Central Extensions of Coverings of Symplectomorphism Groups

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Abstract. Each even dimensional submanifold of a symplectic manifold defines a Lie algebra 2-cocycle on the Lie algebra of symplectic vector fields. We study its integrability to the group of symplectic diffeomorphisms. When the submanifold is symplectic, we describe a coadjoint orbit of the corresponding extension.

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1. Introduction

In this paper we study central extensions of the group of symplectic diffeomorphisms $\text{Symp}(M,\omega)$ of a symplectic manifold $(M,\omega)$, that are defined by a $2k$-dimensional submanifold $N$ of $M$. The corresponding Lie algebra extension is described by the 2-cocycle $\sigma_N$ on the Lie algebra of symplectic vector fields of $(M,\omega)$ by

$$\sigma_N(X,Y) = \int_N i_X i_Y \omega^{k+1}.$$

The main result is that for $H^{2k+1}(M,\mathbb{R}) = 0$ and when the cohomology class $[\omega]^{k+1} \in H^{2k+2}(M,\mathbb{R})$ is integral, such a central extension of $\text{Symp}(M,\omega)$ exists. Moreover, when the submanifold $N$ is symplectic, then a coadjoint orbit of this extension is the connected component containing $N$ of the space of $2k$-dimensional symplectic submanifolds of $M$, the non-linear symplectic Grassmannian $SG_{2k}(M)$.

For $k = 0$ we recover some results from [5], where a group 2-cocycle on $\text{Symp}(M,\omega)$, integrating the Lie algebra 2-cocycle $(X,Y) \mapsto \omega(X,Y)(x)$, $x \in M$, is given under the assumption that the symplectic form $\omega$ is exact. A similar 2-cocycle defined by a $2k$-dimensional submanifold $N$ of $M$ is given in Section 3. Other types of group 2-cocycles on $\text{Symp}(M,\omega)$ were considered in [12] and [13]. In [12] Neretin constructs a group 2-cocycle on the group of compactly supported symplectic diffeomorphisms with the help of a group 2-cocycle on $\text{Sp}(2n,\mathbb{R})$. 

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In general it defines a non-trivial cohomology class, while its restriction to the connected component of the identity is a coboundary. In [13] Reznikov uses the closed $Sp(2n, \mathbb{R})$-invariant closed 2-form on $Sp(2n, \mathbb{R})/U(n)$ due to Borel to define a group 2-cocycle on $\text{Symp}(M, \omega)$.

This work is a natural continuation of [2], since it uses the same objects: the non-linear Grassmannian $Gr_n(M)$ of $n$-dimensional submanifolds of $M$ with forms induced by the tilda map from forms on $M$, and same tools: Kostant’s central extension associated to a prequantizable presymplectic manifold (presymplectic manifold means a manifold with a closed 2-form).

A 2-cocycle similar to $\sigma_N$ is the Lichnerowicz 2-cocycle $\lambda_N$ on the Lie algebra of divergence free vector fields of an $m$-dimensional manifold $M$ with volume form $\nu$ [8] [14]:

$$\lambda_N(X, Y) = \int_N i_X i_Y \nu.$$ 

In [4] Section 25 it is shown that when $\nu$ is an integral volume form, the 2-cocycle $\lambda_N$ is integrable to the group of exact volume preserving diffeomorphisms of $M$. The geometric construction uses Kostant’s extension associated to the codimension two non-linear Grassmannian ($Gr_{m-2}(M), \tilde{\nu}$), which is a prequantizable symplectic manifold with a hamiltonian action of the group of exact volume preserving diffeomorphisms. The connected component containing $N$ is a coadjoint orbit of the central extended lie group [2]. The integrability of $\lambda_N$ to the group of all volume preserving diffeomorphisms is studied in [10] and [15].

Unlike $\lambda_N$, whose cohomology class vanishes only for 0-homologous $N$, the restriction of $\sigma_N$ to the Lie algebra of hamiltonian vector fields is a coboundary, hence the integrability to the group of hamiltonian diffeomorphisms of $(M, \omega)$ is obvious. When the cohomology class $[\omega]^{k+1} \in H^{2k+2}(M, \mathbb{R})$ is integral, the presymplectic manifold $(Gr_{2k}(M), \omega^{k+1})$ is prequantizable. The $\text{Symp}(M, \omega)$-action on it is hamiltonian if and only if $H^{2k+1}(M, \mathbb{R}) = 0$. Pulling back Kostant’s central extension by this action, we get a central Lie group extension integrating $\sigma_N$. When $H^{2k+1}(M, \mathbb{R}) \neq 0$, we pass to covering spaces of $\text{Symp}(M, \omega)$ and $Gr_{2k}(M)$. The minimal covering group $\widetilde{\text{Symp}}(M, \omega)$ of $\text{Symp}(M, \omega)$ on which $\sigma_N$ can be integrated is also determined.

In [2] it is shown that all connected components of the non-linear symplectic Grassmannian $\text{SGr}_{2k}(M)$ appear as coadjoint orbits of the group of Hamiltonian diffeomorphisms. For $H^{2k+1}(M, \mathbb{R}) = 0$, the connected component containing $N$ is a coadjoint orbit of the central extension of $\text{Symp}(M, \omega)$ integrating $\sigma_N$. For $H^{2k+1}(M, \mathbb{R}) \neq 0$, a certain covering space of it is a coadjoint orbit of the central extensions of $\widetilde{\text{Symp}}(M, \omega)$ and $\widetilde{\text{Symp}}(M, \omega)$ integrating $\sigma_N$.

We are grateful to Stefan Haller for suggesting to consider minimal covering groups admitting central extensions and Rui Loja Fernandes for the impulse given to look for coadjoint orbits of groups of symplectic diffeomorphisms.

2. Non-linear Grassmannians

We start by collecting some facts from [2] about the non-linear Grassmannian and the non-linear symplectic Grassmannian.
Let $\text{Gr}_n(M)$ be the non-linear Grassmannian of oriented compact $n$–dimensional submanifolds without boundary of a smooth manifold $M$. It is a Fréchet manifold in a natural way, see [7] Section 44. Suppose $N \in \text{Gr}_n(M)$. Then the tangent space of $\text{Gr}_n(M)$ at $N$ can naturally be identified with the space of smooth sections of the normal bundle $TN^\perp := (TN|_N)/TN$.

