Journal of Lie Theory
Volume **19** (2009) 1–27

Composition Series for a Family of Modules of Nongraded Hamiltonial Type Lie Algebras

Yufeng Zhao*

Communicated by E. Zelmanov

Abstract. Nongraded infinite-dimensional simple Lie algebras of Hamiltonial type were constructed by Xu related to pairs of locally-finite derivations on certain commutative associative algebras. In this paper, we construct a family of modules with parameters for nongraded Hamiltonial type Lie algebras based on finite-dimensional irreducible modules of symplectic Lie algebras. When the corresponding modules of symplectic Lie algebras are finite-dimensional irreducible weight modules whose weight spaces are all one-dimensional, we get a composition series for these modules and an explicit construction of the composition series are also given by means of the exterior algebra powers.
Mathematics Classification 2000: 17B10, 17B65.
Key Words and Phrases: Nongraded Hamiltonial type Lie algebras, representations of nongraded Hamiltonial type Lie algebras.

1. Introduction

A Lie algebra $\mathcal{G}$ is called *graded* by an abelian group Δ if

$$\mathcal{G} = \bigoplus_{x\in\Delta} \mathcal{G}_x, \qquad [\mathcal{G}_y, \mathcal{G}_z] \subseteq \mathcal{G}_{y+z}, \qquad \forall\ y,\ z \in \Delta. \tag{1}$$

A composition series for a Lie algebra $\mathcal{G}$- module $\mathcal{M}$ is a series of submodules

$$\{0\} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_n = \mathcal{M} \tag{2}$$

with $\mathcal{M}_i/\mathcal{M}_{i-1}$ irreducible.

The Lie algebras associated with vertex algebras ([3], [4]) and the Lie algebras generated by conformal algebras are in general nongraded Lie algebras. The algebraic aspect of quantum field theory is a certain new representation theory of the Lie algebras generated by conformal algebras. However, not much work has been done on the representation theory of nongraded Lie algebras. Kac and Radul ([5], [6]) classified all the quasifinite irreducible highest weight modules for the $W_{1+\infty}$ Lie algebra, a nongraded Lie algebra of differential operators. Xu [20]

*Research supported by NSFC Grant 10701002

ISSN 0949–5932 / \$2.50

constructed irreducible modules of centrally-extended classical Lie algebras over left ideals of the algebra of differential operators on the circle, through certain irreducible modules of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries. Motivated from his classification of quadratic conformal algebras corresponding to certain Hamiltonian pairs in integrable systems, Xu ([16-19]) found seven types of nongraded infinite-dimensional simple Lie algebras related to a finite set of locally-finite derivations on certain commutative associative algebras. Su and Zhou [15] classified all the irreducible generalized weight modules with multiplicity one for nongraded Lie algebras of Witt type. In [21], we have constructed a family of irreducible generalized weight modules from finite-dimensional irreducible modules of general linear Lie algebra for nongraded Witt type Lie algebras, which are nongraded generalizations of Rao's work on irreducible representations of the derivation Lie algebra of the algebra of Laurent polynomials in several variables ([10], [11]) and Lin-Tan' work on irreducible representations of the Lie algebra of derivations for quantum torus [7]. We [22] have also constructed a large family of irreducible modules for two-derivation nongraded Block type Lie algebras via imbedding into the corresponding algebras of Weyl type. In this paper, we construct a family of modules with parameters for nongraded Hamiltonial type Lie algebras based on finite-dimensional irreducible modules of symplectic Lie algebras. When the corresponding modules of symplectic Lie algebras are finite-dimensional irreducible weight modules whose weight spaces are all one-dimensional, we get a composition series for these modules and an explicit construction of the composition series are also given by means of the exterior algebra powers. Below we will give more details of the backgrounds and our results.

Throughout this paper, let $\mathbb{F}$ be a field with characteristic 0. All the vector spaces are assumed over $\mathbb{F}$. Denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the additive semigroup of nonnegative integers. Let n be a positive integer. For any $1 \leq s \leq 2n$, let

$$\varsigma(s) = \begin{cases} 1, & \text{if } 1 \leq s \leq n, \\ -1, & \text{if } n+1 \leq s \leq 2n. \end{cases} \qquad s' = s + \varsigma(s)n. \tag{3}$$

Assume $\mathcal{A}$ is a commutative associative algebra over $\mathbb{F}$ and $\{\partial_1, \cdots, \partial_{2n}\}$ is a set of mutually commutative derivations of $\mathcal{A}$. We define the following Lie bracket $[\cdot,\cdot]$ on $\mathcal{A}$:

$$[u,v] = \sum_{i=1}^{n} [\partial_i(u)\partial_{i'}(v) - \partial_{i'}(u)\partial_i(v)] \qquad \text{for } u, v \in \mathcal{A}. \tag{4}$$

The vector space

$$\mathcal{H} = \{\sum_{i=1}^{2n} a_i\partial_i \mid a_i \in \mathcal{A},\ \sum_{i=1}^{2n} \partial_i(a_i) = 0\} \tag{5}$$

is an associative subalgebra of the associative algebra End$\mathcal{A}$ of linear transformations on $\mathcal{A}$, which can be viewed as a Lie algebra with the commutator as the Lie bracket. We call $\mathcal{H}$ a *Lie algebra of Hamiltonial type* (cf. [8], [9], [12]).

Let $E_{i,j}$ be the $2n \times 2n$ matrix with (i,j)-th entry 1 and 0 elsewhere. Then the symplectic Lie algebra $sp(2n,\mathbb{F})$ is a vector space with a basis $\{E_{i,i}-E_{i',i'}(1 \le i \le n),\ E_{i,j}-E_{i',j'}(1 \le i \ne j \le n),\ E_{i,i'}(1 \le i \le 2n),\ E_{i,j'}+E_{j,i'}(1 \le i \ne j \le 2n)\}$ and the Lie bracket:

$$[E_{i,j}, E_{k,l}] = \delta_{j,k}E_{i,l} - \delta_{l,i}E_{k,j}. \tag{6}$$

Note that the subspace η with a basis $\{h_i = E_{i,i} - E_{i',i'} \mid 1 \le i \le n\}$ is a Cartan subalgebra of $sp(2n,\mathbb{F})$. Denote the dual vector space of η by η^* and let $\omega_1, \cdots, \omega_n$ be the fundamental dominant weights in η^* defined by $\omega_i(h_j) = \delta_{i,j}$.

Now we assume $V(\psi)$ is a finite-dimensional irreducible module with highest weight ψ for symplectic Lie algebra $sp(2n,\mathbb{F})$. Set

$$\overline{V}(\psi) = \mathcal{A} \bigotimes_{\mathbb{F}} V(\psi) \tag{7}$$

and define the following $\mathcal{H}$-module structure on $\overline{V}(\psi)$ by:

$$\rho : \mathcal{H} \to gl(\overline{V}(\psi))$$

$$\rho(\sum_{i=1}^{2n} a_i\partial_i)(f \otimes v) = \sum_{i=1}^{2n}(a_i\partial_i(f) + \xi_i a_i f) \otimes v + \sum_{i,j=1}^{2n} f\partial_i(a_j) \otimes E_{i,j}v, \tag{8}$$

for $f, a_i \in \mathcal{A}, v \in V(\psi)$, where $\xi = (\xi_1, \cdots, \xi_{2n}) \in \mathbb{F}^{2n}$ is a fixed vector. Furthermore, we define the following imbedding σ of the Lie algebra $(\mathcal{A}, [\cdot,\cdot])$ into $\mathcal{H}$ by:

$$\sigma : \mathcal{A} \to \mathcal{H};\ u \mapsto \sum_{i=1}^{n}[\partial_i(u)\partial_{i'} - \partial_{i'}(u)\partial_i] \quad \text{for any} \quad u \in \mathcal{A} \tag{9}$$

and set

$$\varrho = \rho \circ \sigma. \tag{10}$$

It is easy to verify that $(\varrho, \overline{V}(\psi))$ is a representation for Lie algebra $(\mathcal{A}, [\cdot,\cdot])$. The following result is our main theorem in this paper:

Theorem 1.1. *Assume that $\mathcal{A}$ is the semigroup algebra defined in [16] and ∂_i $(1 \le i \le 2n)$ are the locally-finite derivations in [16]. Then $(\mathcal{A}, [\cdot,\cdot])$-module $\overline{V}(\psi)$ has a composition series if $sp(2n,\mathbb{F})$- module $V(\psi)$ is a finite-dimensional irreducible module whose weight spaces are all one-dimensional.*

Remark 1.2. Via the imbedding σ defined by (9), $(\mathcal{A}, [\cdot,\cdot])$-module $\overline{V}(\psi)$ can be viewed as restricted module of $\mathcal{H}$-module $\overline{V}(\psi)$. When the locally-finite derivations ∂_i $(1 \le i \le 2n)$ are all graded operators, graded $\mathcal{H}$-module $\overline{V}(\psi)$ has been studied by Shen in ([12-14]).

The paper is organized as follows. In Section 2, we will first give detailed construction of nongraded Lie algebra of Hamiltonial type. Then we will explicitly construct the composition series for its modules defined by (10) under the condition that the corresponding modules of symplectic Lie algebras are finite-dimensional irreducible weight modules whose weight spaces are all one-dimensional. In section 3, 4, 5 and 6, we will prove Theorem 1.1 in four cases respectively.

2. Constructions of nongraded Hamiltonial type Lie algebra and composition series for its modules

In this section, we will first give detailed construction of nongraded Lie algebra of Hamiltonial type. Then we will explicitly construct the composition series for its modules defined by (10) under the condition that the corresponding modules of symplectic Lie algebras are finite-dimensional irreducible weight modules with multiplicity one.

A linear transformation $\mathcal{T}$ on a vector space V is called *locally-finite* if $\dim(\sum_{i=0}^{\infty}\mathbb{F}\mathcal{T}^i(v))$ is finite for any $v\in V$. Let $\{\varphi_p \mid 1\le p\le 2n\}$ be the coordinate mapping from $\mathbb{F}^{2n}$ to $\mathbb{F}$. Pick

$$J_p\in\{\{0\},\ \mathbb{N}\}\quad \text{for } 1\le p\le 2n \tag{11}$$

and set

$$J=J_1\times J_2\times\cdots\times J_{2n}, \tag{12}$$

where the addition on J is defined componentwisely. Take an additive subgroup Γ of $\mathbb{F}^{2n}$ such that

$$\Gamma\bigcap(\bigcap_{1\le p\ne q\le 2n}\ker\varphi_q\backslash\ker\varphi_p)\ne\emptyset\qquad \text{if } J_p=\{0\}\quad\text{for } 1\le p\le 2n. \tag{13}$$

Denote by $\mathcal{A}$ the semigroup algebra of $\Gamma\times J$ with a basis $\{x^{\alpha,\mathbf{i}}\mid(\alpha,\mathbf{i})\in\Gamma\times J\}$ and the algebraic operation defined by

$$x^{\alpha,\mathbf{i}}.x^{\beta,\mathbf{j}}=x^{\alpha+\beta,\mathbf{i}+\mathbf{j}}\quad \text{for } (\alpha,\mathbf{i}),\ (\beta,\mathbf{j})\in\Gamma\times J. \tag{14}$$

Moreover, we denote

$$i_{[p]}=(0,\cdots,\overset{p}{i},0,\cdots 0)\quad \text{for } i\in J_p;\ \tau_{[s]}=(0,\cdots,\overset{s}{\tau},\cdots,0)\quad \text{for } \tau\in\varphi_s(\Gamma). \tag{15}$$

For $1\le p\le 2n$, we define $\partial_p\in\mathrm{End}\mathcal{A}$ by :

$$\partial_p(x^{\alpha,\mathbf{i}})=\alpha_p x^{\alpha,\mathbf{i}}+i_p x^{\alpha,\mathbf{i}-\mathbf{1}_{[\mathbf{p}]}}\qquad \text{for } (\alpha,\mathbf{i})\in\Gamma\times\mathbf{J}. \tag{16}$$

It can be verified that $\{\partial_p\mid 1\le p\le 2n\}$ are linearly independent and commuting locally-finite derivations on $\mathcal{A}$. Xu (cf. [17]) introduced the following Lie bracket $[\cdot,\cdot]$ on $\mathcal{A}$:

$$[u,v]=\sum_{i=1}^{n}[\partial_i(u)\partial_{i'}(v)-\partial_{i'}(u)\partial_i(v)]\qquad \text{for } u,\ v\in\mathcal{A}. \tag{17}$$

We call $(\mathcal{A},[\cdot,\cdot])$ a *nongraded Lie algera of Hamiltonial type.*

For convenience, we denote

$$\beta'_i=\beta_i+\xi_i\quad \text{for } \beta=(\beta_1,\cdots,\beta_{2n})\in\Gamma. \tag{18}$$

By (8-10), $(\mathcal{A},[\cdot,\cdot])$-module $\overline{V}(\psi)$ can be explicitly given by:

$$\begin{aligned}
& x^{\alpha,\mathbf{p}}.x^{\beta,\mathbf{q}}\otimes v \\
= & \sum_{i=1}^{n}[(\alpha_i\beta'_{i'}-\alpha_{i'}\beta'_i)x^{\alpha+\beta,\mathbf{p}+\mathbf{q}}+(p_i\beta'_{i'}-q_i\alpha_{i'})x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i]}} \\
+ & (q_{i'}\alpha_i-p_{i'}\beta'_i)x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i']}}+(p_iq_{i'}-q_ip_{i'})x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i]}-1_{[i']}}]\otimes v \\
+ & \sum_{i,j=1}^{n}[x^{\alpha+\beta,\mathbf{p}+\mathbf{q}}\otimes(-\alpha_i\alpha_{j'}E_{i,j}.v+\alpha_i\alpha_jE_{i,j'}.v-\alpha_{i'}\alpha_{j'}E_{i',j}.v \\
+ & \alpha_{i'}\alpha_jE_{i',j'}.v)+x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i]}}\otimes(p_i\alpha_jE_{i,j'}.v-p_i\alpha_{j'}E_{i,j}.v) \\
+ & x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[j]}}\otimes(p_j\alpha_iE_{i,j'}.v+p_j\alpha_{i'}E_{i',j'}.v) \\
+ & x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[j']}}\otimes(-p_{j'}\alpha_iE_{i,j}.v-p_{j'}\alpha_{i'}E_{i',j}.v) \\
+ & x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i']}}\otimes(-p_{i'}\alpha_{j'}E_{i',j}.v+p_{i'}\alpha_jE_{i',j'}.v) \\
+ & x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[j]}-1_{[i']}}\otimes p_jp_{i'}E_{i',j'}.v-x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i]}-1_{[j']}}\otimes p_ip_{j'}E_{i,j}.v \\
+ & x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i]}-1_{[j]}}\otimes p_j(p_i-\delta_{i,j})E_{i,j'}.v \\
- & x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i']}-1_{[j']}}\otimes p_{j'}(p_{i'}-\delta_{i,j})E_{i',j}.v],
\end{aligned} \tag{19}$$

for $(\alpha,\mathbf{p}),\ (\beta,\mathbf{q})\in\Gamma\times J,\ v\in V(\psi)$. In particular, we have

$$x^{0,1_{[s]}}.x^{\beta,\mathbf{q}}\otimes v=\varsigma(s)(\beta'_{s'}x^{\beta,\mathbf{q}}\otimes v+q_{s'}x^{\beta,\mathbf{q}-\mathbf{1}_{[\mathbf{s}']}}\otimes v)\quad \text{if } J_s=\mathbb{N}. \tag{20}$$

