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Hom-Algebras and Homology

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Abstract. Classes of G-Hom-associative algebras are constructed as deformations of G-associative algebras along algebra endomorphisms. As special cases, we obtain Hom-associative and Hom-Lie algebras as deformations of associative and Lie algebras, respectively, along algebra endomorphisms. Chevalley-Eilenberg type homology for Hom-Lie algebras are also constructed.
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1. Introduction

In [14] Hartwig, Larsson, and Silvestrov introduced Hom-Lie algebras as part of a study of deformations of the Witt and the Virasoro algebras. Closely related algebras also appeared earlier in the work of Liu [20] and Hu [16]. A *Hom-Lie algebra* is a triple $(L, [-,-], \alpha)$, in which L is a vector space, α is a linear self-map of L, and the skew-symmetric bilinear bracket satisfies an α-twisted variant of the Jacobi identity, called the *Hom-Jacobi identity* (3). Lie algebras are special cases of Hom-Lie algebras in which α is the identity map. Some q-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra [14]. Hom-Lie algebras are closely related to discrete and deformed vector fields and differential calculus [14, 18, 19].

Hom-Lie algebras are also useful in mathematical physics. In [30, 31], applications of Hom-Lie algebras to a generalization of the Yang-Baxter equation (YBE) [6, 7, 25] and to braid group representations [3, 4] are discussed. In particular, for a vector space M and a linear self-map α on M, the *Hom-Yang-Baxter equation* (HYBE) for (M, α) is the equation

$$(\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B) = (B \otimes \alpha) \circ (\alpha \otimes B) \circ (B \otimes \alpha),$$

where $B\colon M^{\otimes 2} \to M^{\otimes 2}$ is a bilinear map that commutes with $\alpha^{\otimes 2}$. The YBE is the special case of the HYBE with $\alpha = Id$. It is proved in [30, Theorem 1.1] that every Hom-Lie algebra $(L, [-,-], \alpha)$ gives rise to a solution B_α of the HYBE for

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$(\mathbb{K} \oplus L, Id \oplus \alpha)$. More precisely, this operator B_α is defined as

$$B_\alpha((a,x) \otimes (b,y)) = (b, \alpha(y)) \otimes (a, \alpha(x)) + (1,0) \otimes (0, [x,y])$$

for $a, b \in \mathbb{K}$ (the characteristic 0 ground field) and $x, y \in L$. If in addition α is invertible, then so is B_α. It is also shown in [30, Theorem 1.4] that every solution of the HYBE for (M, α) gives rise to operators B_i $(1 \leq i \leq n-1)$ on $M^{\otimes n}$ that satisfy the braid relations. In particular, if α is invertible in the Hom-Lie algebra $(L, [-,-], \alpha)$, then these operators B_i on $(\mathbb{K} \oplus L)^{\otimes n}$ satisfy the braid relations and are invertible. So we obtain a representation of the braid group on n strands on the linear automorphism group of $(\mathbb{K} \oplus L)^{\otimes n}$. It is, therefore, useful to have concrete examples of Hom-Lie algebras.

Meanwhile in [21], Makhlouf and Silvestrov introduced the notion of a *Hom-associative algebra* (A, μ, α), in which α is a linear self-map of the vector space A and the bilinear operation μ satisfies an α-twisted version of associativity (2). Associative algebras are special cases of Hom-associative algebras in which α is the identity map. A Hom-associative algebra A gives rise to a Hom-Lie algebra $HLie(A)$ via the commutator bracket [21]. In this sense, Hom-associative algebras play the role of associative algebras in the Hom-Lie setting. Moreover, any Hom-Lie algebra L has a corresponding enveloping Hom-associative algebra $U_{HLie}(L)$ in such a way that $HLie$ and U_{HLie} are adjoint functors [26]. In fact, a unital version of $U_{HLie}(L)$ has the structure of a Hom-bialgebra [27, Theorem 3.12], generalizing the bialgebra structure on the usual enveloping algebra of a Lie algebra. Besides [21, 26, 27, 30, 31], Hom-algebras have been further studied in [2, 5, 8, 10, 11, 12, 17, 22, 23, 24, 29, 32, 33, 34].

There are two main purposes of this paper:

1. We show how certain Hom-algebras arise naturally from classical algebras. In particular, we show how arbitrary associative and Lie algebras deform into Hom-associative and Hom-Lie algebras, respectively, via any algebra endomorphisms. This construction actually applies more generally to G-Hom-associative algebras (Theorem 2.3), which are introduced in [21]. This gives a systematic method for constructing many different types of Hom-algebras, including Hom-Lie algebras.

