Journal of Lie Theory
Volume **19** (2009) 735–766

Pairs of Lie Algebras and their Self-Normalizing Reductive Subalgebras

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Communicated by S. Gindikin

Abstract. We consider a class $\mathcal{P}$ of pairs $(\mathfrak{g}, \mathfrak{g}_1)$ of $\mathbb{K}$-Lie algebras $\mathfrak{g}_1 \subset \mathfrak{g}$ satisfying certain "rigidity conditions"; here $\mathbb{K}$ is a field of characteristic 0, $\mathfrak{g}$ is semisimple, and $\mathfrak{g}_1$ is reductive. We provide some further evidence that $\mathcal{P}$ contains a number of nonsymmetric pairs that are worth studying; e.g., in some branching problems, and for the purposes of the geometry of orbits. In particular, for an infinite series $(\mathfrak{g}, \mathfrak{g}_1) = (\mathfrak{sl}(n+1), \mathfrak{sl}(2))$ we show that it is in $\mathcal{P}$, and precisely describe a $\mathfrak{g}_1$-module structure of the Killing-orthogonal $\mathfrak{p}(n)$ of $\mathfrak{g}_1$ in $\mathfrak{g}$. Using this and the Kostant's philosophy concerning the exponents for (complex) Lie algebras, we obtain two more results. First; suppose $\mathbb{K}$ is algebraically closed, $\mathfrak{g}$ is semisimple all of whose factors are classical, and $\mathfrak{s}$ is a principal TDS. Then $(\mathfrak{g}, \mathfrak{s})$ belongs to $\mathcal{P}$. Second; suppose $(\mathfrak{g}, \mathfrak{g}_1)$ is a pair satisfying certain technical condition (**C**), and there exists a semisimple $\mathfrak{s} \subseteq \mathfrak{g}_1$ such that $(\mathfrak{g}, \mathfrak{s})$ is from $\mathcal{P}$ (e.g., $\mathfrak{s}$ is a principal TDS). Then $(\mathfrak{g}, \mathfrak{g}_1)$ is from $\mathcal{P}$ as well. Finally, given a pair $(\mathfrak{g}, \mathfrak{g}_1)$, we have some useful observations concerning the relationship between the coadjoint orbits corresponding to $\mathfrak{g}$ and $\mathfrak{g}_1$, respectively.
Mathematics Subject Classification 2000: 17B05, 17B10, 17B20.
Key Words and Phrases: Pair of Lie algebras, semisimple Lie algebra, reductive subalgebra, self-normalizing subalgebra, principal nilpotent element, principal TDS, trivial extension.

Introduction

Unless specified otherwise, throughout this paper all Lie algebras are finite-dimensional and defined over a field $\mathbb{K}$ of characteristic zero. Let now $\mathfrak{g}$ be a Lie algebra; by $B_{\mathfrak{g}}$ we denote its Killing form. Let $\mathfrak{g}_1$ be a proper subalgebra such that we have the following:

(**C**) *The restriction of $B_{\mathfrak{g}}$ to $\mathfrak{g}_1$ is nondegenerate.*

There are many instances when such pairs arise; in particular when $\mathfrak{g}$ is semisimple and $\mathfrak{g}_1$ is reductive. For example, in some branching problems, when we want

* The author was supported in part by the Ministry of Science, Education and Sports, Republic of Croatia, Grant No. 037-0372781-2811.

ISSN 0949–5932 / \$2.50

to decompose certain restrictions, to $\mathfrak{g}_1$, of some (irreducible) representations of $\mathfrak{g}$. Also, concerning the geometry of orbits, it is interesting to understand a relationship between the (nilpotent) (co)adjoint orbits for the pairs $(\mathfrak{g}, \mathfrak{g}_1)$ under consideration. Some other questions are of structure-theoretic nature. We would like to better understand the embedding $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$, to find some interesting intermediate subalgebras $\mathfrak{g}_1 \subseteq \mathfrak{r} \subseteq \mathfrak{g}$, or just subalgebras of $\mathfrak{g}$ that are, loosely speaking, germane to $\mathfrak{g}_1$; cf. Proposition 5.1. But for $(\mathfrak{g}, \mathfrak{g}_1)$ as above, it is clear that in order to be able to say something interesting about the correspondences of the considered objects related to $\mathfrak{g}$ and to $\mathfrak{g}_1$ (e.g., representations and orbits) the embedding of $\mathfrak{g}_1$ in $\mathfrak{g}$ must be in a certain sense "rigid". It turns out that the following two additional conditions will define a class of pairs $(\mathfrak{g}, \mathfrak{g}_1)$ which is suitable for research:

(**Q1**) *For any Cartan subalgebra* $\mathfrak{h}_1$ *of* $\mathfrak{g}_1$ *there exists a* unique *Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{h}_1 \subseteq \mathfrak{h}$;

(**Q2**) $\mathfrak{g}_1$ *is* self-normalizing *in* $\mathfrak{g}$.

For our needs here we will denote this class by $\boldsymbol{\mathcal{P}}$. There are many interesting pairs in it. First, it is well known that $\boldsymbol{\mathcal{P}}$ contains all the pairs $(\mathfrak{g}, \mathfrak{g}_1)$, where $\mathfrak{g}$ is semisimple, which are either symmetric or such that $\mathfrak{g}_1$ is a Cartan subalgebra of $\mathfrak{g}$. Also, suppose that σ is an automorphism, of a semisimple $\mathfrak{g}$, of *prime* order m, and that the field $\mathbb{K}$ contains a primitive m-th root of unity (e.g., $\mathbb{K}$ algebraically closed). Define $\mathfrak{g}_1$ to be the fixed point algebra for σ. Then the main result of [Š1] can be formulated as follows: $(\mathfrak{g}, \mathfrak{g}_1)$ *belongs to* $\boldsymbol{\mathcal{P}}$. This generalizes the case of symmetric pairs, i.e., the case $m = 2$. But the most interesting fact about our class of pairs is an observation which in a rough form states that there are many other (nonsymmetric) pairs within $\boldsymbol{\mathcal{P}}$ that are worth studying. This observation, in a more or less implicit form, seems to be first noticed and explored by R. K. Brylinski, B. Kostant, T. Levasseur, S. P. Smith and D. A. Vogan; see [BK], [LS], [V1] and [V2].

As we already mentioned, a number of pairs $(\mathfrak{g}, \mathfrak{g}_1)$ arise in branching problems. Very often such pairs are symmetric and/or such that $\mathfrak{g}_1$ is a maximal reductive subalgebra of $\mathfrak{g}$. For some recent important results about branching, both for Lie groups and Lie algebras, see [EHW], [HTW], [Kn2], [Ko1], [Ko2], [Ks4] and [V2].

For $(\mathfrak{g}, \mathfrak{g}_1)$ satisfying the condition (**C**), define $\mathfrak{p}$ to be the Killing-orthogonal of $\mathfrak{g}_1$ in $\mathfrak{g}$; thus we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$. Let now $\mathfrak{g}^n = \mathfrak{sl}(n+1)$, and $\rho_n : \mathfrak{sl}(2) \to \mathfrak{g}^n$ be the unique $(n+1)$-dimensional irreducible representation. Define $\mathfrak{g}_1^n$ to be the image of ρ_n, a Lie algebra isomorphic to $\mathfrak{sl}(2)$. The theorem given below, which is our first main result, is concerned with the infinite series of pairs $(\mathfrak{g}, \mathfrak{g}_1) = (\mathfrak{g}^n, \mathfrak{g}_1^n)$.

Theorem 0.1. *We have the following:*

(I) All the pairs $(\mathfrak{g}^n, \mathfrak{g}_1^n)$ *are from the class* $\boldsymbol{\mathcal{P}}$.

(II) A $\mathfrak{g}_1^n$*-module structure of* $\mathfrak{p}(n)$ *is given by*

$$\mathfrak{p}(n) = V^1_{4\varpi} \oplus \cdots \oplus V^1_{2n\varpi} = V^1_{(n+2)\varpi} \otimes V^1_{(n-2)\varpi}.$$

For the notation in the theorem, we have: $\mathfrak{g}^n = \mathfrak{g}_1^n \oplus \mathfrak{p}(n)$ is the corresponding decomposition, ϖ is the fundamental weight for $\mathfrak{sl}(2)$, and $V_{k\varpi}^1$ is a simple finite-dimensional $\mathfrak{g}_1^n$-module with highest weight $k\varpi$. Notice also that for $n > 2$ our pairs are both *nonsymmetric* and such that $\mathfrak{g}_1$ is *not* a maximal subalgebra of $\mathfrak{g}$.

The claim (II) above might be understood as a part of the first step toward a more general problem of decomposing the restrictions $\rho_{|\mathfrak{g}_1}$ of any (finite-dimensional) irreducible representation ρ of $\mathfrak{g}$, for various pairs $(\mathfrak{g}, \mathfrak{g}_1)$ in $\boldsymbol{\mathcal{P}}$; see Remark 3.3. In particular, for $\mathbb{K} = \mathbb{C}$, it computes the exponents of $\mathrm{SL}(n, \mathbb{C})$. Notice that these exponents, and much more, has already been found in the seminal Kostant's paper [Ks1]; see also Sect. 4.4 in [CM]. But we think that our constructive, and purely algebraic, approach might result with a new insight in branching problems for various pairs $(\mathfrak{g}, \mathfrak{g}_1)$. (Notice also that quite analogous conclusions might be obtained for a base field $\mathbb{K}$ of characteristic $p > 0$, with p big enough; see Sect. 2 in [Š4], and Remarks 1.4 and 3.3 in the present paper.) For a circle of related ideas here we have to mention another fundamental work of Kostant [Ks2] in which he, among other things, studied the generalized exponents. From the vast literature concerning these let us mention just [Br], [JLZ] and [L], where one can find some important ideas and/or results.

The following theorem, which relies on the previous one, is our second main result. It is a generalization of the part (I) above, for the case of algebraically closed base field. Let it be said how we strongly believe that the same statement holds without the phrase “all of whose simple factors are classical”, as it is the case when $\mathbb{K} = \mathbb{C}$; see Theorem 4.6, and also the paragraph preceding Question 4.8. (Notice that Theorem 4.6 is in fact more or less an easy consequence of the profound Kostant's research in [Ks1].)

Theorem 0.2. *Suppose $\mathbb{K}$ is algebraically closed. Let $\mathfrak{g}$ be semisimple all of whose simple factors are classical Lie algebras, and let $\mathfrak{s}$ be a principal TDS. Then the pair $(\mathfrak{g}, \mathfrak{s})$ is from the class $\boldsymbol{\mathcal{P}}$.*

The next theorem, which is our third main result, explains how to find some new pairs within $\boldsymbol{\mathcal{P}}$, via those we already have. (For just one simple situation when the theorem applies see Examples 3.6 and 3.7; now we have $\mathfrak{g} = \mathfrak{sl}(4)$, $\mathfrak{g}_1 \cong \mathfrak{sp}(4)$ and $\mathfrak{s} \cong \mathfrak{sl}(2)$.) Notice how this theorem gives more credit to the previous one; and in particular it emphasize the role of principal TDS (in accordance with the general Kostant's philosophy). Roughly speaking, the theorem says the following: Having a “convenient” pair $(\mathfrak{g}, \mathfrak{g}_1)$, in order to see that it is in $\boldsymbol{\mathcal{P}}$, we have to find a “small” subalgebra $\mathfrak{s} \subseteq \mathfrak{g}_1$ such that $(\mathfrak{g}, \mathfrak{s})$ belongs to $\boldsymbol{\mathcal{P}}$.

Theorem 0.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{g}_1$ a reductive subalgebra satisfying the following two assumptions:*

(1) The pair $(\mathfrak{g}, \mathfrak{g}_1)$ satisfies the condition $(\mathbf{C})$ (e.g., $\mathfrak{g}_1$ is absolutely simple);

(2) There exists a semisimple subalgebra $\mathfrak{s} \subseteq \mathfrak{g}_1$ such that the pair $(\mathfrak{g}, \mathfrak{s})$ satisfies both $(\mathbf{Q1})$ and $(\mathbf{Q2})$.

Then the pair $(\mathfrak{g}, \mathfrak{g}_1)$ is from the class $\boldsymbol{\mathcal{P}}$.

In order to find even more pairs of Lie algebras that belong to the class $\boldsymbol{\mathcal{P}}$, we first have to find out which pairs satisfy the above condition ($\mathbf{C}$). (Although not very restrictive, as we will see, it is in fact a very strong requirement imposed on pairs.) Suppose now that $\mathbb{K}$ is algebraically closed. Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra. Let $\mathfrak{h} \subseteq \mathfrak{q} \subseteq \mathfrak{g}$ be a (standard) parabolic subalgebra. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a Levi decomposition, with $\mathfrak{l}$ reductive and $\mathfrak{u}$ nilpotent. It is clear that the latter direct sum is in fact $B_{\mathfrak{g}}$-orthogonal. But moreover we have the following auxiliary result which points out at some new pairs satisfying ($\mathbf{C}$). It is an immediate consequence of Proposition 2.2 and Lemma 1.2; see also [B2], Ch. VII, Sect. 2.1, Cor. 4.

Proposition 0.4. *The restriction of the Killing form $B_{\mathfrak{q}}$ to $\mathfrak{l}$ is nondegenerate. Therefore, the sum $\mathfrak{l} \oplus \mathfrak{u}$ is also $B_{\mathfrak{q}}$-orthogonal. Furthermore, the pair $(\mathfrak{q}, \mathfrak{l})$ is from the class $\boldsymbol{\mathcal{P}}$.*

Let us now explain the organization of the paper. Section 1 is preliminary. There we first recall some facts about pairs $(\mathfrak{g}, \mathfrak{g}_1)$ that we consider. Then we introduce a natural map $\mathcal{E} : \mathfrak{g}_1^* \to \mathfrak{g}^*$, which we call the trivial extension, and then provide a useful auxiliary observation concerning the relationship between the ad^*-orbits on $\mathfrak{g}_1^*$ and $\mathfrak{g}^*$, respectively. This is of course a precursor for the corresponding result on the coadjoint orbits for groups. In Section 2 we give more details concerning the condition ($\mathbf{C}$). The central result there is Proposition 2.2. Section 3 studies the pairs $(\mathfrak{sl}(n+1), \mathfrak{sl}(2))$. There we provide a (detailed) proof of Theorem 3.2; it is a more precise version of Theorem 0.1. We also include a useful observation on compatible Borel subalgebras; see Examples 3.6 and 3.7. The aim of Section 4 is to prove the last two theorems. First, after one result which is interesting in its own right (Proposition 4.1), we prove Theorem 0.3; see also Definition 4.5. Subsections 4.1, 4.2 and 4.3, which among other things discuss Question 4.8, provide a slightly more than we need for a proof of Theorem 0.2. As an illustration we also consider one “exceptional situation”; i.e., in Subsection 4.4 we treat the case when $\mathfrak{g}$ is of type G_2. Section 5 contains two further results. Firstly, given a pair $(\mathfrak{g}, \mathfrak{g}_1)$, for any $\mu \in \mathfrak{g}_1^*$ we define a certain subspace $\mathfrak{s}(\mu)$ of $\mathfrak{g}$. It turns out that $\mathfrak{s}(\mu)$ is a subalgebra of $\mathfrak{g}$; see Proposition 5.1. The parametrized family $(\mathfrak{s}(\mu) : \mu \in \mathfrak{g}_1^*)$ of subalgebras can be useful while studying both the representations and orbits corresponding to the considered pair $(\mathfrak{g}, \mathfrak{g}_1)$. The second result, Proposition 5.4, states that the trivial extension $\mathcal{E}$ preserves both the semisimple and nilpotent functionals.