And

$$\begin{aligned}
& x^{-2\tau_{[s]},\mathbf{0}}.x^{\tau_{[s]},\mathbf{0}}.x^{\tau_{[s]},\mathbf{0}}.x^{\beta,\mathbf{q}}\otimes v \\
= & 2\tau^3\varsigma(s')[x^{\beta,\mathbf{q}}\otimes(\beta'^3_{s'}v-3\tau^2\beta'_{s'}E^2_{s,s'}.v-2\tau^3E^3_{s,s'}.v) \\
+ & 3q_{s'}(q_{s'}-1)x^{\beta,\mathbf{q}-\mathbf{2}_{[\mathbf{s}']}}\otimes\beta_{s'}v+3q_{s'}x^{\beta,\mathbf{q}-\mathbf{1}_{[\mathbf{s}']}}\otimes(\beta'^2_{s'}v-\tau^2E^2_{s,s'}.v) \\
+ & q_{s'}(q_{s'}-1)(q_{s'}-2)x^{\beta,\mathbf{q}-\mathbf{3}_{[\mathbf{s}']}}\otimes v]\qquad \text{if } J_s=\{0\}.
\end{aligned} \tag{21}$$

Lemma 2.1. *(cf. Proposition 3.6 in [1]) The highest weight module $V(\psi)$ for symplectic Lie algebra $sp(2n,\mathbb{F})$ is an irreducible finite-dimensional module whose weight spaces are all one-dimensional if and only if $\psi=0,\omega_1$, or ω_n when $n=2$ or 3, where ω_i are the fundamental dominant weights of $sp(2n,\mathbb{F})$.*

In the rest of this section, we will construct the composition series for $\mathcal{A}$-module $\overline{V}(\psi)$ explicitly under the condition $\psi=0,\ \omega_1,$ or ω_n when $n=2$ or 3 respectively.

Suppose v is a nonzero vector of $V(0)$. It is straightforward to verify that $\mathbb{F}(x^{-\xi,\mathbf{0}}\otimes v)$ is a submodule of $\overline{V}(0)$ if $\xi\in\Gamma$.

Theorem 2.2. *If $\xi\in\Gamma$, then $\mathcal{A}$-module $\overline{V}(0)/(\mathbb{F}x^{-\xi,\mathbf{0}}\otimes v)$ is irreducible. If $\xi\notin\Gamma$, then $\mathcal{A}$-module $\overline{V}(0)$ is irreducible.*

It is known that $sp(2n,\mathbb{F})$ -module $V(\omega_1)$ is isomorphic to the natural $2n$-dimensional module $\mathbb{F}^{2n}$ with standard basis $\{e_1,\cdots,e_{2n}\}$, i.e. $e_i=(0,\cdots,\overset{i}{1},\cdots,0)$ for $1\le i\le 2n$ (cf. Proposition 13.24 in [2]). For any $(\alpha,\mathbf{i})\in\Gamma\times\mathbf{J}$, we denote

$$\partial'_s(x^{\alpha,\mathbf{i}}\otimes v)=(\alpha'_sx^{\alpha,\mathbf{i}}+i_sx^{\alpha,\mathbf{i}-1_{[s]}})\otimes v,\ \text{for } 1\le s\le 2n,\ v\in V(\psi), \tag{22}$$

$$x(\alpha, \mathbf{i}) = \sum_{s=1}^{2n} \partial'_s(x^{\alpha,\mathbf{i}} \otimes e_s); \tag{23}$$

$$\begin{aligned} x_{s,t}(\alpha, \mathbf{i}) &= \partial'_{t'}(x^{\alpha,\mathbf{i}} \otimes e_s) - \partial'_{s'}(x^{\alpha,\mathbf{i}} \otimes e_t), \\ x_{s',t'}(\alpha, \mathbf{i}) &= \partial'_t(x^{\alpha,\mathbf{i}} \otimes e_{s'}) - \partial'_s(x^{\alpha,\mathbf{i}} \otimes e_{t'}),\ 1 \le s < t \le n; \\ x_{s,t'}(\alpha, \mathbf{i}) &= \partial'_t(x^{\alpha,\mathbf{i}} \otimes e_s) + \partial'_{s'}(x^{\alpha,\mathbf{i}} \otimes e_{t'}), \quad 1 \le s, t \le n. \end{aligned} \tag{24}$$

Set

$$\begin{aligned} U(\omega_1) &= \sum_{(\alpha,\mathbf{i})\in\Gamma\times\mathbf{J},\ 1\le s<t\le 2n} \mathbb{F}x_{s,t}(\alpha, \mathbf{i}), \\ \tilde{U}(\omega_1) &= U(\omega_1) \bigcup (x^{-\xi,\mathbf{0}} \otimes V(\omega_1)) \quad \text{if } \xi \in \Gamma; \\ W(\omega_1) &= \sum_{(\alpha,\mathbf{i})\in\Gamma\times\mathbf{J}} \mathbb{F}x(\alpha, \mathbf{i}), \\ \tilde{W}(\omega_1) &= W(\omega_1) \bigcup (x^{-\xi,\mathbf{0}} \otimes V(\omega_1)) \quad \text{if } \xi \in \Gamma. \end{aligned} \tag{25}$$

It is straightforward to verify that $\mathcal{A}$-module $\overline{V}(\omega_1)$ has the following series of submodules:

$$\{0\} \subset W(\omega_1) \subset \tilde{W}(\omega_1) \subset \tilde{U}(\omega_1) \subset \overline{V}(\omega_1) \quad \text{if } \xi \in \Gamma; \tag{26}$$

$$\{0\} \subset W(\omega_1) \subset U(\omega_1) \subset \overline{V}(\omega_1) \quad \text{if } \xi \notin \Gamma. \tag{27}$$

Theorem 2.3. *If $\xi \in \Gamma$ (resp. $\xi \notin \Gamma$), then (26) (resp. (27)) is a composition series for $\mathcal{A}$-module $\overline{V}(\omega_1)$.*

Consider the exterior product

$$E^k(\mathbb{F}^{2n}) = \mathbb{F}^{2n} \wedge \mathbb{F}^{2n} \wedge \cdots \wedge \mathbb{F}^{2n} \ (k \text{ copies }). \tag{28}$$

From Proposition 13.28 in [2], we know that $sp(4, \mathbb{F})$ -module $V(\omega_2)$ is isomorphic to $\text{Span}_{\mathbb{F}}\{e_1 \wedge e_2,\ e_1 \wedge e_4,\ e_2 \wedge e_3,\ e_3 \wedge e_4,\ e_1 \wedge e_3 - e_2 \wedge e_4\}$ with the action $X.(e_{i_1} \wedge e_{i_2}) = (X.e_{i_1}) \wedge e_{i_2} + e_{i_1} \wedge (X.e_{i_2})$, $\forall\ X \in sp(4, \mathbb{F})$. For any $(\alpha, \mathbf{i}) \in \Gamma \times \mathbf{J}$, we denote

$$\begin{aligned} y_1(\alpha, \mathbf{i}) &= (\partial'_2\partial'_4 - \partial'_1\partial'_3)(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_2) + \partial'^2_4(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_4) + \partial'^2_3(x^{\alpha,\mathbf{i}} \otimes e_2 \wedge e_3), \\ y_2(\alpha, \mathbf{i}) &= \partial'^2_2(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_2) + (\partial'_1\partial'_3 + \partial'_2\partial'_4)(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_4) + \partial'^2_3(x^{\alpha,\mathbf{i}} \otimes e_3 \wedge e_4), \\ y_3(\alpha, \mathbf{i}) &= 2\partial'_2(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_2) + 2\partial'_4(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_4) + \partial'_3(x^{\alpha,\mathbf{i}} \otimes (e_1 \wedge e_3 - e_2 \wedge e_4)), \\ y_4(\alpha, \mathbf{i}) &= -\partial'^2_1(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_2) + (\partial'_1\partial'_3 + \partial'_2\partial'_4)(x^{\alpha,\mathbf{i}} \otimes e_2 \wedge e_3) - \partial'^2_4(x^{\alpha,\mathbf{i}} \otimes e_3 \wedge e_4), \\ y_5(\alpha, \mathbf{i}) &= -2\partial'_1(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_2) + 2\partial'_3(x^{\alpha,\mathbf{i}} \otimes e_2 \wedge e_3) - \partial'_4(x^{\alpha,\mathbf{i}} \otimes (e_1 \wedge e_3 - e_2 \wedge e_4)), \\ y_6(\alpha, \mathbf{i}) &= 2\partial'_2(x^{\alpha,\mathbf{i}} \otimes e_2 \wedge e_3) - 2\partial'_4(x^{\alpha,\mathbf{i}} \otimes e_3 \wedge e_4) + \partial'_1(x^{\alpha,\mathbf{i}} \otimes (e_1 \wedge e_3 - e_2 \wedge e_4)), \\ y_7(\alpha, \mathbf{i}) &= 2\partial'_1(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_4) + 2\partial'_3(x^{\alpha,\mathbf{i}} \otimes e_3 \wedge e_4) - \partial'_2(x^{\alpha,\mathbf{i}} \otimes (e_1 \wedge e_3 - e_2 \wedge e_4)), \\ y_8(\alpha, \mathbf{i}) &= \partial'^2_1(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_4) + \partial'^2_2(x^{\alpha,\mathbf{i}} \otimes e_2 \wedge e_3) + (\partial'_1\partial'_3 - \partial'_2\partial'_4)(x^{\alpha,\mathbf{i}} \otimes e_3 \wedge e_4), \end{aligned}$$

$$\begin{aligned} y_9(\alpha, \mathbf{i}) \ &= \ 2\partial'_1\partial'_4(x^{\alpha,\mathbf{i}} \otimes e_1 \wedge e_4) + 2\partial'_2\partial'_3(x^{\alpha,\mathbf{i}} \otimes e_2 \wedge e_3) \\ &- \ (\partial'_2\partial'_4 - \partial'_1\partial'_3)(x^{\alpha,\mathbf{i}} \otimes (e_1 \wedge e_3 - e_2 \wedge e_4)), \end{aligned}$$

$$\begin{aligned} y_{10}(\alpha,\mathbf{i}) &= -2\partial_1'\partial_2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2)+2\partial_3'\partial_4'(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4)\\ &- (\partial_1'\partial_3'+\partial_2'\partial_4')(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3-e_2\wedge e_4)); \end{aligned}$$

$$\begin{aligned} z_1(\alpha,\mathbf{i}) &= (\partial_1'\partial_3'+\partial_2'\partial_4')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2)+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_4)\\ &- \partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3)+\partial_3'\partial_4'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3-e_2\wedge e_4)), \end{aligned}$$

$$\begin{aligned} z_2(\alpha,\mathbf{i}) &= \partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2)-(\partial_1'\partial_3'-\partial_2'\partial_4')(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3)\\ &- \partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4)+\partial_1'\partial_4'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3-e_2\wedge e_4)), \end{aligned}$$

$$\begin{aligned} z_3(\alpha,\mathbf{i}) &= -\partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2)+(\partial_1'\partial_3'-\partial_2'\partial_4')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_4)\\ &+ \partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4)-\partial_2'\partial_3'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3-e_2\wedge e_4)), \end{aligned}$$

$$\begin{aligned} z_4(\alpha,\mathbf{i}) &= -\partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_4)-(\partial_1'\partial_3'+\partial_2'\partial_4')(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4)\\ &+ \partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3)+\partial_1'\partial_2'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3-e_2\wedge e_4)). \end{aligned} \tag{29}$$

Set

$$U(\omega_2)=\sum_{1\le k\le 10,\ (\alpha,\mathbf{i})\in\Gamma\times\mathbf{J}}\mathbb{F}y_k(\alpha,\mathbf{i}),$$

$$\tilde U(\omega_2)=U(\omega_2)\bigcup(x^{-\xi,\mathbf{0}}\otimes V(\omega_2))\quad\text{if }\xi\in\Gamma;$$

$$W(\omega_2)=\sum_{1\le k\le 4,\ (\alpha,\mathbf{i})\in\Gamma\times\mathbf{J}}z_k(\alpha,\mathbf{i}),$$

$$\tilde W(\omega_2)=W(\omega_2)\bigcup(x^{-\xi,\mathbf{0}}\otimes V(\omega_2))\quad\text{if }\xi\in\Gamma. \tag{30}$$

It is straightforward to verify that $\mathcal{A}$-module $\overline{V}(\omega_2)$ has the following series of submodules:

$$\{0\}\subset W(\omega_2)\subset\tilde W(\omega_2)\subset\tilde U(\omega_2)\subset\overline{V}(\omega_2)\quad\text{if }\xi\in\Gamma; \tag{31}$$

$$\{0\}\subset W(\omega_2)\subset U(\omega_2)\subset\overline{V}(\omega_2)\quad\text{if }\xi\notin\Gamma. \tag{32}$$

Theorem 2.4. *If $\xi\in\Gamma$ (resp. $\xi\notin\Gamma$) , then (31) (resp. (32)) is a composition series for $\mathcal{A}$-module $\overline{V}(\omega_2)$.*