2. We lay the foundation of a homology theory for Hom-Lie algebras. In particular, we construct a Chevalley-Eilenberg type homology theory for Hom-Lie algebras with non-trivial coefficients. When applied to a Lie algebra L, our homology of L coincides with the usual Chevalley-Eilenberg homology of L [9]. The corresponding cohomology theory for Hom-Lie algebras was studied in [23].

The rest of this paper is organized as follows.

In the next section, basic definitions about G-Hom-associative algebras are recalled. It is then shown that G-associative algebras deform into G-Hom-associative algebras via an algebra endomorphism (Theorem 2.3). The desired deformations of associative and Lie algebras into their Hom counterparts are special cases of this result (Corollary 2.5). Examples of such Hom-associative and

Hom-Lie deformations are then given (Examples 2.6 - 2.14). Note that, since the appearance of an earlier version of this paper [28], Theorem 2.3 has been applied and generalized in [2, Theorem 2.7], [5, Theorems 1.7 and 2.6], [11, Section 2], [12, Proposition 1], [24, Theorem 3.15 and Proposition 3.30], [27, Example 3.7 and Proposition 4.2], and [29]–[34].

In Section 3, the homology of a Hom-Lie algebra with non-trivial coefficients is constructed. An interpretation of the 0th homology module is given.

2. G-Hom-associative algebras as deformations of G-associative algebras

The purposes of this section are to recall some basic definitions about G-Hom-associative algebras and to show that G-associative algebras deform into G-Hom-associative algebras via algebra endomorphisms (Theorem 2.3 and Examples 2.6 - 2.14).

Throughout the rest of this paper, we work over a fixed field $\mathbb{K}$ of characteristic 0. Tensor products, Hom, modules, and chain complexes are all meant over $\mathbb{K}$, unless otherwise specified.

Definition 2.1. A *Hom-module* is a pair (M, α_M) consisting of (i) a vector space M and (ii) a linear self-map $\alpha_M\colon M \to M$. A *morphism* $f\colon (M, \alpha_M) \to (N, \alpha_N)$ of Hom-modules is a linear map $f\colon M \to N$ such that $f \circ \alpha_M = \alpha_N \circ f$.

Definition 2.2. Let G be a subgroup of Σ_3, the symmetric group on three letters. A **G-Hom-associative algebra** [21] is a triple (A, μ, α) in which A is a vector space, $\mu\colon A^{\otimes 2} \to A$ is a bilinear map, and $\alpha\colon A \to A$ is a linear map, satisfying the following *G-Hom-associativity* axiom:

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \left\{ (x_{\sigma(1)} x_{\sigma(2)}) \alpha(x_{\sigma(3)}) - \alpha(x_{\sigma(1)}) (x_{\sigma(2)} x_{\sigma(3)}) \right\} = 0 \tag{1}$$

for $x_i \in A$, where $\varepsilon(\sigma)$ is the signature of σ. A **G-associative algebra** is a not-necessarily associative algebra (A, μ), satisfying (1) with $\alpha = Id$. Here and in what follows, we use the abbreviation xy for $\mu_A(x, y)$.

In some statements in this article the multiplicativity of α is essential. In all such cases, this will be explicitly indicated.

A *morphism* $f\colon (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B)$ of G-Hom-associative algebras is a morphism $f\colon (A, \alpha_A) \to (B, \alpha_B)$ of Hom-modules such that $f \circ \mu_A = \mu_B \circ f^{\otimes 2}$.

Special cases of G-Hom-associative algebras include the following:

1. A **Hom-associative algebra** is a G-Hom-associative algebra in which G is the trivial subgroup $\{e\}$. The G-Hom-associativity axiom (1) now takes the form
$$(xy)\alpha(z) = \alpha(x)(yz), \tag{2}$$
which we call *Hom-associativity*.

2. A **Hom-Lie algebra** is a G-Hom-associative algebra (A, μ, α) in which $\mu = [-,-]$ is skew-symmetric and G is the three-element subgroup A_3 of Σ_3. The A_3-Hom-associativity axiom (1) is equivalent to

$$[\alpha(x), [y, z]] + [\alpha(z), [x, y]] + [\alpha(y), [z, x]] = 0, \tag{3}$$

called the *Hom-Jacobi identity.*

3. A **Hom-left-symmetric algebra** is a G-Hom-associative algebra in which $G = \{e, (1\ 2)\}$. The $\{e, (1\ 2)\}$-Hom-associativity axiom (1) is equivalent to

$$(xy)\alpha(z) - \alpha(x)(yz) = (yx)\alpha(z) - \alpha(y)(xz). \tag{4}$$

Left-symmetric algebras (also called left pre-Lie algebras and Vinberg algebras) are exactly the $\{e, (1\ 2)\}$-associative algebras. In other words, left-symmetric algebras are the algebras that satisfy (4) with $\alpha = Id$.

