Journal of Lie Theory
Volume **20** (2010) 113–125

Dunkl Operators on $\mathbb{R}^d$ and Uncentered Maximal Function

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Communicated by J. Faraut

Abstract. We consider Dunkl theory on $\mathbb{R}^d$ associated to a finite reflection group G. We establish first an estimate for the Dunkl translation of the characteristic function of a closed ball with radius $\varepsilon > 0$ centered at 0. Second, we prove that the uncentered maximal function associated with the Dunkl operators is of weak-type $(1,1)$. We finally obtain that it is strong-type (p,p) for $1 < p \leq +\infty$.
Mathematics Subject Classification 2000: 42B10, 42B25, 44A15, 44A35.
Key Words and Phrases: Dunkl operators, Dunkl transform, Dunkl translations, uncentered maximal function.

1. Introduction

We consider Dunkl operators $T_i, 1 \leq i \leq d$, on $\mathbb{R}^d$, associated to an arbitrary finite reflection group G and a non negative multiplicity function k, introduced twenty years ago by C. F. Dunkl in [9] (see next section). Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^d$ and was further developed by several mathematicians (see [2, 3, 8, 10, 11, 12, 16, 17, 19, 20, 24, 27]). The Dunkl kernel E_k has been introduced by C.F. Dunkl in [10]. This kernel is used to define the Dunkl transform $\mathcal{F}_k$. K. Trimèche has introduced in [28] the Dunkl translation operators τ_x, $x \in \mathbb{R}^d$, on the space of infinitely differentiable functions on $\mathbb{R}^d$. At the moment an explicit formula for the Dunkl translation $\tau_x(f)$ of a function f is unknown in general. However, such formula is known in two cases: when the function f is radial (see next section) and when $G = \mathbb{Z}_2^d$. In particular, the boundedness of τ_x is established in these cases. As a result one obtains a formula for the convolution $*_k$.

The Hardy-Littlewood maximal function was first introduced by Hardy and Littlewood in 1930 for functions defined on the circle (see [14]). Later it was extended to various Lie groups, symmetric spaces, some weighted measure spaces (see [5, 13, 21, 23, 24, 25]), and various hypergroups (see[6, 7, 22]).

In this paper, we establish first an estimate for the Dunkl translation of the characteristic function of the closed ball with radius ε centered at 0, $\tau_x(\chi_{B(0,\varepsilon)})$,

*The author thanks the referees for their remarks and suggestions. The author is supported by the DGRST research project 04/UR/15-02 and the program CMCU 07G 1501.

ISSN 0949–5932 / \$2.50

$x \in \mathbb{R}^d$, $x \neq 0$, based on the inversion formula. This estimate plays a key role to prove the weak-type $(1,1)$ of the uncentered maximal function $M_k f$ associated with the Dunkl operators defined for each integrable function f on $(\mathbb{R}, d\nu_k)$ by

$$M_k f(x) = \sup_{\varepsilon>0,\ z\in C(x,\varepsilon)} \frac{1}{\nu_k(B(0,\varepsilon))} \left| f *_k \chi_{B(0,\varepsilon)}(z) \right|, \quad x \in \mathbb{R}^d,$$

where $C(x,\varepsilon) = \Big\{ \xi \in \mathbb{R}^d : \max\{0, \|x\| - \varepsilon\} \leq \|\xi\| < \|x\| + \varepsilon \Big\}$ and ν_k is a weighted measure on $\mathbb{R}^d$ (see next section). We obtain finally that M_k is bounded on L^p for $1 < p \leq +\infty$. This is an extension to the higher dimension for an arbitrary finite reflection group G of the results obtained in the rank-one case for $G = \mathbb{Z}_2$ in [1]. Previous results were obtained in [24] for the centered maximal function (i.e. using $\{x\}$ instead of $C(x,\varepsilon)$ in the sup).

The contents of this paper are as follows.
In section 2, we recall some basic definitions, notations and results in Dunkl theory. In section 3, we establish estimates for $\tau_x(\chi_{B(0,\varepsilon)})$, $x \in \mathbb{R}^d$, $x \neq 0$, and we prove that the uncentered maximal function $M_k f$ is of weak-type $(1,1)$ and strong-type (p,p) for $1 < p \leq +\infty$.

Along this paper we denote by $\langle .,. \rangle$ the usual Euclidean inner product in $\mathbb{R}^d$ as well as its extension to $\mathbb{C}^d \times \mathbb{C}^d$, we write for $x \in \mathbb{R}^d$, $\|x\| = \sqrt{\langle x, x \rangle}$ and we use c to denote a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

- $\mathcal{E}(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$.
- $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of functions in $\mathcal{E}(\mathbb{R}^d)$ which are rapidly decreasing as well as their derivatives.
- $\mathcal{D}(\mathbb{R}^d)$ the subspace of $\mathcal{E}(\mathbb{R}^d)$ of compactly supported functions.

