sigma)^\sigma = a$, and for $(a,b) \in D_L$, we have $(a,b)^\sigma = (b^\sigma, a^\sigma)$. We also have

$$\mathrm{End}_{D_L}(V_L) = \mathrm{End}_{D_{L_\circ,1}}(V_{L_\circ,1}) \times \mathrm{End}_{D_{L_\circ,2}}(V_{L_\circ,2})$$

whith $\mathrm{End}_{D_{L_\circ,i}}(V_{L_\circ,i}) = \xi_i \mathrm{End}_{D_L}(V_L)$. Moreover the map $v_1 \mapsto h_L(v_1, \cdot)$ induces a σ-isomorphism φ from the right $D_{L_\circ,1}$-module $V_{L_\circ,1}$ to the left $D_{L_\circ,2}$-module $\mathrm{Hom}_{D_{L_\circ,2}}(V_{L_\circ,2}, D_{L_\circ,2})$. For $g = (g_1, g_2) \in \widetilde{G}_L = \mathrm{Aut}_{D_{L_\circ,1}}(V_{L_\circ,1}) \times \mathrm{Aut}_{D_{L_\circ,2}}(V_{L_\circ,2})$, we have $g \in G_{L_\circ}$ if and only if

$$g_2 = {}^{\mathrm{t}}\varphi^{-1} \circ {}^{\mathrm{t}}g_1^{-1} \circ \varphi^{\mathrm{t}};$$

so the map $\widetilde{G}_L \to \mathrm{Aut}_{D_{L_\circ,1}}(V_{L_\circ,1})$, $(g_1, g_2) \mapsto g_1$ by restriction gives an isomorphism of groups $\iota : G_{L_\circ} \to \mathrm{Aut}_{D_{L_\circ,0}}(V_{L_\circ,1})$. Let us choose a simple *right* $D_{L_\circ,1}$-module M_1 (since $D_{L_\circ,1}$ is a quotient of D_L, M is also a simple D_L-module). Put $V_1 = \mathrm{Hom}_{D_{L_\circ,1}}(M, V_{L_\circ,1})$. From the previous case, we have a canonical identification of $G_1 = \mathrm{Aut}_{L_\circ}(V_1)$ with $\mathrm{Aut}_{D_{L_\circ,0}}(V_{L_\circ,1})$. Hence we obtain an isomorphism of groups $\iota : G_{L_\circ} \to G_1$ which is defined over $L_\circ$.

4.2. Unramified descent: buildings. Suppose moreover that the extension $L_\circ/F_\circ$ is unramified (we could also suppose that the reductive $L_\circ$-group $\mathbf{G}^\circ \times_{F_\circ} L_\circ$ is quasi-split, or even residually split, but this is not necessary). Hence L is field if and only if the extension $F/F_\circ$ is trivial or totally ramified. Let $\mathcal{I} = \mathcal{I}(\mathbf{G}, F_\circ)$ be the (non-enlarged) building of G — it can be viewed as the building of a valuated root datum as in 1.5, cf. [BT4] 1.14, 1.15 —, and let $\mathcal{I}_{L_\circ} = \mathcal{I}(\mathbf{G} \times_{F_\circ} L_\circ, L_\circ)$ be that of $G_{L_\circ}$. From [BT4] 4.1, there exists a unique G-equivariant and affine map $\mathcal{I} \to \mathcal{I}_{L_\circ}$ whose image is the subset $(\mathcal{I}_{L_\circ})^\Gamma$ formed of those points which are fixed by the Galois group $\Gamma = \mathrm{Gal}(L_\circ/F_\circ)$.

Let $\widetilde{\mathcal{I}}^1$ be the enlarged building of $\widetilde{G}$, $\mathcal{N}^1$ be the set of (D-)norms on V ([BT3] 1.1), and $\mathcal{L}^1$ be the set of ($\mathfrak{o}_D$)-lattice-functions in V ([BL] 2.1), where $\mathfrak{o}_D$ denotes the ring of integers of D. From [BT4] and [BL], by replacing F by D, the results of 1.6 remain true. In particular, there exists a $\widetilde{G}$-equivariant affine map $\tilde{\mathcal{I}}^1 \to \mathcal{L}^1$, $p \mapsto \Lambda_p$, which is bijective and unique up to translation by a real number; where the affine structure on $\mathcal{L}^1$ is given by that on $\mathcal{N}^1$ via the $\widetilde{G}$-equivariant bijection $\mathcal{N}^1 \to \mathcal{L}^1$, $\alpha \mapsto \Lambda_\alpha$ defined by