4. A **Hom-Lie-admissible algebras** is a Σ_3-Hom-associativity algebra. Every G-Hom-associative algebra is also a Hom-Lie-admissible algebra. Moreover, if (A, μ, α) is a Hom-Lie-admissible algebra, then $(A, [-,-], \alpha)$ is a Hom-Lie algebra [21, Section 2], where $[-,-]$ is the commutator bracket defined by μ. A Lie-admissible algebra is exactly a Σ_3-associative algebra, i.e., a Hom-Lie-admissible algebra in which $\alpha = Id$. Equivalently, a Lie-admissible algebra is an algebra whose commutator bracket satisfies the Jacobi identity.

The following result says that G-associative algebras deform into G-Hom-associative algebras along any algebra endomorphism.

Theorem 2.3. *Let (A, μ) be a G-associative algebra and $\alpha\colon A \to A$ be a linear map such that $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$. Then $(A, \mu_\alpha = \alpha \circ \mu, \alpha)$ is a G-Hom-associative algebra. Moreover, α is multiplicative with respect to μ_α, i.e., $\alpha \circ \mu_\alpha = \mu_\alpha \circ \alpha^{\otimes 2}$.*

Suppose that (B, μ') is another G-associative algebra and that $\alpha'\colon B \to B$ is a linear map such that $\alpha' \circ \mu' = \mu' \circ \alpha'^{\otimes 2}$. If $f\colon A \to B$ is an algebra morphism (i.e., $f \circ \mu = \mu' \circ f^{\otimes 2}$) that satisfies $f \circ \alpha = \alpha' \circ f$, then $f\colon (A, \mu_\alpha, \alpha) \to (B, \mu'_{\alpha'} = \alpha' \circ \mu', \alpha')$ is a morphism of G-Hom-associative algebras.

We will use the following observations in the proof of Theorem 2.3.

Lemma 2.4. *Let $A = (A, \mu)$ be a not-necessarily associative algebra and $\alpha\colon A \to A$ be an algebra morphism. Then the multiplication $\mu_\alpha = \alpha \circ \mu$ satisfies*

$$\mu_\alpha(\mu_\alpha(x, y), \alpha(z)) = \alpha^2((xy)z) \quad \text{and} \quad \mu_\alpha(\alpha(x), \mu_\alpha(y, z)) = \alpha^2(x(yz)) \tag{5}$$

for $x, y, z \in A$, where $\alpha^2 = \alpha \circ \alpha$. Moreover, α is multiplicative with respect to μ_α, i.e., $\alpha \circ \mu_\alpha = \mu_\alpha \circ \alpha^{\otimes 2}$.

Proof. Using the hypothesis that α is an algebra morphism, we have

$$\mu_\alpha(\mu_\alpha(x,y),\alpha(z)) = \alpha(\alpha(xy)\alpha(z)) = \alpha^2((xy)z),$$

proving the first assertion in (5). The other assertion in (5) is proved similarly. For the last assertion, observe that both $\alpha\circ\mu_\alpha$ and $\mu_\alpha\circ\alpha^{\otimes 2}$ are equal to $\alpha\circ\mu\circ\alpha^{\otimes 2}$. ∎

We now give the proof of Theorem 2.3

Proof. By Lemma 2.4, α is multiplicative with respect to μ_α. Next we check (1) with the multiplication $\mu_\alpha = \alpha\circ\mu$. We compute as follows:

$$\begin{aligned}&\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}\left\{\mu_\alpha(\mu_\alpha(x_{\sigma(1)},x_{\sigma(2)}),\alpha(x_{\sigma(3)})) - \mu_\alpha(\alpha(x_{\sigma(1)}),\mu_\alpha(x_{\sigma(2)},x_{\sigma(3)}))\right\}\\ &\quad = \alpha^2\left\{\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}\left\{(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)} - x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)})\right\}\right\} = 0.\end{aligned}$$

The first equality follows from Lemma 2.4 and the linearity of α. The second equality follows from the hypothesis that (A,μ) is G-associative.

Finally, f is a morphism of G-Hom-associative algebras because $f\circ\alpha = \alpha'\circ f$ by hypothesis and

$$f\circ\mu_\alpha = f\circ\alpha\circ\mu = \alpha'\circ f\circ\mu = \mu'_{\alpha'}\circ f^{\otimes 2}$$

by the assumption that f is an algebra morphism. ∎

If we take G to be the subgroups $\{e\}$, A_3, $\{e,(1\ 2)\}$, and Σ_3, respectively, in Theorem 2.3, we obtain the following result.