1. Notation, conventions and preliminaries

For $N \in \mathbb{N}$, by E_{ij} (or $E_{i,j}$) we denote the N-by-N matrix having 1 in the $(i,j)^{\text{th}}$ place and 0 elsewhere. For a field $\mathbb{K}$, by $\overline{\mathbb{K}}$ we denote its algebraic closure.

Suppose that $\mathfrak{g}$ is a $\mathbb{K}$-Lie algebra. If $\mathfrak{s}$ is its subalgebra, by $N_{\mathfrak{g}}(\mathfrak{s})$ we denote the normalizer of $\mathfrak{s}$ in $\mathfrak{g}$. Also, for a $\mathfrak{g}$-module V, by $V^{\mathfrak{s}}$ we denote the subspace of $\mathfrak{s}$-invariants in V. If $\mathfrak{g}$ is reductive and $\mathfrak{h}$ is a split Cartan subalgebra of $\mathfrak{g}$, by $\Delta(\mathfrak{g}, \mathfrak{h})$ we denote the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For every root

γ, by X_γ we denote a nonzero vector from the corresponding root subspace $\mathfrak{g}_\gamma$.

Given $\mathfrak{g}$ as above, we define $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \overline{\mathbb{K}}$. Recall also that a Lie algebra $\mathfrak{g}$ is called *absolutely simple* if $\overline{\mathfrak{g}}$ is simple.

Suppose that G is a (connected) linear algebraic $\mathbb{K}$-group. Let $\mathfrak{g}$ be the Lie algebra of G. For the coadjoint representation $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$, and its derived representation $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$, we will use the usual "dot-notation". That is, given $g \in G$, $X \in \mathfrak{g}$ and $\varphi \in \mathfrak{g}^*$ we write $g.\varphi$ and $X.\varphi$ for the corresponding coadjoint actions.

By $\{\boldsymbol{h}, \boldsymbol{e}, \boldsymbol{f}\}$ we will denote the usual basis of $\mathfrak{sl}(2, \mathbb{K})$, i.e.,

$$\boldsymbol{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In what follows we will consider certain pairs of $\mathbb{K}$-Lie algebras $(\mathfrak{g}, \mathfrak{g}_1)$. Here $\mathfrak{g}_1$ is a proper subalgebra of $\mathfrak{g}$. Since the trivial situations $(\mathfrak{s} \times \mathfrak{s}', \mathfrak{s})$, where $\mathfrak{s}$ and $\mathfrak{s}'$ are semisimple Lie algebras, will not be considered, we assume the following condition for pairs $(\mathfrak{g}, \mathfrak{g}_1)$:

(**P**) $\mathfrak{g}_1$ *is not an ideal of* $\mathfrak{g}$.

In particular such a pair is called *symmetric* if $\mathfrak{g}_1 = \mathfrak{g}^\theta$, for some involutive automorphism θ of $\mathfrak{g}$; and *nonsymmetric* otherwise. (Let it be said that our symmetric pairs might be with $\mathfrak{g}$ non-semisimple; see, e.g., Example 2.1.) We then say that $\mathfrak{g}_1$ is a (non)symmetric subalgebra of $\mathfrak{g}$. Now, denote by $\boldsymbol{\beta}$ the restriction of the Killing form $B_\mathfrak{g}$ to $\mathfrak{g}_1$. As it was mentioned before, we are interested in such pairs $(\mathfrak{g}, \mathfrak{g}_1)$ which also satisfy the condition (**C**): $\boldsymbol{\beta}$ *is a nondegenerate form.*

As we already stated in the Introduction, there are many pairs satisfying the condition (**C**); i.e., it is not very restrictive. The following easy statement will be useful below ([Š1], Cor. 1.; cf. [B1], Ch. I, §6, Sect. 10).

Lemma 1.1. *Let $(\mathfrak{g}, \mathfrak{g}_1)$ be a pair where $\mathfrak{g}_1$ is an absolutely simple Lie algebra. Then there exists a nonzero $q \in \mathbb{K}$ such that $\boldsymbol{\beta} = qB_{\mathfrak{g}_1}$; in particular, $(\mathfrak{g}, \mathfrak{g}_1)$ satisfies the condition* (**C**).

For $(\mathfrak{g}, \mathfrak{g}_1)$ as above, let $r : \mathfrak{g}^* \to \mathfrak{g}_1^*$ be the restriction map between the duals. Let $\kappa : \mathfrak{g} \to \mathfrak{g}^*$ be the Killing homomorphism. Define also an isomorphism $\kappa_1 : \mathfrak{g}_1 \to \mathfrak{g}_1^*$ given by $\kappa_1(x_1) = \boldsymbol{\beta}(x_1, .)$, for $x_1 \in \mathfrak{g}_1$. Then define a (linear) map $\pi : \mathfrak{g} \to \mathfrak{g}_1$ satisfying $\kappa_1 \circ \pi = r \circ \kappa$. This π is a $\mathfrak{g}_1$-module homomorphism; we call it the *associated homomorphism* of $(\mathfrak{g}, \mathfrak{g}_1)$. For the convenience of the reader and later needs we state the following two lemmas; they recall some basic observations from [Š3] and [Š4].

Lemma 1.2. *Define a vector subspace $\mathfrak{p} = \ker \pi$. Then $\mathfrak{p}$ is the Killing-orthogonal of $\mathfrak{g}_1$ in $\mathfrak{g}$, and thus we have a direct sum decomposition*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}.$$

We also have $[\mathfrak{g}_1, \mathfrak{p}] \subseteq \mathfrak{p}$.

A pair $(\mathfrak{g}, \mathfrak{g}_1)$, with $\mathfrak{g}$ semisimple, will be called an *irreducible* pair if $\mathfrak{p}$ is simple under $\operatorname{ad} \mathfrak{g}_1$-action; otherwise it will be called *reducible.*

Lemma 1.3. *We have the following:*

(i) A pair $(\mathfrak{g}, \mathfrak{g}_1)$ is symmetric if and only if $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{g}_1$.

(ii) Supposing the Killing form $B_{\mathfrak{g}}$ is nondegenerate, we have $\mathfrak{p}^{\mathfrak{g}_1} = 0$ if and only if $N_{\mathfrak{g}}(\mathfrak{g}_1) = \mathfrak{g}_1$.

(iii) If $(\mathfrak{g}, \mathfrak{g}_1)$ is irreducible, then $N_{\mathfrak{g}}(\mathfrak{g}_1) = \mathfrak{g}_1$ and $\mathfrak{g}_1$ is a maximal proper subalgebra of $\mathfrak{g}$.

Remark 1.4. (1) A notion of symmetric/nonsymmetric pairs can be generalized in a straightforward manner even for $\mathrm{char}(\mathbb{K}) > 0$; see [Š4] for more details. More precisely, one can again consider a class of pairs $(\mathfrak{g}, \mathfrak{g}_1)$ of $\mathbb{K}$-Lie algebras that satisfy only the conditions $(\mathbf{C})$ and $(\mathbf{P})$. But now, in the positive characteristic setting, we have to be more careful. For instance, if we consider a pair $(\mathfrak{g}, \mathfrak{g}_1) = (\mathfrak{g}^{\langle n \rangle}, \mathfrak{g}_1^{\langle n \rangle})$ defined as in Section 3 below, then for n such that $p = n+1$ is a prime, and $\mathrm{char}(\mathbb{K}) = p$, we have the following: The form $B_{\mathfrak{g}}$ is nondegenerate, while $\boldsymbol{\beta}$ is degenerate.

(2) Of course, in $\mathrm{char}(\mathbb{K}) = 0$, $\mathfrak{g}$ is semisimple if and only if $B_{\mathfrak{g}}$ is nondegenerate. The part (ii) of the lemma is formulated so that it holds and for $\mathbb{K}$ of positive characteristic.

Given a pair $(\mathfrak{g}, \mathfrak{g}_1)$, an interesting question is how are related the coadjoint orbits on the corresponding duals. For what follows it will be helpful to start with the next easy lemma. It is a generalized version of the "trivial implication" of Theorem 2.2 in [BK].

Lemma 1.5. *Let $\mathfrak{G}$ be the semidirect product of a subalgebra $\mathfrak{G}_1$ and an ideal $\mathfrak{A}$ of $\mathfrak{G}$. Suppose that $\mathfrak{A}^{\mathfrak{G}_1} = 0$; this is equivalent to $N_{\mathfrak{G}}(\mathfrak{G}_1) = \mathfrak{G}_1$. Then for any $\gamma \in \mathfrak{G}^*$, such that the restriction $\gamma_{|\mathfrak{A}} = 0$, we have $\mathfrak{G}.\gamma = \mathfrak{G}_1.\gamma$.*

Proof. Any $X \in \mathfrak{G}$ decompose as $X = X_1 + A$, where $X_1 \in \mathfrak{G}_1$ and $A \in \mathfrak{A}$. Now it is easy to check that $X.\gamma = X_1.\gamma$. ■

Consider now a pair $(\mathfrak{g}, \mathfrak{g}_1)$, where $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$ and $\mathfrak{g}$ is semisimple. We can ask whether it is possible to find some (nontrivial) $\gamma \in \mathfrak{g}^*$, satisfying $\gamma_{|\mathfrak{p}} = 0$, such that $\mathfrak{g}.\gamma = \mathfrak{g}_1.\gamma$.

Lemma 1.6. *(i) The vector subspaces $\pi([\mathfrak{p}, \mathfrak{p}])$ and $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ are ideals of $\mathfrak{g}_1$ and $\mathfrak{g}$, respectively.*

(ii) Suppose that $\mathfrak{g}$ or $\mathfrak{g}_1$ is simple. Then there is no nontrivial $\gamma \in \mathfrak{g}^*$, satisfying $\gamma_{|\mathfrak{p}} = 0$, such that $\mathfrak{g}.\gamma = \mathfrak{g}_1.\gamma$.

Proof. (i) Let $\mathfrak{a} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$. Then for $z_1, z_2 \in \mathfrak{p}$ decompose $[z_1, z_2] = y + w$, $y \in \mathfrak{g}_1$ and $w \in \mathfrak{p}$, and note that $y = \pi([z_1, z_2])$. Hence it is clear that $\mathfrak{a} = \mathfrak{p} + \pi([\mathfrak{p}, \mathfrak{p}])$. Now, using the Jacobi identity, we have

$$[\mathfrak{g}_1, \mathfrak{a}] \subseteq \mathfrak{p} + [\mathfrak{g}_1, \pi([\mathfrak{p}, \mathfrak{p}])] \subseteq \mathfrak{p} + \pi([\mathfrak{g}_1, [\mathfrak{p}, \mathfrak{p}]]) \subseteq \mathfrak{a},$$

and

$$[\mathfrak{p}, \mathfrak{a}] \subseteq [\mathfrak{p}, \mathfrak{p}] + [\mathfrak{p}, \mathfrak{g}_1] \subseteq \mathfrak{a}.$$

(ii) Suppose to the contrary, that such γ exists. Then for any $x \in \mathfrak{g}$ we can find $x_1 \in \mathfrak{g}_1$ so that $x.\gamma = x_1.\gamma$. Write $x = y + z$, $y \in \mathfrak{g}_1$ and $z \in \mathfrak{p}$. Then for any $w \in \mathfrak{p}$, where $x.\gamma(w) = x_1.\gamma(w)$ if and only if $\gamma([z, w]) = 0$. This means that we have $\gamma([\mathfrak{p}, \mathfrak{p}]) = 0$.

If $\mathfrak{g}$ is simple, then by (i) we have $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] = \mathfrak{g}$. Therefore $[\mathfrak{p}, \mathfrak{p}] \supseteq \mathfrak{g}_1$, and so $\gamma = 0$; a contradiction. Assume now that $\mathfrak{g}_1$ is simple. By (i), again, either $\pi([\mathfrak{p}, \mathfrak{p}])$ equals $\mathfrak{g}_1$ or 0. In the first case, $[\mathfrak{p}, \mathfrak{p}] \supseteq \mathfrak{g}_1$, as before. In the second case we conclude that $\mathfrak{p}$ is an ideal of $\mathfrak{g}$. Hence, $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$ as well, yielding to a contradiction with (**P**). ■

The previous lemma shows that the above question, asking for some γ's such that $\mathfrak{g}.\gamma = \mathfrak{g}_1.\gamma$, should be modified. Before we do this let us introduce the following useful map.

Definition 1.7. Given a pair $(\mathfrak{g}, \mathfrak{g}_1)$, define the *trivial extension*

$$\mathcal{E} : \mathfrak{g}_1^* \to \mathfrak{g}^*, \qquad \mathcal{E}(\mu)_{|\mathfrak{p}} = 0;$$

i.e., $\mathcal{E}$ extends every $\mu \in \mathfrak{g}_1^*$ trivially on $\mathfrak{p}$.

Let $\mu \in \mathfrak{g}_1^*$ be arbitrary. Define a vector subspace $T(\mu) \leq \mathfrak{p}$ by

$$T(\mu) = \mathfrak{p}^{\mathcal{E}(\mu)} \cap \mathfrak{p};$$

i.e., $T(\mu)$ is the set of all $z \in \mathfrak{p}$ satisfying $\mathcal{E}(\mu)([z, \mathfrak{p}]) = 0$.

Proposition 1.8. *For an arbitrary $\mu \in \mathfrak{g}_1^*$, we have*

$$\mathfrak{g}.\mathcal{E}(\mu) \cap \mathfrak{g}_1^* = \mathfrak{g}_1.\mu. \tag{1}$$

Proof. Define

$$\Omega = \{x \in \mathfrak{g} \mid x.\mathcal{E}(\mu)_{|\mathfrak{p}} = 0\}.$$

Clearly, Ω is a vector subspace of $\mathfrak{g}$ containing $\mathfrak{g}_1$. Let $x \in \Omega$ be arbitrary, and write $x = x_1 + z$, $x_1 \in \mathfrak{g}_1$ and $z \in \mathfrak{p}$. Then $0 = z.\mathcal{E}(\mu)(\mathfrak{p})$, i.e., $z \in T(\mu)$. Thus we have proved the equality $\Omega = \mathfrak{g}_1 \oplus T(\mu)$. Let now x and x_1 be as above. Then for any $y \in \mathfrak{g}_1$, we have $x.\mathcal{E}(\mu)(y) = \mathcal{E}(\mu)([x_1, y])$ and $x_1.\mu(y) = \mu([x_1, y])$. This proves the equality $x.\mathcal{E}(\mu)_{|\mathfrak{g}_1} = x_1.\mu$. Thus we have the inclusion from left to right in (1). The opposite one is obvious. ■

2. Pairs satisfying the condition (C)

In the previous section we said that only pairs $(\mathfrak{g}, \mathfrak{g}_1)$ satisfying the condition (C) will be interesting for us. (Notice that it is in fact a very strong condition imposed on pairs of Lie algebras.) For such pairs recall the meaning of $\mathfrak{p}$, and a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$. As a well known fact we have that there are a number of pairs $(\mathfrak{g}, \mathfrak{g}_1)$ satisfying (C). Numerous examples are available when $\mathrm{char}(\mathbb{K}) = 0$, and in particular when both $\mathfrak{g}$ and $\mathfrak{g}_1$ are semisimple; for more information and/or details see, e.g., [BK], [Ks3], [LS] and [Š3]. Here we would like to point out at some new pairs. For that purpose it might be a good idea first to look at one trivial example.