$$\Lambda_\alpha(r) = \{v \in V : \alpha(v) \geq r\}, \quad r \in \mathbb{R}.$$

This induces a bijective, G-equivariant and affine map $\mathcal{I} \to \mathcal{L}^1_{h,q}$, which is the unique G-equivariant affine map from $\mathcal{I}$ to $\mathcal{L}^1_{h,q}$; where $\mathcal{L}^1_{h,q}$ is the G-stable convex subset of $\mathcal{L}^1$ corresponding to the set of MM-norms for (h,q) via the bijection $\mathcal{N}^1 \to \mathcal{L}^1$, $\alpha \mapsto \Lambda_\alpha$. In particular, $\mathcal{I}$ identifies with a G-stable convex subset of the building $\tilde{\mathcal{I}}^1$.

Let denote by $\mathcal{I}_{L_\circ,1} = \mathcal{I}(\mathbf{G}_1, L_\circ)$ the building of G_1.

If L is a field, the identification $G_{L_\circ} = G_1$ gives an identification $\mathcal{I}_{L_\circ} = \mathcal{I}_{L_\circ,1}$. The actions of Γ on $G_{L_\circ}$ and $\mathcal{I}_{L_\circ}$ give some actions on G_1 and $\mathcal{I}_{L_\circ,1}$. Note that we also have an action of Γ on $\mathrm{End}_{D_1}(V_1) = \mathrm{End}_{D_L}(V_L)$ — even if Γ does not act on V_1, nor on D_1.

Now if $L = L_\circ^2$, the isomorphism $\iota : G_{L_\circ} \to G_1$ gives a bijection $\mathcal{I}_{L_\circ} \to \mathcal{I}_{1,L_\circ}$, still denoted by ι. The action of Γ on $G_{L_\circ} \subset \mathrm{Aut}_{D_{L_\circ,1}}(V_{L_\circ,1}) \times \mathrm{Aut}_{D_{L_\circ,2}}(V_{L_\circ,2})$ is described as in Remark 2.1 (cf. [BT4] 1.13, Remarque). In particular, the subgroup $\Gamma' = \mathrm{Gal}(L_\circ/F)$ of Γ acts on $G_1 = \mathrm{Aut}_{D_{L_\circ,1}}(V_{L_\circ,1})$, and for $\gamma \in \Gamma \smallsetminus \Gamma'$ and $g = (g_1, g_2) \in G_{L_\circ}$, we have

$$g^\gamma = (g_2^\gamma, g_1^\gamma) = (({}^{\mathrm{t}}\varphi^{-1} \circ {}^{\mathrm{t}}g_1{}^{-1} \circ \varphi^{\mathrm{t}})^\gamma, g_1^\gamma);$$

here the automorphism $\gamma \otimes \mathrm{Id}$ of $\mathrm{End}_{D_L}(V_L) = L_\circ \otimes_{F_\circ} \mathrm{End}_D(V)$, denoted by $g \mapsto g^\gamma$, induces two γ-isomorphisms $\mathrm{End}_{D_{L_\circ,1}}(V_{L_\circ,1}) \to \mathrm{End}_{D_{L_\circ,2}}(V_{L_\circ,2})$ and $\mathrm{End}_{D_{L_\circ,2}}(V_{L_\circ,2}) \to \mathrm{End}_{D_{L_\circ,1}}(V_{L_\circ,1})$, still denoted by $g \mapsto g^\gamma$. This makes the action of Γ on G_1, identified with $G_{L_\circ}$ via ι, explicit. We also have an action of Γ on $\mathcal{I}_{L_\circ,1}$, identified with $\mathcal{I}_{L_\circ}$ via ι.

In both cases, we obtain a bijection $\mathcal{I} \to (\mathcal{I}_{L_\circ,1})^\Gamma$, which is the unique G-equivariant and affine map $\mathcal{I} \to \mathcal{I}_{L_\circ,1}$, and can be described in terms of norms (or lattice-functions) as in 2.3.