To any $k$-form $\alpha$ the tilda map associates a $(k - n)$-form $\tilde{\alpha}$ on $\text{Gr}_n(M)$ by:

$$\tilde{\alpha}(Y_1, \ldots, Y_{k-n}) := \int_N i_{Y_{k-n}} \cdots i_{Y_1} \alpha.$$ 

Here, all $Y_j$ are tangent vectors at $N \in \text{Gr}_n(M)$, i.e. sections of $TN^\perp$. There is a natural action of the group $\text{Diff}(M)$ on $\text{Gr}_n(M)$ by $\varphi \cdot N = \varphi(N)$. For every vector field $X \in \mathfrak{X}(M)$ on $M$, the fundamental vector field $\zeta_X$ on $\text{Gr}_n(M)$ is $\zeta_X(N) = X|_N$, viewed as a section of $TN^\perp$. One can verify that

$$\begin{align*}
\tilde{d}\alpha &= d\tilde{\alpha} \\
L_{\zeta_X} \tilde{\alpha} &= \tilde{L}_X \alpha \\
i_{\zeta_X} \tilde{\alpha} &= \tilde{i}_X \alpha \\
\varphi^* \tilde{\alpha} &= \tilde{\alpha}.
\end{align*}$$

Theorem 1 from [2] shows that, if $[\alpha] \in H^k(M, \mathbb{Z})$, then $(\text{Gr}_{k-2}(M), \tilde{\alpha})$ is prequantizable, i.e. there exist a principal $S^1$–bundle $P \to \text{Gr}_{k-2}(M)$ and a principal connection 1-form $\eta \in \Omega^1(P)$ whose curvature form is $\tilde{\alpha}$.

Let $p : \tilde{\text{Gr}}_n(M) \to \text{Gr}_n(M)$ denote the universal covering projection. The elements in $\tilde{\text{Gr}}_n(M)$ are seen as homotopy classes $[N_i]$ of curves $t \mapsto N_t$ of $n$-dimensional submanifolds of $M$, starting at a fixed point $N_0 \in \text{Gr}_n(M)$. Any closed form $\alpha \in \Omega^{n+1}(M)$ gives rise to a closed 1-form $\tilde{\alpha}$ on $\text{Gr}_n(M)$, hence to a smooth function $\tilde{\alpha}$ on $\tilde{\text{Gr}}_n(M)$, uniquely defined by the conditions

$$p^* \tilde{\alpha} = d\tilde{\alpha} \quad \tilde{\alpha}([N_0]) = 0,$$

$[N_0]$ denoting the homotopy class of the constant curve.

The value of $\tilde{\alpha}$ at $[N_i]$ can be obtained by integrating the $(n + 1)$-form $\alpha$ over an $(n + 1)$-chain $c$ in $M$ obtained from the class $[N_i]$. This chain is not unique, it depends on a representative curve $N_t$ and on a curve $f_t$ of embeddings $N_0 \hookrightarrow M$ with $f_t(N_0) = N_t$, namely $c : (t, x) \in I \times N_0 \mapsto f_t(x) \in M$. But the integral $\int_c \alpha$ does not depend on these choices and equals $\tilde{\alpha}([N_i])$. In the sequel we will use also the notation $\int_{[N_i]} \alpha$.

Suppose $(M, \omega)$ is a connected $2n$–dimensional symplectic manifold. We denote by $\text{SG}^{2k}(M) \subset \text{Gr}^{2k}(M)$ the open subset of symplectic submanifolds of $M$ and we call it the non-linear symplectic Grassmannian. The 2-form $\tilde{\omega}^{k+1}$ is a symplectic form on $\text{SG}^{2k}(M)$. This symplectic manifold is prequantizable if $[\omega]^{k+1} \in H^{2k+2}(M, \mathbb{Z})$.

**Theorem 2.1.** [2] All connected components of the non-linear symplectic Grassmannian $(\text{SG}^{2k}(M), \omega^{k+1})$ are coadjoint orbits of the group of Hamiltonian diffeomorphisms.
3. A 2-cocycle on the symplectomorphism group

In this section we generalize the 2-cocycle defined in [5] on the symplectomorphism group of an exact symplectic manifold $M$ with $H^1(M, \mathbb{R}) = 0$.

The space of normalized locally smooth $p$-cochains on a Lie group $G$ is the space $C^p_1(G)$ of all maps $c : G^k \to \mathbb{R}$, locally smooth in a neighborhood of the identity, which satisfy $c(g_1, \ldots, g_p) = 0$ whenever some $g_j = e$. It is a differential complex with the differential

$$(d(gc)(g_0, \ldots, g_p) = c(g_1, \ldots, g_p) + \sum_{j=1}^{p} (-1)^j c(g_0, \ldots, g_{j-1}g_j, \ldots, g_p)$$

$$+ (-1)^{p+1} c(g_0, \ldots, g_{p-1}).$$

The second locally smooth group cohomology space $H^2_2(G)$ is in bijection with equivalence classes of central Lie group extensions of $G$ by $\mathbb{R}$ possessing a smooth local section.

Let $M$ be a non-compact connected manifold with an exact symplectic form $\omega = da$. For $k \neq 0$, every oriented compact $2k$-dimensional submanifold without boundary $N_0$ of $M$ (i.e. $N_0 \in \text{Gr}_{2k}(M)$) defines a normalized group 2-cocycle $c_{N_0}$ on $\text{Symp}(M, \omega)$, the connected component of the identity of the symplectomorphism group, by

$$c_{N_0}(\varphi_1, \varphi_2) = \int_{\varphi_2(N_0)} (\alpha - \varphi_1^* \alpha) \wedge \omega^k. \quad (1)$$

The integral is taken over any bordism with boundary $\varphi_2(N_0) - N_0$, which exists since $\varphi_2$ can be connected to the identity by a diffeotopy. This integral does not depend on the chosen bordism since the $(2k+1)$-form $(\alpha - \varphi_1^* \alpha) \wedge \omega^k$ is exact.