2. Preliminaries

In this section, we recall some notations and results in Dunkl theory and we refer for more details to the articles [8, 9, 24] or to the survey [20].

Let G be a finite reflection group on $\mathbb{R}^d$, associated with a root system R. For $\alpha \in R$, we denote by H_α the hyperplane orthogonal to α and we set $\check{\mathbb{R}}^d = \mathbb{R}^d \backslash H$ where $H = \bigcup_{\alpha \in R} H_\alpha$. For a given $\beta \in \check{\mathbb{R}}^d$, we fix a positive subsystem $R_+ = \{\alpha \in R : \langle \alpha, \beta \rangle > 0\}$. We denote by k a nonnegative multiplicity function defined on R with the property that k is G-invariant. We associate with k the index

$$\gamma = \gamma(R) = \sum_{\xi \in R_+} k(\xi) > 0,$$

and the weighted measure ν_k defined by

$$d\nu_k(x) := w_k(x)dx \quad \text{where} \quad w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d.$$

w_k is G-invariant and homogeneous of degree 2γ.

Further, we introduce the Mehta-type constant c_k by

$$c_k = \left(\int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} d\nu_k(x) \right)^{-1}.$$

For every $1 \leq p \leq +\infty$, we denote by $L^p_k(\mathbb{R}^d)$ the space $L^p(\mathbb{R}^d, d\nu_k(x))$, $L^p_k(\mathbb{R}^d)^{rad}$ the subspace of those $f \in L^p_k(\mathbb{R}^d)$ that are radial and we use $\| \ \|_{p,k}$ as a shorthand for $\| \ \|_{L^p_k(\mathbb{R}^d)}$.

By using the homogeneity of w_k, it is shown in [17] that for $f \in L^1_k(\mathbb{R}^d)^{rad}$, there exists a function F on $[0, +\infty)$ such that $f(x) = F(\|x\|)$, for all $x \in \mathbb{R}^d$. The function F is integrable with respect to the measure $r^{2\gamma+d-1}dr$ on $[0, +\infty)$ and we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)\, d\nu_k(x) &= \int_0^{+\infty} \Big(\int_{S^{d-1}} w_k(ry) d\sigma(y) \Big) F(r) r^{d-1} dr \\ &= d_k \int_0^{+\infty} F(r) r^{2\gamma+d-1} dr, \end{aligned} \tag{1}$$

where S^{d-1} is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$ and

$$d_k = \int_{S^{d-1}} w_k(x) d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})}.$$

The Dunkl operators T_j, $1 \leq j \leq d$, on $\mathbb{R}^d$ associated to the reflection group G and the multiplicity function k are the first-order differential-difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in \mathcal{E}(\mathbb{R}^d), \quad x \in \mathbb{R}^d,$$

where σ_α is the reflection with respect to the hyperplane H_α orthogonal to α and $\alpha_j = \langle \alpha, e_j \rangle$, $(e_1, e_2, \ldots, e_d)$ being the canonical basis of $\mathbb{R}^d$.
In the case $k = 0$, the T_j reduce to the corresponding partial derivatives.

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x,y) = y_j\, u(x,y), & 1 \leq j \leq d, \\ u(0,y) = 1. \end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, denoted by $E_k(x,y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. M. Rösler has proved in [18] the following integral representation for the Dunkl Kernel

$$E_k(x,z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\mu_x^k(y), \qquad x \in \mathbb{R}^d, \quad z \in \mathbb{C}^d, \tag{2}$$

where μ_x^k is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$.
We have for all $\lambda \in \mathbb{C}$ and $z, z' \in \mathbb{C}^d$,

$$E_k(z, z') = E_k(z', z)\,,\ E_k(z, 0) = 1\,,\ E_k(\lambda z, z') = E_k(z, \lambda z')$$

and for $x, y \in \mathbb{R}^d$,

$$|E_k(x, iy)| \leq 1. \tag{3}$$

For fixed $x, y \in \mathbb{R}^d \backslash \{0\}$ and $r > 0$, we put

$$\mathcal{I}(x, y, r) = d_k^{-1} \int_{S^{d-1}} E_k(irx, z)\, E_k(-iry, z)\, w_k(z) d\sigma(z).$$