Corollary 2.5. *Let $A=(A,\mu)$ be a not-necessarily associative algebra and $\alpha\colon A\to A$ be an algebra morphism. Write A_α for the triple $(A,\mu_\alpha=\alpha\circ\mu,\alpha)$.*

1. *If A is an associative algebra, then A_α is a Hom-associative algebra.*
2. *If A is a Lie algebra, then A_α is a Hom-Lie algebra.*
3. *If A is a left-symmetric algebra, then A_α is a Hom-left-symmetric algebra.*
4. *If A is a Lie-admissible algebra, then A_α is a Hom-Lie-admissible algebra.*

In view of Theorem 2.3, we think of the G-Hom-associative algebra $A_\alpha = (A,\mu_\alpha,\alpha)$ as a deformation of the G-associative algebra A that reduces to A when $\alpha = \mathrm{Id}_A$. In the rest of this section, we give several examples of this kind of Hom-associative and Hom-Lie deformations.

Example 2.6 (Polynomial Hom-associative algebras). Consider the polynomial algebra $A=\mathbb{K}[x_1,\ldots,x_n]$ in n variables. Then an algebra endomorphism α of A is uniquely determined by the n polynomials $\alpha(x_i)=\sum\lambda_{i;r_1,\ldots,r_n}x_1^{r_1}\cdots x_n^{r_n}$ for $1\le i\le n$. Define μ_α by

$$\mu_\alpha(f,g) = f(\alpha(x_1),\ldots,\alpha(x_n))g(\alpha(x_1),\ldots,\alpha(x_n))$$

for f and g in A. By Corollary 2.5, $A_\alpha = (A, \mu_\alpha, \alpha)$ is a Hom-associative algebra that reduces to the original polynomial algebra A when $\alpha(x_i) = x_i$ for $1 \le i \le n$, i.e., $\alpha = \mathrm{Id}$. We think of the collection $\{A_\alpha \colon \alpha \text{ an algebra endomorphism of } A\}$ as a family of deformations of the polynomial algebra A into Hom-associative algebras. A generalization of this example is considered in [24, Example 3.32]. ■

Example 2.7 (**Group Hom-associative algebras**). Let $A = \mathbb{K}[G]$ be the group-algebra over a group G. If $\alpha \colon G \to G$ is a group morphism, then it can be extended to an algebra endomorphism of A by setting $\alpha\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g \alpha(g)$. By Corollary 2.5, $A_\alpha = (A, \mu_\alpha, \alpha)$ is a Hom-associative algebra in which

$$\mu_\alpha\left(\sum a_g g, \sum b_g g\right) = \sum c_g \alpha(g),$$

where $(\sum a_g g)(\sum b_g g) = \sum c_g g$. We think of the collection

$$\{A_\alpha \colon \alpha \colon G \to G \text{ a group morphism}\}$$

as a family of deformations of the group-algebra A into Hom-associative algebras. A generalization of this example is considered in [24, Example 3.31]. ■

Example 2.8 (**Hom-associative deformations by inner automorphisms**). Let A be a unital associative algebra. Suppose that $u \in A$ is an invertible element. Then the map $\alpha(u) \colon A \to A$ defined by $\alpha(u)(x) = uxu^{-1}$ for $x \in A$ is an algebra automorphism. In this case, we have

$$\mu_{\alpha(u)}(x, y) = uxyu^{-1}$$

for $x, y \in A$. By Corollary 2.5, the triple $A_u = (A, \mu_{\alpha(u)}, \alpha(u))$ is a Hom-associative algebra. We think of the collection $\{A_u \colon u \in A \text{ invertible}\}$ as a 1-parameter family of deformations of A into Hom-associative algebras. ■

Example 2.9 (**Hom-associative deformations by nilpotent derivations**). Let A be an associative algebra. Recall that a *derivation on* A is a linear self-map D on A that satisfies the Leibniz identity, $D(xy) = D(x)y + xD(y)$, for $x, y \in A$. Such a derivation is said to be *nilpotent* if $D^n = 0$ for some $n \ge 1$. For example, if $x \in A$ is a nilpotent element, say, $x^n = 0$, then the linear self-map $\mathrm{ad}(x)$ on A defined by $\mathrm{ad}(x)(y) = xy - yx$ is a nilpotent derivation on A. Given a nilpotent derivation D on A (with, say, $D^n = 0$), the linear self-map

$$\exp D = \mathrm{Id}_A + D + \frac{1}{2}D^2 + \cdots + \frac{1}{(n-1)!}D^{n-1}$$

is actually an algebra automorphism of A (see, e.g., [1, p.26]). With $\mu_{\exp D}$ defined as $\mu_{\exp D}(x, y) = (\exp D)(xy)$, Corollary 2.5 shows that we have a Hom-associative algebra $A_D = (A, \mu_{\exp D}, \exp D)$. We think of the collection $\{A_D \colon D \text{ is a nilpotent derivation on } A\}$ as a 1-parameter family of deformations of A into Hom-associative algebras. ■