Example 2.1. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$, $\mathrm{char}(\mathbb{K}) \neq 2$. Put $\mathfrak{h} = \mathbb{K}\boldsymbol{h}$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathbb{K}\boldsymbol{e}$. Then the pair $(\mathfrak{b}, \mathfrak{h})$ satisfies the condition (C); and moreover, this pair is symmetric. For it we just have to note that $B_{\mathfrak{b}}(\boldsymbol{h}, \boldsymbol{e}) = 0 = B_{\mathfrak{b}}(\boldsymbol{e}, \boldsymbol{e})$ and $B_{\mathfrak{b}}(\boldsymbol{h}, \boldsymbol{h}) = 4$. Thus we have $\mathfrak{p} = \mathbb{K}\boldsymbol{e}$, and $\theta \in \mathrm{Aut}\,\mathfrak{b}$ is given by $\theta(\boldsymbol{h}) = \boldsymbol{h}$ and $\theta(\boldsymbol{e}) = -\boldsymbol{e}$; cf. Lemma 1.3(i).

Suppose now, for simplicity, that $\mathbb{K}$ is algebraically closed of characteristic zero. (Let us emphasize, as it will be clear from our computations below, that this assumption on $\mathbb{K}$ can be in fact relaxed. More precisely, for many concrete pairs of Lie algebras the field $\mathbb{K}$ under consideration may be arbitrary of characteristic $p \geq 0$; p odd, when positive, and sometimes sufficiently big.) Let $\mathfrak{g}$ be a semisimple $\mathbb{K}$-Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Consider a parabolic subalgebra $\mathfrak{q}$, where

$$\mathfrak{h} \subseteq \mathfrak{q} \subseteq \mathfrak{g}.$$

A more direct realization of $\mathfrak{q}$ is as follows; see, e.g., [Kn1], Ch. V, Sect. 7. For the corresponding root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, and a choice of positive roots Δ^+, take a convenient closed subset $\Delta^+ \subseteq \Gamma \subseteq \Delta$ such that

$$\mathfrak{q} = \mathfrak{q}(\Gamma) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha.$$

Furthermore, let

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u},$$

where $\mathfrak{l}$ is the Levi factor and $\mathfrak{u}$ is the nilpotent radical of $\mathfrak{q}$. The main purpose of this section is to prove the following result. (Let us emphasize that although perhaps at first glance the statement seems to be more or less obvious, the given proof shows that we have to be careful; cf. Claim 2 in Example 2.3, and Lemma 2.4, below.)

Proposition 2.2. *The restriction of $B_{\mathfrak{q}}$ to $\mathfrak{l}$ is nondegenerate; i.e., $(\mathfrak{q}, \mathfrak{l})$ satisfies the condition (C).*

Before we give a proof of this proposition, it will be instructive to look at one more example.

Example 2.3. Let $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$, and then take a Cartan subalgebra $\mathfrak{h} = \mathbb{C}h_1 \oplus \mathbb{C}h_2$, where $h_1 = \operatorname{diag}(1,-1,0)$ and $h_2 = \operatorname{diag}(0,1,-1)$. Let e_i be defined on the space of 3-by-3 diagonal matrices as $e_i(E_{jj}) = \delta_{ij}$. Put $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$; and then $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$, and $\Delta = \Delta^+ \cup \Delta^-$. For further needs note that $\alpha_1(h_1) = \alpha_2(h_2) = 2$ and $\alpha_1(h_2)\alpha_2(h_1) = -1$. Next define $\Gamma = \Delta^+ \cup \{-\alpha_1\}$, and then $\mathfrak{q} = \mathfrak{q}(\Gamma)$. A straightforward computation shows that for $B_{\mathfrak{q}}$ we have the following

Claim 1. The restriction of $B_{\mathfrak{q}}$ to $\mathfrak{h}$ is a nondegenerate form.

It is also interesting to note here the following fact:

Claim 2. There is no $c \in \mathbb{C}$ such that

$$B_{\mathfrak{q}}(u,v) = c\operatorname{Tr}(uv), \qquad \text{for } u,v \in \mathfrak{q}.$$

For this one just has to check that $B_{\mathfrak{q}}(h_1,h_1) = 10$ and $B_{\mathfrak{q}}(h_2,h_2) = 7$, while at the same time $\operatorname{Tr}(h_ih_i) = 2$ for $i = 1,2$.

Let us proceed with our example. For that purpose put $\mathfrak{l} = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1}$, the Levi factor of $\mathfrak{q}$. Also, choose $X_{\alpha_1} = E_{12}$, $X_{\alpha_2} = E_{23}$, $X_{\alpha_1+\alpha_2} = E_{13}$ and $X_{-\alpha_1} = E_{21}$. Now, for the operator $\Omega = \operatorname{ad} X_{-\alpha_1} \operatorname{ad} X_{\alpha_1}$, we have:

$$\begin{aligned}
\Omega(h_i) &= \alpha_1(h_i)h_1, \qquad \text{for } i = 1,2,\\
\Omega(X_{\alpha_1}) &= \Omega(X_{\alpha_1+\alpha_2}) = 0,\\
\Omega(X_{\alpha_2}) &= X_{\alpha_2},\\
\Omega(X_{-\alpha_1}) &= \alpha_1(h_1)X_{-\alpha_1}.
\end{aligned}$$

As a consequence it follows that

$$B_{\mathfrak{q}}(X_{-\alpha_1}, X_{\alpha_1}) = 5. \tag{2}$$

A similar computation shows that

$$B_{\mathfrak{q}}(\mathfrak{h}, \mathfrak{g}_{\alpha_1}) = 0 = B_{\mathfrak{q}}(\mathfrak{h}, \mathfrak{g}_{-\alpha_1}). \tag{3}$$

Let now $u = c^- X_{-\alpha_1} + w + c^+ X_{\alpha_1} \in \mathfrak{l}$, where $w \in \mathfrak{h}$ and $c^{\pm} \in \mathbb{C}$, be such that $B_{\mathfrak{q}}(u,\mathfrak{l}) = 0$. Using (3) and Claim 1, we deduce that $w = 0$. Hence, by (2) and the fact that $B_{\mathfrak{q}}(X_{\pm\alpha_1}, X_{\pm\alpha_1}) = 0$, it follows that

$$0 = B_{\mathfrak{q}}(u, X_{\alpha_1}) = 5c^-,$$

and thus $c^- = 0$. Similarly, $c^+ = 0$. This shows that Proposition 2.2 holds for this particular example of $\mathfrak{g}$ and $\mathfrak{q}$.

Concerning the Claim 2 above it is also useful to be aware of the following fact; for completeness and later needs we include a short argument. It clearly emphasize the difference between the Borel subalgebras and arbitrary parabolic subalgebras; for the later ones the situation is of course more complicated.

Lemma 2.4. *Let $\mathbb{K}$, $\mathfrak{g}$ and $\mathfrak{h}$ be as in the paragraph before Proposition 2.2. Let $\mathfrak{b} \supseteq \mathfrak{h}$ be a Borel subalgebra of $\mathfrak{g}$. Then, for the Killing forms of $\mathfrak{b}$ and $\mathfrak{g}$, we have*

$$B_{\mathfrak{b}} = 1/2\, B_{\mathfrak{g}}.$$

Proof. Let $\{h_1, \dots h_l\}$ be a basis of $\mathfrak{h}$. For $\Omega_{ij} = \operatorname{ad} h_i \operatorname{ad} h_j$ we have $\Omega_{ij}(h_k) = 0$ for every k, and

$$\Omega_{ij}(X_{\pm\varphi}) = \varphi(h_j)\varphi(h_i)X_{\pm\varphi}, \qquad \text{for } \varphi \in \Delta^+;$$

that is,

$$B_{\mathfrak{b}}(h_i, h_j) = \sum_{\varphi \in \Delta^+} \varphi(h_i)\varphi(h_j).$$

Also, for $\Omega_{i\varphi} = \operatorname{ad} h_i \operatorname{ad} X_\varphi$, we have

$$\Omega_{i\varphi}(h_k) = -\varphi(h_k)\varphi(h_i)X_\varphi \quad \text{ and } \quad \Omega_{i\varphi}(X_\omega) \in \mathfrak{g}_{\varphi+\omega};$$

here $\varphi \in \Delta^+$ and $\omega \in \Delta$. Thus the trace of the restriction of $\Omega_{i\varphi}$ to $\mathfrak{b}$ is equal to zero for every $\varphi \in \Delta^+$; i.e., $B_{\mathfrak{b}}(h_i, X_\varphi) = 0$. Analogously, $B_{\mathfrak{b}}(X_\varphi, X_\psi) = 0$ for every $\varphi, \psi \in \Delta^+$. As a resume of the above we have

$$B_{\mathfrak{g}}(h_i, h_j) = \sum_{\varphi \in \Delta^+ \cup \Delta^-} \varphi(h_j)\varphi(h_i) = 2 \sum_{\varphi \in \Delta^+} \varphi(h_j)\varphi(h_i) = 2B_{\mathfrak{b}}(h_i, h_j),$$

while at the same time $B_{\mathfrak{g}}(h_i, X_\varphi) = B_{\mathfrak{g}}(X_\varphi, X_\psi) = 0$, for all i and $\varphi, \psi \in \Delta^+$. Thus we are done. ■

Now we are ready to prove Proposition 2.2.

Proof. Take first any $\varphi, \psi \in \Gamma$ such that $\varphi + \psi \neq 0$, and put $\Omega = \operatorname{ad} X_\varphi \operatorname{ad} X_\psi$. We have

$$\Omega(h) = -\psi(h)[X_\varphi, X_\psi], \qquad \text{for } h \in \mathfrak{h}.$$

Thus the contribution to the trace of the restriction of Ω to $\mathfrak{q}$, coming from $\mathfrak{h}$, equals 0. The contribution to the trace coming from X_α, for any $\alpha \in \Gamma$, is equal to 0 as well (cf. Lemma 2.4). As a consequence we have $B_{\mathfrak{q}}(X_\varphi, X_\psi) = 0$.

Take now any $\varphi \in \Gamma$ and $H \in \mathfrak{h}$, and put $\Omega = \operatorname{ad} X_\varphi \operatorname{ad} H$. Similarly as above we have $\Omega(h) = 0$ for any $h \in \mathfrak{h}$, and $\Omega(X_\alpha) \in \mathfrak{g}_{\varphi+\alpha}$ for any $\alpha \in \Gamma$. Thus, $B_{\mathfrak{q}}(X_\varphi, H) = 0$.

As a consequence of the above observations we deduce that

$$B_{\mathfrak{q}}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \qquad \text{for } \alpha, \beta \in \Gamma \cup \{0\}, \quad \alpha + \beta \neq 0.$$

Now we are going to show the only "tricky fact", i.e., the following

Claim. The restriction of $B_{\mathfrak{q}}$ to $\mathfrak{g}_{-\alpha} \times \mathfrak{g}_\alpha$ is nondegenerate, for any $\alpha \in \Delta^+$ satisfying $-\alpha \in \Gamma$.

[[*Proof.* To see this define an operator

$$\Omega = \operatorname{ad} X_{-\alpha} \operatorname{ad} X_\alpha.$$

We want to compute $\Omega(X_\beta)$, for any $\beta \in \Gamma$. Let us consider these two possibilities for $\beta \neq \pm\alpha$:

(M1) $\beta \in \Delta^+$, *and* $-\beta \notin \Gamma$;

(M2) $\beta \in \Delta^+$, *and* $-\beta \in \Gamma$.

Suppose (M1). Let $\beta - p\alpha, \dots, \beta + q\alpha$ be the α-string containing β. If $\beta + \alpha \notin \Delta$, then necessarily $q = 0$, and therefore $\Omega(X_\beta) = 0$. If $\beta + \alpha \in \Delta$, then $q > 0$, and also $\Omega(X_\beta) = \nu X_\beta$, where $\nu = q(p+1) > 0$.

Suppose now (M2). For the α-string containing β as above, we have that $-\beta - q\alpha, \dots, -\beta + p\alpha$ is the α-string containing $-\beta$. It follows that $\Omega(X_{-\beta}) = p(q+1)X_{-\beta}$. Also, as for (M1), $\Omega(X_\beta) = q(p+1)X_\beta$. We conclude that the contribution to the trace of Ω on $\mathfrak{q}$, coming from $X_{-\beta}$ and X_β, is equal to $2pq + p + q$.

Now, as we may assume that $[X_\alpha, X_{-\alpha}] = h_\alpha$, where h_α satisfies $\alpha(h_\alpha) = 2$, it immediately follows that $\Omega(X_{-\alpha}) = 2X_{-\alpha}$. Thus the contribution to the trace of Ω, coming from $X_{-\alpha}$ and X_α, is equal to 2.

It remained to consider $\Omega(h_i)$; here $\{h_1, \dots, h_l\}$ is a basis of $\mathfrak{h}$ as before. For that we need a little preparation. Given a basis $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of Δ, it is well known that $\Pi^\vee$ is a basis of the dual root system $\Delta^\vee$. Furthermore, the map $\Psi : \mathfrak{h} \to \mathfrak{h}^*$ defined by $\Psi(h, h') = B_\mathfrak{g}(h, h')$ is a vector space isomorphism. We have in particular $\Psi(h_\alpha) = \alpha^\vee$, for every $\alpha \in \Delta$. Next, write for any such α, $\alpha^\vee = n_1\alpha_1^\vee + \cdots + n_l\alpha_l^\vee$. If we have chosen $\Pi \subseteq \Delta^+$, then an easy application of Ψ gives that all n_i are nonnegative. Now, we may also assume that $h_i = h_{\alpha_i}$, for $i = 1, \dots, l$. From all noted above it immediately follows that

$$h_\alpha = \sum_{j=1}^{l} n_j h_j. \tag{4}$$

Finally, using (4), we have

$$\Omega(h_i) = \alpha(h_i)h_\alpha = \sum_{j=1}^{l} \alpha(h_i) n_j h_j.$$

Hence, we clearly have that the contribution to the trace of Ω coming from $\mathfrak{h}$ is equal to

$$\sum_{j=1}^{l} n_j \alpha(h_j) = \alpha\Big(\sum_{j=1}^{l} n_j h_j\Big) = 2; \tag{5}$$

here for the last equality we use (4) again.

As a conclusion we have that $B_\mathfrak{q}(X_{-\alpha}, X_\alpha) \neq 0$; more precisely, the latter scalar is ≥ 4.]]

Thus we clearly have our proposition proved. ∎

Remark 2.5. The given proof of our proposition is in a sense constructive. Therefore for any setting we have it is possible to compute the trace of the operator $\Omega = \operatorname{ad} X_{-\alpha_1} \operatorname{ad} X_{\alpha_1}$ on $\mathfrak{q}$, for all roots α such that $\pm\alpha \in \Gamma$. To illustrate this, consider the setting of Example 2.3 again, and let us use the notation of the above proof. Put $\alpha = \alpha_1$. We will estimate the contributions to the trace of Ω, coming from various roots $\beta \in \Gamma$, and the one coming from $\mathfrak{h}$. First, there are two roots β satisfying (M1): $\beta = \alpha_2$ and $\beta = \alpha_1 + \alpha_2$. For $\beta = \alpha_2$ we have $p = 0$, $q = 1$, and therefore $\nu = 1$. For $\beta = \alpha_1 + \alpha_2$ we have $\beta + \alpha \notin \Delta$; thus $\nu = 0$. We conclude that the contribution, coming from the roots satisfying (M1), is equal to $1+0 = 1$. Next, there are no roots satisfying (M2). Furthermore, the contribution coming from the roots $\pm\alpha$ is equal to 2. The contribution coming from $\mathfrak{h}$ is equal to 2 too. As a conclusion we have that $B_{\mathfrak{q}}(X_{-\alpha_1}, X_{\alpha_1}) = 1 + 2 + 2 = 5$, as we already noted in (2).