It is shown in [17, 19] that

$$\int_{S^{d-1}} E_k(ix, z)\, w_k(z) d\sigma(z) = d_k\ j_{\gamma+\frac{d}{2}-1}(\|x\|). \tag{4}$$

and

$$\mathcal{I}(x, y, r) = \int_{\mathbb{R}^d} j_{\gamma+\frac{d}{2}-1}\Big(r\sqrt{\|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\langle\frac{x}{\|x\|}, \eta\rangle}\Big) d\mu^k_{\frac{y}{\|y\|}}(\eta) \tag{5}$$

where $\mu^k_{\frac{y}{\|y\|}}$ is the measure given by the relation (2) with $\|\eta\| \leq \left\|\frac{y}{\|y\|}\right\| = 1$ and $j_{\gamma+\frac{d}{2}-1}$ the normalized Bessel function of the first kind and order $\gamma + \frac{d}{2} - 1$ given by

$$j_{\gamma+\frac{d}{2}-1}(\lambda x) = \begin{cases} 2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma + \frac{d}{2})\frac{J_{\gamma+\frac{d}{2}-1}(\lambda x)}{(\lambda x)^{\gamma+\frac{d}{2}-1}} & \text{if } \lambda x \neq 0 \\ 1 & \text{if } \lambda x = 0 \end{cases}$$

$\lambda \in \mathbb{C},\ x \in \mathbb{R}$, with $J_{\gamma+\frac{d}{2}-1}$ the Bessel function of first kind and order $\gamma + \frac{d}{2} - 1$.
The product formula for the function $j_{\gamma+\frac{d}{2}-1}$ is given by,

$$j_{\gamma+\frac{d}{2}-1}(r\|x\|) j_{\gamma+\frac{d}{2}-1}(r\|y\|) = c_{\gamma,d} \int_0^\pi j_{\gamma+\frac{d}{2}-1}(r\xi)\, (\sin\theta)^{2(\gamma+\frac{d}{2}-1)} d\theta$$

where $\xi = r\sqrt{\|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta}$ and $c_{\gamma,d} = \dfrac{\Gamma(\gamma + \frac{d}{2})}{\sqrt{\pi}\Gamma(\gamma + \frac{d}{2} - \frac{1}{2})}$ with $c_{\gamma,d} \displaystyle\int_0^\pi (\sin\theta)^{2(\gamma+\frac{d}{2}-1)} d\theta = 1$ (see [26]).

The Dunkl transform $\mathcal{F}_k$ is defined for $f \in \mathcal{D}(\mathbb{R}^d)$ by

$$\mathcal{F}_k(f)(x) = c_k \int_{\mathbb{R}^d} f(y) E_k(-ix, y)\, d\nu_k(y), \quad x \in \mathbb{R}^d.$$

We list some known properties of this transform :

i) The Dunkl transform of a function $f \in L^1_k(\mathbb{R}^d)$ has the following basic property

$$\|\mathcal{F}_k(f)\|_{\infty,k} \le \|f\|_{1,k} .$$

ii) The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is invariant under the Dunkl transform $\mathcal{F}_k$.

iii) When both f and $\mathcal{F}_k(f)$ are in $L^1_k(\mathbb{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(y) E_k(ix,y)\, d\nu_k(y), \quad x \in \mathbb{R}^d.$$

iv) (Plancherel's theorem) The Dunkl transform on $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric automorphism of $L^2_k(\mathbb{R}^d)$.

The Dunkl transform of a function in $L^1_k(\mathbb{R}^d)^{rad}$ is also radial and can be expressed via the Hankel transform. More precisely, according to [17], we have the following results:

$$\begin{aligned}\mathcal{F}_k(f)(x) &= \int_0^{+\infty} \Big(\int_{S^{d-1}} E_k(-ix,y) w_k(y) d\sigma(y) \Big) F(r) r^{2\gamma+d-1} dr \\ &= \mathcal{H}_{\gamma+\frac{d}{2}-1}(F)(\|x\|), \quad x \in \mathbb{R}^d, \end{aligned} \tag{6}$$

where F is the function defined on $[0,+\infty)$ by $F(\|x\|) = f(x)$, $x \in \mathbb{R}^d$ and $\mathcal{H}_{\gamma+\frac{d}{2}-1}$ is the Hankel transform of order $\gamma + \frac{d}{2} - 1$.