Example 2.10 (**Hom-Lie** $\mathsf{sl}(2,\mathbb{C})$)**.** Consider the complex Lie algebra $\mathsf{sl}(2,\mathbb{C})$ of 2×2 matrices with trace 0. A standard linear basis of $\mathsf{sl}(2,\mathbb{C})$ consists of the elements

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which satisfy the relations $[h,e] = 2e$, $[h,f] = -2f$, and $[e,f] = h$. Let $\lambda \neq 0$ be a scalar in $\mathbb{C}$. Consider the linear map $\alpha_\lambda \colon \mathsf{sl}(2,\mathbb{C}) \to \mathsf{sl}(2,\mathbb{C})$ defined by

$$\alpha_\lambda(h) = h, \quad \alpha_\lambda(e) = \lambda e, \quad \text{and} \quad \alpha_\lambda(f) = \lambda^{-1} f$$

on the basis elements. The map α_λ is actually a Lie algebra morphism. In fact, it suffices to check this on the basis elements, which is immediate from the definition of α_λ. By Corollary 2.5, we have a Hom-Lie algebra $\mathsf{sl}(2,\mathbb{C})_\lambda = (\mathsf{sl}(2,\mathbb{C}), [-,-]_{\alpha_\lambda}, \alpha_\lambda)$. The Hom-Lie algebra bracket $[-,-]_{\alpha_\lambda}$ on the basis elements is given by

$$[h,e]_{\alpha_\lambda} = 2\lambda e, \quad [h,f]_{\alpha_\lambda} = -2\lambda^{-1} f, \quad \text{and} \quad [e,f]_{\alpha_\lambda} = h.$$

We think of the collection $\{\mathsf{sl}(2,\mathbb{C})_\lambda \colon \lambda \neq 0 \text{ in } \mathbb{C}\}$ as a one-parameter family of deformations of $\mathsf{sl}(2,\mathbb{C})$ into Hom-Lie algebras. ■

Example 2.11 (**Hom-Lie** $\mathsf{sl}(\mathsf{n},\mathbb{C})$)**.** This is a generalization of the previous example to $n > 2$. Let $\mathsf{sl}(\mathsf{n},\mathbb{C})$ be the complex Lie algebra of $n \times n$ matrices with trace 0. It is generated as a Lie algebra by the elements

$$e_i = E_{i,i+1}, \quad f_i = E_{i+1,i}, \quad \text{and} \quad h_i = E_{ii} - E_{i+1,i+1}$$

for $1 \leq i \leq n-1$, where E_{ij} denotes the matrix with 1 in the (i,j)-entry and 0 everywhere else. These elements satisfy some relations similar to those of $\mathsf{sl}(2,\mathbb{C})$ (see, e.g., [15, p.9]).

Let $\lambda_1, \ldots, \lambda_{n-1}$ be non-zero scalars in $\mathbb{C}$. Consider the map

$$\alpha_{\lambda_1,\ldots,\lambda_{n-1}} \colon \mathsf{sl}(\mathsf{n},\mathbb{C}) \to \mathsf{sl}(\mathsf{n},\mathbb{C})$$

defined on the generators by

$$\alpha_{\lambda_1,\ldots,\lambda_{n-1}}(e_i) = \lambda_i e_i, \quad \alpha_{\lambda_1,\ldots,\lambda_{n-1}}(f_i) = \lambda_i^{-1} f_i, \quad \text{and} \quad \alpha_{\lambda_1,\ldots,\lambda_{n-1}}(h_i) = h_i$$

for $1 \leq i \leq n-1$. It is easy to check that $\alpha_{\lambda_1,\ldots,\lambda_{n-1}}$ actually defines a Lie algebra morphism. By Corollary 2.5, we have a Hom-Lie algebra

$$\mathsf{sl}(\mathsf{n},\mathbb{C})_{\lambda_1,\ldots,\lambda_{n-1}} = (\mathsf{sl}(\mathsf{n},\mathbb{C}), [-,-]_{\alpha_{\lambda_1,\ldots,\lambda_{n-1}}}, \alpha_{\lambda_1,\ldots,\lambda_{n-1}}).$$

We think of the collection $\left\{\mathsf{sl}(\mathsf{n},\mathbb{C})_{\lambda_1,\ldots,\lambda_{n-1}} \colon \lambda_1, \ldots, \lambda_{n-1} \neq 0 \text{ in } \mathbb{C}\right\}$ as an $(n-1)$-parameter family of deformations of $\mathsf{sl}(\mathsf{n},\mathbb{C})$ into Hom-Lie algebras. ■

Example 2.12 (Hom-Lie Heisenberg algebra). Let H be the 3-dimensional Heisenberg Lie algebra, which consists of the strictly upper-triangular complex 3×3 matrices. It has a standard linear basis consisting of the elements

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Heisenberg relation $[e, f] = h$ is satisfied, and $[e, h] = 0 = [f, h]$ are the other two relations for the basis elements.