**Proposition 3.1.** The cohomology class of $c_{N_0}$ on $\text{Symp}(M, \omega)$ does not depend on the choice of the 1-form $\alpha$ satisfying $\omega = da$ and on the choice of the submanifold $N_0$ in a connected component of $\text{Gr}_{2k}(M)$, if $k \neq 0$.

**Proof.** Let $\omega = d\tilde{\alpha}$ and let $\tilde{c}_{N_0}$ be the group 2-cocycle defined with $\tilde{\alpha}$. The $(2k+1)$-form $(\tilde{\alpha} - \alpha) \wedge \omega^k$ is exact. A $2k$-form $\beta$ such that $(\tilde{\alpha} - \alpha) \wedge \omega^k = d\beta$ defines a group 1-cochain $a(\varphi) = \int_{N_0} (\varphi^* \beta - \beta)$. Then $\tilde{c}_{N_0} - c_{N_0} = da$.

Given $N_0$ and $N_1$ in the same connected component of $\text{Gr}_{2k}(M)$, there is a bordism $\tau$ in $M$ with boundary $N_1 - N_0$. Then $c_{N_0} - c_{N_1} = dGb$ for the group 1-cochain $b(\varphi) = \int_{\tau} (\alpha - \varphi^* \alpha) \wedge \omega^k$, $\varphi \in \text{Symp}(M, \omega)$. 

**Remark 3.2.** There is a geometric interpretation of the central Lie group extension of $\text{Symp}(M, \omega)$ given by the 2-cocycle $c_{N_0}$. Let $\mathcal{M}$ be the connected component of $N_0$ in $\text{Gr}_{2k}(M)$ and the exact 2-form $\Omega = \omega^{k+1} = d\omega \wedge \omega^k$ on $\mathcal{M}$. On the trivial bundle $\mathcal{M} \times \mathbb{R} \to \mathcal{M}$ we consider the connection 1-form $\theta = \tilde{\alpha} \wedge \omega^k + dt$ with curvature $\Omega$. Then the central extension defined by $c_{N_0}$, denoted $\text{Symp}(M, \omega) \times_{c_{N_0}} \mathbb{R}$, acts on $\mathcal{M} \times \mathbb{R}$ preserving $\theta$. The action is

$$(\varphi, a) \cdot (N, t) = (\varphi(N), \int_{N_0}^{N} (\varphi^* \alpha - \alpha) \wedge \omega^k + t + a),$$

where $(\varphi, a)$ acts on $N$ and leaves $t$ fixed.
where the integral is taken over any bordism in \( M \) with boundary \( N - N_0 \).

This central extension can be identified with the group of connection preserving automorphisms of \( \mathcal{M} \times \mathbb{R} \) projectable to diffeomorphisms of \( \mathcal{M} \) coming from the canonical \( \text{Symp}(M, \omega) \)-action.

**Remark 3.3.** For \( k = 0 \) we obtain the construction from [5], but in this case the condition \( H^1(\mathcal{M}, \mathbb{R}) = 0 \) has to be imposed to assure that the 1-form \( \alpha - \varphi^* \alpha \) is exact: \( N_0 \) is a point \( x_0 \in M \), \( c_{x_0}(\varphi_1, \varphi_2) = \int_{x_0}^{x_2(x_0)} (\alpha - \varphi_1^* \alpha) \), \( \mathcal{M} = M \) and \( \Omega = \omega \).

**Remark 3.4.** Let \( M = G/K \) be a non-compact symmetric space admitting a \( G \)-invariant complex structure and \( \omega \) the symplectic form defined by the Hermitian metric. It is shown in [5] that the restriction of \( c_{x_0} \) to \( G \) is cohomologous to the Guichardet-Wigner 2-cocycle on \( G \). The latter is known to define a non-trivial cohomology class. We define a generalized Guichardet-Wigner 2-cocycle on \( G \) corresponding to a \( 2k \)-dimensional submanifold \( N_0 \) of \( M \).

Given \( g_1, g_2 \in G \), we denote by \( (x, g_1(x), g_1g_2(x)) \) the oriented geodesic cone having as vertex the point \( x \) and as base the geodesic segment from \( g_1(x) \) to \( g_1g_2(x) \), and by \( (N_0, g_1(N_0), g_1g_2(N_0)) \) the \( (2k + 2) \)-chain \( \bigcup_{x \in N_0} (x, g_1(x), g_1g_2(x)) \). Generalizing the Guichardet-Wigner construction we define

\[
w_{N_0}(g_1, g_2) = \int_{(N_0, g_1(N_0), g_1g_2(N_0))} \omega^{k+1},
\]

which is a group 2-cocycle on \( G \).

The restriction of \( c_{N_0} \) to \( G \) is cohomologous to \( w_{N_0} \). Indeed, given \( g \in G \), we denote by \( (x, g(x)) \) the geodesic segment from \( x \) to \( g(x) \) and by \( (N_0, g(N_0)) = \bigcup_{x \in N_0} (x, g(x)) \) the geodesic cylinder with boundary \( g(N_0) - N_0 \). The group 1-cochain \( a_{N_0}(g) = \int_{(N_0, g(N_0))} \alpha \wedge \omega^k \) on \( G \) has the property \( da_{N_0} = w_{N_0} - c_{N_0}|_{G \times G} \).

### 4. 2-cocycles on the Lie algebra of symplectic vector fields

The space of continuous Lie algebra cochains on the topological Lie algebra \( \mathfrak{g} \) is \( C^p_c(\mathfrak{g}) = \{ \sigma : \mathfrak{g}^p \to \mathbb{R}|\sigma \text{ continuous alternating} \} \). With the differential

\[
(d_\sigma)(X_0, \ldots, X_p) = \sum_{i \leq j} (-1)^{i+j} \sigma([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)
\]

it becomes a differential complex. Its cohomology is the continuous Lie algebra cohomology \( H^p_c(\mathfrak{g}) \). The second continuous cohomology space \( H^2_c(\mathfrak{g}) \) is in bijection with equivalence classes of central topological Lie algebra extensions of \( \mathfrak{g} \) by \( \mathbb{R} \).