3. $(\mathfrak{sl}(n+1), \mathfrak{sl}(2))$-pairs

The main purpose of this section is to study the pairs $(\mathfrak{g}, \mathfrak{g}_1) = (\mathfrak{sl}(n+1), \mathfrak{sl}(2))$ obtained as is explained below. First we set up the notation which will be used throughout this section. Define the standard invariant bilinear form $\langle x, y\rangle = \operatorname{Tr}(xy)$ on $\mathfrak{gl}(N)$. Let $\mathfrak{g} = \mathfrak{sl}(n+1)$, and let

$$\mathfrak{h} = \{\text{diagonal matrices in } \mathfrak{g}\}$$

be its Cartan subalgebra. Let $e_i \in \mathfrak{h}^*$ be defined by $e_i(\sum_j h_j E_{jj}) = h_i$. Recall that the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is

$$\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) = \{e_i - e_j \mid i \neq j\}.$$

Let $\Pi = \Pi(\mathfrak{g}, \mathfrak{h}) = \{\alpha_1, \ldots, \alpha_n\}$, the associated set of simple roots, where $\alpha_i = e_i - e_{i+1}$; by $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ we denote the corresponding positive roots. For later use note that, for $\vartheta \in \mathfrak{h}$,

$$[\vartheta, E_{ij}] = (e_i - e_j)(\vartheta) E_{ij}. \tag{6}$$

Let $\rho : \mathfrak{sl}(2) \to \mathfrak{g} \hookrightarrow \mathfrak{gl}(n+1)$ be the unique $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}(2)$. Define $\mathfrak{g}_1 = \rho(\mathfrak{sl}(2))$, and consider the pairs $(\mathfrak{g}, \mathfrak{g}_1)$; loosely speaking, we consider pairs $(\mathfrak{sl}(n+1), \mathfrak{sl}(2))$. Note that these pairs satisfy the condition (**C**); cf. Lemma 4.3 below. Therefore we have the usual decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{p}(n)$ is the (Killing-)orthogonal of $\mathfrak{g}_1$ in $\mathfrak{g}$. Our first task here is to describe the vector spaces $\mathfrak{p}$ in a convenient way. For the basis $\{\boldsymbol{h}, \boldsymbol{e}, \boldsymbol{f}\}$ of $\mathfrak{sl}(2)$ define $H = \rho(\boldsymbol{h})$, $E = \rho(\boldsymbol{e})$ and $F = \rho(\boldsymbol{f})$. More precisely,

$$H = \sum_{i=1}^{n+1} (n+2-2i) E_{ii}, \qquad E = \sum_{i=1}^{n} m_i E_{i,i+1}, \qquad F = \sum_{i=1}^{n} E_{i+1,i};$$

here

$$m_i = i(n-i+1), \qquad \text{for } i = 1, \ldots, n.$$

Also define $\mathfrak{h}_1 = \mathbb{K}H$, a Cartan subalgebra of $\mathfrak{g}_1$. Now for an $(n+1)$-by-$(n+1)$ matrix $M = (x_{ij})$ we have the following equivalences: $\langle M, F\rangle = 0$ if and only if

$\sum_{i=1}^{n} x_{i,i+1} = 0$; $\langle M, H\rangle = 0$ if and only if $\sum_{i=1}^{n+1}(n+2-2i)x_{ii} = 0$; $\langle M, E\rangle = 0$ if and only if $\sum_{i=1}^{n} m_i x_{i+1,i} = 0$. Hence it immediately follows that we have a direct sum decomposition

$$\mathfrak{p} = \mathfrak{l} \oplus \Big\{ (x_{ij}) \ \mid \ x_{ii} = 0 \quad \text{and} \quad \sum_{i=1}^{n} x_{i,i+1} = 0 = \sum_{i=1}^{n} m_i x_{i+1,i} \Big\},$$

where

$$\mathfrak{l} = \mathfrak{l}(n) = \{ M \in \mathfrak{h} \mid \langle M, \mathfrak{h}_1\rangle = 0\},$$

the orthogonal of $\mathfrak{h}_1$ in $\mathfrak{h}$. If we define

$$\boldsymbol{X}_i = -E_{12} + E_{i+1,i+2}, \qquad \boldsymbol{Y}_i = -(m_{i+1}/m_1)E_{21} + E_{i+2,i+1},$$

for $1 \le i \le n-1$, then we have the following explicit description of $\mathfrak{p}$.

Lemma 3.1.

$$\mathfrak{p}(n) = \mathfrak{l}(n) + \sum_{|i-j|\ge 2} \mathbb{K}E_{ij} + \sum_{i=1}^{n-1} \mathbb{K}\boldsymbol{X}_i + \sum_{i=1}^{n-1} \mathbb{K}\boldsymbol{Y}_i.$$

Note. In what follows, if there will be a danger of ambiguity, we will write the superscript "n", as $\langle n\rangle$, in all the symbols we have defined; e.g., $\mathfrak{g}^{\langle n\rangle}$, $H^{\langle n\rangle}$, $m_i^{\langle n\rangle}$, $\boldsymbol{X}_i^{\langle n\rangle}$, etc.

The purpose of the following theorem is to gather a number of interesting facts about pairs $(\mathfrak{sl}(n+1), \mathfrak{sl}(2))$. Concerning the claim (i), recall the following general fact; see, e.g., Corollary 5.31 in [Kn1]. Suppose $\mathfrak{G}$ is a semisimple Lie algebra, $\mathfrak{H}$ is a split Cartan subalgebra, $\Delta = \Delta(\mathfrak{G}, \mathfrak{H})$ and Δ^+ is a choice of positive roots. Define a nilpotent subalgebra $\mathfrak{N} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{G}_\alpha$. If $\mathcal{V}$ is a finite dimensional $\mathfrak{G}$-module, then the subspace $\mathcal{V}^{\mathfrak{N}}$ of $\mathfrak{N}$-invariants has an $\mathfrak{H}$-module structure which determines $\mathcal{V}$ up to equivalence.

Theorem 3.2. *(i) As a $\mathfrak{g}_1$-module, $\mathfrak{p}(n)$ is simple if and only if $n = 2$. More precisely, the $\mathfrak{g}_1$-module structure of $\mathfrak{p}(n)$ is given by*

$$\begin{aligned} \mathfrak{p}(n) &= V^1_{4\varpi} \oplus V^1_{6\varpi} \oplus \cdots \oplus V^1_{2n\varpi} \\ &= V^1_{(n+2)\varpi} \otimes V^1_{(n-2)\varpi}; \end{aligned}$$

here ϖ denotes the fundamental weight for $\mathfrak{sl}(2)$.

(ii) The pair $(\mathfrak{g}, \mathfrak{g}_1)$ is symmetric if and only if $n = 2$.

(iii) For $n \ge 3$ we have

$$[\mathfrak{p}(n), \mathfrak{p}(n)] = \mathfrak{g}.$$

(iv) $\mathfrak{g}_1$ is self-normalizing in $\mathfrak{g}$.

(v) $\mathfrak{h}$ is the unique Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}_1$.

Remark 3.3. As we already said in the Introduction, (i) of the theorem is the first step of a more general branching problem. We should also mention here the work of Osinovskaya who consider a similar problem but in a different setting. More precisely, she consider the problem of decomposability of restrictions of representations ρ for classical groups G to certain naturally embedded subgroups H of small rank; in particular when H is of type A_1 or A_2 (see [O1], [O2]). But let us also emphasize that the fact that H is naturally embedded has as a consequence that $\rho_{|H}$ is "more decomposable" than in the situations which we aim to study (cf. Exercise 2 on p. 34 in [Hu]); i.e., the setting of pairs $(\mathfrak{g}, \mathfrak{g}_1)$ within the class $\boldsymbol{\mathcal{P}}$. As an illustration of the claim that sometimes the restrictions $\rho_{|\mathfrak{g}_1}$ of some irreducible representations ρ of $\mathfrak{g}$ will have a "small number" of irreducible constituents we point out at the pair $(\mathfrak{so}(7), \mathfrak{G}_2)$ from $\boldsymbol{\mathcal{P}}$. A famous fact, which is due to Levasseur and Smith ([LS], Sect. 3), says that there is a particular interesting infinite-dimensional irreducible $\mathfrak{so}(7)$-representation which is also as a $\mathfrak{G}_2$-representation irreducible.

Proof. (i) We will show that the $\mathfrak{h}_1$-module $\mathfrak{p}(n)^E$ of E-invariants is of dimension $n-1$. The second step is more precise; i.e., this module is multiplicity free with the set of $\mathfrak{h}_1$-weights $\{2k\varpi \mid 2 \leq k \leq n\}$. For that first note that

$$\mathfrak{l} = \left\{ \sum_{i=1}^{n+1} d_i E_{ii} \mid \sum_{i=1}^{n+1} (n+2-2i) d_i = 0 = \sum_{i=1}^{n+1} d_i \right\}.$$

Next, it is easy to check that the following holds:

$$\begin{aligned} [E, E_{ij}] &= m_{i-1} E_{i-1,j} - m_j E_{i,j+1} \qquad \text{for } 1 \leq i, j \leq n+1, \\ [E, \boldsymbol{X}_i] &= m_2 E_{13} + m_i E_{i,i+2} - m_{i+2} E_{i+1,i+3} \qquad \text{for } 1 \leq i \leq n-1, \\ [E, \boldsymbol{Y}_i] &= m_{i+1}(E_{22} - E_{11} + E_{i+1,i+1} - E_{i+2,i+2}) \qquad \text{for } 1 \leq i \leq n-1; \end{aligned} \tag{7}$$

of course, we understand that $E_{kl} = 0$ if $k < 1$ or $l > n+1$.

Let $M \in \mathfrak{p}(n)^E$, and decompose it in accordance with Lemma 3.1 as

$$M = l + \sum_{i=1}^{n-1} s_i \boldsymbol{X}_i + \sum_{i=1}^{n-1} t_i \boldsymbol{Y}_i + \sum_{|i-j| \geq 2} c_{ij} E_{ij}, \tag{8}$$

where $l \in \mathfrak{l}$ and $s_i, t_i, c_{ij} \in \mathbb{K}$. Then

$$0 = [E, M] = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5, \tag{9}$$

where $\mathcal{S}_1 = [E, l]$, $\mathcal{S}_2 = \sum_i s_i [E, \boldsymbol{X}_i]$, $\mathcal{S}_3 = \sum_i t_i [E, \boldsymbol{Y}_i]$, $\mathcal{S}_4 = \sum_{i-j \geq 2} c_{ij} [E, E_{ij}]$ and $\mathcal{S}_5 = \sum_{j-i \geq 2} c_{ij} [E, E_{ij}]$. Obviously, taking into account the above formulas for commutators (7), we see that the equality (9) holds if and only if $\mathcal{S}_i = 0$ for all i. Thus in particular, since $\mathcal{S}_1 = -\sum_{i=1}^{n} m_i \alpha_i(l) E_{i,i+1}$, it is straightforward that $l = 0$.

Now we treat a more complicated term, i.e., the equality $\mathcal{S}_2 = 0$. In order to see what's going on let us first take for example $n = 4$. Now

$$\begin{aligned} 0 = \mathcal{S}_2 = (s_1(m_1 + m_2) + s_2 m_2 + s_3 m_2)\ &E_{13} \\ +(-s_1 m_3 + s_2 m_2)\ &E_{24} \\ +(-s_2 m_4 + s_3 m_3)\ &E_{35}; \end{aligned}$$

here the summands are arranged so that the corresponding E_{ij}'s are lexicographically ordered. Then for $k \geq 4$ define a matrix

$$A_4^k = \begin{pmatrix} m_1^k + m_2^k & m_2^k & m_2^k \\ -m_3^k & m_2^k & 0 \\ 0 & -m_4^k & m_3^k \end{pmatrix}.$$

Analogously for arbitrary $k \geq n \geq 3$ we will have matrices A_n^k; in particular, A_3^k is obtained by deleting the 3rd row and 3rd column in A_4^k. It is easy to check by induction that

$$A_{n+1}^k = \begin{pmatrix} A_n^k & C_{n+1}^k \\ R_{n+1}^k & m_n^k \end{pmatrix} \qquad \text{for } k \geq n+1,$$

where $R_{n+1}^k = \begin{pmatrix} 0 & \cdots & 0 & -m_{n+1}^k \end{pmatrix}$ and $C_{n+1}^k = \begin{pmatrix} m_2^k & 0 & \cdots & 0 \end{pmatrix}^t$ are row and column vectors of length $n-1$, respectively ("t" denotes the transpose). Hence, by expansion according to the n^{th} column,

$$\det(A_{n+1}^k) = \prod_{i=1}^{n} m_{i+1}^k + m_n^k \det A_n^k.$$

In particular, $\det(A_n^k) > 0$ for all n and $k \geq n$, and therefore $s_1 = \cdots = s_{n-1} = 0$. The case $\mathcal{S}_3 = 0$ is easy; here one also has that $t_1 = \cdots = t_{n-1} = 0$. In order to treat the equality $\mathcal{S}_4 = 0$, we just have to rewrite

$$\mathcal{S}_4 = \sum_{i=3}^{n+1} \sum_{j=1}^{i-2} c_{ij}(m_{i-1}E_{i-1,j} - m_j E_{i,j+1}).$$

Hence it obviously follows that $c_{ij} = 0$ for all (i,j) such that $i \geq j+2$. It remained to treat the most complicated situation, i.e., $\mathcal{S}_5 = 0$. Again, as for $\mathcal{S}_2 = 0$, it will be instructive to first see what we have for small n's. For $n = 3$, $\mathcal{S}_5 = 0$ is equivalent to the system-equation

$$(\Sigma_3) \qquad -m_3 c_{13} + m_1 c_{24} = 0$$

in the unknowns c_{13}, c_{14}, c_{24}. Also for $n = 4$ we obtain the system

$$(\Sigma_4) \qquad \begin{aligned} -m_3 c_{13} + m_1 c_{24} &= 0 \\ -m_4 c_{14} + m_1 c_{25} &= 0 \\ -m_4 c_{24} + m_2 c_{35} &= 0 \end{aligned}$$

in the unknowns c_{13}, c_{14}, c_{15}, c_{24}, c_{25}, c_{35}. For general $n \geq 3$ we will have a system (Σ_n), consisting of $(n-2)(n-1)/2$ equations in $(n-1)n/2$ unknowns c_{ij} for $1 \leq i \leq j-2 \leq n-1$, given as follows:

$$(\Sigma_n) \qquad \begin{aligned} -m_j c_{1j} + m_1 c_{2,j+1} &= 0 \qquad \text{for } j = 3, \ldots, n \\ \cdots \qquad & \cdots \\ -m_j c_{kj} + m_k c_{k+1,j+1} &= 0 \qquad \text{for } j = k+2, \ldots, n \\ \cdots \qquad & \cdots \\ -m_n c_{n-2,n} + m_{n-2} c_{n-1,n+1} &= 0. \end{aligned}$$

We claim that the solution of (Σ_n) is an $(n-1)$-dimensional vector space. In other words, $\dim \mathfrak{p}(n)^E = n-1$. To see this define vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}$ by

$$\boldsymbol{v}_i = E_{i,i+2} + \frac{m_{i+2}}{m_1} E_{2,i+3} + \frac{m_{i+2}}{m_1}\frac{m_{i+3}}{m_2} E_{3,i+4} + \cdots ;$$

more precisely, the above sum defining $\boldsymbol{v}_i$ has $n-i$ summands where the $(k+1)^{\text{th}}$ one, for $1 \le k < n-i$, equals $\prod_{j=1}^{k}(m_{j+i+1}/m_j)E_{k+1,k+i+2}$. Now it is easy to check that

$$\mathfrak{p}(n)^E = \operatorname{span}_{\mathbb{K}}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}\}.$$

Furthermore, as (6) implies

$$[H, E_{ij}] = 2(j-i)E_{ij}, \tag{10}$$

we then obviously have $[H, \boldsymbol{v}_i] = 2(i+1)\boldsymbol{v}_i$. Thus $\mathfrak{p}(n)^E$ is an $\mathfrak{h}_1$-module where every $\boldsymbol{v}_i$ is an eigenvector for $\mathfrak{h}_1$-action with the eigenvalue $2(i+1)$. This means that we have an equality of $\mathfrak{g}_1$-modules

$$\mathfrak{p}(n) = V^1_{4\varpi} \oplus V^1_{6\varpi} \oplus \cdots \oplus V^1_{2n\varpi} \qquad \text{for } n \in \mathbb{N}.$$

As a special case of the Littlewood-Richardson rule for $\mathfrak{sl}(2)$ we have that $\mathfrak{p}(n)$ can be written as the tensor product $V^1_{(n+2)\varpi} \otimes V^1_{(n-2)\varpi}$.