Let λ_1 and λ_2 be non-zero scalars in $\mathbb{C}$. Consider the map $\alpha_{\lambda_1,\lambda_2} \colon \mathsf{H} \to \mathsf{H}$ defined on the basis elements by

$$\alpha_{\lambda_1,\lambda_2}(e) = \lambda_1 e, \quad \alpha_{\lambda_1,\lambda_2}(f) = \lambda_2 f, \quad \text{and} \quad \alpha_{\lambda_1,\lambda_2}(h) = \lambda_1\lambda_2 h.$$

It is straightforward to check that $\alpha_{\lambda_1,\lambda_2}$ defines a Lie algebra morphism. By Corollary 2.5, we have a Hom-Lie algebra $\mathsf{H}_{\lambda_1,\lambda_2} = (\mathsf{H}, [-,-]_{\alpha_{\lambda_1,\lambda_2}}, \alpha_{\lambda_1,\lambda_2})$, whose bracket satisfies the *twisted Heisenberg relation*

$$[e, f]_{\alpha_{\lambda_1,\lambda_2}} = \lambda_1\lambda_2 h.$$

We think of the collection $\{\mathsf{H}_{\lambda_1,\lambda_2} \colon \lambda_1, \lambda_2 \in \mathbb{C}\}$ as a 2-parameter family of deformations of H into Hom-Lie algebras. ∎

Example 2.13 (Matrix Hom-Lie algebras). Let G be a matrix Lie group (e.g., $\mathsf{GL}(\mathsf{n}, \mathbb{C})$, $\mathsf{SL}(\mathsf{n}, \mathbb{C})$, $\mathsf{U}(\mathsf{n})$, $\mathsf{O}(\mathsf{n})$, and $\mathsf{Sp}(\mathsf{n})$), and let $\mathfrak{g}$ be the Lie algebra of G. Given any element $x \in G$, it is well-known that the map $Ad_x \colon \mathfrak{g} \to \mathfrak{g}$ defined by $Ad_x(g) = xgx^{-1}$ is a Lie algebra morphism (see, e.g., [13, Proposition 2.23]). By Corollary 2.5, we have a Hom-Lie algebra

$$\mathfrak{g}_x = (\mathfrak{g}, [-,-]_{Ad_x}, Ad_x)$$

in which

$$[g_1, g_2]_{Ad_x} = x(g_1g_2 - g_2g_1)x^{-1}$$

for $g_1, g_2 \in \mathfrak{g}$. We think of the collection $\{\mathfrak{g}_x \colon x \in G\}$ as a 1-parameter family of deformations of $\mathfrak{g}$ into Hom-Lie algebras. ∎

Example 2.14 (Hom-Lie Witt algebra). The Witt algebra W is the complex Lie algebra of derivations on the Laurent polynomial algebra $\mathbb{C}[t^{\pm 1}]$. It can be regarded as the one-dimensional $\mathbb{C}[t^{\pm 1}]$-module

$$W = \mathbb{C}[t^{\pm 1}] \cdot \frac{d}{dt},$$

whose Lie bracket is given by

$$\left[f \cdot \frac{d}{dt}, g \cdot \frac{d}{dt}\right] = \left(f\frac{dg}{dt} - g\frac{df}{dt}\right) \cdot \frac{d}{dt}$$

for $f, g \in \mathbb{C}[t^{\pm 1}]$ (see, e.g., [14, Example 11]). Given any scalar $\lambda \in \mathbb{C}$, the map $\alpha_\lambda \colon W \to W$ defined by

$$\alpha_\lambda\left(f \cdot \frac{d}{dt}\right) = f(\lambda + t) \cdot \frac{d}{dt}$$

is easily seen to be a Lie algebra morphism. By Corollary 2.5, we have a Hom-Lie algebra $W_\lambda = (W, [-,-]_{\alpha_\lambda}, \alpha_\lambda)$, in which the bracket is given by

$$\left[f \cdot \frac{d}{dt}, g \cdot \frac{d}{dt}\right]_{\alpha_\lambda} = \left(f(\lambda + t)\frac{dg}{dt}(\lambda + t) - g(\lambda + t)\frac{df}{dt}(\lambda + t)\right) \cdot \frac{d}{dt}.$$

We think of the collection $\{W_\lambda \colon \lambda \in \mathbb{C}\}$ as a 1-parameter family of deformations of the Witt algebra W into Hom-Lie algebras. ■

3. Homology for Hom-Lie algebras

The purpose of this section is to construct the homology for a Hom-Lie algebra. We begin by defining the coefficients.