The natural homomorphism from the complex of \( \mathbb{R} \)-valued locally smooth group cochains to the complex of \( \mathbb{R} \)-valued continuous Lie algebra cochains in degree two has the following form:

\[
c'(X, Y) = \frac{\partial^2}{\partial t \partial s} \bigg|_{(0,0)} (c(\exp tX, \exp sY) - c(\exp sY, \exp tX)).
\]
For the exact symplectic manifold \((M, \omega = d\alpha)\) we compute the 2-cocycle on the Lie algebra \(\text{symp}(M, \omega)\) of symplectic vector fields corresponding to the 2-cocycle \(c_{N_0}\) on \(\text{Symp}(M, \omega)\):

\[
c'_{N_0}(X, Y) = -\frac{\partial}{\partial s} \int_{N_0}^{\text{Fl}^s(N_0)} L_X \alpha \wedge \omega^k + \frac{\partial}{\partial t} \int_{N_0}^{\text{Fl}_t(N_0)} L_Y \alpha \wedge \omega^k
\]

\[
= \int_{N_0} (i_X L_Y - i_Y L_X) \alpha \wedge \omega^k = \int_{N_0} (i_{[X,Y]} + i_X i_Y d - d i_X i_Y) \alpha \wedge \omega^k
\]

\[
= \int_{N_0} i_X i_Y \omega^{k+1} + \int_{N_0} i_{[X,Y]}(\alpha \wedge \omega^k).
\]

This proves that:

**Proposition 4.1.** The Lie algebra 2-cocycle associated to the group 2-cocycle \(c_{N_0}\) is cohomologous to the Lie algebra 2-cocycle

\[
\sigma_{N_0}(X, Y) = \int_{N_0} i_X i_Y \omega^{k+1}.
\]  

(3)

**Remark 4.2.** For a non-exact symplectic form \(\omega\) on \(M\), \(\sigma_{N_0}\) is still a continuous Lie algebra 2-cocycle on \(\text{symp}(M, \omega)\).

**Remark 4.3.** When \(k = 0\), so \(N_0 = x_0\) is a point of \(M\), we get \(\sigma_{\sigma_0}(X, Y) = \omega(X, Y) (x_0)\). The Lie algebra 2-cocycle \(\sigma_{\sigma_0}\) defines a non-trivial cohomology class in \(H^2_c(\text{symp}(M, \omega))\) if either \(M\) is non-compact, or \(M\) is compact with non-zero linear map \(a \in H^1(M, \mathbb{R}) \rightarrow a \wedge \omega^{n-1} \in H^{2n-1}(M, \mathbb{R})\) [5].

For the rest of this section \(M\) is assumed to be compact. In the exact sequence of Lie algebra homomorphisms

\[
0 \rightarrow \text{ham}(M, \omega) \rightarrow \text{symp}(M, \omega) \xrightarrow{s_\omega} H^1(M, \mathbb{R}) \rightarrow 0,
\]

for \(s_\omega(X) = [i_X \omega]\), the ideal \(\text{ham}(M, \omega)\) of Hamiltonian vector fields is perfect [1]. Therefore the pull-back by \(s_\omega\) is an injective homomorphism in continuous Lie algebra cohomology: \(s_\omega^*: H^2_c(H^1(M, \mathbb{R})) = \Lambda^2 H^1(M, \mathbb{R})^* \rightarrow H^2_c(\text{symp}(M, \omega))\).

**Remark 4.4.** The continuous Lie algebra cohomology space \(H^2_c(\text{ham}(M, \omega))\) is isomorphic to \(H^1(M, \mathbb{R})\) [14]. For a surface of genus greater than 2, the Hamiltonian group is contractible and its universal central extension by \(H^1(M, \mathbb{R})\) is constructed in [4] as an inductive limit of groups.

**Remark 4.5.** The continuous cohomology space \(H^2_c(\text{symp}(M, \omega))\) is isomorphic to \(\text{Ker}(t) \oplus \text{Im}(s_\omega^*)\) [16] for the transgression map

\[
t : H^2_c(\text{ham}(M, \omega)) = H^1(M, \mathbb{R}) \rightarrow H^3_c(H^1(M, \mathbb{R})) = \Lambda^3 H^1(M, \mathbb{R})^*
\]

\[
t(a)(b_1, b_2, b_3) = n(n-1) \int_M a \wedge b_1 \wedge b_2 \wedge b_3 \wedge \omega^{n-2} - \frac{n^2}{\text{Vol}(M)} \sum_{cycd} \langle a, b_1 \rangle_M \langle b_2, b_3 \rangle_M.
\]
Here \( \langle a, b \rangle_M = \int_M a \wedge b \wedge \omega^{n-1} \) denotes the symplectic pairing in \( H^1(M, \mathbb{R}) \) and \( \text{Vol}(M) = \int_M \omega^n \) the symplectic volume.

One can easily see that the restriction of the Lie algebra 2-cocycle \( \sigma_{N_0} \) to the ideal \( \mathfrak{ham}(M, \omega) \) of Hamiltonian vector fields is a coboundary. This follows also from the lemma below.

**Lemma 4.6.** The cohomology class of the 2-cocycle \( \sigma_{N_0}(X,Y) = \int_{N_0} i_X i_Y \omega^{k+1} \) on \( \mathfrak{symp}(M, \omega) \) is in \( \text{Im}(s_*^\omega) \). More precisely \( [\sigma_{N_0}] = s_*^\omega [\rho] \) where \( \rho \in \Lambda^2 H^1(M, \mathbb{R})^* \) is given by

\[
\rho(a, b) = (k + 1) \left( k \langle a, b \rangle_{N_0} - n \frac{\text{Vol}(N_0)}{\text{Vol}(M)} \langle a, b \rangle_M \right), \quad a, b \in H^1(M, \mathbb{R}),
\]

\( \text{Vol} \) denoting the symplectic volume and \( \langle a, b \rangle_{N_0} = \int_{N_0} a \wedge b \wedge \omega^{k-1} \).