(ii) and (iii) We have $(\Delta^2)^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$. Choose the root vectors $X_{\alpha_1} = E_{12}$, $X_{\alpha_2} = E_{23}$, $X_{\alpha_1+\alpha_2} = E_{13}$ and $X_{-\alpha} = (X_\alpha)^t$ for $\alpha \in (\Delta^2)^+$ ("t" denotes the transpose). Since $\mathfrak{l}(2) = \mathbb{K}L$, where $L = E_{11} - 2E_{22} + E_{33}$, we have (see Lemma 3.1)

$$\mathfrak{p}(2) = \mathbb{K}L + \mathbb{K}\boldsymbol{X}_1 + \mathbb{K}\boldsymbol{Y}_1 + \mathbb{K}X_{\alpha_1+\alpha_2} + \mathbb{K}X_{-\alpha_1-\alpha_2};$$

note that $\boldsymbol{X}_1 = -X_{\alpha_1} + X_{\alpha_2}$ and $\boldsymbol{Y}_1 = -X_{-\alpha_1} + X_{-\alpha_2}$. Since

$$\begin{aligned}
[\boldsymbol{X}_1, X_{-\alpha_1-\alpha_2}] &= F,\\
[X_{\alpha_1+\alpha_2}, X_{-\alpha_1-\alpha_2}] &= H/2 = [\boldsymbol{X}_1, \boldsymbol{Y}_1],\\
[X_{\alpha_1+\alpha_2}, \boldsymbol{Y}_1] &= E/2,
\end{aligned}$$

and

$$[L, \boldsymbol{Y}_1] = 3F, \qquad [\boldsymbol{X}_1, L] = 3F/2, \qquad [X_{\pm(\alpha_1+\alpha_2)}, L] = 0,$$

we conclude that $[\mathfrak{p}(2), \mathfrak{p}(2)] \subseteq \mathfrak{g}_1^{\langle 2\rangle}$. By Lemma 1.3(i) we have that $(\mathfrak{g}^{\langle 2\rangle}, \mathfrak{g}_1^{\langle 2\rangle})$ is a symmetric pair.

Let $n \ge 3$. Then first note that

$$[\boldsymbol{X}_i, \boldsymbol{Y}_j] = \frac{m_{j+1}}{m_1}(E_{11} - E_{22}) + \delta_{ij}(E_{i+1,i+1} - E_{i+2,i+2}).$$

Hence in particular $[\boldsymbol{X}_2, \boldsymbol{Y}_1] \in [\mathfrak{p}(n), \mathfrak{p}(n)]$, and so $E_{11} - E_{22} \in [\mathfrak{p}(n), \mathfrak{p}(n)]$. Furthermore it is clear that $E_{i+1,i+1} - E_{i+2,i+2} \in [\mathfrak{p}(n), \mathfrak{p}(n)]$ as well, for all i. Thus $\mathfrak{h} \subseteq [\mathfrak{p}(n), \mathfrak{p}(n)]$. Now define

$$\lambda_k = \begin{cases} E_{11} - nE_{nn} + (n-1)E_{n+1,n+1} & \text{if } k = 1,\\ -\left(\frac{n-k+1}{n}\right)E_{11} + E_{kk} + \left(\frac{1-k}{n}\right)E_{n+1,n+1} & \text{if } 2 \le k \le n. \end{cases}$$

Clearly, $\lambda_k \in \mathfrak{l}(n)$. For $1 < i < j$ we have

$$[\lambda_i, E_{ij}] = \begin{cases} E_{ij} & \text{if } j \le n, \\ \left(\frac{n+i-1}{n}\right) E_{i,n+1} & \text{if } j = n+1, \end{cases}$$

and also

$$[\lambda_1, E_{1j}] = \begin{cases} E_{1j} & \text{if } 2 \le j < n, \\ (n+1)E_{1n} & \text{if } j = n, \\ (2-n)E_{1,n+1} & \text{if } j = n+1. \end{cases}$$

We conclude that $E_{ij} \in [\mathfrak{p}(n), \mathfrak{p}(n)]$ for all $i < j$. The same conclusion for i, j such that $i > j$ follows so that one transposes the above commutators. Thus we have both (ii) and (iii) proved.

(iv) First note that for all i we have

$$[H, \boldsymbol{X}_i] = 2\boldsymbol{X}_i, \qquad [H, \boldsymbol{Y}_i] = -2\boldsymbol{Y}_i. \tag{11}$$

(But $\{H, \boldsymbol{X}_i, \boldsymbol{Y}_i\}$ is not a standard triple.) Now write an arbitrary $M \in \mathfrak{p}(n)^{\mathfrak{g}_1}$ as in (8). By (10) and (11), from $[H, M] = 0$ it clearly follows that $M = l \in \mathfrak{l}$. Furthermore, from $[E, l] = 0$ we easily conclude that $l = 0$. It remained to take into account Lemma 1.3(ii).

(v) It is sufficient to show that $\alpha(H) \neq 0$ for every root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$; see Lemma 4.2 below. But this is clear by (10). ■

Remark 3.4. Concerning the claim (ii) above it is interesting to note the following. The map $\theta : \mathfrak{g}^{\langle 2\rangle} \to \mathfrak{g}^{\langle 2\rangle}$, $\theta(x_1 + p) = x_1 - p$, for $x_1 \in \mathfrak{g}_1^{\langle 2\rangle}$ and $p \in \mathfrak{p}(2)$, is explicitly given by

$$\theta \begin{pmatrix} d_1 & x_1 & x_3 \\ y_1 & d_2 & x_2 \\ y_3 & y_2 & d_3 \end{pmatrix} = \begin{pmatrix} -d_3 & x_2 & -x_3 \\ y_2 & -d_2 & x_1 \\ -y_3 & y_1 & -d_1 \end{pmatrix};$$

obviously $\theta^2 = 1_{\mathfrak{g}^{\langle 2\rangle}}$, and by Lemma 1.3(i) we know that $\theta \in \operatorname{Aut} \mathfrak{g}^{\langle 2\rangle}$. (Of course, the latter fact can be checked straightforwardly too.)

Suppose for the moment that $\mathbb{K}$ is algebraically closed, and $(\mathfrak{g}, \mathfrak{g}_1)$ is any pair from $\boldsymbol{\mathcal{P}}$. For a Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{g}_1$, consider the corresponding pair $(\mathfrak{h}, \mathfrak{h}_1)$. Also, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ and $\Delta_1 = \Delta(\mathfrak{g}_1, \mathfrak{h}_1)$. From the point of view of representation theory it is interesting to know how Δ and Δ_1 are mutually related. More precisely, we often have to know how to write a particular $X_\beta \in (\mathfrak{g}_1)_\beta$, $\beta \in \Delta_1$, as a linear combination of $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta$ (see [LS]). Concerning this it is also useful to find various pairs of compatible Borel subalgebras (see [BK] and [Ks3]), and more generally, pairs of compatible parabolic subalgebras; note in particular that for a chosen Borel subalgebra $\mathfrak{b}_1$ of $\mathfrak{g}_1$ there is a unique Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that the pair $(\mathfrak{b}, \mathfrak{b}_1)$ is compatible. Here "compatible" has the following meaning.

Definition 3.5. Under the above setting, we say that a pair $(\mathfrak{q}, \mathfrak{q}_1)$ of parabolic subalgebras $\mathfrak{h} \le \mathfrak{q} \le \mathfrak{g}$ and $\mathfrak{h}_1 \le \mathfrak{q}_1 \le \mathfrak{g}_1$ is *compatible* with $(\mathfrak{h}, \mathfrak{h}_1)$, or just compatible, if $\mathfrak{q}_1 \le \mathfrak{q}$.

In general, given a reducible pair $(\mathfrak{g}, \mathfrak{g}_1)$ such that moreover $\mathfrak{g}_1$ is self-normalizing in $\mathfrak{g}$, one can ask whether $\mathfrak{g}_1$ is a maximal subalgebra of $\mathfrak{g}$ (cf. Lemma 1.3(iii) and [Š3, Sect. 5; in particular, Ex. 5.10]). Concerning this we give the following instructive example.

Example 3.6. Consider the pair $(\mathfrak{g}, \mathfrak{g}_1) = (\mathfrak{sl}(4), \mathfrak{sl}(2))$. Define $\widetilde{\mathfrak{c}} = \mathbb{K}H + \mathbb{K}\vartheta$, where $\vartheta = E_{22} - E_{33}$; note that $H = 3(E_{11} - E_{44}) + E_{22} - E_{33}$. The fact is that there is a subalgebra $\mathfrak{g}_1 \subseteq \widetilde{\mathfrak{s}} \subseteq \mathfrak{g}$, $\widetilde{\mathfrak{s}} \cong \mathfrak{sp}(4)$, such that $\widetilde{\mathfrak{c}}$ is its Cartan subalgebra. For these $\widetilde{\mathfrak{s}}$ and $\widetilde{\mathfrak{c}}$, the simple roots are $\Pi(\widetilde{\mathfrak{s}}, \widetilde{\mathfrak{c}}) = \{\widetilde{\alpha}, \widetilde{\beta}\}$ and the positive roots are $\Delta^+(\widetilde{\mathfrak{s}}, \widetilde{\mathfrak{c}}) = \{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\alpha} + \widetilde{\beta}, \widetilde{\alpha} + 2\widetilde{\beta}\}$, where $\widetilde{\alpha} = \alpha_{2|\mathfrak{c}}$ and $\widetilde{\beta} = \alpha_{3|\mathfrak{c}}$; recall that $\Pi(\mathfrak{g}, \mathfrak{h}) = \{\alpha_1, \alpha_2, \alpha_3\}$. Now it is an easy exercise to check that $\widetilde{\mathfrak{s}} = \mathfrak{u}^- \oplus \widetilde{\mathfrak{c}} \oplus \mathfrak{u}^+$, where $\mathfrak{u}^\pm = \bigoplus_{\gamma \in \Delta^\pm(\widetilde{\mathfrak{s}}, \widetilde{\mathfrak{c}})} X_\gamma$, and

$$X_{\widetilde{\alpha}} = E_{23}, \qquad X_{\widetilde{\beta}} = E_{12} + E_{34}$$
$$X_{\widetilde{\alpha}+\widetilde{\beta}} = [X_{\widetilde{\beta}}, X_{\widetilde{\alpha}}] = E_{13} - E_{24}, \qquad X_{\widetilde{\alpha}+2\widetilde{\beta}} = [X_{\widetilde{\beta}}, X_{\widetilde{\alpha}+\widetilde{\beta}}] = -2E_{14}.$$

Let $\mathfrak{b}_1 = \mathfrak{h}_1 \oplus \mathbb{K}E$, $\widetilde{\mathfrak{b}} = \widetilde{\mathfrak{c}} + \mathfrak{u}^+$ and $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\delta \in \Delta^+} X_\delta$ be the corresponding Borel subalgebras of $\mathfrak{g}_1$, $\widetilde{\mathfrak{s}}$ and $\mathfrak{g}$, respectively, determined by the sets of positive roots; here $E = 3E_{12} + 4E_{23} + 3E_{34}$. For later use note the following

Observation. A pair of embeddings $\mathfrak{g}_1 \subseteq \widetilde{\mathfrak{s}} \subseteq \mathfrak{g}$ is compatible in the above sense; i.e.,

$$\mathfrak{b}_1 \subseteq \widetilde{\mathfrak{b}} \subseteq \mathfrak{b}.$$

Example 3.7. (i) Given matrices $X, Y, Z, T \in M_n(\mathbb{K})$, define a block-matrix $M = \left(\begin{smallmatrix} X & Y \\ Z & T \end{smallmatrix}\right) \in M_{2n}(\mathbb{K})$. Let $M \mapsto M^\sharp$ be the *symplectic involution*, where $M^\sharp = \left(\begin{smallmatrix} T^t & -Y^t \\ -Z^t & X^t \end{smallmatrix}\right)$. Then $\mathfrak{sp}(2n) = \{M \mid M^\sharp = -M\}$; i.e., $\mathfrak{sp}(2n)$ consists of all matrices of the form $M = \left(\begin{smallmatrix} X & Y \\ Z & -X^t \end{smallmatrix}\right)$, where $Y^t = Y$ and $Z^t = Z$. This is a standard embedding of $\mathfrak{sp}(2n)$ into $\mathfrak{sl}(2n)$; see [Š2], and also [Š4] for the prime characteristic setting. In particular, for $\mathfrak{s}' = \mathfrak{sp}(4)$ and its Cartan subalgebra $\mathfrak{c}' = \mathbb{K}(E_{11} - E_{33}) + \mathbb{K}(E_{22} - E_{44})$, we have $\Pi = \{\alpha, \beta\}$ and $\Delta^+ = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta\}$, where $\alpha = 2e_2$ and $\beta = e_1 - e_2$. Then for the corresponding root vectors we can take the following:

$$X_\alpha = E_{24}, \qquad X_\beta = E_{12} - E_{43}$$
$$X_{\alpha+\beta} = E_{14} + E_{23}, \qquad X_{\alpha+2\beta} = E_{13}.$$

Concerning the previous example and included observation we have the following fact. For $\mathfrak{b}_1, \mathfrak{b}$ being as before, and a Borel subalgebra $\mathfrak{b}' = \mathfrak{c}' \oplus \mathfrak{n}'$ of $\mathfrak{s}'$, where $\mathfrak{n}' = \bigoplus_{\gamma \in \Delta^+} X_\gamma$, we have $\mathfrak{h}_1 \not\subseteq \mathfrak{c}'$; and also $\mathfrak{b}_1 \not\subseteq \mathfrak{s}'$. At the same time $\mathfrak{c}' \subseteq \mathfrak{h}$, but the pair $(\mathfrak{b}, \mathfrak{b}')$ is not compatible.

(ii) A slightly different, realization of $\mathfrak{sp}(2n)$ within $\mathfrak{sl}(2n)$ is given by $\mathfrak{sp}(2n) = \{M \mid M^\dagger = -M\}$, where $M_{2n}(\mathbb{K}) \ni M \mapsto M^\dagger$ is the same map as in Subsection 4 below. Thus $\mathfrak{sp}(2n)$ consists of all matrices $M = \left(\begin{smallmatrix} X & Y \\ Z & -X^\tau \end{smallmatrix}\right)$, where $Y^\tau = Y$ and $Z^\tau = Z$. In particular for $\mathfrak{s}'' = \mathfrak{sp}(4)$ and its Cartan subalgebra $\mathfrak{c}'' = \mathbb{K}(E_{11} - E_{44}) + \mathbb{K}(E_{22} - E_{33})$ we have $\mathfrak{c}'' = \widetilde{\mathfrak{c}}$, with $\widetilde{\mathfrak{c}}$ being as in Example 3.6.