It is known from Proposition 13.28 in [2] that $sp(6,\mathbb{F})$ -module $V(\omega_3)$ is isomorphic to $\mathrm{Span}_{\mathbb{F}}\{e_1\wedge e_2\wedge e_3,\ e_1\wedge e_2\wedge e_6,\ e_1\wedge e_3\wedge e_5,\ e_1\wedge e_5\wedge e_6,\ e_2\wedge e_3\wedge e_4,\ e_2\wedge e_4\wedge e_6,\ e_3\wedge e_4\wedge e_5,\ e_4\wedge e_5\wedge e_6,\ e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6,\ e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4,\ e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5,\ e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6,\ e_2\wedge e_4\wedge e_5-e_3\wedge e_4\wedge e_6,\ e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6\}$ with the action $X.(e_{i_1}\wedge e_{i_2}\wedge e_{i_3})=(X.e_{i_1})\wedge e_{i_2}\wedge e_{i_3}+e_{i_1}\wedge(X.e_{i_2})\wedge e_{i_3}+e_{i_1}\wedge e_{i_2}\wedge(X.e_{i_3})$, $\forall\ X\in sp(6,\mathbb{F})$. For any $(\alpha,\mathbf{i})\in\Gamma\times\mathbf{J}$, we denote

$$\begin{aligned} &o_1(\alpha,\mathbf{i})\\ &=2\partial_3'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)+\partial_5'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\ &+2\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)+\partial_4'(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4)), \end{aligned}$$

$$\begin{aligned}&o_2(\alpha,\mathbf{i})\\&=-2\partial_2'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)-\partial_6'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\&+2\partial_5'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)+\partial_4'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5)),\end{aligned}$$

$$\begin{aligned}&o_3(\alpha,\mathbf{i})\\&=-2\partial_1'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)-\partial_6'(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))\\&-2\partial_4'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)+\partial_5'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5)),\end{aligned}$$

$$\begin{aligned}&o_4(\alpha,\mathbf{i})\\&=2\partial_3'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)+\partial_2'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\&-2\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)+\partial_4'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6)),\end{aligned}$$

$$\begin{aligned}&o_5(\alpha,\mathbf{i})\\&=-2\partial_2'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)+\partial_3'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\&-2\partial_5'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)-\partial_4'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6)),\end{aligned}$$

$$\begin{aligned}&o_6(\alpha,\mathbf{i})\\&=-2\partial_3'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)+\partial_1'(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))\\&+2\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)+\partial_5'(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_4\wedge e_5-e_3\wedge e_4\wedge e_6)),\end{aligned}$$

$$\begin{aligned}&o_7(\alpha,\mathbf{i})\\&=-2\partial_1'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)+\partial_3'(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))\\&+2\partial_4'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)-\partial_5'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6)),\end{aligned}$$

$$\begin{aligned}&o_8(\alpha,\mathbf{i})\\&=2\partial_2'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)+\partial_1'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5))\\&+2\partial_5'(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)-\partial_6'(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_4\wedge e_5-e_3\wedge e_4\wedge e_6)),\end{aligned}$$

$$\begin{aligned}&o_9(\alpha,\mathbf{i})\\&=-2\partial_1'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)+\partial_2'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5))\\&+2\partial_4'(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)-\partial_6'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6)),\end{aligned}$$

$$\begin{aligned}&o_{10}(\alpha,\mathbf{i})\\&=2\partial_3'(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)+\partial_1'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6))\\&+2\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)+\partial_2'(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_4\wedge e_5-e_3\wedge e_4\wedge e_6)),\end{aligned}$$

$$\begin{aligned}&o_{11}(\alpha,\mathbf{i})\\&=2\partial_2'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)-\partial_3'(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_4\wedge e_5-e_3\wedge e_4\wedge e_6))\\&-2\partial_5'(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)+\partial_1'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6)),\end{aligned}$$

$$o_{12}(\alpha,\mathbf{i})$$
$$= -2\partial_1'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6) - \partial_3'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_4\wedge e_5 + e_3\wedge e_5\wedge e_6))$$
$$-2\partial_4'(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6) + \partial_2'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_4\wedge e_6 - e_2\wedge e_5\wedge e_6)),$$

$$o_{13}(\alpha,\mathbf{i})$$
$$= \partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3) - \partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)$$
$$+\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5) - \partial_5'\partial_6'(x^{\alpha,\mathbf{i}}\otimes (e_2\wedge e_4\wedge e_5 - e_3\wedge e_4\wedge e_6))$$
$$+(\partial_1'\partial_4' + \partial_2'\partial_5' + \partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4),$$

$$o_{14}(\alpha,\mathbf{i})$$
$$= \partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6) - \partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)$$
$$+\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6) + \partial_3'\partial_5'(x^{\alpha,\mathbf{i}}\otimes (e_2\wedge e_4\wedge e_5 - e_3\wedge e_4\wedge e_6))$$
$$+(-\partial_1'\partial_4' - \partial_2'\partial_5' + \partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6),$$

$$o_{15}(\alpha,\mathbf{i})$$
$$= \partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5) + \partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)$$
$$-\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6) - \partial_2'\partial_6'(x^{\alpha,\mathbf{i}}\otimes (e_2\wedge e_4\wedge e_5 - e_3\wedge e_4\wedge e_6))$$
$$+(-\partial_1'\partial_4' + \partial_2'\partial_5' - \partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5),$$

$$o_{16}(\alpha,\mathbf{i})$$
$$= \partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6) + \partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)$$
$$-\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5) - \partial_2'\partial_3'(x^{\alpha,\mathbf{i}}\otimes (e_2\wedge e_4\wedge e_5 - e_3\wedge e_4\wedge e_6))$$
$$+(\partial_1'\partial_4' - \partial_2'\partial_5' - \partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6),$$

$$o_{17}(\alpha,\mathbf{i})$$
$$= \partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3) + \partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)$$
$$+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5) - \partial_4'\partial_6'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_4\wedge e_5 + e_3\wedge e_5\wedge e_6))$$
$$-(\partial_1'\partial_4' + \partial_2'\partial_5' + \partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5),$$

$$o_{18}(\alpha,\mathbf{i})$$
$$= \partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6) + \partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)$$
$$+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6) + \partial_3'\partial_4'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_4\wedge e_5 + e_3\wedge e_5\wedge e_6))$$
$$+(\partial_1'\partial_4' + \partial_2'\partial_5' - \partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6),$$

$$o_{19}(\alpha,\mathbf{i})$$
$$= \partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5) + \partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)$$
$$+\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6) + \partial_1'\partial_6'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_4\wedge e_5 + e_3\wedge e_5\wedge e_6))$$
$$+(-\partial_1'\partial_4' + \partial_2'\partial_5' + \partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5),$$

$$
\begin{aligned}
&o_{20}(\alpha,\mathbf{i})\\
&=\partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)+\partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)\\
&+\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)+\partial_1'\partial_3'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6))\\
&+(\partial_1'\partial_4'-\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6),
\end{aligned}
$$

$$
\begin{aligned}
&o_{21}(\alpha,\mathbf{i})\\
&=\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)+\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)\\
&-\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)+\partial_4'\partial_5'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6))\\
&+(\partial_1'\partial_4'+\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6),
\end{aligned}
$$

$$
\begin{aligned}
&o_{22}(\alpha,\mathbf{i})\\
&=\partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)+\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)\\
&-\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)+\partial_2'\partial_4'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6))\\
&+(-\partial_1'\partial_4'+\partial_2'\partial_5'-\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6),
\end{aligned}
$$

$$
\begin{aligned}
&o_{23}(\alpha,\mathbf{i})\\
&=-\partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)+\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)\\
&+\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)-\partial_1'\partial_5'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6))\\
&+(\partial_1'\partial_4'-\partial_2'\partial_5'-\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6),
\end{aligned}
$$

$$
\begin{aligned}
&o_{24}(\alpha,\mathbf{i})\\
&=-\partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)+\partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)\\
&+\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)+\partial_1'\partial_2'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6))\\
&+(-\partial_1'\partial_4'-\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6),
\end{aligned}
$$

$$
\begin{aligned}
&o_{25}(\alpha,\mathbf{i})\\
&=-\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)+\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)\\
&+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)-\partial_3'\partial_5'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\
&+(-\partial_1'\partial_4'+\partial_2'\partial_5'-\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6),
\end{aligned}
$$

$$
\begin{aligned}
&o_{26}(\alpha,\mathbf{i})\\
&=\partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)-\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)\\
&+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)+\partial_2'\partial_6'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\
&+(-\partial_1'\partial_4'-\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5),
\end{aligned}
$$

$$
\begin{aligned}
&o_{27}(\alpha,\mathbf{i})\\
&=\partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)-\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)\\
&+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)-\partial_2'\partial_3'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\
&+(\partial_1'\partial_4'+\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6),
\end{aligned}
$$

$$
\begin{aligned}
&o_{28}(\alpha,\mathbf{i})\\
&=\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)-\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)\\
&+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)-\partial_5'\partial_6'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\
&+(\partial_1'\partial_4'-\partial_2'\partial_5'-\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3),
\end{aligned}
$$

$$
\begin{aligned}
&o_{29}(\alpha,\mathbf{i})\\
&=\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)+\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)\\
&+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)+\partial_4'\partial_6'(x^{\alpha,\mathbf{i}}\otimes (e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))\\
&+(\partial_1'\partial_4'-\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3),
\end{aligned}
$$

$$
\begin{aligned}
&o_{30}(\alpha,\mathbf{i})\\
&=\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)+\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)\\
&+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)+\partial_3'\partial_4'(x^{\alpha,\mathbf{i}}\otimes (e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))\\
&+(-\partial_1'\partial_4'+\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6),
\end{aligned}
$$

$$
\begin{aligned}
&o_{31}(\alpha,\mathbf{i})\\
&=\partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)+\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)\\
&+\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)+\partial_1'\partial_6'(x^{\alpha,\mathbf{i}}\otimes (e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))\\
&+(\partial_1'\partial_4'+\partial_2'\partial_5'-\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4),
\end{aligned}
$$

$$
\begin{aligned}
&o_{32}(\alpha,\mathbf{i})\\
&=\partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)+\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)\\
&+\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)-\partial_1'\partial_3'(x^{\alpha,\mathbf{i}}\otimes (e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))\\
&-(\partial_1'\partial_4'+\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6),
\end{aligned}
$$

$$
\begin{aligned}
&o_{33}(\alpha,\mathbf{i})\\
&=\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)\\
&-\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)\\
&-\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)-\partial_4'\partial_5'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5))\\
&+(\partial_1'\partial_4'+\partial_2'\partial_5'-\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3),
\end{aligned}
$$

$$
\begin{aligned}
&o_{34}(\alpha,\mathbf{i})\\
&=\partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)+\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)\\
&-\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)-\partial_2'\partial_4'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5))\\
&+(\partial_1'\partial_4'-\partial_2'\partial_5'-\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5),
\end{aligned}
$$

$$
\begin{aligned}
&o_{35}(\alpha,\mathbf{i})\\
&=\partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)-\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)\\
&-\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)-\partial_1'\partial_5'(x^{\alpha,\mathbf{i}}\otimes (e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5))\\
&+(\partial_1'\partial_4'-\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4),
\end{aligned}
$$

$$\begin{aligned}
&o_{36}(\alpha,\mathbf{i})\\
&=\partial_2'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)-\partial_1'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)\\
&+\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)+\partial_1'\partial_2'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5))\\
&+(\partial_1'\partial_4'+\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5),
\end{aligned}$$

$$\begin{aligned}
&o_{37}(\alpha,\mathbf{i})\\
&=2\partial_1'\partial_2'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)-2\partial_1'\partial_5'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)\\
&-\partial_3'\partial_6'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5))+2\partial_2'\partial_4'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)\\
&+2\partial_4'\partial_5'(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)+\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6)),
\end{aligned}$$

$$\begin{aligned}
&o_{38}(\alpha,\mathbf{i})\\
&=2\partial_2'\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)-2\partial_3'\partial_5'(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)\\
&-\partial_1'\partial_4'(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_4\wedge e_5-e_3\wedge e_4\wedge e_6))-2\partial_2'\partial_3'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)\\
&-2\partial_5'\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)+\partial_1'^2(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6)),
\end{aligned}$$

$$\begin{aligned}
&o_{39}(\alpha,\mathbf{i})\\
&=2\partial_1'\partial_3'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)-2\partial_1'\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)\\
&-\partial_2'\partial_5'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6))-2\partial_3'\partial_4'(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)\\
&-2\partial_4'\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)+\partial_2'^2(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4)),
\end{aligned}$$

$$\begin{aligned}
&o_{40}(\alpha,\mathbf{i})=2\partial_1'\partial_2'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)+2\partial_1'\partial_5'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)\\
&-\partial_3'\partial_6'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6))-2\partial_2'\partial_4'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)\\
&+2\partial_4'\partial_5'(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)+\partial_3'^2(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5)),
\end{aligned}$$

$$\begin{aligned}
&o_{41}(\alpha,\mathbf{i})\\
&=-2\partial_2'\partial_3'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)-2\partial_2'\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)\\
&-\partial_1'\partial_4'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))+2\partial_3'\partial_5'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5))\\
&-2\partial_5'\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_4\wedge e_5-e_3\wedge e_4\wedge e_6)),
\end{aligned}$$

$$\begin{aligned}
&o_{42}(\alpha,\mathbf{i})=-2\partial_1'\partial_3'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)-2\partial_1'\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)\\
&-\partial_1'\partial_5'(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))-2\partial_3'\partial_4'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)\\
&+2\partial_4'\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)+\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6));
\end{aligned}$$

$$\begin{aligned}
&p_1(\alpha,\mathbf{i})\\
&=(\partial_1'\partial_4'+\partial_2'\partial_5'+\partial_3'\partial_6')(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)+\partial_6'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)\\
&-\partial_5'^2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)+\partial_5'\partial_6'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\
&+\partial_4'^2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)+\partial_4'\partial_6'(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4)\\
&-\partial_4'\partial_5'(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5)),
\end{aligned}$$

$$
\begin{aligned}
&p_2(\alpha,\mathbf{i})\\
&=(\partial'_1\partial'_4-\partial'_2\partial'_5-\partial'_3\partial'_6)(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)+\partial'^2_1(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)\\
&+\partial'^2_6(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)+\partial'_1\partial'_6(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))\\
&-\partial'^2_5(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)+\partial'_5\partial'_6(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_4\wedge e_6-e_3\wedge e_4\wedge e_6))\\
&-\partial'_1\partial'_5(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5)),
\end{aligned}
$$

$$
\begin{aligned}
&p_3(\alpha,\mathbf{i})\\
&=(\partial'_3\partial'_6-\partial'_2\partial'_5-\partial'_1\partial'_4)(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)+\partial'^2_3(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)\\
&+\partial'^2_4(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)+\partial'_3\partial'_5(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\
&-\partial'^2_5(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)+\partial'_3\partial'_4(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))\\
&-\partial'_4\partial'_5(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6)),
\end{aligned}
$$

$$
\begin{aligned}
&p_4(\alpha,\mathbf{i})\\
&=(\partial'_1\partial'_4-\partial'_2\partial'_5-\partial'_3\partial'_6)(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)+\partial'^2_3(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)\\
&-\partial'^2_2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)+\partial'_2\partial'_3(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\
&+\partial'^2_4(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)+\partial'_3\partial'_4(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6))\\
&-\partial'_2\partial'_4(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6)),
\end{aligned}
$$

$$
\begin{aligned}
&p_5(\alpha,\mathbf{i})\\
&=(\partial'_1\partial'_4+\partial'_2\partial'_5+\partial'_3\partial'_6)(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)+\partial'^2_3(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)\\
&-\partial'^2_2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)+\partial'_1\partial'_3(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6))\\
&+\partial'^2_1(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)+\partial'_2\partial'_3(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_4\wedge e_6-e_3\wedge e_4\wedge e_6))\\
&-\partial'_1\partial'_2(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6)),
\end{aligned}
$$