K. Trimèche has introduced in [28] the Dunkl translation operators τ_x, $x \in \mathbb{R}^d$, on $\mathcal{E}(\mathbb{R}^d)$. For $f \in L^2_k(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, we have

$$\mathcal{F}_k(\tau_x(f))(y) = E_k(ix,y)\mathcal{F}_k(f)(y). \tag{7}$$

By [24], if both f and $\mathcal{F}_k(f)$ are in $L^1_k(\mathbb{R}^d)$, we get

$$\tau_x(f)(y) = c_k \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\xi) E_k(ix,\xi) E_k(-iy,\xi)\, d\nu_k(\xi). \tag{8}$$

Notice that for all $x, y \in \mathbb{R}^d$, $\tau_x(f)(y) = \tau_y(f)(x)$ and for fixed $x \in \mathbb{R}^d$, τ_x is a continuous linear mapping from $\mathcal{E}(\mathbb{R}^d)$ into $\mathcal{E}(\mathbb{R}^d)$.
M. Rösler has proved in ([19], Theorem 5.1) the following integral representation of $\tau_x(f)$ when f is a radial function in $\mathcal{E}(\mathbb{R}^d)$

$$\tau_x(f)(y) = \int_{\mathbb{R}^d} f(\xi) d\rho^k_{x,y}(\xi), \tag{9}$$

where $\rho^k_{x,y}$ is a compactly supported radial probability measure on $\mathbb{R}^d$.
As an operator on $L^2_k(\mathbb{R}^d)$, τ_x is bounded. A priori it is not at all clear whether the translation operator can be defined for L^p- functions with p different from 2. However, according to Theorem 3.7 in [24], we obtain that $\tau_x(g) \ge 0$ for a bounded

and nonnegative function g in $L^1_k(\mathbb{R}^d)^{rad}$, the operator τ_x can be extended to $L^p_k(\mathbb{R}^d)^{rad}$, $1 \leq p \leq 2$, and we have

$$\|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k} \quad \text{for} \quad f \in L^p_k(\mathbb{R}^d)^{rad}.$$

For fixed $x, y \in \mathbb{R}^d$, it was shown in [4] that the mapping $f \mapsto \tau_x(f)(y)$ defines a supported distribution with support contained in the spherical shell

$$\{\, z \in \mathbb{R}^d \,:\, |\, \|x\| - \|y\|\, | \leq \|z\| \leq \|x\| + \|y\|\,\}. \tag{10}$$

The Dunkl convolution product $*_k$ of two functions f and g in $L^2_k(\mathbb{R}^d)$ is given by

$$(f \ *_k g)(x) = \int_{\mathbb{R}^d} \tau_x(f)(-y) g(y)\, d\nu_k(y), \quad x \in \mathbb{R}^d.$$

The Dunkl convolution product is commutative and for $f,\, g \in L^2_k(\mathbb{R}^d)$ we have

$$\mathcal{F}_k(f \ *_k \ g) = \mathcal{F}_k(f)\mathcal{F}_k(g). \tag{11}$$

It was shown in ([24], Theorem 4.1) that when g is a bounded function in $L^1_k(\mathbb{R}^d)^{rad}$, then

$$(f \ *_k g)(x) = \int_{\mathbb{R}^d} f(y)\tau_x(g)(-y)\, d\nu_k(y), \quad x \in \mathbb{R}^d,$$

initially defined on the intersection of $L^1_k(\mathbb{R}^d)$ and $L^2_k(\mathbb{R}^d)$ extends to all $L^p_k(\mathbb{R}^d)$, $1 \leq p \leq +\infty$ as a bounded operator. In particular,

$$\|f *_k g\|_{p,k} \leq \|f\|_{p,k} \|g\|_{1,k}\,.$$

3. Dunkl translations and uncentered maximal function

In this section we establish estimates of $\tau_x(\chi_{B(0,\varepsilon)})$, $x \in \mathbb{R}^d$, $x \neq 0$, where $\chi_{B(0,\varepsilon)}$ is the characteristic function of the closed ball $B(0,\varepsilon)$. We prove the weak-type $(1,1)$ of the uncentered maximal function $M_k f$ and the L^p-boundedness of M_k for $1 < p \leq +\infty$.

Lemma 3.1. *There exists a positive constant c such that for any $\varepsilon > 0$ and $x \in \mathbb{R}^d$, $x \neq 0$, one has*

$$0 \leq \tau_x(\chi_{B(0,\varepsilon)})(y) \leq c\left(\frac{\varepsilon}{\|x\|}\right)^{2\gamma+d-1}, \ a.e\, y \in \mathbb{R}^d. \tag{12}$$

Here $x, y \in \check{\mathbb{R}}^d$.

