

Homogeneous Toric Varieties

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Abstract. A description of transitive actions of a semisimple algebraic group G on toric varieties is obtained. Every toric variety admitting such an action lies between a product of punctured affine spaces and a product of projective spaces. The result is based on the Cox realization of a toric variety as a quotient space of an open subset of a vector space V by a quasitorus action and on investigation of the G -module structure of V .

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1. Introduction

We study toric varieties X equipped with a transitive action of a connected semisimple algebraic group G . In this case X is called a *homogeneous toric variety*. The ground field \mathbb{K} is algebraically closed and of characteristic zero.

Consider a quasiaffine variety

$$\mathcal{X} = \mathcal{X}(n_1, \dots, n_m) := (\mathbb{K}^{n_1} \setminus \{0\}) \times \dots \times (\mathbb{K}^{n_m} \setminus \{0\})$$

with $n_i \geq 2$. The group $G = G_1 \times \dots \times G_m$, where every component G_i is either $\mathrm{SL}(n_i)$ or $\mathrm{Sp}(n_i)$, and n_i is even in the second case, acts on \mathcal{X} transitively and effectively. Let $\mathbb{S} = (\mathbb{K}^\times)^m$ be an algebraic torus acting on \mathcal{X} by component-wise scalar multiplication, and

$$p: \mathcal{X} \rightarrow \mathcal{Y} := \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_m-1}$$

be the quotient morphism. Fix a closed subgroup $S \subseteq \mathbb{S}$. The action of the group S on \mathcal{X} admits a geometric quotient $p_X: \mathcal{X} \rightarrow X := \mathcal{X}/S$. The variety X is toric, it carries the induced action of the quotient group \mathbb{S}/S , and there is a quotient morphism $p^X: X \rightarrow \mathcal{Y}$ for this action closing the commutative diagram

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$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{p_X} & X \\
 & \searrow p & \swarrow p^X \\
 & \mathcal{Y} &
 \end{array}$$

The induced action of the group G on X is transitive and locally effective. We say that the G -variety X is obtained from \mathcal{X} by *central factorization*. The following theorem gives a classification of transitive actions of semisimple groups on toric varieties up to a twist by a diagram automorphism of the acting group.

Theorem 1.1. *Let X be a toric variety with a transitive locally effective action of a connected simply connected semisimple algebraic group G . Then $G = G_1 \times \dots \times G_m$, where every simple component G_i is either $\mathrm{SL}(n_i)$ or $\mathrm{Sp}(n_i)$, and the variety X is obtained from $\mathcal{X} = \mathcal{X}(n_1, \dots, n_m)$ by central factorization. Conversely, any variety obtained from \mathcal{X} by central factorization is a homogeneous toric variety.*

Theorem 1.1 also describes homogeneous spaces of a semisimple group that have a toric structure. It is natural to apply the Cox realization of a variety in order to search for toric varieties in a given class of varieties. This idea is already used in [8], where toric affine $\mathrm{SL}(2)$ -embeddings are characterized.

In Section 2 we recall basic facts on the Cox realization and its generalization. Criteria of existence of an open G -orbit on X in terms of G - and $(G \times S)$ -actions on the total coordinate space Z are also given there. In Section 3 we prove Theorem 1.1. The next section is devoted to special classes of toric homogeneous varieties and to a characterization of their fans. In the last section we consider transitive actions of reductive groups on toric varieties.

Our results are closely connected with the results of E.B. Vinberg [17], where algebraic transformation groups of maximal rank were classified. Recall that an *algebraic transformation group of maximal rank* is an effective generically transitive (i.e., with an open orbit) action of an algebraic group \mathcal{G} on an algebraic variety X such that $\dim X = \mathrm{rk} \mathcal{G}$, where $\mathrm{rk} \mathcal{G}$ is the rank of a maximal torus T of the group \mathcal{G} . In this situation the induced action of the torus T on X is effective and generically transitive, see [6]. If the group \mathcal{G} is semisimple, then an open \mathcal{G} -orbit on X is a homogeneous toric variety. It turns out that in this case X is a product of projective spaces and \mathcal{G} acts on X transitively. Theorem 1.1 implies that every homogeneous toric variety determines a reductive transformation group of maximal rank; here \mathcal{G} is the quotient group $(\mathrm{GL}(n_1) \times \dots \times \mathrm{GL}(n_m))/S$.