$$
\begin{aligned}
&p_6(\alpha,\mathbf{i})\\
&=(\partial'_2\partial'_5-\partial'_1\partial'_4-\partial'_3\partial'_6)(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_4\wedge e_6)+\partial'^2_3(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)\\
&-\partial'^2_5(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)+\partial'_1\partial'_5(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6))\\
&+\partial'^2_1(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_6)-\partial'_1\partial'_3(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4))\\
&-\partial'_3\partial'_5(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_4\wedge e_6-e_3\wedge e_4\wedge e_6)),
\end{aligned}
$$

$$
\begin{aligned}
&p_7(\alpha,\mathbf{i})\\
&=(\partial'_2\partial'_5+\partial'_1\partial'_4-\partial'_3\partial'_6)(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)+\partial'^2_2(x^{\alpha,\mathbf{i}}\otimes e_2\wedge e_3\wedge e_4)\\
&-\partial'^2_1(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)+\partial'_1\partial'_2(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5))\\
&-\partial'^2_6(x^{\alpha,\mathbf{i}}\otimes e_4\wedge e_5\wedge e_6)-\partial'_1\partial'_6(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5+e_3\wedge e_5\wedge e_6))\\
&-\partial'_2\partial'_6(x^{\alpha,\mathbf{i}}\otimes(e_2\wedge e_4\wedge e_6-e_3\wedge e_4\wedge e_6)),
\end{aligned}
$$

$$
\begin{aligned}
&p_8(\alpha,\mathbf{i})\\
&=(\partial'_1\partial'_4+\partial'_3\partial'_6-\partial'_2\partial'_5)(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_3\wedge e_5)+\partial'^2_2(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_2\wedge e_3)\\
&-\partial'^2_6(x^{\alpha,\mathbf{i}}\otimes e_1\wedge e_5\wedge e_6)+\partial'_2\partial'_6(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6))\\
&-\partial'^2_4(x^{\alpha,\mathbf{i}}\otimes e_3\wedge e_4\wedge e_5)-\partial'_2\partial'_4(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5))\\
&+\partial'_4\partial'_6(x^{\alpha,\mathbf{i}}\otimes(e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6)). \qquad (33)
\end{aligned}
$$

Set

$$U(\omega_3)=\sum_{1\le k\le 42,\ (\alpha,\mathbf{i})\in\Gamma\times\mathbf{J}}\mathbb{F}o_k(\alpha,\mathbf{i}),$$

$$\tilde{U}(\omega_3)=U(\omega_3)\bigcup(x^{-\xi,\mathbf{0}}\otimes V(\omega_3))\quad\text{if }\xi\in\Gamma;$$

$$W(\omega_3)=\sum_{1\le k\le 8,\ (\alpha,\mathbf{i})\in\Gamma\times\mathbf{J}}\mathbb{F}p_k(\alpha,\mathbf{i}),$$

$$\tilde{W}(\omega_3)=W(\omega_3)\bigcup(x^{-\xi,\mathbf{0}}\otimes V(\omega_3))\quad\text{if }\xi\in\Gamma. \tag{34}$$

It is straightforward to verify that $\mathcal{A}$-module $\overline{V}(\omega_3)$ has the following series of submodules:

$$\{0\}\subset W(\omega_3)\subset\tilde{W}(\omega_3)\subset\tilde{U}(\omega_3)\subset\overline{V}(\omega_3)\quad\text{if }\xi\in\Gamma; \tag{35}$$

$$\{0\}\subset W(\omega_3)\subset U(\omega_3)\subset\overline{V}(\omega_3)\quad\text{if }\xi\notin\Gamma. \tag{36}$$

Theorem 2.5. *If $\xi\in\Gamma$ (resp. $\xi\notin\Gamma$), then (35) (resp. (36)) is a composition series for $\mathcal{A}$-module $\overline{V}(\omega_3)$.*

The following fact will be used in the proof of Theorem 2.3, Theorem 2.4 and Theorem 2.5:

Lemma 2.6. *Suppose $\overline{W}$ is any nonzero submodule of $\mathcal{A}$-module $\overline{V}(\psi)$. If $0\ne x^{\beta,\mathbf{0}}\otimes\omega\in\overline{W}$, $\omega\in V(\psi)$, then $\overline{W}$ contains the following vectors:*

(i) $x^{\beta,\mathbf{0}}\otimes[\beta_{s'}'^2E_{t,t'}.\omega-\beta_{t'}'\beta_{s'}'(E_{s,t'}+E_{t,s'}).\omega+\beta_{t'}'^2E_{s,s'}.\omega]$, $\forall\ 1\le s<t\le n$;

(ii) $x^{\beta,\mathbf{0}}\otimes[\beta_t'^2E_{s,s'}.\omega-\beta_{s'}'^2E_{t',s}.\omega-\beta_{s'}'\beta_t'(E_{s,t}-E_{t',s'}).\omega]$, $\forall\ 1\le s,t\le n$;

(iii) $x^{\beta,\mathbf{0}}\otimes[\beta_s'^2E_{t',t}.\omega-\beta_t'\beta_s'(E_{t',s}+E_{s',t}).\omega+\beta_t'^2E_{s',s}.\omega]\in\overline{W}$, $\forall\ 1\le s<t\le n$.

Proof. Choose any $1\le s<t\le n$. We will first prove (i) in the following three cases:

Case 1. $J_s=J_t=\mathbb{N}$.

By (19), we have

$$\begin{aligned}\overline{W}\ &\ni\ x^{0,1_{[s]}+1_{[t]}}.x^{\beta,\mathbf{0}}\otimes\omega\\ &=\ \beta_{s'}'x^{\beta,1_{[t]}}\otimes\omega+\beta_{t'}'x^{\beta,1_{[s]}}\otimes\omega+x^{\beta,\mathbf{0}}\otimes(E_{s,t'}+E_{t,s'}).\omega,\end{aligned} \tag{37}$$

$$\overline{W}\ni x^{0,2_{[s]}}.x^{\beta,\mathbf{0}}\otimes\omega=2(\beta_{s'}'x^{\beta,1_{[s]}}\otimes\omega+x^{\beta,\mathbf{0}}\otimes E_{s,s'}.\omega). \tag{38}$$

Eliminating $x^{\beta,1_{[t]}}\otimes\omega$ and $x^{\beta,1_{[s]}}\otimes\omega$ in (37) through (38), we finally get (i).

Case 2. $J_s=J_t=\{0\}$.

For any $0 \neq \tau_{[s]} \in \Gamma$, $0 \neq \tau'_{[t]} \in \Gamma$ and $a \in \mathbb{Z}$, we have

$$\begin{aligned}
\overline{W} &\ni x^{\tau_{[s]}+a\tau'_{[t]},\mathbf{0}}.x^{(1-a)\tau'_{[t]},\mathbf{0}}.x^{\beta,\mathbf{0}} \otimes \omega \\
&= x^{\beta+\tau_{[s]}+\tau'_{[t]},\mathbf{0}} \otimes [(1-a)\tau'\beta'_{t'}(\tau\beta'_{s'} + a\tau'\beta'_{t'})\omega \\
&+ a(1-a)\tau\tau'^2\beta'_{t'}(E_{s,t'} + E_{t,s'}).\omega \\
&+ (1-a)\tau'^2((1-a)\tau\beta'_{s'} + a\tau'\beta'_{t'})E_{t,t'}.\omega \\
&+ a(1-a)^2\tau\tau'^3(E_{s,t'} + E_{t,s'}).E_{t,t'}.\omega \\
&+ (1-a)\tau'\tau^2\beta'_{t'}E_{s,s'}.\omega \\
&+ (1-a)^2\tau^2\tau'^2E_{s,s'}.E_{t,t'}.\omega + (1-a)^2a^2\tau'^4E^2_{t,t'}.\omega]; \qquad (39)
\end{aligned}$$

$$\begin{aligned}
\overline{W} &\ni x^{a\tau_{[1]}+\tau'_{[2]},\mathbf{0}}.x^{(1-a)\tau'_{[1]},\mathbf{0}}.x^{\beta,\mathbf{0}} \otimes \omega \\
&= x^{\beta+\tau_{[s]}+\tau'_{[t]},\mathbf{0}} \otimes [(1-a)\tau\beta'_3(\tau'\beta'_{t'} + a\tau\beta'_{s'})\omega \\
&+ a(1-a)\tau'\tau^2\beta'_{s'}(E_{s,t'} + E_{t,s'}).\omega \\
&+ (1-a)\tau^2((1-a)\tau'\beta'_{t'} + a\tau\beta'_{s'})E_{s,s'}.\omega \\
&+ a(1-a)^2\tau'\tau^3(E_{s,t'} + E_{t,s'}).E_{s,s'}.\omega \\
&+ (1-a)\tau\tau'^2\beta'_{s'}E_{t,t'}.\omega + (1-a)^2\tau^2\tau'^2E_{s,s'}.E_{t,t'}.\omega \\
&+ (1-a)^2a^2\tau^4E^2_{t,t'}.\omega]; \qquad (40)
\end{aligned}$$

The coefficients of a^2 in (39) and (40) imply that

$$\begin{aligned}
&x^{\beta+\tau_{[s]}+\tau'_{[t]},\mathbf{0}} \otimes [-\beta'^2_{t'}\omega - \tau\beta'_{t'}(E_{s,t'} + E_{t,s'}).\omega \\
+ \; &(\tau\beta'_{s'} - \tau'\beta'_{t'})E_{t,t'}.\omega + \tau^2E_{s,s'}.E_{t,t'}.\omega] \in \overline{W}, \qquad (41)
\end{aligned}$$

$$\begin{aligned}
&x^{\beta+\tau_{[s]}+\tau'_{[t]},\mathbf{0}} \otimes [-\beta'^2_{s'}\omega - \tau'\beta'_{s'}(E_{s,t'} + E_{t,s'}).\omega \\
- \; &(\tau\beta'_{s'} - \tau'\beta'_{t'})E_{s,s'}.\omega + \tau'^2E_{s,s'}.E_{t,t'}.\omega] \in \overline{W}. \qquad (42)
\end{aligned}$$

Letting $a = 0$ in (39) and eliminating $x^{\beta+\tau_{[s]}+\tau'_{[t]},\mathbf{0}} \otimes \omega$ through (41) and (42), we finally obtain

$$\begin{aligned}
x^{\beta+\tau_{[s]}+\tau'_{[t]},\mathbf{0}} \otimes \omega' &= x^{\beta+\tau_{[s]}+\tau'_{[t]},\mathbf{0}} \otimes [\beta'^2_{s'}E_{t,t'}.\omega - \beta'_{s'}\beta'_{t'}(E_{s,t'} + E_{t,s'}).\omega \\
&+ \beta'^2_{t'}E_{s,s'}.\omega + (\tau\beta'_{s'} + \tau'\beta'_{t'})E_{s,s'}.E_{t,t'}.\omega] \in \overline{W}. \qquad (43)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&x^{-\tau_{[s]}-\tau'_{[t]},\mathbf{0}}.x^{\beta+\tau_{[s]}+\tau'_{[t]},\mathbf{0}} \otimes \omega' \\
= \; &x^{\beta,\mathbf{0}} \otimes [(-\tau\beta'_{s'} - \tau'\beta'_{t'})\omega' + \tau^2E_{s,s'}.\omega' + \tau\tau'(E_{s,t'} + E_{t,s'}).\omega' + \tau'^2E_{t,t'}.\omega'], \qquad (44)
\end{aligned}$$

From considering the coefficients of τ^2 or τ'^2 in (44), we can also get (i).

Case 3. $J_s = \mathbb{N}$, $J_t = \{0\}$.

For any $a \in \mathbb{Z}$ and $0 \neq \tau_{[t]} \in \Gamma$, we have

$$\overline{W} \ni x^{\tau_{[t]},\mathbf{0}}.x^{\beta,\mathbf{0}} \otimes w = x^{\beta+\tau_{[t]},\mathbf{0}} \otimes \tau(\beta'_{t'}w + \tau E_{t,t'}.w), \qquad (45)$$

$$\begin{aligned}
\overline{W} &\ni x^{a\tau_{[t]},\mathbf{0}}.x^{(1-a)\tau_{[t]},2_{[s]}}.x^{\beta,\mathbf{0}}\otimes w\\
&= x^{\beta+\tau_{[t]},2_{[s]}}\otimes[a(1-a)\tau^2\beta_{t'}^{\prime 2}w+(1-a)a\tau^3\beta'_{t'}E_{t,t'}.w\\
&+ (1-a)^2a^2\tau^4E_{t,t'}^2.w]+x^{\beta+\tau_{[t]},1_{[s]}}\otimes[2a\beta'_{s'}\beta'_{t'}\tau w\\
&+ 2a^2\tau^2\beta'_{s'}E_{t,t'}.w+2a(1-a)\tau^2\beta'_{t'}(E_{s,t'}+E_{t,s'}).w\\
&+ 2(1-a)a^2\tau^3E_{t,t'}(E_{s,t'}+E_{t,s'}).w]\\
&+ x^{\beta+\tau_{[t]},\mathbf{0}}\otimes(2a\tau\beta'_{t'}E_{s,s'}+2a^2\tau^2E_{t,t'}.E_{s,s'}.w),
\end{aligned}\tag{46}$$

$$\begin{aligned}
\overline{W} &\ni x^{(1-a)\tau_{[t]},1_{[s]}}.x^{a\tau_{[t]},1_{[s]}}.x^{\beta,\mathbf{0}}\otimes w\\
&= x^{\beta+\tau_{[t]},2_{[s]}}\otimes[(1-a)a\tau^2\beta_{t'}^{\prime 2}w+a(1-a)\tau^3\beta'_{t'}E_{t,t'}.w\\
&+ (1-a)^2a^2\tau^4E_{t,t'}^2.w]+x^{\beta+\tau_{[t]},1_{[s]}}\otimes[\beta'_{s'}\beta'_{t'}\tau w\\
&+ (1-2a+2a^2)\tau^2\beta'_{n+s}E_{t,n+t}.w+2a(1-a)\tau^2\beta'_{t'}(E_{s,t'}+E_{t,s'}).w\\
&+ (1-a)a\tau^3E_{t,t'}(E_{s,t'}+E_{t,s'}).w]+x^{\beta+\tau_{[t]},\mathbf{0}}\otimes[\beta_{s'}^{\prime 2}w+\tau\beta'_{s'}(E_{s,t'}\\
&+ E_{t,s'}).w+a(1-a)\tau^2(E_{s,t'}+E_{t,s'})^2.w],
\end{aligned}\tag{47}$$