**Proof.** The Lie bracket of two symplectic vector fields \( X \) and \( Y \) is a Hamiltonian vector field with Hamiltonian function \(-\omega(X, Y)\). Therefore the unique Hamiltonian function with zero integral on \( M \) for \([X, Y]\) is

\[
h_{[X,Y]} = -\omega(X, Y) + \frac{n}{\text{Vol}(M)} ([i_X \omega], [i_Y \omega])_M.
\]

Then

\[
\sigma_{N_0}(X,Y) = \int_{N_0} i_X i_Y \omega^{k+1} = k(k + 1) \int_{N_0} i_X \omega \wedge i_Y \omega \wedge \omega^{k-1} - (k + 1) \int_{N_0} \omega(X,Y) \omega^k
\]

\[
= k(k + 1) ([i_X \omega], [i_Y \omega])_{N_0} - (k + 1) \int_{N_0} (-h_{[X,Y]} + \frac{n}{\text{Vol}(M)} ([i_X \omega], [i_Y \omega])_M) \omega^k
\]

\[
= \rho(s_{[X,Y]}(X), s_{[Y]}(Y)) + (k + 1) \int_{N_0} h_{[X,Y]} \omega^k.
\]

With the help of a continuous linear retraction \( r : \mathfrak{symp}(M, \omega) \to \mathfrak{ham}(M, \omega) \), we write \( \sigma_{N_0} = s^\omega \rho + d\tau \) for the Lie algebra 1-cochain \( \tau(X) = -(k + 1) \int_{N_0} h_r(X) \omega^k \) on \( \mathfrak{symp}(M, \omega) \). Hence \( [\sigma_{N_0}] = s^\omega_* [\rho] \).

All the elements in \( \text{Im}(s_*^\omega) \) can be integrated to \( \widehat{\text{Symp}}(M, \omega) \). The corresponding central extensions are the pull-backs of central extensions of \( H^1(M, \mathbb{R}) \) by the flux homomorphism \( S_\omega : \widehat{\text{Symp}}(M, \omega) \to H^1(M, \mathbb{R}), \quad S_\omega([\varphi_t]) = \int_0^1 [i_{\delta t \varphi_t} \omega] dt \).

Here the right logarithmic derivative is \( \delta^r \varphi_t = \frac{d}{dt} \varphi_t \circ \varphi_t^{-1} \).

5. **Geometric constructions of central extensions**

There is a geometric construction of central Lie group extensions using Kostant’s extension. The ingredients are a connected Lie group \( G \), a prequantizable presymplectic manifold \( (\mathcal{M}, \Omega) \) and a Hamiltonian action of \( G \) on \( \mathcal{M} \). This means that
for all fundamental vector fields $\zeta_X$, $X \in \mathfrak{g}$, the 1-forms $i_{\zeta_X}\Omega$ are exact. Let $\mathcal{P} \to \mathcal{M}$ be the principal $S^1$-bundle with connection 1-form $\eta$ and curvature $\Omega$. We denote by Aut($\mathcal{P}, \eta$) the group of quantomorphisms, i.e. the connected component of the group of equivariant connection preserving diffeomorphisms of $\mathcal{P}$, and by Ham($\mathcal{M}, \Omega$) the group of Hamiltonian diffeomorphisms of $\mathcal{M}$. Kostant’s central extension [6] associated to ($\mathcal{M}, \Omega$) is

$$1 \to S^1 \to \text{Aut}(\mathcal{P}, \eta) \to \text{Ham}(\mathcal{M}, \Omega) \to 1. \quad (4)$$

Its pull-back to $G$ by the Hamiltonian action leads to a 1-dimensional central Lie group extension of $G$, even if $\mathcal{M}$ is infinite dimensional [11].

Let $(M, \omega)$ be a compact connected $2n$-dimensional symplectic manifold with integral $[\omega^{k+1}] \in H^{2k+2}(M, \mathbb{R})$. We fix a closed $2k$-dimensional submanifold $N_0$ of $M$ and denote by $\mathcal{M}$ the connected component of $\text{Gr}_2k(M)$ containing $N_0$. The integrality of $[\omega^{k+1}]$ assures (see Section 2) the existence of a principal circle bundle $\mathcal{P} \to \mathcal{M}$ and a principal connection 1-form $\eta$ having curvature $\Omega$. To this data one can associate Kostant’s central group extension (4). The canonical action of Symp($M, \omega$) on $\mathcal{M}$ preserves $\Omega$, hence it is a symplectic action.

**Theorem 5.1.** If $H^{2k+1}(M, \mathbb{R}) = 0$, then Symp($M, \omega$) acts in a Hamiltonian way on $(\mathcal{M}, \Omega)$. The pull-back of Kostant’s extension (4) is a central Lie group extension of Symp($M, \omega$) integrating $\sigma_{N_0}$.

**Proof.** For $X \in \mathfrak{sym}(M, \omega)$, the $(2k+1)$-form $i_X\omega^{k+1}$ is closed, hence exact. Let $\gamma \in \Omega^{2k}(M)$ such that $i_X\omega^{k+1} = d\gamma$. Then $i_{\zeta_X}\Omega = i_X\omega^{k+1} = d\gamma = d\tilde{\gamma}$ is exact too, ensuring that the Symp($M, \omega$)-action is Hamiltonian.

The pull-back of Kostant’s extension is a Lie group extension by Proposition 3.4 in [11]. Its corresponding Lie algebra 2-cocycle is $(X, Y) \mapsto -\Omega(\zeta_X, \zeta_Y)(N_0) = -\int_{N_0} i_Yi_X\omega^{k+1} = \sigma_{N_0}(X, Y)$ (see Section 3 in [2]).

A momentum map $\mu : \mathcal{M} \to \mathfrak{sym}(M, \omega)^*$ for the Hamiltonian action of Symp($M, \omega$) on $\mathcal{M}$ is $\mu(\varphi)(X) = \int_{N_0} i_X\omega^{k+1}$. The group 1-cocycle measuring its non-equivariance is

$$\kappa : \text{Symp}(M, \omega) \to \mathfrak{sym}(M, \omega)^* \quad \kappa(\varphi)(X) = \mu(\varphi(N_0))(X) = \int_{N_0} i_X\omega^{k+1}. \quad (5)$$

When $k = 0$, then $\mathcal{M} = M$, $\Omega = \omega$, $H^1(M, \mathbb{R}) = 0$ and Ham($M, \omega$) = Symp($M, \omega$), so Theorem 5.1 gives just Kostant’s central group extension. Its corresponding Lie algebra extension is trivial.