¿From now on, $(L, [-,-], \alpha_L)$ will denote a Hom-Lie algebra in which α_L is multiplicative with respect to $[-,-]$, unless otherwise specified.

Definition 3.1. By a *(right) Hom-L-module*, we mean a Hom-module (M, α_M) that comes equipped with a right L-action, $\rho \colon M \otimes L \to M$ $(m \otimes x \mapsto mx)$, such that the following two conditions are satisfied for $m \in M$ and $x, y \in L$:

$$\begin{aligned} \alpha_M(m)[x,y] &= (mx)\alpha_L(y) - (my)\alpha_L(x), \\ \alpha_M(mx) &= \alpha_M(m)\alpha_L(x) \end{aligned} \tag{6}$$

Example 3.2. Here are some examples of Hom-L-modules.

1. One can consider L itself as a Hom-L-module in which the L-action is the bracket $[-,-]$.
2. If $\mathfrak{g}$ is a Lie algebra and M is a right $\mathfrak{g}$-module in the usual sense, then (M, Id_M) is a Hom-$\mathfrak{g}$-module. ■

For the rest of this section, (M, α_M) will denote a fixed Hom-L-module, where L is a Hom-Lie algebra. For $n \geq 0$, let $\Lambda^n L$ denote the nth exterior power of L, with $\Lambda^0 L = \mathbb{K}$. A typical generator in $\Lambda^n L$ is denoted by $x_1 \wedge \cdots \wedge x_n$ with each $x_i \in L$. We will use the following abbreviations:

$$\begin{aligned} x_1 \cdots \widehat{x_i} \cdots x_n &= x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_n, \\ \alpha_L(x_1 \cdots \widehat{x_i} \cdots x_n) &= \alpha_L(x_1) \wedge \cdots \wedge \alpha_L(x_{i-1}) \wedge \alpha_L(x_{i+1}) \wedge \cdots \wedge \alpha_L(x_n). \end{aligned}$$

Likewise, the symbols $x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_n$, $\alpha_L(x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_n)$, and so forth mean that the terms $\widehat{x_i}$, $\widehat{x_j}$, etc., are omitted.

Define the *module of n-chains of L with coefficients in M* as

$$CE_n^\alpha(L, M) = M \otimes \Lambda^n L.$$

For $p \geq 1$, define a linear map $d_p\colon CE^{\alpha}_p(L,M) \to CE^{\alpha}_{p-1}(L,M)$ by setting (for $m \in M$, $x_i \in L$)

$$d_p(m \otimes x_1 \wedge \cdots \wedge x_p) = \eta_1 + \eta_2, \tag{7}$$

where

$$\eta_1 = \sum_{i=1}^{p} (-1)^{i+1} m x_i \otimes \alpha_L(x_1 \cdots \widehat{x_i} \cdots x_p)$$

and

$$\eta_2 = \sum_{i<j} (-1)^{i+j} \alpha_M(m) \otimes [x_i, x_j] \wedge \alpha_L(x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_p).$$

Theorem 3.3. *The data $(CE^{\alpha}_*(L,M), d)$ forms a chain complex.*

Proof. Using the notations in (7), we have

$$\begin{aligned} d^2(m \otimes x_1 \wedge \cdots \wedge x_p) &= d(\eta_1) + d(\eta_2) \\ &= (\eta_{11} + \eta_{12}) + (\eta_{21} + \eta_{22}). \end{aligned}$$

Therefore, to prove the Theorem, it suffices to show that

$$\eta_{11} + \eta_{12} + \eta_{21} = 0 \tag{8}$$

and

$$\eta_{22} = 0. \tag{9}$$

To prove (8), first note that η_{21} is a sum of $p-1$ terms, the first of which is

$$\sum_{i<j} (-1)^{i+j} \left(\alpha_M(m)[x_i, x_j]\right) \otimes \alpha_L^2(x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_p). \tag{10}$$

On the other hand, we have that

$$\begin{aligned} \eta_{11} &= \sum_{i=1}^{p} (-1)^{i+1} \sum_{j<i} (-1)^{j+1} (m x_i)\alpha_L(x_j) \otimes \alpha_L^2(x_1 \cdots \widehat{x_j} \cdots \widehat{x_i} \cdots x_p) \\ &\quad + \sum_{i=1}^{p} (-1)^{i+1} \sum_{j>i} (-1)^{(j-1)+1} (m x_i)\alpha_L(x_j) \otimes \alpha_L^2(x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_p) \\ &= -\sum_{i<j} (-1)^{i+j} \left((m x_i)\alpha_L(x_j) - (m x_j)\alpha_L(x_i)\right) \otimes \alpha_L^2(x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_p). \end{aligned}$$

By the first Hom-L-module axiom (6), the last line is equal to (10) with a minus sign.