Subtracting equation (46) from (47), we obtain

$$\begin{aligned}
\overline{W} &\ni x^{\beta+\tau_{[t]},1_{[s]}}\otimes[(1-2a)\tau\beta'_{s'}\beta'_{t'}w+(1-2a)\tau^2\beta'_{s'}E_{t,t'}.w\\
&+ a(1-a)(1-2a)\tau^3E_{t,t'}.(E_{s,t'}+E_{t,s'}).w]\\
&+ x^{\beta+\tau_{[t]},\mathbf{0}}\otimes[-2a\tau\beta'_{t'}E_{s,s'}.w-2a^2\tau^2E_{t,t'}.E_{s,s'}.w+\beta_{s'}^{\prime 2}w\\
&+ \tau\beta'_{s'}(E_{s,t'}+E_{t,s'}).w+a(1-a)\tau^2(E_{s,t'}+E_{t,s'})^2.w],
\end{aligned}\tag{48}$$

The coefficients of a^3, a^2 and a in (48) imply

$$\begin{aligned}
\overline{W} &\ni -x^{\beta+\tau_{[t]},1_{[s]}}\otimes 2\tau\beta'_{s'}(\beta'_{t'}\omega+\tau E_{t,t'}.\omega)\\
&+ x^{\beta+\tau_{[t]},\mathbf{0}}\otimes[\tau^2(E_{s,t'}+E_{t,s'})^2.w-2\tau\beta'_{t'}E_{s,s'}.\omega],
\end{aligned}\tag{49}$$

$$\begin{aligned}
\overline{W} &\ni x^{\beta+\tau_{[t]},1_{[s]}}\otimes\tau\beta'_{s'}(\beta'_{t'}\omega+\tau E_{t,t'}.\omega)\\
&+ x^{\beta+\tau_{[t]},\mathbf{0}}\otimes[\beta_{s'}^{\prime 2}.\omega+\tau\beta'_{s'}(E_{s,t'}+E_{t,s'}).w],
\end{aligned}\tag{50}$$

Hence

$$\begin{aligned}
\overline{W} &\ni x^{\beta+\tau_{[t]},\mathbf{0}}\otimes[2\beta_{s'}^{\prime 2}.\omega-2\tau\beta'_{t'}E_{s,s'}\\
&+ 2\tau\beta'_{s'}(E_{s,t'}+E_{t,s'}).w+\tau^2(E_{s,t'}+E_{t,s'})^2.w],
\end{aligned}\tag{51}$$

Eliminating $x^{\beta+\tau_{[t]},\mathbf{0}}\otimes\omega$ in (51) through (45), we finally get

$$\begin{aligned}
x^{\beta+\tau_{[t]},\mathbf{0}}\otimes\omega' &= x^{\beta+\tau_{[t]},\mathbf{0}}\otimes[2\beta_{t'}^{\prime 2}E_{s,s'}.\omega-2\beta'_{s'}\beta'_{t'}(E_{s,t'}+E_{t,s'}).w\\
&+ 2\beta_{s'}^{\prime 2}E_{t,t'}.\omega-\tau\beta'_{t'}(E_{s,t'}+E_{t,s'})^2.w]\in\overline{W}.
\end{aligned}\tag{52}$$

Furthermore,

$$x^{-\tau_{[t]},\mathbf{0}}.x^{\beta+\tau_{[t]},\mathbf{0}}\otimes\omega'=x^{\beta,\mathbf{0}}\otimes(-\tau\beta'_{t'}\omega'+\tau^2E_{t,t'}.\omega'),\tag{53}$$

$$\begin{aligned}
x^{-\tau_{[t]},1_{[1]}}.x^{\beta+\tau_{[t]},\mathbf{0}}\otimes\omega' &= x^{\beta,1_{[1]}}\otimes(-\tau\beta'_{t'}\omega'+\tau^2E_{t,t'}.\omega')\\
&+ x^{\beta,\mathbf{0}}\otimes[\beta'_{s'}\omega'-\tau(E_{s,t'}+E_{t,s'}).w]\in\overline{W}.
\end{aligned}\tag{54}$$

From consider the coefficients of τ in (53) or (54), we finally get (i). The case $J_s=\{0\}$, $J_t=\mathbb{N}$ is similar with Case 3. When $1\le s\le n<t\le 2n$ (resp. $n+1\le s<t\le 2n$), we can prove (ii) (resp. (iii)) by the similar arguments as in Case 1, Case 2 and Case 3. ∎

3. Proof of Theorem 2.2

In this section, we will give the proof of Theorem 2.2.

Proof. Suppose v is a nonzero vector of $V(0)$. By (19), $\mathcal{A}$-module $\overline{V}(0)$ is explicitly given by:

$$\begin{aligned} & x^{\alpha,\mathbf{p}}.x^{\beta,\mathbf{q}}\otimes v \\ = & \sum_{i=1}^{n}[(\alpha_i\beta'_{i'}-\alpha_{i'}\beta'_i)x^{\alpha+\beta,\mathbf{p}+\mathbf{q}}+(p_i\beta'_{i'}-q_i\alpha_{i'})x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i]}} \\ + & (q_{i'}\alpha_i-p_{i'}\beta'_i)x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i]}-1_{[i']}}+(p_iq_{i'}-q_ip_{i'})x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[i']}}]\otimes v, \end{aligned} \tag{55}$$

for $(\alpha,\mathbf{p}),\ (\beta,\mathbf{q})\in\Gamma\times J$. Let $\overline{W}$ be any nonzero submodule of $\overline{V}(0)$ that strictly contains $\mathbb{F}x^{-\xi,\mathbf{0}}\otimes v$. We treat $x^{-\xi,\mathbf{0}}\otimes v$ as 0 if $\xi\notin\Gamma$. For any $\mathbf{i}\in\mathbf{J}$, we define

$$|\mathbf{i}|=\sum_{p=1}^{2n}i_p. \tag{56}$$

Set

$$\overline{V}_k=\mathrm{span}\{x^{\alpha,\mathbf{i}}\otimes v\mid\alpha\in\Gamma,\ |\mathbf{i}|\le k\}\quad\text{for }k\in\mathbb{N}. \tag{57}$$

Moreover, we define

$$\hat{k}=\min\{k\in\mathbb{N}\mid(\overline{V}_k\bigcap\overline{W})\backslash\mathbb{F}(x^{-\xi,\mathbf{0}}\otimes v)\neq\emptyset\}. \tag{58}$$

For any $0\neq u\in(\overline{V}_k\bigcap\overline{W})\backslash\mathbb{F}(x^{-\xi,\mathbf{0}}\otimes v)$, we can write it in the following form:

$$u=\sum_{(-\xi,\mathbf{0})\neq(\alpha,\mathbf{i})\in\Gamma\times\mathbf{J},\ |\mathbf{i}|=\hat{k}}a_{\alpha,\mathbf{i}}x^{\alpha,\mathbf{i}}\otimes v+u', \tag{59}$$

where $a_{\alpha,\mathbf{i}}\in\mathbb{F},\ u'\in\overline{V}_{\hat{k}-1}+\mathbb{F}(x^{-\xi,\mathbf{0}}\otimes v)$, and define

$$l(u)=|\{\alpha\in\Gamma\mid a_{\alpha,\mathbf{i}}\neq0\text{ for }\mathbf{i}\in\mathbf{J},\ |\mathbf{i}|=\hat{k}\}|. \tag{60}$$

Set

$$l=\min\{l(v)\mid0\neq v\in(\overline{V}_k\bigcap\overline{W})\backslash\mathbb{F}(x^{-\xi,\mathbf{0}}\otimes v)\}. \tag{61}$$

Choose $0\neq u_1\in(\overline{V}_k\bigcap\overline{W})\backslash\mathbb{F}(x^{-\xi,\mathbf{0}}\otimes v)$ such that $l(u_1)=l$ and write it in the form (59). For any $1\le s\le2n$. Assume $J_s=\{0\}$. By (13) and (21), we have

$$\begin{aligned} & x^{-2\tau_{[s]},\mathbf{0}}.x^{\tau_{[s]},\mathbf{0}}.x^{\tau_{[s]},\mathbf{0}}.u_1 \\ = & \sum_{(-\xi,\mathbf{0})\neq(\alpha,\mathbf{i})\in\Gamma\times\mathbf{J},\ |\mathbf{i}|=\hat{k}}2\varsigma(s')a_{\alpha,\mathbf{i}}\alpha'^3_{s'}\tau^3x^{\alpha,\mathbf{i}}\otimes v(\mathrm{mod}\overline{V}_{\hat{k}-1}), \end{aligned} \tag{62}$$

Since τ takes an infinite number of elements in $\mathbb{F}$ for $0\neq\tau_{[s]}\in\Gamma$, the coefficients of τ^3 show

$$\alpha'_{s'}=\beta'_{s'}\quad\text{whenever }a_{\alpha,\mathbf{i}}.a_{\beta,\mathbf{j}}\neq0 \tag{63}$$

by the minimality of $l(u_1)$. Suppose $J_s=\mathbb{N}$. Then we have

$$x^{0,1_{[s]}}.u_1=\sum_{(-\xi,\mathbf{0})\neq(\alpha,\mathbf{i})\in\Gamma\times\mathbf{J},\ |\mathbf{i}|=\hat{k}}a_{\alpha,\mathbf{i}}\varsigma(s)\alpha'_{s'}x^{\alpha,\mathbf{i}}\otimes v(\mathrm{mod}\overline{V}_{\hat{k}-1}). \tag{64}$$

Thus we can also get (63). Therefore u_1 must take the following form

$$u_1=\sum_{(-\xi,\mathbf{0})\neq(\alpha,\mathbf{i})\in\Gamma\times\mathbf{J},\ |\mathbf{i}|=\hat{k}} a_{\mathbf{i}}x^{\alpha,\mathbf{i}}\otimes v(\mathrm{mod}\overline{V}_{\hat{k}-1}). \tag{65}$$

Suppose $\hat{k}>0$. Assume $i_p>0$ for some $a_{\mathbf{i}}\neq 0$. If $J_{p'}=\mathbb{N}$, then

$$0\neq x^{0,1_{[p']}}.u_1-\varsigma(p')\alpha'_p u_1\in(\overline{V}_{\hat{k}-1}\bigcap\overline{W})\backslash\mathbb{F}(x^{-\xi,\mathbf{0}}\otimes v). \tag{66}$$

Assume $J_{p'}=\{0\}$. If $\alpha'_p=0$ and $i_p=1$, we have

$$x^{-\tau_{[p']},\mathbf{0}}.x^{\tau_{[p']},2_{[p]}}.u_1=\sum_{(-\xi,\mathbf{0})\neq(\alpha,\mathbf{i})\in\Gamma\times\mathbf{J},\ |\mathbf{i}|=\hat{k}} 2\tau(2\alpha'_{p'}-\tau)a_{\mathbf{i}}x^{\alpha,\mathbf{i}}\otimes v(\mathrm{mod}\overline{V}_{\hat{k}-1}). \tag{67}$$

If $\alpha'_p\neq 0$ or $i_p\neq 1$, we have

$$x^{-\tau_{[p']},\mathbf{0}}.x^{\tau_{[p']},\mathbf{0}}.u_1=\sum_{(-\xi,\mathbf{0})\neq(\alpha,\mathbf{i})\in\Gamma\times\mathbf{J},\ |\mathbf{i}|=\hat{k}} -\tau^2 a_{\mathbf{i}}\partial'^2_p(x^{\alpha,\mathbf{i}}\otimes v)(\mathrm{mod}\overline{V}_{\hat{k}-1}). \tag{68}$$

It follows from (67) and (68) that $(\overline{V}_{\hat{k}-1}\bigcap\overline{W})\backslash\mathbb{F}(x^{-\xi,\mathbf{0}}\otimes v)\neq\{0\}$. This contradicts the definition of $\hat{k}$. Hence, $\hat{k}=0$, i.e. $\exists\ x^{\beta,\mathbf{0}}\otimes v\in\overline{W},\ \beta\neq-\xi$. Since the coefficients of α_i^2 or q_i^2 in $x^{\gamma-\alpha,\mathbf{p}-\mathbf{q}}.x^{\alpha-\beta,\mathbf{q}}.x^{\beta,\mathbf{0}}\otimes v$ is $\varsigma(i')\beta'_{i'}(\gamma'_{i'}x^{\gamma,\mathbf{q}}\otimes v+q_{i'}x^{\gamma,\mathbf{q}-1_{[i']}}\otimes v)$ ($1\leq i\leq 2n$), we have

$$\gamma'_{i'}x^{\gamma,\mathbf{p}}\otimes v+p_{i'}x^{\gamma,\mathbf{p}-1_{[i']}}\otimes v\in\overline{W},\ \forall\ (\gamma,\mathbf{p})\in\Gamma\times\mathbf{J}.$$

Thus we get Theorem 2.2. ∎

4. Proof of Theorem 2.3

In this section, we will give the proof of Theorem 2.3.