Proof. For $\varepsilon > 0$, we take a radial function φ in $\mathcal{D}(\mathbb{R}^d)$ such that

$$0 \leq \varphi(\xi) \leq 1, \text{ for } \|\xi\| > \varepsilon \quad \text{and} \quad \varphi(\xi) = 1 \text{ for } \|\xi\| \leq \varepsilon.$$

The inequality $0 \leq \chi_{B(0,\varepsilon)} \leq \varphi$ together with the nonnegativity of τ_x, $x \in \mathbb{R}^d$ on radial functions in $L^1_k(\mathbb{R}^d) \cap L^\infty_k(\mathbb{R}^d)$ (see [24], Theorem 3.7), shows that

$$0 \leq \tau_x(\chi_{B(0,\varepsilon)})(y) \leq \tau_x(\varphi)(y), \quad y \in \mathbb{R}^d.$$

Using (9), we get

$$\tau_x(\varphi)(y) \le \|\varphi\|_{\infty,k} \le 1\,, \quad \text{a.e}\ y \in \mathbb{R}^d,$$

hence we deduce

$$0 \le \tau_x(\chi_{B(0,\varepsilon)})(y) \le 1, \quad x \in \mathbb{R}^d,\ \text{a.e}\ y \in \mathbb{R}^d,$$

and we obtain (12) for $\|x\| \in]0, 2\varepsilon]$. Therefore we can assume now that $\|x\| > 2\varepsilon$ and in view of (10) that $y \in \mathbb{R}^d$ satisfies $|\,\|x\| - \|y\|\,| < \varepsilon$.
Let q_t, $t > 0$ be the heat kernel given by $q_t(x) = (2t)^{-(\gamma+\frac{d}{2})}e^{-t\|x\|^2}$, $x \in \mathbb{R}^d$. Clearly $\|q_t\|_{1,k} = c_k^{-1}$. Fom ([24], Theorem 3.6), both $\chi_{B(0,\varepsilon)} *_k q_t$ and $\mathcal{F}_k(\chi_{B(0,\varepsilon)} *_k q_t)$ are in $L^1_k(\mathbb{R}^d)^{rad}$, $\tau_x(\chi_{B(0,\varepsilon)} *_k q_t)(y) \ge 0$ and

$$\lim_{t\to 0^+} \tau_x(\chi_{B(0,\varepsilon)} *_k q_t)(y) = \tau_x(\chi_{B(0,\varepsilon)})(y) \ge 0\,,\ \text{a.e}\ \ y \in \mathbb{R}^d. \tag{13}$$

By (7), (8) and (11), we have
$\tau_x(\chi_{B(0,\varepsilon)} *_k q_t)(y) =$

$$\int_{\mathbb{R}^d} E_k(ix,\lambda)E_k(-iy,\lambda)\mathcal{F}_k(\chi_{B(0,\varepsilon)})(\lambda)\mathcal{F}_k(q_t)(\lambda)d\nu_k(\lambda). \tag{14}$$

For $\lambda \in \mathbb{R}^d$, put $\|\lambda\| = r$. Thus using (4) and (6), we can see that

$$\begin{aligned}\mathcal{F}_k(\chi_{B(0,\varepsilon)})(\lambda) &= \frac{c_k^{-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})}\int_0^\varepsilon j_{\gamma+\frac{d}{2}-1}(rt)\,t^{2\gamma+d-1}dt \\ &= \frac{c_k^{-1}}{2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)}\,\varepsilon^{2(\gamma+\frac{d}{2})}j_{\gamma+\frac{d}{2}}(r\varepsilon).\end{aligned}$$

Since $|j_{\gamma+\frac{d}{2}}(r\varepsilon)| \le 1$, we get

$$|\mathcal{F}_k(\chi_{B(0,\varepsilon)})(\lambda)| \le c\,\varepsilon^{2(\gamma+\frac{d}{2})}. \tag{15}$$

We have

$$|\mathcal{F}_k(q_t)(\lambda)| \le \|\mathcal{F}_k(q_t)\|_{\infty,k} \le \|q_t\|_{1,k} \le c, \quad \text{a.e}\ \ \lambda \in \mathbb{R}^d. \tag{16}$$

We write (14) in the form

$$\begin{aligned}\tau_x(\chi_{B(0,\varepsilon)} *_k q_t)(y) &= \int_{\|\lambda\|\le\|x\|^{-1}} + \int_{\|x\|^{-1}\le\|\lambda\|\le\varepsilon^{-1}} + \int_{\varepsilon^{-1}\le\|\lambda\|} \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{17}$$