Note that here $\mathfrak{h}_1 \subseteq \mathfrak{c}'' \subseteq \mathfrak{h}$. Next define $\mathfrak{b}'' = \mathfrak{c}'' \oplus \mathfrak{n}''$, where

$$\mathfrak{n}'' = \mathrm{span}_{\mathbb{K}}\{E_{12} - E_{34}, E_{13} + E_{24}, E_{14}, E_{23}\},$$

the nilpotent radical taken in the obvious way. Clearly, the pair of Borel subalgebras $(\mathfrak{b}, \mathfrak{b}'')$ is compatible. But at the same time $E \notin \mathfrak{b}''$, and so the pair $(\mathfrak{b}'', \mathfrak{b}_1)$ is not compatible.

4. The role of principal TDS

Throughout this section we assume that the reader is familiar with [Ks1]. In particular, we use the Kostant's terminology. Let us begin by the following proposition. Although quite simple it provides a crucial observation in our approach.

Proposition 4.1. *Suppose we have a pair $(\mathfrak{g}, \mathfrak{g}_1)$ satisfying the condition $(\mathbf{C})$. Suppose also that a subalgebra $\mathfrak{a} \subseteq \mathfrak{g}_1$ is such that $\mathfrak{a}$ is self-normalizing in $\mathfrak{g}$. Then $\mathfrak{g}_1$ is self-normalizing in $\mathfrak{g}$ as well.*

Proof. Take some $x \in N_{\mathfrak{g}}(\mathfrak{g}_1)$. According to the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$, write $x = x_1 + p$. Using the inclusion $[\mathfrak{g}_1, \mathfrak{p}] \subseteq \mathfrak{p}$ (Lemma 1.2), it immediately follows that

$$[p, \mathfrak{g}_1] = 0.$$

Thus in particular $[p, \mathfrak{a}] = 0$. Hence, as $\mathfrak{a}$ is self-normalizing in $\mathfrak{g}$, we have that $p \in \mathfrak{a}$. By the fact $\mathfrak{a} \subseteq \mathfrak{g}_1$, we conclude that $p \in \mathfrak{p} \cap \mathfrak{g}_1 = 0$. This means that $x = x_1 \in \mathfrak{g}_1$. ■

Concerning the condition $(\mathbf{Q1})$, stated in the Introduction, for the convenience of the reader and later use we formulate the following result. It might be understood as a "weak version" of Theorem 3.5 in [Š3]; see also Remark 3.8 there.

Lemma 4.2. *Suppose $\mathfrak{g}$ is a semisimple $\mathbb{K}$-Lie algebra, and $\mathfrak{g}_1$ is a subalgebra reductive in $\mathfrak{g}$. Let $\mathfrak{c}_1$ and $\mathfrak{c}$ be any Cartan subalgebras of $\overline{\mathfrak{g}_1}$ and $\overline{\mathfrak{g}}$, respectively, such that $\mathfrak{c}_1 \subseteq \mathfrak{c}$. Suppose that for every root $\phi \in \Delta(\overline{\mathfrak{g}}, \mathfrak{c})$, the restriction*

$$\phi_{|\mathfrak{c}_1} \neq 0.$$

Then the condition $(\mathbf{Q1})$ holds. More precisely, for $\mathfrak{h}_1$ given, the subalgebra $\mathfrak{h}$ is equal to the centralizer $C_{\mathfrak{g}}(\mathfrak{h}_1)$.

We also state the following auxiliary result. It is an easy consequence of Lemma 1.1.

Lemma 4.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{s}$ any TDS. Then the pair $(\mathfrak{g}, \mathfrak{s})$ satisfies the condition $(\mathbf{C})$.*

Let again $\mathfrak{g}$ be a semisimple Lie algebra. Suppose $\mathfrak{h}$ is a split Cartan subalgebra. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. For a choice of a basis Π of Δ, let $\Delta^{\pm}$ be the set of positive/negative roots. Given a root $\phi \in \Delta^{\pm}$, we know that $\phi = \sum_{\alpha\in\Pi} n_\alpha \alpha$, where $\pm n_\alpha \in \mathbb{N}_0$ for all α. Recall that the *level*, or *order*, of ϕ is given as

$$o(\phi) = \sum_{\alpha\in\Pi} n_\alpha.$$

Under the above setting we introduce the following terminology.

Definition 4.4. A nilpotent element

$$\mathsf{e} = \sum_{\phi\in\Delta^+} c_\phi X_\phi \in \bigoplus_{\phi\in\Delta^+} \mathfrak{g}_\phi, \qquad c_\phi \in \mathbb{K},$$

is called a (positive) *principal nilpotent* element if

$$c_\phi \neq 0, \qquad \forall \phi \in \Pi.$$

A subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, defined by

$$\mathfrak{s} = \mathrm{span}_{\mathbb{K}}\{\mathsf{f}, \mathsf{h}, \mathsf{e}\},$$

is called a *principal* TDS if e is a principal nilpotent element.

Now we are ready for a proof of our third main result, i.e. Theorem 0.3.

Proof. First notice, by a careful inspection of the argument which follows, how we may assume that the base field $\mathbb{K}$ is algebraically closed; the details for checking this will be left to the reader.

By Proposition 4.1, we know that $(\mathfrak{g}, \mathfrak{g}_1)$ satisfies $(\mathbf{Q2})$. We have to see that it satisfies $(\mathbf{Q1})$ as well. For that purpose suppose to the contrary; i.e., let $\mathfrak{h}_1$ be a Cartan subalgebra of $\mathfrak{g}_1$ for which there are two distinct Cartan subalgebras $\mathfrak{h}$, $\mathfrak{h}'$ of $\mathfrak{g}$ that both contain $\mathfrak{h}_1$.

Let now $\mathfrak{c}'$ be an arbitrary Cartan subalgebra of $\mathfrak{s}$. Then $\mathfrak{c}'$ is a commutative subalgebra of $\mathfrak{g}_1$ all of whose elements are semisimple, when understood as elements of $\mathfrak{g}_1$; here we use that $\mathfrak{s}$ is reductive in $\mathfrak{g}_1$. So there exists a Cartan subalgebra $\mathfrak{h}_1'$ of $\mathfrak{g}_1$ such that $\mathfrak{c}' \subseteq \mathfrak{h}_1'$. Now, if we denote by G_1 the adjoint group of $\mathfrak{g}_1$, there exists $g \in G_1$ such that $g.\mathfrak{h}_1' = \mathfrak{h}_1$. Define $\mathfrak{c} = g.\mathfrak{c}'$. It is clear that $\mathfrak{c} \subseteq \mathfrak{h}_1$. Furthermore, $\mathfrak{c}$ is a Cartan subalgebra of $\widetilde{\mathfrak{s}} = g.\mathfrak{s}$.

Finally, as a conclusion of all that we noticed, it follows that the pair $(\mathfrak{g}, \widetilde{\mathfrak{s}})$ does not satisfy $(\mathbf{Q1})$; a contradiction. ∎

Theorem 0.3 suggests that it would be useful to study the set of all subalgebras, of a given (complex) Lie algebra $\mathfrak{g}$, that satisfy the following definition.

Definition 4.5. Let $\mathfrak{g}$ be a semisimple Lie algebra. A reductive subalgebra $\mathfrak{g}_1$ is a *principal subalgebra* of $\mathfrak{g}$ if there exists a principal TDS $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{s} \subseteq \mathfrak{g}_1$.

Suppose now that $\mathfrak{g}$ is a complex simple Lie algebra, and $\mathfrak{s} \subseteq \mathfrak{g}$ is a principal TDS. Then, by Lemma 4.3, the pair $(\mathfrak{g}, \mathfrak{s})$ satisfies $(\mathbf{C})$. Next we have $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is as usual. By a Kostant's analysis in [Ks1] we in particular know that $\mathfrak{p}$, as an $\mathfrak{s}$-module for the adjoint representation, is given as follows:

$$\mathfrak{p} = V_1 \oplus \cdots \oplus V_{n-1}.$$

Here every V_i is a simple module, and $\dim V_i$ is odd and ≥ 5. Moreover, n is the rank of $\mathfrak{g}$, and $\dim V_i \neq \dim V_j$ whenever $i \neq j$. Using this it is clear that $N_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{s}$; i.e., $(\mathfrak{g}, \mathfrak{s})$ satisfies $(\mathbf{Q2})$. Furthermore, it turns out that this pair satisfies $(\mathbf{Q1})$ as well. Thus we have the following.

Theorem 4.6. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{s}$ be a principal TDS. Then the pair $(\mathfrak{g}, \mathfrak{s})$ is from the class $\boldsymbol{\mathcal{P}}$.*

Let now $\mathfrak{g}$ be a semisimple $\mathbb{K}$-Lie algebra, where $\mathbb{K}$ is any field of characteristic zero. Let $\mathfrak{s}$ be a principal TDS. The purpose of the rest of this section is to give a strong support to the following, somewhat loosely formulated, observation.

Observation 4.7. A number of such pairs $(\mathfrak{g}, \mathfrak{s})$ belong to the class $\boldsymbol{\mathcal{P}}$.

In particular the presented arguments will be sufficient to obtain Theorem 0.2.

Kostant's approach in [Ks1] relies, at some crucial points, on the fact that the base field $\mathbb{K}$ equals $\mathbb{C}$. Although it is very likely that the main conclusions of that paper hold for any algebraically closed field $\mathbb{K}$ of characteristic zero, we do not know whether this is indeed the case. So in order to deduce Theorem 0.2 we present another approach, which is in a sense "roundabout"; and do not intend to imitate the full force of Kostant's deep insights. Besides, it provides particular information concerning the following general question which is interesting in its own right.

Question 4.8. Given any semisimple Lie algebras $\mathfrak{g} \subseteq \mathfrak{G}$, when we have the following:

($\bullet$) *There exists a principal nilpotent element $\mathsf{e} \in \mathfrak{g}$ so that at the same time e is principal nilpotent when it is considered as an element of $\mathfrak{G}$?*

Our strategy, concerning Observation 4.7, is as follows. First, with no loss of generality, we will assume that $\mathfrak{g}$ is simple. Then we will search for a convenient embedding

$$\mathfrak{g} \hookrightarrow \mathfrak{G} = \mathfrak{G}_m = \mathfrak{sl}(m, \mathbb{K})$$

which *has* an appropriate e as in ($\bullet$) of Question 4.8. Now having such e we will find $\mathsf{h}, \mathsf{f} \in \mathfrak{g}$ so that $\{\mathsf{f}, \mathsf{h}, \mathsf{e}\}$ is a standard triple. Thus $\mathfrak{s} = \mathrm{span}_{\mathbb{K}}\{\mathsf{f}, \mathsf{h}, \mathsf{e}\}$ will be a principal TDS of $\mathfrak{g}$. But as $\mathfrak{s}$ is also a principal TDS of $\mathfrak{G}$, we have that in particular $N_{\mathfrak{G}}(\mathfrak{s}) = \mathfrak{s}$; here we use Theorem 0.1(I). Of course, as a consequence, $\mathfrak{s}$ is self-normalizing in $\mathfrak{g}$ as well; i.e., $(\mathbf{Q2})$ holds. By Lemmas 4.2 and 4.3, the conditions $(\mathbf{C})$ and $(\mathbf{Q2})$ will hold as well.

For further needs we also formulate the following obvious lemma.

Lemma 4.9. *Suppose that $\mathbb{K}$ is algebraically closed. Let $\mathfrak{g}$ be a Lie algebra, and G be its adjoint group. Let $\mathfrak{s}$ and $\mathfrak{s}'$ be any G-conjugated subalgebras of $\mathfrak{g}$; i.e., $\mathfrak{s}' = g \cdot \mathfrak{s}$, for some $g \in G$. Then the normalizers $N_{\mathfrak{g}}(\mathfrak{s})$ and $N_{\mathfrak{g}}(\mathfrak{s}')$ are G-conjugated as well. In particular, $\mathfrak{s}$ is self-normalizing in $\mathfrak{g}$ if and only if $\mathfrak{s}'$ is of the same kind.*

As it will be shown below, the above explained will work when $\mathfrak{g}$ is classical of type C_n or B_n. But it will not work for $\mathfrak{g}$ of type D_n; see Remark 4.11. Now we will prove by direct computation that the corresponding pairs $(\mathfrak{g}, \mathfrak{s})$ belong to $\boldsymbol{\mathcal{P}}$. Finally, as an illustration we will also show the latter claim when $\mathfrak{g}$ is an exceptional Lie algebra of type G_2. Again our proof will be a straightforward computation.

Let for the moment $\mathbb{K}$ be an arbitrary field. Given $i \in \mathbb{N}$, define $\boldsymbol{s}_i$ to be the i-by-i matrix having 1 on the skew diagonal and 0 elsewhere. Now for a matrix $M \in M_{kl}(\mathbb{K})$ we define

$$M^\tau = \boldsymbol{s}_l M^t \boldsymbol{s}_k;$$

the map $M \mapsto M^\tau$ is the *skew transpose.* Notice this obvious fact: For matrices $M_1 \in M_{jk}(\mathbb{K})$ and $M_2 \in M_{kl}(\mathbb{K})$ we have

$$(M_1 M_2)^\tau = M_2^\tau M_1^\tau.$$

4.1. $\mathfrak{g}$ of type C_n.

Let now $\mathfrak{G} = \mathfrak{G}_{2n} = \mathfrak{sl}(2n, \mathbb{K})$. As in Sect. 2 of [Š4], for $\varepsilon = \pm 1$ define a map $A \mapsto A^\dagger = A^\dagger_\varepsilon$, on block-matrices, given by

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = A \longmapsto A^\dagger = \begin{pmatrix} T^\tau & \varepsilon Y^\tau \\ \varepsilon Z^\tau & X^\tau \end{pmatrix}.$$

Next define a Lie algebra

$$\begin{aligned} \mathfrak{g}^\varepsilon &= \{A \in \mathfrak{G} \mid A^\dagger = -A\} \\ &= \left\{ \begin{pmatrix} X & Y \\ Z & -X^\tau \end{pmatrix} \in M_{2n}(\mathbb{K}) \;\middle|\; Y + \varepsilon Y^\tau = Z + \varepsilon Z^\tau = 0 \right\}; \end{aligned}$$

of course, $\mathfrak{g}^- = \mathfrak{sp}(2n, \mathbb{K})$ and $\mathfrak{g}^+ = \mathfrak{so}(2n, \mathbb{K})$. As the map $\Theta = \Theta^\varepsilon$, given by $A \mapsto -A^\dagger_\varepsilon$, is an involutive automorphism of $\mathfrak{G}$, we have the following (cf. [Š3], Corollary 4.5 and Theorem 4.6]).