Proof. Denote

$$\overline{U}(\omega_1)=\begin{cases} U(\omega_1) & \text{if } \xi\notin\Gamma,\\ \tilde{U}(\omega_1), & \text{if } \xi\in\Gamma.\end{cases} \tag{69}$$

Let $\overline{P}_1(\omega_1)$ be any nonzero submodule of $\overline{V}(\omega_1)/\overline{U}(\omega_1)$. Denote $x^{\alpha,\mathbf{p}}\otimes e_k+\overline{U}(\omega_1)$ by $x^{\alpha,\mathbf{p}}\overline{\otimes}e_k$ for $(\alpha,\mathbf{p})\in\Gamma\times\mathbf{J}$. Then by (19), we have

$$\begin{aligned} & x^{\alpha,\mathbf{p}}.x^{\beta,\mathbf{q}}\overline{\otimes}e_k\\ = \ & \varsigma(k)\beta'_{k'}(\sum_{s=1}^{2n}\alpha_s x^{\alpha+\beta,\mathbf{p}+\mathbf{q}}\overline{\otimes}e_s+\sum_{s=1}^{2n}p_s x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[s]}}\overline{\otimes}e_s)\\ + \ & \varsigma(k)q_{k'}(\sum_{s=1}^{2n}\alpha_s x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[k']}}\overline{\otimes}e_s+\sum_{s=1}^{2n}p_s x^{\alpha+\beta,\mathbf{p}+\mathbf{q}-1_{[k']}-1_{[s]}}\overline{\otimes}e_s), \end{aligned} \tag{70}$$

for $(\alpha,\mathbf{p}),(\beta,\mathbf{q})\in\Gamma\times\mathbf{J},1\leq k\leq 2n$. By (21), (22) and similar argument as in the proof of Theorem 2.2, we can prove that

$$\exists\ 0\neq x^{\beta,\mathbf{0}}\overline{\otimes}\sum_{k=1}^{2n}a_k e_k\in\overline{P}_1(\omega_1),\ \beta\neq-\xi. \tag{71}$$

We may assume $\beta'_s \neq 0$ for some $1 \leq s \leq 2n$, since $\beta \neq -\xi$. Denote

$$\mathcal{W} = \{(x_1, \cdots, x_{2n}) \in \mathbb{F}^{2n} | \sum_{k=1}^{2n} x_k \varsigma(k)\beta'_{k'} = 0\}. \tag{72}$$

Gauss-Jordan elimination in solving homogeneous linear equations implies that $\dim_{\mathbb{F}} \mathcal{W} = 2n-1$ and $\{\beta'_s e_i + \varsigma(s)\varsigma(i)\beta'_{i'} e_{s'} \mid 1 \leq s' \neq i \leq 2n\}$ is a basis of $\mathcal{W}$. If $\sum\limits_{k=1}^{2n} a_k \varsigma(k)\beta'_{k'} = 0$, then

$$(a_1, \cdots, a_{2n}) = \sum_{1 \leq s' \neq i \leq 2n} b_i(\beta'_s e_i + \varsigma(s)\varsigma(i)\beta'_{i'} e_{s'}) \text{ for some } b_i \in \mathbb{F}. \tag{73}$$

Therefore

$$\sum_{k=1}^{2n} a_k e_k = \sum_{1 \leq s' \neq i \leq 2n} b_i(\beta'_s e_i + \varsigma(s)\varsigma(i)\beta'_{i'} e_{s'}) \in U(\omega_1), \tag{74}$$

which contradicts (71). So $\sum_{k=1}^{2n} a_k \varsigma(k)\beta'_{k'} \neq 0$. If $J_{s'} = \{0\}$, then we choose $0 \neq \tau_{[s']} \in \Gamma$ and have

$$0 \neq x^{\tau_{[s']},\mathbf{0}}.x^{\beta,\mathbf{0}} \overline{\otimes} \sum_{k=1}^{2n} a_k e_k = \tau \sum_{k=1}^{2n} a_k \varsigma(k)\beta'_{k'} x^{\beta+\tau_{[s']},\mathbf{0}} \overline{\otimes} e_{s'} \in \overline{P}_1(\omega_1). \tag{75}$$

If $J_{s'} = \mathbb{N}$, then we have

$$x^{0,1_{[s']}}.x^{\beta,\mathbf{0}} \overline{\otimes} \sum_{k=1}^{2n} a_k e_k = \sum_{k=1}^{2n} a_k \varsigma(k)\beta'_{k'} x^{\beta,\mathbf{0}} \overline{\otimes} e_{s'} \in \overline{P}_1(\omega_1). \tag{76}$$

By (71), (75) and (76), we always have some

$$0 \neq x^{\alpha,\mathbf{0}} \overline{\otimes} e_{s'} \in \overline{P}_1(\omega_1), \ \alpha \neq -\xi, \ \alpha'_s \neq 0. \tag{77}$$

Furthermore,

$$\begin{aligned}
\overline{P}_1(\omega_1) \ &\ni \ x^{\gamma-\beta,\mathbf{i}}.x^{\beta-\alpha,\mathbf{0}}.x^{\alpha,\mathbf{0}} \overline{\otimes} e_{s'} \\
&= \ \varsigma(s)\alpha'_{s'} \sum_{l,m=1}^{2n} (\beta_l - \alpha_l)\varsigma(l)\beta'_{l'}(\gamma_m - \beta_m) x^{\gamma,\mathbf{i}} \overline{\otimes} e_m \\
&+ \ \varsigma(s)\alpha'_{s'} \sum_{l,m=1}^{2n} i_m(\beta_l - \alpha_l) x^{\gamma,\mathbf{i}-1_{[m]}} \overline{\otimes} e_m.
\end{aligned} \tag{78}$$

Assume $\gamma_{s'} \neq \{0\}$. Then by considering the coefficients of $\beta^2_{s'}$ in (78), we get

$$x^{\gamma,\mathbf{i}} \overline{\otimes} e_{s'} \in \overline{P}_1(\omega_1) \quad \text{for any } (\gamma, \mathbf{i}) \in \Gamma \times \mathbf{J}. \tag{79}$$

If $\gamma_{s'} = \{0\}$, then we have

$$\begin{aligned}
\overline{P}_1(\omega_1) \ &\ni \ x^{\gamma-\alpha,\mathbf{p}-(i-1)_{[s']}}.x^{0,i_{[s']}}.x^{\alpha,\mathbf{0}} \overline{\otimes} e_{s'} \\
&= \ i\alpha'^2_s(\sum_{k=1}^{2n} (\gamma_k - \alpha_k) x^{\gamma,\mathbf{p}} \overline{\otimes} e_k + \sum_{k \neq s'} p_k x^{\gamma,\mathbf{p}-1_{[k]}} \overline{\otimes} e_k \\
&+ \ (p_{s'} - i + 1) x^{\gamma,\mathbf{p}-1_{[s']}} \overline{\otimes} e_{s'}).
\end{aligned} \tag{80}$$

Considering the coefficients of i^2 in (80), we also get (79). For any $1 \le s' \ne k \le 2n$, we have

$$x^{\tau_{[k]},\mathbf{0}}.x^{\beta,\mathbf{0}}\overline{\otimes}e_{s'} = \beta'_s\varsigma(s')\tau x^{\tau_{[k]}+\beta,\mathbf{0}}\overline{\otimes}e_k \in \overline{P}_1(\omega_1), \tag{81}$$

$$x^{0,1_{[k]}}.x^{\beta,1_{[k']}}\overline{\otimes}e_{s'} = \varsigma(s')\beta'_s x^{\beta,1_{[k']}}\overline{\otimes}e_k \in \overline{P}_1(\omega_1). \tag{82}$$

From (78)-(82), we can obtain $\overline{P}_1(\omega_1) = \overline{V}(\omega_1)/\overline{U}(\omega_1)$.

Let $\overline{P}_2(\omega_1)$ be any nonzero submodule of $U(\omega_1)/W(\omega_1)$. By (20) and (21), we claim that

$$\begin{aligned}\exists\ \ u_2 + W(\omega_1) \ &= \ x^{\beta,\mathbf{0}}\otimes \sum_{1\le s<t\le n} a_{s,t}x_{s,t}(\beta,\mathbf{0}) + \sum_{1\le s<t\le n} b_{s,t}x_{s',t'}(\beta,\mathbf{0}) \\ &\quad + \sum_{1\le s,t\le n} c_{s,t}x_{s,t'}(\beta,\mathbf{0}) + W(\omega_1) \in \overline{P}_2(\omega_1), \\ &\quad \text{where } u_2 \in U(\omega_1)\backslash W(\omega_1),\ \ \beta \ne -\xi. \end{aligned}\tag{83}$$

For $1 \le s \le n$, denote

$$x_s = (\delta_{s,1}-1)\sum_{k=1}^{s-1} a_{k,s}\beta'_{k'} + \sum_{k=s+1}^{n} a_{s,k}\beta'_{k'} + \sum_{k=1}^{n} c_{s,k}\beta'_k, \tag{84}$$

$$x_{s'} = \sum_{k=1}^{n} b_{s,k}\beta'_k + (\delta_{s,1}-1)\sum_{k=1}^{s-1} b_{k,s}\beta'_k + \sum_{k=1}^{n} c_{k,s}\beta'_{k'}. \tag{85}$$

Obviously, $u_2 = \sum\limits_{s=1}^{n}(x_se_s + x_{s'}e_{s'})$. Set

$$\begin{aligned}\mathcal{W}' \ = \ &\{(x_1,\cdots,x_{2n}) \in \mathbb{F}^{2n} \mid x_{s'}\beta'_{t'} - x_{t'}\beta'_{s'} = 0,\ x_s\beta'_t - x_t\beta'_s = 0 \\ &(\forall\ 1 \le s < t \le n);\ x_{s'}\beta'_t - x_t\beta'_{s'} = 0\ (\forall\ 1 \le s,t \le n)\ \}.\end{aligned}\tag{86}$$

By Gauss-Jordan elimination in solving homogeneous linear equations, we can verify that $\dim_{\mathbb{F}}\mathcal{W}' = 1$ and $(\beta'_1,\cdots,\beta'_{2n})$ is a basis of $\mathcal{W}'$. Since $u_2 \notin W(\omega_1)$, at least one of the following statements holds:

$$\exists\ 1 \le s < t \le n \quad \text{s. t.} \quad x_{s'}\beta'_{t'} - x_{t'}\beta'_{s'} \ne 0 \quad \text{or} \quad x_s\beta'_t - x_t\beta'_s \ne 0;$$

$$\text{or} \quad \exists \quad 1 \le s,t \le n \quad \text{s. t.} \quad x_{s'}\beta'_t - x_t\beta'_{s'} \ne 0. \tag{87}$$

Moreover, Lemma 2.6 implies that

$$\begin{aligned} x_{s,t}(\beta,\mathbf{0}) \in \overline{P}_2(\omega_1) &\qquad \text{if } x_{s'}\beta'_{t'} - x_{t'}\beta'_{s'} \ne 0,\ 1 \le s < t \le n; \\ x_{s,t'}(\beta,\mathbf{0}) \in \overline{P}_2(\omega_1) &\qquad \text{if } x_{s'}\beta'_t - x_t\beta'_{s'} \ne 0,\ 1 \le s,t \le n; \\ x_{t',s'}(\beta,\mathbf{0}) \in \overline{P}_2(\omega_1) &\qquad \text{if } x_s\beta'_t - x_t\beta'_s \ne 0,\ \ 1 \le s < t \le n. \end{aligned}\tag{88}$$

Since the coefficients of α_i^2 or q_i^2 in $x^{\gamma-\alpha,\mathbf{p}-\mathbf{q}}.x^{\alpha-\beta,\mathbf{q}}.x_{s,t}(\beta,\mathbf{0})$ is $\varsigma(i')\beta'^2_{i'}x_{s,t}(\gamma,\mathbf{p})$ (where $i \in \{s,t\}$, $1 \le s < t \le 2n$), (87) and (88) imply there exists some $1 \le s < t \le 2n$ such that

$$x_{s,t}(\gamma,\mathbf{p}) \in \overline{P}_2(\omega_1) \quad \text{for any} \quad (\gamma,\mathbf{p}) \in \Gamma \times \mathbf{J}. \tag{89}$$

Then applying Lemma 2.6, we can prove that $\overline{P}_2(\omega_1) = U(\omega_1)/W(\omega_1)$.

Let $\overline{P}_3(\omega_1)$ be any nonzero submodule of $W(\omega_1)$. By (20), (21) and similar argument as in the proof of Theorem 2.2, we can prove that

$$\exists \quad x(\beta, \mathbf{0}) \in \overline{P}_3(\omega_1), \ \beta \neq -\xi. \tag{90}$$

Since the coefficients of α_i^2 or p_i^2 in $x^{\gamma-\alpha,\mathbf{p}-\mathbf{q}}.x^{\alpha-\beta,\mathbf{q}}.x(\beta, \mathbf{0})$ is $\varsigma(i')\beta_{i'}'^2 x(\gamma, \mathbf{q})$ (where $1 \le i \le 2n$), we have

$$x(\gamma, \mathbf{p}) \in \overline{P}_3(\omega_1), \ \forall \ (\gamma, \mathbf{p}) \in \Gamma \times \mathbf{J}. \tag{91}$$

Thus we get $\overline{P}_3(\omega_1) = W(\omega_1)$. ■

5. Proof of Theorem 2.4

In this section, we will give the proof of Theorem 2.4.

Proof. Denote

$$\overline{U}(\omega_2) = \begin{cases} U(\omega_2) & \text{if } \xi \notin \Gamma, \\ \tilde{U}(\omega_2) & \text{if } \xi \in \Gamma. \end{cases} \tag{92}$$

Let $\overline{P}_1(\omega_2)$ be any nonzero submodule of $\overline{V}(\omega_2)/\overline{U}(\omega_2)$. Then by (10), (11) and similar argument as in the proof of Theorem 2.2, we can prove that there exists some

$$\begin{aligned} & u_3 + \overline{U}(\omega_2) \\ = \ & x^{\beta,\mathbf{0}} \otimes [a_1 e_1 \wedge e_2 + a_2 e_1 \wedge e_4 + a_3 e_2 \wedge e_3 + a_4 e_3 \wedge e_4 \\ + \ & a_5(e_1 \wedge e_3 - e_2 \wedge e_4)] + \overline{U}(\omega_2) \\ \in \ & \overline{P}_1(\omega_2), \ \text{where } u_3 \in \overline{V}(\omega_2) \backslash \overline{U}(\omega_2), \ \beta \neq -\xi. \end{aligned} \tag{93}$$

Set

$$\begin{aligned} f_1(\mathbf{a}) &= a_2\beta_3'^2 - a_3\beta_4'^2 - a_4(\beta_1'\beta_3' + \beta_2'\beta_4') - 2a_5\beta_3'\beta_4', \\ f_2(\mathbf{a}) &= -a_1\beta_3'^2 + a_3(\beta_2'\beta_4' - \beta_1'\beta_3') + a_4\beta_2'^2 + 2a_5\beta_2'\beta_3', \\ f_3(\mathbf{a}) &= a_1\beta_4'^2 + a_2(\beta_1'\beta_3' - \beta_2'\beta_4') - a_4\beta_1'^2 - 2a_5\beta_1'\beta_4', \\ f_4(\mathbf{a}) &= -a_1(\beta_1'\beta_3' + \beta_2'\beta_4') + a_2\beta_2'^2 - a_3\beta_1'^2 + 2a_5\beta_1'\beta_2'. \end{aligned} \tag{94}$$

Assume $f_i(\mathbf{a}) = 0$ $(1 \le i \le 4)$. By Gauss-Jordan elimination in solving homogeneous linear equations, we can verify that u_3 is a linear combination of $y_j(\beta, \mathbf{0})$, $y_k(\beta, \mathbf{0})$, $y_l(\beta, \mathbf{0})$ if $\beta_s' \neq 0$, where $(s, \ j, \ k, \ l) \in \{(1, \ 4, \ 6, \ 8), \ (2, \ 2, \ 7, \ 8), \ (3, \ 1, \ 2, \ 3), \ (4, \ 1, \ 4, \ 5)\}$. This contradicts $u_3 \notin \overline{U}(\omega_2)$. Hence $f_i(\mathbf{a}) \neq 0$ for some $i \in \overline{1,4}$. Moreover, Lemma 2.6 implies that

$$x^{\beta,\mathbf{0}} \otimes e_i \wedge e_j + \overline{U}(\omega_2) \text{ if } f_k(\mathbf{a}) \neq 0, \tag{95}$$

where $(i, \ j, \ k) \in \{(1, \ 2, \ 1), \ (1, \ 4, \ 2), \ (2, \ 3, \ 3), \ (3, \ 4, \ 4)\}$. Since the coefficients of α_s^2 or p_s^2 in $x^{\gamma-\alpha,\mathbf{p}}.x^{\alpha-\beta,\mathbf{q}-\mathbf{p}}.(x^{\beta,\mathbf{0}} \otimes e_i \wedge e_j + \overline{U}(\omega_2))$ is $\varsigma(s')\beta_{s'}'^2(x^{\gamma,\mathbf{p}} \otimes e_i \wedge e_j + \overline{U}(\omega_2))$ (where $s \in \{i, \ j\}$), there exists some $(i, \ j) \in \{(1, \ 2), \ (1, \ 4), \ (2, \ 3), \ (3, \ 4)\}$

such that $x^{\gamma,\mathbf{p}} \otimes e_i \wedge e_j + \overline{U}(\omega_2) \in \overline{P}_1(\omega_2)$ for any $(\gamma, \mathbf{p}) \in \Gamma \times \mathbf{J}$. Then applying Lemma 2.6 repeatedly, we can get $\overline{P}_1(\omega_2) = \overline{V}(\omega_2)/\overline{U}(\omega_2)$.