When $H^{2k+1}(M, \mathbb{R}) \neq 0$, the Symp($M, \omega$)-action is no longer Hamiltonian. Passing to universal covering spaces, we get a Hamiltonian action of $\widetilde{\text{Symp}}(M, \omega)$ on ($\tilde{\mathcal{M}}, p^*\Omega$), where $p : \tilde{\mathcal{M}} \to \mathcal{M}$ denotes the universal covering projection. Given $X \in \mathfrak{sym}(M, \omega)$, the fundamental vector field $\tilde{\zeta}_X$ on $\tilde{\mathcal{M}}$ satisfies $Tp\tilde{\zeta}_X = \zeta_X$, so

$$i_{\tilde{\zeta}_X}p^*\Omega = p^*i_{\zeta_X}\Omega = p^*(i_X\omega^{k+1}) = d(i_X\omega^{k+1}),$$

where $\zeta_X$ is defined by

$$i_{\zeta_X}\Omega = i_{\zeta_X}\omega^{k+1} = i_X\omega^{k+1} = d\gamma = d\tilde{\gamma},$$

for all fundamental vector fields $\zeta_X$, $X \in \mathfrak{g}$, the 1-forms $i_{\zeta_X}\Omega$ are exact.
By restricting \( \tilde{\omega}^{k+1} \) is a smooth map on \( \hat{M} \) defined as in Section 2. The momentum map is in this case

\[
\tilde{\mu} : \hat{M} \to \mathfrak{symp}(M, \omega)^*, \quad \tilde{\mu}([N_i]) = i_X \omega^{k+1}([N_i]) = \int_{[N_i]} i_X \omega^{k+1}.
\]

The 1-cocycle \( \tilde{\kappa} : \hat{\text{Symp}}(M, \omega) \to \mathfrak{symp}(M, \omega)^* \) which measures the failure of \( \tilde{\mu} \) to be equivariant is

\[
\tilde{\kappa}([\varphi_t]) = \tilde{\mu}([\varphi_t(N_0)]) = \int_{[\varphi_t(N_0)]} i_X \omega^{k+1}.
\]

**Proposition 5.2.** The pull-back of Kostant’s central extension (4) associated to the prequantizable presymplectic manifold \( (M, \mathfrak{p}^*\Omega) \) by the canonical Hamiltonian action of \( \text{Symp}(M, \omega) \) is a central Lie group extension integrating the Lie algebra 2-cocycle \( \sigma_{N_0} \).

**Proof.** Knowing the \( \text{Symp}(M, \omega) \)-action on \( \hat{M} \) is Hamiltonian, we have just to compute:

\[
-p^*\Omega(\tilde{\zeta}_X, \tilde{\zeta}_Y)([N_0]) = -\Omega(\zeta_X, \zeta_Y)(N_0) = -\int_{N_0} i_Y i_X \omega^{k+1} = \sigma_{N_0}(X, Y).
\]

Theorem 3.4 from [11] ensures that we indeed get a Lie group extension. \( \blacksquare \)

6. Minimal covering groups for Lie algebra 2-cocycles

Let \( \mathfrak{g} \) be a Lie algebra, \( \mathfrak{z} \) a Mackey complete locally convex space and \( \omega \) a continuous \( \mathfrak{z} \)-valued Lie algebra 2-cocycle on \( \mathfrak{g} \). Then the infinitesimal flux cocycle \( f_\omega : X \in \mathfrak{g} \mapsto i_X \omega \in C^1_c(\mathfrak{g}, \mathfrak{z}) \) is a Lie algebra 1-cocycle on \( \mathfrak{g} \) with values in the \( \mathfrak{g} \)-module \( C^1_c(\mathfrak{g}, \mathfrak{z}) \) of continuous linear maps from \( \mathfrak{g} \) to \( \mathfrak{z} \).

Let \( G \) be a connected Lie group with Lie algebra \( \mathfrak{g} \) and \( \tilde{G} \) its universal covering group. We denote by \( X^* \) the right invariant vector field on \( G \) defined by \( X \in \mathfrak{g} \) and by \( \omega^l \) the (closed) left invariant 2-form on \( G \) defined by \( \omega \in Z^2_c(\mathfrak{g}, \mathfrak{z}) \). The abstract flux 1-cocycle \( F_\omega : \tilde{G} \to C^1_c(\mathfrak{g}, \mathfrak{z}) \) associated to \( \omega \) is defined by \( F_\omega([\gamma])(X) = -\int_{[\gamma]} i_X \omega^l \) [9]. Here \( [\gamma] \in \tilde{G} \) denotes the homotopy class of a path \( \gamma \) in \( G \) starting at the identity. Another expression for the flux 1-cocycle is [10]

\[
F_\omega([\gamma])(X) = \int_0^1 \omega(\gamma(t)^{-1}\gamma'(t), \text{Ad}(\gamma(t)^{-1}X))dt.
\]

By restricting \( \tilde{F}_\omega \) to \( \pi_1(G) \) we get the flux homomorphism \( F_\omega : \pi_1(G) \to H^1(\mathfrak{g}, \mathfrak{z}) \).

Let \( \Gamma_\omega \) be the period group of \( \omega \), i.e. the image of the period map \([\beta] \in \pi_2(G) \mapsto \int_{\mathfrak{g}^2} \beta^*\omega^l \in \mathfrak{z} \).