Lemma 4.10. *The pairs $(\mathfrak{G}, \mathfrak{g}^\varepsilon)$ are symmetric, and thus in particular they are from the class $\boldsymbol{\mathcal{P}}$.*

Symplectic case; i.e., $\varepsilon = -1$. Define diagonal matrices

$$H_i = E_{ii} - E_{2n+1-i, 2n+1-i}, \qquad 1 \le i \le n.$$

Let

$$\mathfrak{h} = \operatorname{span}_{\mathbb{K}}\{H_1, \ldots, H_n\};$$

$\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{g}^-$. Let $\epsilon_i \in \mathfrak{h}^*$ define the dual basis, i.e., $\epsilon_i(H_j) = \delta_{ij}$. For a basis of the corresponding root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ we take

$$\Pi = \{\alpha_1, \ldots, \alpha_n\},$$

where $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $1 \leq i < n$, and $\alpha_n = \alpha_n^- = 2\epsilon_n$. We also choose

$$X_{\alpha_i} = \begin{cases} E_{i,i+1} - E_{2n-i,2n-i+1} & \text{for } 1 \leq i < n, \\ E_{n,n+1} & \text{for } i = n. \end{cases}$$

Put

$$\mathsf{e} = \sum_{i=1}^{n} X_{\alpha_i} \in \mathfrak{g}.$$

For this e we clearly have the condition $(\bullet)$ of Question 4.8 fulfilled.

Remark 4.11. Analogously as above, we can consider the *even orthogonal case*, i.e., $\varepsilon = 1$. We have the same Cartan subalgebra $\mathfrak{h}$, and dual basis (ϵ_i). Also, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, with only difference that now $\alpha_n = \alpha_n^+ = \epsilon_{n-1} + \epsilon_n$; and correspondingly,

$$X_{\alpha_n} = E_{n-1,n+1} - E_{n,n+2}.$$

But a problem here is that there is no $\mathsf{e} \in \mathfrak{g} = \mathfrak{g}^+$ which is principal nilpotent so that at the same time e is principal nilpotent while considered as an element of $\mathfrak{G}$. Thus, as we already said, the case when $\mathfrak{g}$ is of type D_n must be treated in a different way.

4.2. $\mathfrak{g}$ of type B_n.

Let now $m = 2n+1$ and $\mathfrak{G} = \mathfrak{G}_m = \mathfrak{sl}(m, \mathbb{K})$. Similarly as in Sect. 2 of [Š5], define a map $A \mapsto A^\dagger$. This map, on block-matrices, is given by

$$\begin{pmatrix} a & \boldsymbol{r_1} & \boldsymbol{r_2} \\ \boldsymbol{c_1} & X & Y \\ \boldsymbol{c_2} & Z & T \end{pmatrix} = A \longmapsto A^\dagger = \begin{pmatrix} a & \boldsymbol{c_2}^\tau & \boldsymbol{c_1}^\tau \\ \boldsymbol{r_2}^\tau & T^\tau & Y^\tau \\ \boldsymbol{r_1}^\tau & Z^\tau & X^\tau \end{pmatrix};$$

here: $a \in \mathbb{K}$, $\boldsymbol{r_i} \in M_{1n}(\mathbb{K})$, $\boldsymbol{c_i} \in M_{n1}(\mathbb{K})$ and $X, Y, Z, T \in M_n(\mathbb{K})$. Define a Lie algebra

$$\mathfrak{g} = \{A \in \mathfrak{G} \mid A^\dagger = -A\};$$

an easy computation shows that $A \in \mathfrak{g}$ if and only if

$$A = \begin{pmatrix} 0 & \boldsymbol{r_1} & \boldsymbol{r_2} \\ -\boldsymbol{r_2}^\tau & X & Y \\ -\boldsymbol{r_1}^\tau & Z & -X^\tau \end{pmatrix},$$

where $Y^\tau + Y = Z^\tau + Z = 0$. Analogously as in the previous subsection, $\Theta : A \mapsto -A^\dagger$ is an involutive automorphism of $\mathfrak{G}$. Hence the following analogue of Lemma 4.10.

Lemma 4.12. *The pair $(\mathfrak{G}, \mathfrak{g})$ is symmetric, and thus in particular it is from the class $\boldsymbol{\mathcal{P}}$.*

Let us now agree to count the rows and columns of matrices $A \in \mathfrak{g}$ as $0, 1, \ldots, 2n$. Then define diagonal matrices $H_1, \ldots, H_n$ to be the same as for the symplectic case. Let also $\mathfrak{h}$, (ϵ_i), $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and X_{α_i} be as there, with only difference that now $\alpha_n = \epsilon_n$; and correspondingly,

$$X_{\alpha_n} = -E_{n,n+1} + E_{n+1,n+2}.$$

Again the element e, given as in the previous subsection, satisfies $(\bullet)$ of Question 4.8.

4.3. $\mathfrak{g}$ of type D_n.

Let $\mathfrak{g} = \mathfrak{g}^+ \subseteq \mathfrak{G} = \mathfrak{G}_{2n}$, and H_i, ϵ_i, α_i and X_{α_i}, for $1 \leq i \leq n$, be as in Subsection 4; see Remark 4.11. Let also $\mathfrak{h}$, Π and e be the same as there. In particular e is principal nilpotent, as an element of $\mathfrak{g}$. We need an element $\mathsf{h} \in \mathfrak{h}$ so that $[\mathsf{h}, \mathsf{e}] = 2\mathsf{e}$; i.e., $\alpha_i(\mathsf{h}) = 2$, for $1 \leq i \leq n$. It immediately follows that

$$\mathsf{h} = 2\sum_{i=1}^{n-1}(n-i)H_i.$$

Next we need an element $\mathsf{f} \in \mathfrak{g}$ so that $[\mathsf{e}, \mathsf{f}] = \mathsf{h}$; it is clear that then $[\mathsf{h}, \mathsf{f}] = -2\mathsf{f}$ as well. Writing

$$\mathsf{f} = \sum_{i=1}^{n} a_i X_{-\alpha_i}, \qquad a_i \in \mathbb{K},$$

we obtain

$$[\mathsf{e}, \mathsf{f}] = \sum_{i=1}^{n-1} a_i(H_{i+1} - H_i) - a_n(H_n + H_{n-1}).$$

Hence an easy computation gives that

$$a_{n-1} = a_n = -\sum_{j=1}^{n-1} j \qquad \text{and} \qquad a_i = -2\sum_{j=1}^{i}(n-j), \quad 1 \leq i \leq n-2.$$

Thus we have a standard triple $\{\mathsf{f}, \mathsf{h}, \mathsf{e}\}$, and so $\mathfrak{s} = \operatorname{span}_{\mathbb{K}}\{\mathsf{f}, \mathsf{h}, \mathsf{e}\}$, a principal TDS.

Claim. The pair $(\mathfrak{g}, \mathfrak{s})$ is from the class $\boldsymbol{\mathcal{P}}$.

Proof. (I) As the first step we will show that $\mathfrak{s}$ is self-normalizing in $\mathfrak{g}$. For that purpose, suppose that $x \in N_{\mathfrak{g}}(\mathfrak{s})$. Let us write $x = x^- + x^0 + x^+$, where $x^0 \in \mathfrak{h}$, and

$$x^{\pm} = \sum_{\phi \in \Delta^+} c_{\pm\phi} X_{\pm\phi}, \qquad c_{\pm\phi} \in \mathbb{K}.$$

Notice that

$$[\mathsf{h}, x] = [\mathsf{h}, x^-] + [\mathsf{h}, x^+] \in \mathfrak{s}.$$

Then we consider

$$[\mathsf{h}, x^+] = \sum_{\phi\in\Delta^+} c_\phi \phi(\mathsf{h}) X_\phi.$$

Suppose that there is some $\phi_0 \in \Delta^+$ such that $o(\phi_0) < 1$ and $c_{\phi_0} \neq 0$. It immediately follows that then $\phi_0(\mathsf{h}) \neq 0$, and as a consequence,

$$x^\pm = \sum_{i=1}^n c_i^\pm X_{\pm\alpha_i}, \qquad c_i^\pm \in \mathbb{K}.$$

Now we have $[x, \mathsf{e}] = \Omega + [x^+, \mathsf{e}] \in \mathfrak{s}$, where $\Omega = [x^-, \mathsf{e}] + [x^0, \mathsf{e}]$, and also

$$[x^+, \mathsf{e}] = \sum_{1\le i<j\le n} (c_i^+ - c_j^+)[X_{\alpha_i}, X_{\alpha_j}].$$

As it holds that

$$\{\phi \in \Delta^+ \mid o(\phi) = 2\} = \{\alpha_i + \alpha_{i+1} \mid 1 \le i \le n-2\} \cup \{\alpha_{n-2} + \alpha_n\},$$

it is easy to deduce that $c_1^+ = \cdots = c_n^+ = c^+$. That is, $x^+ = c^+\mathsf{e}$, for some c^+. Analogously, $x^- = c^-\mathsf{f}$, for some c^-. Hence,

$$x = c^-\mathsf{f} + x^0\mathsf{h} + c^+\mathsf{e}.$$

Notice now how $[x, \mathsf{e}] \in \mathfrak{s}$ implies that

$$[x^0, \mathsf{e}] = \sum_{i=1}^n \alpha_i(x^0) X_{\alpha_i} \in \mathfrak{s}.$$

Obviously,

$$\alpha_1(x^0) = \cdots = \alpha_n(x^0).$$

Denoting the latter element by ω, and supposing that $\omega \neq 0$, it follows at once that $x^0 = c^0\mathsf{h}$, where $c^0 = \omega/2 \in \mathbb{K}$. Thus we have proved that $x \in \mathfrak{s}$.

(II) As we have $\alpha_i(\mathsf{h}) = 2$, for every i, it is clear that $\phi(\mathsf{h}) \in 2\mathbb{N}$, for every $\phi \in \Delta^+$. Define now $\mathfrak{h}_1 = \mathbb{K}\mathsf{h}$, a Cartan subalgebra of $\mathfrak{s}$. By Lemma 4.2 it is then obvious that $\mathfrak{h}$ is indeed a unique Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}_1$. ∎

4.4. $\mathfrak{g}$ of type G_2.

Here we will first recall a well known realization of a simple Lie algebra of type G_2 as a subalgebra of $\mathfrak{so}(7, \mathbb{K})$. More details can be found in Sect. 2 of [LS], and Sect. 3 of [Š1].

Consider a standard embedding $\mathfrak{g} = \mathfrak{so}(8, \mathbb{K}) \hookrightarrow \mathfrak{sl}(8, \mathbb{K})$. Let $\mathfrak{h} = \mathrm{span}_{\mathbb{K}}\{H_1, \ldots, H_4\}$, a split Cartan subalgebra, and (ϵ_i) be the dual basis; see Remark 4.11 again.

Define now $\mathfrak{h}_1 = \mathrm{span}_{\mathbb{K}}\{H_1, H_2, H_3\}$, and let then η_i be the restrictions of ϵ_i to $\mathfrak{h}_1$, for $1 \le i \le 3$. Put

$$\Delta_1^+ = \{\eta_1, \eta_2, \eta_3\} \cup \{\eta_i \pm \eta_j \mid 1 \le i < j \le 3\},$$

and also

$$\Pi_1 = \{\beta_1, \beta_2, \beta_3\}, \qquad \text{where } \beta_1 = \eta_1 - \eta_2, \ \ \beta_2 = \eta_2 - \eta_3, \ \ \beta_3 = \eta_3.$$

For $\Delta_1 = \Delta_1^+ \cup (-\Delta_1^+)$ and conveniently chosen (root) vectors X_ϕ, $\phi \in \Delta_1$, we have that

$$\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \bigoplus_{\phi \in \Delta_1} \mathbb{K} X_\phi$$

is a simple Lie algebra of type B_3; i.e., $\mathfrak{g}_1 = \mathfrak{so}(7, \mathbb{K})$.

Let σ be an order 3 automorphism, of the Dynkin diagram of type D_4, satisfying $\sigma(\epsilon_3 + \epsilon_4) = \epsilon_3 - \epsilon_4$; here D_4-diagram is labeled as in [LS]. By the same letter σ we denote a unique extension to an automorphism of $\mathfrak{g}$. Let $\mathfrak{g}_2 = \mathfrak{g}^\sigma$, the fixed point algebra for σ. Then $\mathfrak{g}_2$ is a simple Lie algebra of type G_2, and $\mathfrak{h}_2 = \mathfrak{g}_2 \cap \mathfrak{h}$ is its split Cartan subalgebra. Let $\Pi_2 = \{\alpha_1, \alpha_2\}$ be a basis of the root system $\Delta_2 = \Delta(\mathfrak{g}_2, \mathfrak{h}_2)$, where α_1 (resp. α_2) is the short (resp. long) simple root. Then in particular for the root vectors, corresponding to α_1 and α_2, we have:

$$X_{\alpha_1} = X_{\beta_1} + X_{\beta_3}, \qquad X_{\alpha_2} = X_{\beta_2}.$$

Therefore

$$\mathsf{e} = X_{\alpha_1} + X_{\alpha_2}$$

is a principal nilpotent element of $\mathfrak{g}_2$. But it is also principal nilpotent as an element of $\mathfrak{g}_1$.

Define now $H_\phi \in \mathfrak{h}_2$, for $\phi \in \Delta_2$, as usual. We want to find $\mathsf{h} = u_1 H_{\alpha_1} + u_2 H_{\alpha_2}$ such that $[\mathsf{h}, \mathsf{e}] = 2\mathsf{e}$. Hence, using that $\alpha_2(H_{\alpha_1}) = -3$ and $\alpha_1(H_{\alpha_2}) = -1$, it follows that $u_1 = 6$ and $u_2 = 10$, i.e.,

$$\mathsf{h} = 6H_{\alpha_1} + 10H_{\alpha_2}.$$

Next we need f such that $[\mathsf{e}, \mathsf{f}] = \mathsf{h}$. Proceeding as for h and using the equalities $[X_\phi, X_{-\phi}] = -H_\phi$, for $\phi \in \Delta_2$, it is easy to see that in fact

$$\mathsf{f} = -6X_{-\alpha_1} - 10X_{-\alpha_2}.$$

Now we consider a principal TDS $\mathfrak{s} = \mathrm{span}_{\mathbb{K}}\{\mathsf{f}, \mathsf{h}, \mathsf{e}\}$, as before. Let $\mathfrak{c} = \mathbb{K}\mathsf{h}$, a Cartan subalgebra. Knowing all the positive roots, it immediately follows that

$$\phi(\mathsf{h}) \in \{\pm 2, \pm 4, \pm 6, \pm 8, \pm 10\}, \qquad \phi \in \Delta_2^+.$$

By Lemma 4.2, $\mathfrak{h}_2$ is a unique Cartan subalgebra of $\mathfrak{g}_2$ containing $\mathfrak{c}$. Thus we have "one half" of the following

Claim. The pair $(\mathfrak{g}_2, \mathfrak{s})$ is from the class $\boldsymbol{\mathcal{P}}$.

For the "second half" we have to see that $N_{\mathfrak{g}_2}(\mathfrak{s}) = \mathfrak{s}$. But as e is principal nilpotent in $\mathfrak{g}_1$ as well, by what we know from Subsection 4, it follows that moreover $N_{\mathfrak{g}_1}(\mathfrak{s}) = \mathfrak{s}$.

5. On trivial extension

Suppose for the moment that $\mathfrak{r}$ is an arbitrary reductive Lie algebra. For any $f \in \mathfrak{r}^*$ define an alternating bilinear form $\beta_f(x,y) = f([x,y])$. For any $S \subseteq \mathfrak{r}$, define a subspace $S^f = \{x \in \mathfrak{r} \mid \beta_f(x,S) = 0\}$. In particular, the radical $\mathfrak{r}^f$ of β_f is a Lie subalgebra of $\mathfrak{r}$. Recall that f is a nilpotent element of $\mathfrak{r}^*$ if $f(\mathfrak{r}^f) = 0$. Also, f is semisimple if $\mathfrak{r}^f$ is reductive in $\mathfrak{r}$. For a nondegenerate symmetric invariant bilinear form ϕ on $\mathfrak{r}$, define an isomorphism $\mathcal{K} = \mathcal{K}_\phi : \mathfrak{r} \to \mathfrak{r}^*$, $\mathcal{K}(x) = \phi(x, .)$. Then $\mathfrak{r}^f = \mathfrak{r}^{\mathcal{K}^{-1}(f)}$, the centralizer of $\mathcal{K}^{-1}(f)$ in $\mathfrak{r}$.