From (1), (3), (15) and (16), we obtain

$$\begin{aligned}|I_1| &\le c\,\varepsilon^{2(\gamma+\frac{d}{2})}\int_{\|\lambda\|\le\|x\|^{-1}} d\nu_k(\lambda) \\ &\le c\,\varepsilon^{2(\gamma+\frac{d}{2})}\int_0^{\|x\|^{-1}} r^{2\gamma+d-1}dr \\ &\le c\,\frac{\varepsilon^{2(\gamma+\frac{d}{2})}}{\|x\|^{2(\gamma+\frac{d}{2})}} \le c\left(\frac{\varepsilon}{\|x\|}\right)^{2\gamma+d-1}, \quad \text{for}\quad \|x\| > 2\varepsilon.\end{aligned} \tag{18}$$

To estimate I_2, we put $q_t(\lambda) = \tilde{q}_t(\|\lambda\|)$, $\lambda \in \mathbb{R}^d$. Using (1), (5), (6), (15) and (16), we can assert that

$$\begin{aligned}|I_2| &= \Big| d_k \int_{\|x\|^{-1}}^{\varepsilon^{-1}} \mathcal{I}(x,y,r)\, \mathcal{H}_{\gamma+\frac{d}{2}-1}(\chi_{[0,\varepsilon]})(r)\, \mathcal{H}_{\gamma+\frac{d}{2}-1}(\tilde{q}_t)(r)\; r^{2\gamma+d-1} dr \Big| \\ &\leq c\, \varepsilon^{2(\gamma+\frac{d}{2})} \int_{\|x\|^{-1}}^{\varepsilon^{-1}} \Big|\mathcal{I}(x,y,r)\Big|\, r^{2\gamma+d-1} dr.\end{aligned}$$

The function $\rho \mapsto \sqrt{\rho} J_{\gamma+\frac{d}{2}-1}(\rho)$ is bounded on $(0,+\infty)$, (19)

so we can write,

$$|\, j_{\gamma+\frac{d}{2}-1}(\rho)| \leq c\, \rho^{-\gamma-\frac{d}{2}+\frac{1}{2}}. \tag{20}$$

Since for $\|x\| > 2\varepsilon$ and $|\,\|x\| - \|y\|\,| < \varepsilon$, we have

$$\frac{1}{2}\|x\| < \|x\| - \varepsilon < \|y\| < \|x\| + \varepsilon < \frac{3}{2}\|x\|,$$

then from (5) and (20), it yields

$$\begin{aligned}\Big|\mathcal{I}(x,y,r)\Big| &\leq c\, r^{-2\gamma-d+1}\, \|x\|^{-\gamma-\frac{d}{2}+\frac{1}{2}}\, \|y\|^{-\gamma-\frac{d}{2}+\frac{1}{2}} \\ &\leq c\, r^{-2\gamma-d+1}\, \|x\|^{-2\gamma-d+1}. \end{aligned} \tag{21}$$

This gives

$$\begin{aligned}|I_2| &\leq c\, \varepsilon^{2(\gamma+\frac{d}{2})}\, \|x\|^{-2\gamma-d+1} \int_{\|x\|^{-1}}^{\varepsilon^{-1}} dr \\ &\leq c \Big(\frac{\varepsilon}{\|x\|}\Big)^{2\gamma+d-1}, \quad \text{for} \quad \|x\| > 2\varepsilon. \end{aligned} \tag{22}$$

To estimate I_3, we need to refine the estimates for $\mathcal{F}_k(\chi_{B(0,\varepsilon)})(\lambda)$ in (15). For $r = \|\lambda\| \in [\varepsilon^{-1}, +\infty[$, we can see from (19) that

$$\begin{aligned}|\mathcal{F}_k(\chi_{B(0,\varepsilon)})(\lambda)| &= \frac{c_k^{-1}}{2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)}\, \varepsilon^{2(\gamma+\frac{d}{2})} |j_{\gamma+\frac{d}{2}}(r\varepsilon)| \\ &\leq c\, \frac{\varepsilon^{\gamma+\frac{d}{2}-\frac{1}{2}}}{r^{\gamma+\frac{d}{2}+\frac{1}{2}}} |\sqrt{r\varepsilon}\, J_{\gamma+\frac{d}{2}}(r\varepsilon)| \\ &\leq c\, \varepsilon^{\gamma+\frac{d}{2}-\frac{1}{2}}\, r^{-(\gamma+\frac{d}{2}+\frac{1}{2})}. \end{aligned} \tag{23}$$

Then using (21) and (23), we find that

$$\begin{aligned}|I_3| &\leq c\, \varepsilon^{\gamma+\frac{d}{2}-\frac{1}{2}} \|x\|^{-2\gamma-d+1} \int_{\varepsilon^{-1}}^{+\infty} r^{-(\gamma+\frac{d}{2}+\frac{1}{2})} dr \\ &\leq c \Big(\frac{\varepsilon}{\|x\|}\Big)^{2\gamma+d-1}, \quad \text{for} \quad \|x\| > 2\varepsilon. \end{aligned} \tag{24}$$