Let $\overline{P}_2(\omega_2)$ be any nonzero submodule of $U(\omega_2)/W(\omega_2)$. We can also prove that

$$\exists\ 0 \neq x^{\beta,\mathbf{0}} \otimes \sum_{k=1}^{10} a_k y_k(\beta, \mathbf{0}) + W(\omega_2) \in \overline{P}_2(\omega_2),\ \beta \neq -\xi. \tag{96}$$

Denote

$$\begin{aligned} f(\mathbf{a}) &= a_1\beta_3'\beta_4' + a_2\beta_2'\beta_3' + a_3\beta_3' + a_4\beta_1'\beta_4' + a_5\beta_4' + a_6(\beta_1'\beta_3' \\ &+ \beta_2'\beta_4') + a_7\beta_2' + a_8\beta_1'\beta_2' + a_9\beta_1' + a_{10}(\beta_2'\beta_4' - \beta_1'\beta_3'). \end{aligned} \tag{97}$$

The fact $x^{\beta,\mathbf{0}} \otimes \sum_{k=1}^{10} a_k y_k(\beta, \mathbf{0}) \notin W(\omega_2)$ yields $f(\mathbf{a}) \neq 0$. By Lemma 2.6, we have

$$x^{\beta,\mathbf{0}} \otimes \frac{1}{\beta_i'} f(\mathbf{a}) y_j(\beta, \mathbf{0}) + W(\omega_2) \in \overline{P}(\omega_2), \tag{98}$$

where $(i,\ j) \in \{(4,\ 5),\ (3,\ 3),\ (2,\ 7),\ (1,\ 9)\}$.The fact $\beta \neq -\xi$ shows that $x^{\beta,\mathbf{0}} \otimes y_i(\beta, \mathbf{0}) + W(\omega_2) \in \overline{P}(\omega_2)$ for some $i \in \{3,\ 5,\ 7,\ 9\}$. Since the coefficients of α_s^2 or q_s^2 in $x^{\gamma-\alpha,\mathbf{p}-\mathbf{q}}.x^{\alpha-\beta,\mathbf{q}}.(x^{\beta,\mathbf{0}} \otimes y_i(\beta, \mathbf{0}) + W(\omega_2))$ is $\varsigma(s')\beta_{s'}'^2(x^{\gamma,\mathbf{p}} \otimes y_i(\gamma, \mathbf{p}) + W(\omega_2))$ (where $(s,\ i) \in \{(1,\ 3),\ (2,\ 5),\ (3,\ 9),\ (4,\ 7)\}$), there exists some $i \in \{3,\ 5,\ 7,\ 9\}$ such that $x^{\gamma,\mathbf{p}} \otimes y_i(\gamma, \mathbf{p}) + W(\omega_2) \in \overline{P}_2(\omega_2)$ for any $(\gamma, \mathbf{p}) \in \Gamma \times \mathbf{J}$. Using Lemma 2.6 repeatedly, we obtain $\overline{P}_2(\omega_2) = U(\omega_2)/W(\omega_2)$.

Let $\overline{P}_3(\omega_2)$ be any nonzero submodule of $W(\omega_2)$. We can prove that

$$\exists\quad 0 \neq x^{\beta,\mathbf{0}} \otimes \sum_{k=1}^{4} a_k z_k(\beta, \mathbf{0}) \in \overline{P}_3(\omega_2),\ \beta \neq -\xi. \tag{99}$$

Denote

$$\begin{aligned} f_1'(\mathbf{a}) &= -a_2\beta_4'^2 + a_3\beta_3'^2 - a_4(\beta_1'\beta_3' + \beta_2'\beta_4'), \\ f_2'(\mathbf{a}) &= a_1\beta_4'^2 + a_3(\beta_1'\beta_3' - \beta_2'\beta_4') - a_4\beta_1'^2, \\ f_3'(\mathbf{a}) &= -a_1\beta_3'^2 + a_2(-\beta_1'\beta_3' + \beta_2'\beta_4') + a_4\beta_2'^2, \\ f_4'(\mathbf{a}) &= -a_1(\beta_1'\beta_3' + \beta_2'\beta_4') - a_2\beta_1'^2 + a_3\beta_2'^2. \end{aligned} \tag{100}$$

By Gauss-Jordan elimination in solving homogeneous linear equations, we can verify that $f_k'(\mathbf{a}) = 0\ (1 \leq k \leq 4)$ implies that $\sum_{k=1}^4 a_k z_k(\beta, \mathbf{0}) = 0$. Hence $f_k'(\mathbf{a}) \neq 0$ for some $1 \leq k \leq 4$. Moreover, Lemma 2.6 yields

$$x^{\beta,\mathbf{0}} \otimes z_k(\beta, \mathbf{0}) \in \overline{P}_3(\omega_2) \quad \text{if } f_k'(\mathbf{a}) \neq 0,\ 1 \leq k \leq 4. \tag{101}$$

Since the coefficients of α_s^2 or q_s^2 in $x^{\gamma-\alpha,\mathbf{p}-\mathbf{q}}.x^{\alpha-\beta,\mathbf{q}}.x^{\beta,\mathbf{0}} \otimes z_i(\beta, \mathbf{0})$ is $\varsigma(s')\beta_{s'}'^2 x^{\gamma,\mathbf{p}} \otimes z_i(\gamma, \mathbf{p})$ (where $(s,\ i) \in \{\{1,\ 2\} \times \{1\},\ \{2,\ 3\} \times \{2\},\ \{1,\ 4\} \times \{3\},\ \{3,\ 4\} \times \{4\}\}$), there exists some $1 \leq i \leq 4$ such that $x^{\gamma,\mathbf{p}} \otimes z_i(\gamma, \mathbf{p}) \in \overline{P}_3(\omega_2)$ for any $(\gamma, \mathbf{p}) \in \Gamma \times \mathbf{J}$. Using Lemma 2.6 repeatedly, we can finally obtain $\overline{P}_3(\omega_2) = W(\omega_2)$. ∎

6. Proof of Theorem 2.5

In this section, we will give the proof of Theorem 2.5.

Proof. Denote

$$\overline{U}(\omega_3)=\begin{cases} U(\omega_3) & \text{if } \xi\notin\Gamma,\\ \tilde{U}(\omega_3), & \text{if } \xi\in\Gamma.\end{cases} \tag{102}$$

Let $\overline{P}_1(\omega_3)$ be any nonzero submodule of $\overline{V}(\omega_3)/\overline{U}(\omega_3)$. Denote $x^{\alpha,\mathbf{p}}\otimes e_i\wedge e_j\wedge e_k+\overline{U}(\omega_3)$ by $x^{\alpha,\mathbf{p}}\overline{\otimes}e_i\wedge e_j\wedge e_k$ for $(\alpha,\mathbf{p})\in\Gamma\times\mathbf{J}$. Then by (20), (21) and similar argument as in the proof of Theorem 2.2, we can prove that there exists some

$$\begin{aligned} & u_4+\overline{U}(\omega_3)\\ = \ & x^{\beta,\mathbf{0}}\overline{\otimes}[a_1e_1\wedge e_2\wedge e_3+a_2e_1\wedge e_2\wedge e_6+a_3e_1\wedge e_3\wedge e_5\\ + \ & a_4e_1\wedge e_5\wedge e_6+a_5e_2\wedge e_3\wedge e_4+a_6e_2\wedge e_4\wedge e_6+a_7e_3\wedge e_4\wedge e_5\\ + \ & a_8e_4\wedge e_5\wedge e_6+a_9(e_1\wedge e_2\wedge e_5-e_1\wedge e_3\wedge e_6)\\ + \ & a_{10}(e_2\wedge e_3\wedge e_6+e_1\wedge e_2\wedge e_4)+a_{11}(e_1\wedge e_3\wedge e_4-e_2\wedge e_3\wedge e_5)\\ + \ & a_{12}(e_1\wedge e_4\wedge e_5+e_3\wedge e_5\wedge e_6)+a_{13}(e_2\wedge e_4\wedge e_5-e_3\wedge e_4\wedge e_6)\\ + \ & a_{14}(e_1\wedge e_4\wedge e_6-e_2\wedge e_5\wedge e_6)]\in\overline{P}_1(\omega_3), \end{aligned} \tag{103}$$

where $u_4\in\overline{V}(\omega_3)\backslash\overline{U}(\omega_3)$, $\beta\neq-\xi$. Denote

$$\begin{aligned} g_1(\mathbf{a}) &= a_4\beta_4'^2-a_6\beta_5'^2+a_7\beta_6'^2-a_8(\beta_1'\beta_4'+\beta_2'\beta_5'+\beta_3'\beta_6')\\ &+ 2a_{12}\beta_4'\beta_6'+2a_{13}\beta_5'\beta_6'-2a_{14}\beta_4'\beta_5', \end{aligned}$$

$$\begin{aligned} g_2(\mathbf{a}) &= a_3\beta_4'^2-a_5\beta_5'^2+a_7(\beta_1'\beta_4'+\beta_2'\beta_5'-\beta_3'\beta_6')+a_8\beta_3'^2\\ &- 2a_{11}\beta_4'\beta_5'-2a_{12}\beta_3'\beta_4'-2a_{13}\beta_3'\beta_5', \end{aligned}$$

$$\begin{aligned} g_3(\mathbf{a}) &= a_2\beta_4'^2+a_5\beta_6'^2+a_6(\beta_1'\beta_4'-\beta_2'\beta_5'+\beta_3'\beta_6')-a_8\beta_2'^2\\ &- 2a_{10}\beta_4'\beta_6'+2a_{13}\beta_2'\beta_6'-2a_{14}\beta_2'\beta_4', \end{aligned}$$

$$\begin{aligned} g_4(\mathbf{a}) &= a_1\beta_4'^2+a_5(-\beta_1'\beta_4'+\beta_2'\beta_5'+\beta_3'\beta_6')+a_6\beta_3'^2-a_7\beta_2'^2\\ &- 2a_{10}\beta_3'\beta_4'+2a_{11}\beta_2'\beta_4'+2a_{13}\beta_2'\beta_3', \end{aligned}$$

$$\begin{aligned} g_5(\mathbf{a}) &= a_2\beta_5'^2-a_3\beta_6'^2+a_4(\beta_1'\beta_4'-\beta_2'\beta_5'-\beta_3'\beta_6')-a_8\beta_1'^2\\ &- 2a_9\beta_5'\beta_6'+2a_{12}\beta_1'\beta_6'-2a_{14}\beta_1'\beta_5', \end{aligned}$$

$$\begin{aligned} g_6(\mathbf{a}) &= -a_1\beta_5'^2+a_3(\beta_1'\beta_4'-\beta_2'\beta_5'+\beta_3'\beta_6')+a_4\beta_3'^2+a_7\beta_1'^2\\ &+ 2a_9\beta_3'\beta_5'-2a_{11}\beta_1'\beta_5'-2a_{12}\beta_1'\beta_3', \end{aligned}$$

$$\begin{aligned} g_7(\mathbf{a}) &= a_1\beta_6'^2+a_2(\beta_1'\beta_4'+\beta_2'\beta_5'-\beta_3'\beta_6')-a_4\beta_2'^2+a_6\beta_1'^2\\ &- 2a_9\beta_2'\beta_6'-2a_{10}\beta_1'\beta_6'-2a_{14}\beta_1'\beta_2', \end{aligned}$$

$$\begin{aligned} g_8(\mathbf{a}) &= -a_1(\beta_1'\beta_4' + \beta_2'\beta_5' + \beta_3'\beta_6') + a_2\beta_3'^2 - a_3\beta_2'^2 + a_5\beta_1'^2 \\ &+ 2a_9\beta_2'\beta_3' + 2a_{10}\beta_1'\beta_3' - 2a_{11}\beta_1'\beta_2'. \end{aligned} \tag{104}$$

$g_i(\mathbf{a}) = 0$ implies u_4 is a linear combination of $o_j(\beta,\mathbf{0}), o_k(\beta,\mathbf{0}), o_l(\beta,\mathbf{0}), o_m(\beta,\mathbf{0}), o_n(\beta,\mathbf{0}), o_q(\beta,\mathbf{0}), o_r(\beta,\mathbf{0}), o_t(\beta,\mathbf{0}), o_u(\beta,\mathbf{0})$ if $\beta_s'\neq 0$, where $1 \le i \le 8, (j,k,l,m,n,q,r,t,u,s) \in \{(6,8,10,11,13,14,15,16,38,1),(4,9,10,12,17,18,19,20,39,2),(5,7,11,12,21,22,23,24,40,3),(1,2,4,5,25,26,27,28,41,4),(1,3,6,7,29,30,31,32,42,5),(2,3,8,9,33,34,35,36,37,6)\}$. This contradicts $u_4 \notin \overline{U}(\omega_3)$. So $g_i(\mathbf{a}) \neq 0$ for some $1 \le i \le 8$. Moreover, Lemma 2.6 implies that

$$x^{\beta,\mathbf{0}}\overline{\otimes}e_i \wedge e_j \wedge e_k \in \overline{P}_1(\omega_3) \text{ if } g_l(\mathbf{a}) \neq 0, \tag{105}$$

where$(i,\ j,\ k,\ l) \in \{\ (1,\ 2,\ 3,\ 1),\ (1,\ 2,\ 6,\ 2),\ (1,3,\ 5,\ 3),\ (1,\ 5,\ 6,\ 4),\ (2,\ 3,\ 4,\ 5),\ (2,\ 4,\ 6,\ 6),\ (3,\ 4,\ 5,\ 7),\ (4,\ 5,\ 6,\ 8)\ \}$. Since the coefficients of α_s^2 (or q_s^2) in $x^{\gamma-\alpha,\mathbf{p}-\mathbf{q}}.x^{\alpha-\beta,\mathbf{q}}.x^{\beta,\mathbf{0}}\overline{\otimes}e_i \wedge e_j \wedge e_k$ is $\varsigma(s')\beta_{s'}'^2 x^{\gamma,\mathbf{p}}\overline{\otimes}e_i \wedge e_j \wedge e_k$ $(s \in \{i,\ j,\ k\})$, there exists some $(i,j,k) \in \{(1,\ 2,\ 3),\ (1,\ 2,\ 6),\ (1,\ 3,\ 5),\ (1,\ 5,6),\ (2,\ 3,\ 4),\ (2,\ 4,\ 6),\ (3,\ 4,\ 5),\ (4,\ 5,\ 6)\}$ such that

$$x^{\gamma,\mathbf{p}}\overline{\otimes}e_i \wedge e_j \wedge e_k \in \overline{P}_1(\omega_3),\ \forall\ (\gamma,\mathbf{p}) \in \Gamma \times \mathbf{J}. \tag{106}$$

Then applying Lemma 2.6 repeatedly, we get $\overline{P}_1(\omega_3) = \overline{V}(\omega_3)/\overline{U}(\omega_3)$.