**Theorem 6.1.** [9] Assuming that the period group \( \Gamma_\omega \) is discrete, the central Lie algebra extension \( \mathfrak{g} = \mathfrak{z} \times_\omega \mathfrak{g} \) integrates to a Lie group extension of \( G \) by the abelian Lie group \( Z = \mathfrak{z}/\Gamma_\omega \) if and only if the flux homomorphism \( F_\omega : \pi_1(G) \to H^1(\mathfrak{g}, \mathfrak{z}) \) vanishes. In particular \( \mathfrak{g} \) always integrates to a Lie group extension of \( \tilde{G} \) by \( Z \).
Proposition 6.2. Let $\Pi$ be the kernel of the flux homomorphism $F_\omega : \pi_1(G) \to H^1(\mathfrak{g}, \mathfrak{g})$ and let $\hat{G}$ be the covering group $\hat{G}/\Pi$ of $G$. Then the central Lie algebra extension $\hat{\mathfrak{g}} = \mathfrak{z} \times \omega \mathfrak{g}$ integrates to a Lie group extension of $G$ by $\mathfrak{z}/\Gamma_\omega$. Moreover the covering group $\hat{G}$ of $G$ is minimal with this property.

Proof. The flux homomorphism for $\hat{G}$ vanishes since it is the restriction of the flux homomorphism $F_\omega$ to $\pi_1(\hat{G}) = \Pi = \text{Ker} F_\omega$. Knowing that $\pi_2(G) = \pi_2(\hat{G}) = \pi_2(G)$, the result follows from the previous theorem.

We apply this proposition to the cocycle $\sigma = \sigma_{N_0}$ on the Lie algebra of symplectic vector fields.

Given $V \subset H^{2k+1}(M, \mathbb{R})$, we denote by $V^o \subset H_{2k+1}(M, \mathbb{R})$ the annihilator of $V$ with respect to the canonical pairing between homology and cohomology.

Corollary 6.3. Let $N_0$ be a fixed 2k-dimensional submanifold of $M$ and let $\sigma$ be the Lie algebra 2-cocycle $\sigma(X, Y) = \int_{N_0} i_X i_Y \omega^{k+1}$ on the Lie algebra of symplectic vector fields on $M$. We consider the subgroup $\Pi_\sigma$ of the fundamental group of the group of symplectic diffeomorphisms defined by

$$\Pi_\sigma = \{ [\varphi_t] \in \pi_1(\text{Symp}(M, \omega)) | [\varphi |_{N_0}] \in (H^1(M, \mathbb{R}) \wedge [\omega]^k)^o \subset H_{2k+1}(M, \mathbb{R}) \}.$$  

where $\varphi |_{N_0}$ is the $(2k+1)$-cycle $(t, x) \in [0, 1] \times N_0 \mapsto \varphi_t(x) \in M$. Then the minimal covering group of $\text{Symp}(M, \omega)$ on which $\sigma$ can be integrated is $\overline{\text{Symp}}(M, \omega) = \text{Symp}(M, \omega)/\Pi_\sigma$.

Proof. The flux homomorphism associated to the cocycle $\sigma$ is

$$F_\sigma : \pi_1(\text{Symp}(M, \omega)) \to H^1_c(s\text{ymp}(M, \omega)), \quad F_\sigma([\varphi_t])(X) = \int_{[\varphi_t(N_0)]} i_X \omega^{k+1}. $$

Indeed, the adjoint action in Diff($M$) is $\text{Ad}(\varphi) X = (\varphi^{-1})^* X$ and the relation between the left logarithmic derivative $\delta^l \varphi_t = T \varphi_t^{-1} \frac{d}{dt} \varphi_t$ and the right logarithmic derivative is $\delta^r \varphi_t = \text{Ad}(\varphi_t) \delta^l \varphi_t = (\varphi_t^{-1})^* \delta^l \varphi_t$. Hence

$$F_\sigma([\varphi_t])(X) = -\int_{\varphi_t} i_X \sigma^t (5) = -\int_0^1 \sigma(\text{Ad}(\varphi_t^{-1}) X, \delta^l \varphi_t) dt$$

$$= -\int_0^1 \int_{N_0} i_{\varphi_t^* X} i_{\delta^l \varphi_t} \omega^{k+1} dt = \int_0^1 \int_{N_0} \varphi_t^* i_{\delta^l \varphi_t} i_X \omega^{k+1} dt = \int_{[\varphi_t(N_0)]} i_X \omega^{k+1}. $$

The commutator Lie algebra of $s\text{ymp}(M, \omega)$ is $\text{ham}(M, \omega) [1]$, so its first cohomology space is $H^1_c(s\text{ymp}(M, \omega)) = H^1(M, \mathbb{R})^*$. Under this identification the flux homomorphism becomes $F_\sigma([\varphi_t])(a) = (k+1)\langle [\varphi_t |_{N_0}], a \wedge [\omega]^k \rangle$ for any $a \in H^1(M, \mathbb{R})$. Then $\Pi_\sigma = \text{Ker} F_\sigma$, so $\text{Symp}(M, \omega)$ is the minimal covering group of $\text{Symp}(M, \omega)$ on which $\sigma$ can be integrated (by Proposition 6.2).

The flux homomorphism $F_\sigma$ vanishes when $H^{2k+1}(M, \mathbb{R}) = 0$. In this case $\overline{\text{Symp}}(M, \omega) = \text{Symp}(M, \omega)$, fact already known from Theorem 5.1.
A geometric construction of the central extension of $\widetilde{\text{Symp}}(M,\omega)$ can be done using the covering space $q: \tilde{M} \to M$ of the connected component $\mathcal{M}$ of $\text{Gr}_{2\mathbb{R}}(M)$ containing $N_0$, defined by $\mathcal{M} = M/\Pi_M$ for

$$\Pi_M = \{ [N_i] \in \pi_1(\mathcal{M}) : \int_{[N_i]} i_X \omega^{k+1} = 0, \text{for all } X \in \text{symp}(M,\omega) \}. \quad (6)$$

**Lemma 6.4.** The groups $\widetilde{\text{Symp}}(M,\omega)$ and $\widetilde{\text{Symp}}(M,\omega)$ act on $(\tilde{M}, q^*\Omega)$ in a Hamiltonian way with momentum map

$$\tilde{\mu}: \tilde{M} \to \text{symp}(M,\omega)^*, \quad \tilde{\mu}([N_i])(X) = \int_{[N_i]} i_X \omega^{k+1}. \quad (7)$$