Thus we obtain by (17), (18), (22) and (24)

$$0 \leq \tau_x(\chi_{B(0,\varepsilon)} *_k q_t)(y) \leq c\left(\frac{\varepsilon}{\|x\|}\right)^{2\gamma+d-1}, \quad \text{for} \quad \|x\| > 2\varepsilon.$$

From (13), it yields

$$0 \leq \tau_x(\chi_{B(0,\varepsilon)})(y) \leq c\left(\frac{\varepsilon}{\|x\|}\right)^{2\gamma+d-1}, \text{ a.e } y \in \mathbb{R}^d,$$

for $\|x\| > 2\varepsilon$ and $|\,\|x\| - \|y\|\,| < \varepsilon$, therefore (12) is established. ■

Notation. For $x \in \mathbb{R}^d$ and $\varepsilon > 0$, we recall that

$$C(x,\varepsilon) = \Big\{z \in \mathbb{R}^d : \max\{0, \|x\| - \varepsilon\} \leq \|z\| < \|x\| + \varepsilon\Big\}.$$

Lemma 3.2. *There exists a positive constant c such that for any $\varepsilon > 0$ and $x \in \mathbb{R}^d$, $x \neq 0$, one has*

$$0 \leq \tau_x(\chi_{B(0,\varepsilon)})(y) \leq c\,\frac{\nu_k(B(0,\varepsilon))}{\nu_k(C(x,\varepsilon))}, \quad a.e\; y \in \mathbb{R}^d. \tag{25}$$

Here $x, y \in \check{\mathbb{R}}^d$.

Proof. Using (1), we have

$$\nu_k(B(0,\varepsilon)) = \int_{B(0,\varepsilon)} d\nu_k(y) = d_k \int_0^{\varepsilon} r^{2\gamma+d-1} dr = \frac{d_k}{2\gamma+d}\,\varepsilon^{2\gamma+d}.$$

Therefore, on the one hand, we have for $\|x\| \leq \varepsilon$,

$$\nu_k(C(x,\varepsilon)) = \int_{C(x,\varepsilon)} d\nu_k(y) = d_k \int_0^{\|x\|+\varepsilon} r^{2\gamma+d-1} dr \leq c\,\nu_k(B(0,\varepsilon)),$$

since by Lemma 3.1, we have

$$0 \leq \tau_x(\chi_{B(0,\varepsilon)})(y) \leq 1, \quad x \in \mathbb{R}^d, \text{ a.e } y \in \mathbb{R}^d,$$

then we obtain (25) for $\|x\| \leq \varepsilon$.
On the other hand, we have for $\|x\| > \varepsilon$,

$$\begin{aligned}
\nu_k(C(x,\varepsilon)) &\leq d_k(\|x\| + \varepsilon)^{2\gamma+d-1} \int_{\|x\|-\varepsilon}^{\|x\|+\varepsilon} dr \\
&\leq c\,\varepsilon^{2\gamma+d}\Big(\frac{\|x\| + \varepsilon}{\varepsilon}\Big)^{2\gamma+d-1} \\
&\leq c\,\nu_k(B(0,\varepsilon))\Big(\frac{\|x\|}{\varepsilon}\Big)^{2\gamma+d-1}.
\end{aligned}$$

Then by (12) we obtain (25) for $\|x\| > \varepsilon$, which proves the result. ■

According to ([21], Lemma 1.6), we have the following Vitali covering lemma.

Lemma 3.3. *Let E be a measurable subset of $\mathbb{R}^d$ (with respect to ν_k) which is covered by the union of a family $\{C_j\}$ where $C_j = C(x_j, \varepsilon_j)$. Then from this family, one can select a subfamily $C_1, C_2, \ldots$ (which may be finite) such that $C_i \cap C_j = \emptyset$ for $i \neq j$ and*

$$\sum_h \nu_k(C_h) \geq c\, \nu_k(E).$$

In order to prove the weak-type $(1,1)$ of $M_k f$, we recall that

$$M_k f(x) = \sup_{\varepsilon>0,\ z\in C(x,\varepsilon)} \frac{1}{\nu_k(B(0,\varepsilon))} \left| f *_k \chi_{B(0,\varepsilon)}(z) \right|, \quad x \in \mathbb{R}^d,$$

for each integrable function f on $(\mathbb{R}, d\nu_k)$. Therefore, we can also write

$$M_k f(x) = \sup_{\varepsilon>0,\ z\in C(x,\varepsilon)} \frac{\left| \int_{\mathbb{R}^d} f(y)\tau_z(\chi_{B(0,\varepsilon)})(-y) d\nu_k(y) \right|}{\int_{B(0,\varepsilon)} d\nu_k(y)}.$$