Let $\overline{P}_2(\omega_3)$ be any nonzero submodule of $U(\omega_3)/W(\omega_3)$. By (2.10), (2.11) and similar argument as in the proof of Theorem 2.2, we can prove that there exists some

$$\begin{aligned} & u_5 + W(\omega_3) \\ = \ & x^{\beta,\mathbf{0}} \otimes [a_1 e_1 \wedge e_2 \wedge e_3 + a_2 e_1 \wedge e_2 \wedge e_6 + a_3 e_1 \wedge e_3 \wedge e_5 \\ + \ & a_4 e_1 \wedge e_5 \wedge e_6 + a_5 e_2 \wedge e_3 \wedge e_4 + a_6 e_2 \wedge e_4 \wedge e_6 \\ + \ & a_7 e_3 \wedge e_4 \wedge e_5 + a_8 e_4 \wedge e_5 \wedge e_6 + a_9(e_1 \wedge e_2 \wedge e_5 - e_1 \wedge e_3 \wedge e_6) \\ + \ & a_{10}(e_2 \wedge e_3 \wedge e_6 + e_1 \wedge e_2 \wedge e_4) + a_{11}(e_1 \wedge e_3 \wedge e_4 - e_2 \wedge e_3 \wedge e_5) \\ + \ & a_{12}(e_1 \wedge e_4 \wedge e_5 + e_3 \wedge e_5 \wedge e_6) + a_{13}(e_2 \wedge e_4 \wedge e_5 - e_3 \wedge e_4 \wedge e_6) \\ + \ & a_{14}(e_1 \wedge e_4 \wedge e_6 - e_2 \wedge e_5 \wedge e_6)] \in \overline{P}_2(\omega_3), \end{aligned} \tag{107}$$

where $u_5 \in U(\omega_3)\backslash W(\omega_3),\ \beta \neq -\xi,\ g_i(\mathbf{a}) = 0\ (1 \le i \le 8)$. Denote

$$h_1(\mathbf{a}) = -a_7\beta_6' + a_8\beta_3' - a_{12}\beta_4' - a_{13}\beta_5',\ h_2(\mathbf{a}) = -a_6\beta_5' - a_8\beta_2' + a_{13}\beta_6' - a_{14}\beta_4',$$

$$h_3(\mathbf{a}) = a_4\beta_4' - a_8\beta_1' + a_{12}\beta_6' - a_{14}\beta_5',\ h_4(\mathbf{a}) = -a_5\beta_6' - a_6\beta_3' + a_{10}\beta_4' - a_{13}\beta_2',$$

$$h_5(\mathbf{a}) = a_5\beta_5' - a_7\beta_2' + a_{11}\beta_4' + a_{13}\beta_3',\ h_6(\mathbf{a}) = a_3\beta_6' + a_4\beta_3' + a_9\beta_5' - a_{12}\beta_1',$$

$$h_7(\mathbf{a}) = -a_3\beta_4' - a_7\beta_1' + a_{11}\beta_5' + a_{12}\beta_3',\ h_8(\mathbf{a}) = -a_2\beta_5' + a_4\beta_2' + a_9\beta_6' + a_{14}\beta_1',$$

$$h_9(\mathbf{a}) = -a_2\beta_4' - a_6\beta_1' + a_{10}\beta_6' + a_{14}\beta_2',\ h_{10}(\mathbf{a}) = a_1\beta_6' - a_2\beta_3' - a_9\beta_2' - a_{10}\beta_1',$$

$$h_{11}(\mathbf{a}) = -a_1\beta_5' - a_3\beta_2' + a_9\beta_3' - a_{11}\beta_1',\ h_{12}(\mathbf{a}) = -a_1\beta_4' + a_5\beta_1' + a_{10}\beta_3' - a_{11}\beta_2'. \tag{108}$$

By Gauss-Jordan elimination in solving homogeneous linear equations, $h_i(\mathbf{a}) = 0\ (1 \le i \le 12)$ implies $u_5 \in W(\omega_3)$. So $h_i(\mathbf{a}) \neq 0$ for some $1 \le i \le 12$. Moreover, it follows from Lemma 2.6 that

$$o_i(\beta,\mathbf{0}) + W(\omega_3) \in \overline{P}_2(\omega_3) \quad \text{if} \quad h_i(\mathbf{a}) \neq 0. \tag{109}$$

Since the coefficients of α_s^2 (or q_s^2) in $x^{\gamma-\alpha,\mathbf{p}-\mathbf{q}}.x^{\alpha-\beta,\mathbf{q}}.(o_i(\beta,\mathbf{0})+W(\omega_3))$ is $\varsigma(s')\beta_{s'}'^2(o_i(\gamma,\mathbf{p})+W(\omega_3))$ (where $(s,i)\in\{\ \{1,\ 2,\ 3,\ 6\}\times\{1\},\ \{1,\ 2,\ 3,\ 5\}\times\{2\},\ \{1,\ 2,\ 3,\ 4\}\times\{3\},\ \{1,\ 3,\ 5,\ 6\}\times\{4\},\ \{1,\ 2,\ 5,\ 6\}\times\{5\},\ \{2,\ 3,\ 4,\ 6\}\times\{6\},\ \{1,\ 2,\ 4,\ 6\}\times\{7\},\ \{2,\ 3,\ 4,\ 5\}\times\{8\},\ \{1,\ 3,\ 4,\ 5\}\times\{9\},\ \{3,\ 4,\ 5,\ 6\}\times\{10\},\ \{2,\ 4,\ 5,\ 6\}\times\{11\},\ \{1,\ 4,\ 5,\ 6\}\times\{12\}\ \}$), there exists some $1\le i\le 12$ such that

$$o_i(\gamma,\mathbf{p})+W(\omega_3)\in\overline{P}_2(\omega_3),\ \forall\ (\gamma,\mathbf{p})\in\Gamma\times\mathbf{J}. \tag{110}$$

Then applying Lemma 2.6 repeatedly, we can obtain $\overline{P}_2(\omega_3)=U(\omega_3)/W(\omega_3)$.

Let $\overline{P}_3(\omega_3)$ be any nonzero submodule of $W(\omega_3)$. By (20), (21) and similar argument as in the proof of Theorem 2.2, we can prove that there exists some

$$0\neq x^{\beta,\mathbf{0}}\otimes\sum_{i=1}^{8}a_ip_i(\beta,\mathbf{0})\in\overline{P}_3(\omega_3),\ \beta\neq-\xi. \tag{111}$$

Denote

$$\begin{aligned}
k_1(\mathbf{a})&=-a_4\beta_4'^2-a_5(\beta_1'\beta_4'+\beta_2'\beta_5'+\beta_3'\beta_6')+a_6\beta_5'^2+a_7\beta_6'^2,\\
k_2(\mathbf{a})&=-a_3\beta_5'^2+a_4(\beta_1'\beta_4'-\beta_2'\beta_5'-\beta_3'\beta_6')+a_5\beta_1'^2-a_8\beta_6'^2,\\
k_3(\mathbf{a})&=-a_2\beta_5'^2+a_5\beta_3'^2+a_7(\beta_1'\beta_4'+\beta_2'\beta_5'-\beta_3'\beta_6')-a_8\beta_4'^2,\\
k_4(\mathbf{a})&=a_1\beta_4'^2+a_2(\beta_1'\beta_4'-\beta_2'\beta_5'-\beta_3'\beta_6')+a_6\beta_3'^2+a_7\beta_2'^2,\\
k_5(\mathbf{a})&=a_1(\beta_1'\beta_4'+\beta_2'\beta_5'+\beta_3'\beta_6')+a_2\beta_1'^2+a_3\beta_3'^2+a_8\beta_2'^2,\\
k_6(\mathbf{a})&=a_1\beta_5'^2-a_4\beta_3'^2+a_7\beta_1'^2-a_8(\beta_1'\beta_4'-\beta_2'\beta_5'+\beta_3'\beta_6'),\\
k_7(\mathbf{a})&=a_1\beta_6'^2+a_3(-\beta_1'\beta_4'-\beta_2'\beta_5'+\beta_3'\beta_6')-a_4\beta_2'^2+a_6\beta_1'^2,\\
k_8(\mathbf{a})&=-a_2\beta_6'^2-a_3\beta_4'^2+a_5\beta_2'^2+a_6(\beta_1'\beta_4'-\beta_2'\beta_5'+\beta_3'\beta_6').
\end{aligned} \tag{112}$$

By Gauss-Jordan elimination in solving homogeneous linear equations, $k_i(\mathbf{a})=0$ $(1\le i\le 8)$ implies that $\sum_{i=1}^{8}a_ip_i(\beta,\mathbf{0})=0$. Hence $k_i(\mathbf{a})\neq 0$ for some $1\le i\le 8$. Moreover, Lemma 2.6 implies that

$$x^{\beta,\mathbf{0}}\otimes p_i(\beta,\mathbf{0})\in\overline{P}_3(\omega_3)\quad\text{if } k_i(\mathbf{a})\neq 0. \tag{113}$$

Since the coefficients of α_i^2 (or q_i^2) in $x^{\gamma-\alpha,\mathbf{p}-\mathbf{q}}.x^{\alpha-\beta,\mathbf{q}}.x^{\beta,\mathbf{0}}\otimes p_k(\beta,\mathbf{0})$ is $\varsigma(i')\beta_{i'}^2x^{\gamma,\mathbf{p}}\otimes p_k(\gamma,\mathbf{p})$ (where $(i,\ k)\in\{\{4,\ 5,\ 6\}\times\{1\}\},\ \{1,\ 5,\ 6\}\times\{2\}\},\ \{3,\ 4,\ 5,\}\times\{3\}\},\ \{2,\ 3,\ 4\}\times\{4\}\},\ \{1,\ 2,\ 3\}\times\{5\}\},\ \{1,\ 3,\ 5\}\times\{6\}\},\ \{1,\ 2,\ 6\}\times\{7\}\},\ \{2,\ 4,\ 6\}\times\{8\}\}$), there exists some $1\le k\le 8$ such that

$$x^{\gamma,\mathbf{p}}\otimes p_k(\gamma,\mathbf{p})\in\overline{P}_3(\omega_3),\ \forall\ (\gamma,\mathbf{p})\in\Gamma\times\mathbf{J}. \tag{114}$$

Then applying Lemma 2.6 repeatedly, we get $\overline{P}_3(\omega_3)=W_3(\omega_3)$. ■

From Lemma 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 2.5, we finally get Theorem 1.1.

References

[1] Benkart, G., D. Britten, and F. Lemire, *Modules with bounded weight multiplicities for simple Lie algebras*, Math. Z., **225** (1997), 333-353.

[2] Carter, R. W., "Lie algebras of finite and Affine type," Cambridge Studies in Advanced Mathematics (**96**), Cambridge University Press, 2005.

[3] Frenkel, I., J. Lepowsky, and A. Meurman, "Vertex operator Algebras and the Monster," Pure and Applied Math Academic Press, 1988.

[4] Kac, V. G., "Vertex algebras for beginners," Univ. Lecture Ser. **10**, Amer. Math. Soc., 1997.

[5] Kac, V., and A. Radul, *Quasifinite highest weight modules over the Lie algebra of differential operators on the Circle*, Commun. Math Phys, **157** (1993), 429–457.

[6] —, *Representation theory of the vertex algebra* $\mathcal{W}_{1+\infty}$, Trans. Groups **1** (1996), 41–70.

[7] Lin, W., and S. Tan, *Representations of the Lie algebra of derivations for quantum torus*, J. Algebra, **275** (2004), 250–274.

[8] Osborn, J. M., *New simple infinite-dimensional Lie algebras of characteristic 0*, J. Alg. **185** (1996), 820–835.

[9] Osborn, J. M., and K. Zhao, *Generalized Poisson brackets and Lie algebras for type H in characteristic* 0, Math. Z., **230** (1999), 107–143.

[10] Rao, S. E., *Irreducible representations of the Lie algebra of the diffeomorphisms of a d-dimensional torus*, J. Algebra **182** (1996), 401–421.

[11] —, *Representations of Witt Algebras*, Publ.Rims, Kyoto Univ. **29** (1994), 191–201.

[12] Shen, G., *Graded modules of graded Lie algebras of Cartan type (I)—mixed product of modules*, Science in China **A 29** (1986), 570–581.

[13] —, *Graded modules of graded Lie algebras of Cartan type (II)—positive and negative graded modules*, Science in China **A 29** (1986), 1009–1019.

[14] —, *Graded modules of graded Lie algebras of Cartan type (III)—irreducible modules*, Chin. Ann. of Math **B 9** (1988), 404–417.

[15] Su, Y., and J. Zhou, *Structure of the Lie algebras related to those of Block*, Comm. Algebra **30** (2002), 3205–3226.

[16] Xu, X., *Generalizations of the Block algebras*, Manuscript math, **100** (1999), 489–518.

[17] —, *New generalized simple Lie algebras of Cartan type over a field with characteristic 0*, J. Algebra, **224** (2000), 23–58.

[18] —, *Quadratic Conformal Superalgebras*, J. Algebra, **231** (2000), 1–38.

[19] —, *Equivalence of conformal superalgebras to Hamiltonian superoperators*, Algebra Colloquium, **8** (2002), 63–92.

[20] —, *Representations of centrally-extended Lie algebras over differential operators and vertex algebras*, J. Pure and Applied Algebra, **212** (2008), 1253–1309.

[21] Zhao, Y., *Irreducible representations of nongraded Witt type Lie algebras*, J. Algebra **298** (2006), 540–562.

[22] —, *Representations of nongraded Lie algebras of Block type*, Manuscripta math. **119** (2006), 183–216.

Yufeng Zhao
LAMA
School of Mathematical Sciences
Peking University
Beijing, 100871, China
yufeng@math.pku.edu.cn

Received Dec 20, 2007
and in final form Sep 12, 2008