The other $p-2$ terms in η_{21} are given by the sum

$$\begin{aligned} \sum_{i<j<k} (-1)^{i+j+k} \big(&\alpha_M(m)\alpha_L(x_i) \otimes \alpha_L([x_j, x_k]) \wedge z - \alpha_M(m)\alpha_L(x_j) \otimes \alpha_L([x_i, x_k]) \wedge z \\ &+ \alpha_M(m)\alpha_L(x_k) \otimes \alpha_L([x_i, x_j]) \wedge z\big), \end{aligned} \tag{11}$$

where

$$z = \alpha_L^2(x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \widehat{x_k} \cdots x_p). \tag{12}$$

Using the second Hom-L-module axiom (6) and the multiplicativity of α_L, we can rewrite (11) as

$$\sum_{i<j<k} (-1)^{i+j+k}(\alpha_M(mx_i) \otimes [\alpha_L(x_j), \alpha_L(x_k)] \wedge z - \alpha_M(mx_j) \otimes [\alpha_L(x_i), \alpha_L(x_k)] \wedge z \\ + \alpha_M(mx_k) \otimes [\alpha_L(x_i), \alpha_L(x_j)] \wedge z). \tag{13}$$

It is straightforward to see that (13) is equal to η_{12} with a minus sign. So far we have proved (8).

To show (9), observe that

$$\eta_{22} = \sum_{i<j<k} (-1)^{i+j+k} \alpha_M^2(m) \otimes y \wedge z + \sum_{i<j<k<l} (-1)^{i+j+k+l} \alpha_M^2(m) \otimes u \wedge w, \tag{14}$$

where z is as in (12) and

$$\begin{aligned} y &= [[x_i, x_j], \alpha_L(x_k)] + [[x_j, x_k], \alpha_L(x_i)] + [[x_k, x_i], \alpha_L(x_j)], \\ u &= [\alpha_L(x_i), \alpha_L(x_j)] \wedge \alpha_L([x_k, x_l]) + [\alpha_L(x_k), \alpha_L(x_l)] \wedge \alpha_L([x_i, x_j]) \\ &\quad - [\alpha_L(x_i), \alpha_L(x_k)] \wedge \alpha_L([x_j, x_l]) - [\alpha_L(x_j), \alpha_L(x_l)] \wedge \alpha_L([x_i, x_k]) \\ &\quad + [\alpha_L(x_i), \alpha_L(x_l)] \wedge \alpha_L([x_j, x_k]) + [\alpha_L(x_j), \alpha_L(x_k)] \wedge \alpha_L([x_i, x_l]), \\ w &= \alpha_L^2(x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \widehat{x_k} \cdots \widehat{x_l} \cdots x_p). \end{aligned}$$

It follows from the Hom-Jacobi identity (3) and the skew-symmetry of $[-,-]$ that

$$y = 0. \tag{15}$$

Likewise, using the multiplicativity of α_L and that $a \wedge b = -b \wedge a$ in an exterior algebra, one infers that

$$u = 0. \tag{16}$$

Combining (14), (15), and (16), it follows that $\eta_{22} = 0$, which proves (9). ■

In view of Theorem 3.3, we define the *nth homology of L with coefficients in M* as

$$H_n^\alpha(L, M) = H_n(CE_*^\alpha(L, M)).$$

Note that for a Lie algebra $\mathfrak{g}$ and a right $\mathfrak{g}$-module M, the chain complex $CE_*^\alpha(\mathfrak{g}, M)$ is exactly the Chevalley-Eilenberg complex [9] that defines the Lie algebra homology of $\mathfrak{g}$ with coefficients in the right $\mathfrak{g}$-module M. This justifies our choice of notation.

Since the differential $d_1 \colon M \otimes \Lambda^1 L = M \otimes L \to M$ is the right L-action map on M, it follows that

$$H_0^\alpha(L, M) = \frac{M}{span_{\mathbb{K}}\{mx \colon m \in M,\, x \in L\}}.$$

In particular, when L is considered as a Hom-L-module via its bracket, we have that

$$H_0^\alpha(L, L) = \frac{L}{[L, L]}, \tag{17}$$

which is the abelianization of L with respect to its bracket.

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