Theorem 3.4. *The uncentered maximal function $M_k f$ is of weak-type $(1,1)$.*

Proof. For $\varepsilon > 0$, $x \in \mathbb{R}^d$, $z \in \check{C}(x,\varepsilon) = C(x,\varepsilon) \cap \check{\mathbb{R}}^d$ and $f \in L^1_k(\mathbb{R}^d)$, we have

$$(f *_k \chi_{B(0,\varepsilon)})(z) = \int_{\mathbb{R}^d} f(y)\tau_z(\chi_{B(0,\varepsilon)})(-y) d\nu_k(y),$$

then using (10) and (25), we obtain

$$\begin{aligned} |(f *_k \chi_{B(0,\varepsilon)})(z)| &\leq \int_{y\in \check{C}(z,\varepsilon)} |\tau_z(\chi_{B(0,\varepsilon)})(-y)||f(y)| d\nu_k(y), \\ &\leq c \left(\int_{y\in \check{C}(z,\varepsilon)} |f(y)| d\nu_k(y) \right) \frac{\nu_k(B(0,\varepsilon))}{\nu_k(C(z,\varepsilon))}. \end{aligned}$$

Hence

$$M_k f(x) \leq c\, \tilde{M}_k f(x) \tag{26}$$

where $\tilde{M}(f)$ is defined by

$$\tilde{M}_k f(x) = \sup_{\varepsilon>0,\ z\in C(x,\varepsilon)} \frac{1}{\nu_k(C(z,\varepsilon))} \int_{y\in C(z,\varepsilon)} |f(y)| d\nu_k(y).$$

For $\lambda > 0$, put

$$\tilde{E}_\lambda = \{x \in \mathbb{R}^d;\ \tilde{M}_k f(x) > \lambda\}.$$

Then for each $x \in \tilde{E}_\lambda$, there exist $\varepsilon_x > 0$ and $z_x \in C(x,\varepsilon)$ such that

$$\int_{y\in C(z_x,\varepsilon_x)} |f(y)| d\nu_k(y) > \lambda\, \nu_k(C(z_x,\varepsilon_x)). \tag{27}$$

Furthermore, note that $x \in C(z_x, \varepsilon_x)$, so that the $(C(z_x, \varepsilon_x))_{x \in \tilde{E}_\lambda}$ form a covering of $\tilde{E}_\lambda$. Thus using Lemma 3.3, we can select a disjoint subsequence $C(z_1, \varepsilon_1)$, $C(z_2, \varepsilon_2), ...$ (which may be finite) such that

$$\sum_h \nu_k(C(z_h, \varepsilon_h)) \geq c\, \nu_k(\tilde{E}_\lambda). \tag{28}$$

We have

$$\int_{y \in \bigcup C(z_h, \varepsilon_h)} |f(y)| d\nu_k(y) \geq \sum_h \int_{y \in C(z_h, \varepsilon_h)} |f(y)| d\nu_k(y) \ ,$$

applying (27) and (28) to each of the mutually disjoint intervals, we get

$$\int_{y \in \bigcup C(z_h, \varepsilon_h)} |f(y)| d\nu_k(y) > \lambda \sum_h \nu_k(C(z_h, \varepsilon_h)) \geq \lambda\, c\, \nu_k(\tilde{E}_\lambda) \ .$$

But since the first member of this inequality is majorized by $\|f\|_{1,k}$, we obtain

$$\nu_k(\tilde{E}_\lambda) \leq c\, \frac{\|f\|_{1,k}}{\lambda}$$

which shows that $\tilde{M}_k f$ is of weak type $(1,1)$ and hence from (26) the same is true for $M_k f$. ■

As consequence of the theorem 3.4, we obtain the following corollary.

Corollary 3.5. *If* $1 < p \leq +\infty$ *and* $f \in L^p_k(\mathbb{R}^d)$, *then one has*

$$M_k f \in L^p_k(\mathbb{R}^d) \quad \textit{and} \quad \|M_k f\|_{p,k} \leq c\, \|f\|_{p,k} \ .$$

Proof. Using Theorem 3.4, ([15], Corollary 21.72) and proceeding in the same manner as in the proof on Euclidean spaces (see for example Theorem 1 in [21], section 1.3), we obtain the results. ■

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Received November 3, 2009
and in final form February 9, 2009