4.3. Filtrations of the Lie algebra. Put $\tilde{\mathfrak{g}} = \mathrm{End}_D(V)$ and $\mathfrak{g} = \mathrm{Lie}(G)$. For $p \in \tilde{\mathcal{I}}^1$, denote by $\tilde{\mathfrak{g}}_{p,\cdot}$ the square $\mathfrak{o}_D$-lattice-function on V defined by (cf. 1.7)

$$\tilde{\mathfrak{g}}_{p,r} = \mathrm{End}(\Lambda_p)(r), \quad r \in \mathbb{R};$$

it depends only on the projection of p to the non-enlarged building $\tilde{\mathcal{I}}$ of $\widetilde{G}$. For $p \in \mathcal{I}$, let $\mathfrak{g}_{p,\cdot}$ be the $\mathfrak{o}_{F_\circ}$-lattice function in $\mathfrak{g}$ defined by

$$\mathfrak{g}_{p,r} = \tilde{\mathfrak{g}}_{p,r} \cap \mathfrak{g}, \quad r \in \mathbb{R},$$

and let $(\mathfrak{g}^{\mathrm{MP}}_{p,r})_{r\in\mathbb{R}}$ the Moy-Prasad filtration of $\mathfrak{g}$ attached to p. We claim that Theorem 1.8 (proved for $D = F$) remains true for $D \neq F$: for all $p \in \mathcal{I}$, we have

$$\mathfrak{g}_{p,r} = \mathfrak{g}^{\mathrm{MP}}_{p,r}, \quad r \in \mathbb{R}. \tag{1}$$

Put $\tilde{\mathfrak{g}}_L = \mathrm{End}_{D_L}(V_L)$ and $\mathfrak{g}_L = \mathrm{Lie}(G_{L_\circ})$. Thus we have $\tilde{\mathfrak{g}}_L = L_\circ \otimes_{F_\circ} \mathfrak{g} = L \otimes_F \mathfrak{g}$ and $\mathfrak{g}_L = L_\circ \otimes_{F_\circ} \mathfrak{g}$. Put also $\tilde{\mathfrak{h}} = \mathrm{End}_{L_\circ}(V_1)$ and $\mathfrak{h} = \mathrm{Lie}(G_1)$. The identification $\tilde{\mathfrak{g}}_L = \tilde{\mathfrak{h}}$ induces an identification $\mathfrak{g}_L = \mathfrak{h}$, and an action of Γ on $\mathfrak{h}$ (cf. 4.2) such that $\mathfrak{h}^\Gamma = \mathfrak{g}$. For $p \in \mathfrak{I}_{L_\circ,1}$, denote by $\mathfrak{h}_{p,\cdot}$ the square $\mathfrak{o}_{L_\circ}$-lattice-function on V_1 defined as in 1.7 if L is a field, and as in [BL] if $L = (L_\circ)^2$. Let also $(\mathfrak{h}_{p,r}^{\mathrm{MP}})_{r\in\mathbb{R}}$ be the filtration of $\mathfrak{h}$ attached to p by Moy and Prasad.

Let p be a point in $\mathfrak{I}$, identified with a point in $(\mathfrak{I}_{L_\circ,1})^\Gamma$ via the canonical bijection $\mathfrak{I} \to (\mathfrak{I}_{L_\circ,1})^\Gamma$ (cf. 4.2). By construction, we have the descent property:

$$(\mathfrak{h}_{p,r}^{\mathrm{MP}})^\Gamma = \mathfrak{g}_{p,r}^{\mathrm{MP}}, \quad r \in \mathbb{R}.$$

From [BL] and Theorem 1.8, we have

$$\mathfrak{h}_{p,r} = \mathfrak{h}_{p,r}^{\mathrm{MP}}, \quad r \in \mathbb{R}$$

So to obtain (1), we just need to prove the descent property:

$$(\mathfrak{h}_{p,r})^\Gamma = \mathfrak{g}_{p,r}, \quad r \in \mathbb{R}. \tag{2}$$

This can be done following 2.4 (the proof is essentially the same, details are left to the reader). So Theorem 1.8 is true even if $D \neq F$.

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