Let, again, $(\mathfrak{g}, \mathfrak{g}_1)$ be a pair where $\mathfrak{g}$ is semisimple. The following proposition is interesting in its own right.

Proposition 5.1. *For any $\mu \in \mathfrak{g}_1^*$, the vector subspace*

$$\mathfrak{s}(\mu) = T(\mu) + [T(\mu), T(\mu)]$$

is a subalgebra of $\mathfrak{g}$, contained in $\mathfrak{g}^{\mathcal{E}(\mu)}$. Furthermore, $\mathcal{E}(\mu)_{|\mathfrak{s}(\mu)} = 0$.

Proof. Let us write $\mathfrak{s} = \mathfrak{s}(\mu)$, and define

$$\mathfrak{s}' = T(\mu) + \pi([T(\mu), T(\mu)]).$$

We will first show that $\mathfrak{s} = \mathfrak{s}'$. For that purpose take any $z_1, z_2 \in T(\mu)$, and then decompose $[z_1, z_2] = y + w$, $y \in \mathfrak{g}_1$ and $w \in \mathfrak{p}$. Note that

$$\begin{aligned}\mathcal{E}(\mu)([w,\mathfrak{p}]) &= \mathcal{E}(\mu)([[z_1,z_2],\mathfrak{p}]) \\ &= \mathcal{E}(\mu)([z_2,[\mathfrak{p},z_1]]) - \mathcal{E}(\mu)([z_1,[\mathfrak{p},z_2]]) = 0.\end{aligned} \tag{12}$$

Thus we conclude that $w \in T(\mu)$, and hence it is clear that

$$[T(\mu), T(\mu)] \subseteq \mathfrak{s}'; \tag{13}$$

i.e., $\mathfrak{s} \subseteq \mathfrak{s}'$. On the other hand, since $y = \pi([z_1, z_2]) = [z_1, z_2] - w \in \mathfrak{s}$, we have $\pi([T(\mu), T(\mu)]) \subseteq \mathfrak{s}$ and so $\mathfrak{s}' \subseteq \mathfrak{s}$, as we claimed.

Let now $z, z_1, z_2 \in T(\mu)$ be arbitrary, and write again $[z_1, z_2] = y + w$. Then we have

$$\begin{aligned}\mathcal{E}(\mu)([[z,y],\mathfrak{p}]) &= -\mathcal{E}(\mu)([[\mathfrak{p},z],y]) = -\mathcal{E}(\mu)([[\mathfrak{p},z],[z_1,z_2]]) \\ &= \mathcal{E}(\mu)([z_2,[[\mathfrak{p},z],z_1]]) - \mathcal{E}(\mu)([z_1,[[\mathfrak{p},z],z_2]]) = 0.\end{aligned}$$

Thus we have $[z,y] \in T(\mu)$, and therefore

$$[T(\mu), \pi([T(\mu), T(\mu)])] \subseteq T(\mu); \tag{14}$$

note that (13) and (14) imply the inclusion

$$[T(\mu), \mathfrak{s}'] \subseteq \mathfrak{s}'. \tag{15}$$

For the next step take $z_1, z_2, z_1', z_2' \in T(\mu)$, and decompose $[z_1', z_2'] = y' + w'$ and $[z_1, z_2] = y + w$ as before. Since we know that $w, w' \in T(\mu)$, (13) implies that $[w, w'] \in \mathfrak{s}'$. Also, by (15), we have $[[z_1, z_2], w'] \in \mathfrak{s}'$; and, analogously, $[w, [z_1', z_2']] \in \mathfrak{s}'$. For the element

$$[[z_1, z_2], [z_1', z_2']] = [[[z_1, z_2], z_1'], z_2'] + [z_1', [[z_1, z_2], z_2']],$$

by considering the right-hand side and using (15), again, we conclude that it is also in $\mathfrak{s}'$. It remained to note that then

$$[y, y'] = [[z_1, z_2], [z_1', z_2']] - [[z_1, z_2], w'] - [w, [z_1', z_2']] + [w, w']$$

is in $\mathfrak{s}'$ as well. Therefore we have the inclusion

$$[\pi([T(\mu), T(\mu)]), \pi([T(\mu), T(\mu)])] \subseteq \mathfrak{s}'. \tag{16}$$

By (15) and (16) we see that $\mathfrak{s}' = \mathfrak{s}(\mu)$ is a subalgebra of $\mathfrak{g}$.

For the inclusion $\mathfrak{s}(\mu) \subseteq \mathfrak{g}^{\mathcal{E}(\mu)}$, first note that by $\mathfrak{g}^{\mathcal{E}(\mu)} \subseteq \mathfrak{g}_1^{\mathcal{E}(\mu)} \cap \mathfrak{p}^{\mathcal{E}(\mu)}$ and the obvious fact

$$T(\mu) = \mathfrak{g}^{\mathcal{E}(\mu)} \cap \mathfrak{p} = \{z \in \mathfrak{p} \mid \mathcal{E}(\mu)([z, \mathfrak{g}]) = 0\},$$

we have

$$\mathfrak{g}^{\mathcal{E}(\mu)} = \mathfrak{g}_1^{\mu} \oplus T(\mu). \tag{17}$$

Let us show that

$$\pi([T(\mu), T(\mu)]) \subseteq \mathfrak{g}_1^{\mu}. \tag{18}$$

To see this take some $z_1, z_2 \in T(\mu)$, and let $y = \pi([z_1, z_2])$. Similarly as in (12), we have

$$\begin{aligned} \mu([y, \mathfrak{g}_1]) &= -\mu(\pi([\mathfrak{g}_1, [z_1, z_2]])) = -\mathcal{E}(\mu)([\mathfrak{g}_1, [z_1, z_2]]) \\ &= \mathcal{E}(\mu)([z_2, [\mathfrak{g}_1, z_1]]) - \mathcal{E}(\mu)([z_1, [\mathfrak{g}_1, z_2]]) = 0. \end{aligned}$$

Thus (18) follows.

To finish the proof of the proposition note that, by definition of $T(\mu)$,

$$\mathcal{E}(\mu)([T(\mu), T(\mu)]) = 0,$$

and thus $\mathcal{E}(\mu)(\mathfrak{s}(\mu)) = 0$ as well. ■

As we will see in the following instructive example, in general, $\mathfrak{s}(\mu)$ and $\mathfrak{g}^{\mathcal{E}(\mu)}$ are not equal. Note also that mostly $\mathfrak{s}(\mu)$ will not be an intermediate subalgebra for the inclusion $\mathfrak{g}_1^{\mu} \subseteq \mathfrak{g}^{\mathcal{E}(\mu)}$.

Example 5.2. Consider the (symmetric) pair $(\mathfrak{g}, \mathfrak{g}_1) = (\mathfrak{sl}(3), \mathfrak{sl}(2))$ as in Section 3. Let H, E, F, L, $\boldsymbol{X} = \boldsymbol{X}_1$, $\boldsymbol{Y} = \boldsymbol{Y}_1$ and $\mathfrak{p} = \mathfrak{p}(2)$ be as there; see the proof of (ii) in Theorem 3.2. Let $0 \neq \mu \in \mathfrak{g}_1^*$ be given by $\mu(H) = a$, $\mu(E) = b$ and $\mu(F) = c$, for some $a, b, c \in \mathbb{K}$. We consider the following four possibilities: (I) $c \neq 0$; (II) $a = c = 0$ and $b \neq 0$; (III) $c = 0$ and $a, b \neq 0$; (IV) $b = c = 0$ and $a \neq 0$. It is easy to check that

$$T(\mu) = \mathbb{K} z_0 \qquad \text{and} \qquad \mathfrak{g}_1^{\mu} = \mathbb{K} y_0,$$

where z_0 and y_0 are given as follows. For (I),

$$z_0 = ((a^2 - 2bc)/12c^2)L - (a/2c)\boldsymbol{X} - (ab/4c^2)\boldsymbol{Y} + E_{13} + (b^2/4c^2)E_{31},$$

and $y_0 = (a/2c)H + E + (b/c)F$. For (II), $z_0 = E_{31}$, and $y_0 = F$. For (III), $z_0 = (a/3b)L - \boldsymbol{Y} + (b/a)E_{31}$, and $y_0 = (a/2b)H + F$. For (IV), $z_0 = L$, and $y_0 = H$. In all the four cases we have $[y_0, z_0] = 0$. Thus $\mathfrak{g}^{\mathcal{E}(\mu)}$ is a 2-dimensional abelian subalgebra of $\mathfrak{g}$. It is also easy to see the following: For (I), μ is nilpotent (resp. semisimple) if $a^2 + 4bc = 0$ (resp. $\neq 0$); For (II), μ is nilpotent; For (III) and (IV), μ is semisimple. Hence it follows that for (III) , (IV) and the case $a^2 + 4bc \neq 0$ in (I), $\mathfrak{g}^{\mathcal{E}(\mu)}$ is a Cartan subalgebra of $\mathfrak{g}$. For the remaining two situations this is not so; now, $\mathfrak{g}^{\mathcal{E}(\mu)}$ is not self-normalizing.

Remark 5.3. Note that

$$\mathfrak{p}^{\mathcal{E}(\mu)} = \mathfrak{g}_1 \oplus T(\mu)$$

is not in general a subalgebra of $\mathfrak{g}$. To see this take some $y \in \mathfrak{g}_1$ and $z \in T(\mu)$. Then $[y, z] \in \mathfrak{p}^{\mathcal{E}(\mu)}$ if and only if $[y, z] \in T(\mu)$ if and only if $\mathcal{E}(\mu)([[\mathfrak{p}, z], y]) = 0$. Let now $(\mathfrak{g}, \mathfrak{g}_1)$ be as in the previous example, and μ as in (II) there. Then, for $z_0 = E_{31}$, $E \in \mathfrak{g}_1$ and $E_{13} \in \mathfrak{p}$, we have $\mathcal{E}(\mu)([[E_{13}, z_0], E]) = \mu(E) \neq 0$. Hence, $[\mathfrak{g}_1, T(\mu)] \not\subseteq \mathfrak{p}^{\mathcal{E}(\mu)}$.

Suppose now that $\mathbb{K}$ is algebraically closed. We conclude by this useful observation concerning the trivial extension.

Proposition 5.4. *Let $(\mathfrak{g}, \mathfrak{g}_1)$ be a pair where $\mathfrak{g}$ is semisimple. Assume that $\mathfrak{g}_1$ is reductive in $\mathfrak{g}$. Then we have the following: $\mu \in \mathfrak{g}_1^*$ is semisimple (resp. nilpotent) if and only if $\mathcal{E}(\mu) \in \mathfrak{g}^*$ is semisimple (resp. nilpotent).*

Proof. We may assume that $\mathfrak{g} \leq \mathfrak{gl}(n, \mathbb{K})$, for some n. Let $G \leq \mathrm{GL}(n, \mathbb{K})$ be a connected semisimple algebraic group such that $\mathfrak{g}$ is its Lie algebra. Also, let $G_1 \leq G$ be a connected reductive group with Lie algebra $\mathfrak{g}_1$. Suppose that $\mu \in \mathfrak{g}_1^*$ is semisimple (resp. nilpotent). This means that there exists some semisimple (resp. nilpotent) element $\mathcal{X}_1 \in \mathfrak{g}_1$ such that μ belongs to the coadjoint orbit $G_1.\kappa_1(\mathcal{X}_1)$; see, e.g., Sect. 1.3 in [CM]. In other words, for some $g_1 \in G_1$ we have $\mu = \mathrm{Ad}^* g_1(\kappa_1(\mathcal{X}_1))$. That is, using the invariance of $\boldsymbol{\beta}$,

$$\mu(y_1) = \boldsymbol{\beta}(\mathrm{Ad}\, g_1(\mathcal{X}_1), y_1), \qquad y_1 \in \mathfrak{g}_1.$$

Hence, for arbitrary $\mathfrak{g} \ni y = y_1 + z$, where $y_1 \in \mathfrak{g}_1$ and $z \in \mathfrak{p}$, we have

$$\begin{aligned}\mathcal{E}(\mu)(y) = \mu(y_1) &= B_{\mathfrak{g}}(\mathrm{Ad}\, g_1(\mathcal{X}_1), y) \\ &= \mathrm{Ad}^* g_1(\kappa(\mathcal{X}_1))(y).\end{aligned}$$

Thus we have shown that

$$\mathcal{E}(\mu) = g_1.\kappa(\mathcal{X}_1).$$

Let now $\mathcal{X} \in \mathfrak{g}$ be such that $\kappa(\mathcal{X}) = \mathcal{E}(\mu)$. Write $\mathcal{X} = y_1 + z$, with y_1 and z as before. Then it immediately follows that $B_{\mathfrak{g}}(z, \mathfrak{p}) = 0$. As $B_{\mathfrak{g}}$ is nondegenerate on $\mathfrak{p}$, we conclude that $z = 0$; i.e., $\mathcal{X} \in \mathfrak{g}_1$. Now for arbitrary $u_1 \in \mathfrak{g}_1$ we have

$$\begin{aligned}\boldsymbol{\beta}(\mathcal{X} - \mathcal{X}_1, u_1) &= B_{\mathfrak{g}}(\mathcal{X}, u_1) - \boldsymbol{\beta}(\mathcal{X}_1, u_1) \\ &= \mathcal{E}(\mu)(u_1) - \mu(u_1) = 0.\end{aligned}$$

By $(\mathbf{C})$, $\mathcal{X} = \mathcal{X}_1$. It remained to note the following: By the assumption that $\mathfrak{g}_1$ is reductive in $\mathfrak{g}$, $\mathcal{X}_1$ is semisimple (resp. nilpotent) as an element of $\mathfrak{g}$. ■

Remark 5.5. (1) The part of the previous proposition concerning nilpotent μ's can be obtained quite easily via the proof of Proposition 5.1. Namely, by definitions of $\mathcal{E}(\mu)$ and $T(\mu)$, from the above noted equality (17) it clearly follows that for $\mu \in \mathfrak{g}_1^*$ we have $\mu(\mathfrak{g}_1^\mu) = 0$ if and only if $\mathcal{E}(\mu)(\mathfrak{g}^{\mathcal{E}(\mu)}) = 0$, i.e., μ is nilpotent if and only if $\mathcal{E}(\mu)$ is nilpotent.

(2) Related to Example 5.2 it is also interesting to note the following simple consequence of the above proposition.

Observation. The subalgebra $\mathfrak{g}^{\mathcal{E}(\mu)}$ is reductive in $\mathfrak{g}$ if and only if $\mu \in \mathfrak{g}_1^*$ is semisimple.

Acknowledgments

The author is grateful to David Vogan for some remarks/corrections in the first version of this paper. He is also deeply grateful to the referee for his/her excellent job; which cannot be overestimated. The referee's observations and suggestions resulted with a significant improvement of the submitted version of the paper. In particular, Theorem 0.2 in fact belongs to him/her. The author is also very grateful to the Editor Simon Gindikin for his understanding of the situation in which the author has been during the preparations of the revised versions.

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Received December 4, 2008
and in final form November 9, 2009