**Proof.** The group $\widetilde{\text{Symp}}(M,\omega)$ acts on $\tilde{M}$ because for any two representing paths $N_i$ and $N'_i$ of the same element $[N_i] = [N'_i] \in \mathcal{M}$ and any $[\varphi_i] \in \text{Symp}(M,\omega)$, the paths $\varphi_i(N_i)$ and $\varphi_i(N'_i)$ represent the same element in $\mathcal{M}$. Indeed, $N_i = N'_i$ and for all $X \in \text{symp}(M,\omega)$

$$\int_{[\varphi_i(N_i)]} i_X \omega^{k+1} = \int_{[N_i]} i_X \omega^{k+1} + \int_{[\varphi_i(N_i)]} i_X \omega^{k+1} = \int_{[N'_i]} i_X \omega^{k+1} + \int_{[\varphi_i(N'_i)]} i_X \omega^{k+1}.$$

The action of $\Pi_\sigma \subset \widetilde{\text{Symp}}(M,\omega)$ on $\tilde{M}$ is trivial. Indeed, let $[\varphi_i] \in \Pi_\sigma$ and $[N_i] \in \mathcal{M}$. Since $i_X \omega^{k+1}$ is closed and $N_0$, $N_1$ cobordant,

$$\int_{[\varphi_i(N_i)]} i_X \omega^{k+1} - \int_{[N_i]} i_X \omega^{k+1} = \int_{[\varphi_i(N_i)]} i_X \omega^{k+1} = \int_{[\varphi_i(N_0)]} i_X \omega^{k+1} = 0,$$

so $[\varphi_i(N_i)] = [\varphi_i] \in \tilde{M}$. Hence the $\text{Symp}(M,\omega)$-action projects to a $\text{Symp}(M,\omega)$-action on $\mathcal{M}$. Given $X \in \text{symp}(M,\omega)$, the fundamental vector field $\tilde{\zeta}_X$ on $\tilde{M}$ satisfies $Tq_\tilde{\zeta}_X = \zeta_X$, so the action is Hamiltonian:

$$i_{\tilde{\zeta}_X} q^*\Omega = q^* i_{\zeta_X} \Omega = q^* i_{i_X \omega^{k+1}} = di_X \omega^{k+1},$$

where $i_{i_X \omega^{k+1}} : [N_i] \to \int_{[N_i]} i_X \omega^{k+1}$ is a well defined function on $\tilde{M}$. Hence a momentum map is $\tilde{\mu}([N_i])(X) = \int_{[N_i]} i_X \omega^{k+1}$. 

Observing that $-q^*\Omega(\tilde{\zeta}_X, \tilde{\zeta}_Y) = \sigma_{N_0}(X, Y)$, and using Lemma 2 together with Proposition 3.4 in [11], we get the following proposition:

**Proposition 6.5.** By pulling back Kostant’s central extension for $(\tilde{M}, q^*\Omega)$, we obtain a geometric construction of a central Lie group extension of $\text{Symp}(M,\omega)$, integrating the Lie algebra cocycle $\sigma_{N_0}$, $\text{Symp}(M,\omega)$ being the minimal covering group of $\text{Symp}(M,\omega)$ on which $\sigma_{N_0}$ can be integrated.
7. Coadjoint orbits of \( \text{Symp}(M, \omega) \)

Let \((M, \omega)\) be a compact symplectic manifold and \(N_0\) a compact \(2k\)-dimensional symplectic submanifold without boundary of \(M\). The group \(\text{Ham}(M, \omega)\) of Hamiltonian diffeomorphisms acts transitively on every connected component of the non-linear symplectic Grassmannian \(\text{SGr}_{2k}(M)\), in particular on the connected component \(\mathcal{S}\) containing \(N_0\).

\(\mathcal{S}\) is an open submanifold of \(\mathcal{M}\), the connected component of \(N_0\) in \(\text{Gr}_{2k}(M)\), and the 2-form \(\Omega = \omega^{k+1}\) restricts to a symplectic form on \(\mathcal{S}\). The symplectic manifold \((\mathcal{S}, \Omega)\) is a coadjoint orbit of \(\text{Ham}(M, \omega)\) by Theorem 2.1. Let \(q : \mathcal{M} \to M\) be the covering from Section 6 and \(\overline{\mathcal{S}} \subset \mathcal{M}\) the connected component of \(q^{-1}(\mathcal{S})\) containing \([N_0]\).

**Proposition 7.1.** The symplectic manifold \((\overline{\mathcal{S}}, q^*\Omega)\) is a coadjoint orbit of the central extensions of \(\text{Symp}(M, \omega)\) and \(\text{Symp}(M, \omega)\) integrating \(\sigma_{N_0}\).

**Proof.** The actions of \(\overset{\sim}{\text{Symp}}(M, \omega)\) and \(\overset{\sim}{\text{Symp}}(M, \omega)\) on \(\overline{\mathcal{S}}\) are Hamiltonian. They lift the transitive action of \(\text{Symp}(M, \omega)\) on \(\mathcal{S}\) [2], hence it is transitive.

A momentum map \(\overline{\mu} : \overline{\mathcal{S}} \to \text{symp}(M, \omega)^*\) is given by (7) and it is injective. Indeed, if \(\overline{\mu}([N_i]) = \overline{\mu}([N'_i])\), then \(\int_{N_i} f \omega^k = \int_{N'_i} f \omega^k\) for any smooth function \(f\) on \(M\) (consider \(i_X \omega = df\)). It follows that \(N_i = N'_i\) and since \(\int_{[N_i]} i_X \omega^{k+1} = \int_{[N'_i]} i_X \omega^{k+1}\), in \(\overline{\mathcal{S}}\) the classes of \([N_i]\) and \([N'_i]\) coincide.

Applying Proposition 1 from [2] to this transitive hamiltonian action, we get the result.

**Corollary 7.2.** For \(H^{2k+1}(M, \mathbb{R}) = 0\), the symplectic manifold \((\mathcal{S}, \Omega)\) is a coadjoint orbit of the central extension of \(\text{Symp}(M, \omega)\) obtained in Theorem 5.1.

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