Finally, let us mention a related result from toric topology. A *torus manifold* is a smooth real even-dimensional manifold M^{2n} with an effective action of a compact torus $(S^1)^n$ such that the set of $(S^1)^n$ -fixed points is nonempty. In [15], homogeneous torus manifolds are studied. The latter are torus manifolds M^{2n} with a transitive action of a compact Lie group K such that the induced action of a maximal torus of K coincides with the given $(S^1)^n$ -action. It is proved that every homogeneous torus manifold may be realized as

$$M = \mathbb{C}\mathbb{P}^{n_1} \times \dots \times \mathbb{C}\mathbb{P}^{n_k} \times (S^{2m_1} \times \dots \times S^{2m_l})/F,$$

where S^{2m} is a sphere of dimension $2m$, F is a subgroup of $\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$ (l copies), and each copy of \mathbb{Z}_2 acts on the corresponding sphere by central symmetry. A compact Lie group

$$K = \mathrm{PSU}(n_1 + 1) \times \dots \times \mathrm{PSU}(n_k + 1) \times \mathrm{SO}(2m_1 + 1) \times \dots \times \mathrm{SO}(2m_l + 1)$$

acts on M transitively. Moreover, the manifold M is orientable if and only if $F \subset \mathrm{SO}(2m_1 + 2m_2 + \dots + 2m_l + l)$.

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2. The Cox construction

A *toric variety* is a normal algebraic variety with an effective generically transitive action of an algebraic torus T . A toric variety X is *non-degenerate* if any invertible regular function on X is constant.

Let $\mathrm{Cl}(X)$ be the divisor class group of the variety X . It is well-known that the group $\mathrm{Cl}(X)$ of a toric variety X is finitely generated, see [7, Section 3.4]. Recall that a *quasitorus* is an affine algebraic group S isomorphic to a direct product of an algebraic torus S^0 and a finite abelian group Γ . Every closed subgroup of a torus is a quasitorus. The group of characters of a quasitorus S is a finitely generated abelian group. The *Neron-Severi quasitorus* of a toric variety X is a quasitorus S whose group of characters is identified with $\mathrm{Cl}(X)$.

We come to a canonical quotient realization of a non-degenerate toric variety X obtained in [5]. Let d be the number of prime T -invariant Weil divisors on X . Consider the vector space \mathbb{K}^d and the torus $\mathbb{T} = (\mathbb{K}^\times)^d$ of all invertible diagonal matrices acting on \mathbb{K}^d . Then there are a closed embedding of the Neron-Severi quasitorus S into \mathbb{T} and an open subset $U \subseteq \mathbb{K}^d$ such that

- the complement $\mathbb{K}^d \setminus U$ is a union of some coordinate subspaces of dimension $\leq d - 2$;
- there exist a categorical quotient $p_X: U \rightarrow U//S$ and an isomorphism $\varphi: X \rightarrow U//S$;
- via isomorphism φ , the T -action on X corresponds to the action of the quotient group \mathbb{T}/S on $U//S$.

Later this realization was generalized to a wider class of normal algebraic varieties, see [11], [4], [10]. One of the conditions that determines this class is finite generation of the divisor class group $\mathrm{Cl}(X)$. This allows to define the Neron-Severi quasitorus S of the variety X . The space \mathbb{K}^d is replaced by an affine factorial (or, more generally, factorially graded, see [1]) S -variety Z . It is called the *total coordinate space* of the variety X . Further, X appears as the quotient space of the categorical quotient $p_X: U \rightarrow U//S$, where U is an open S -invariant subset of Z such that the complement $Z \setminus U$ is of codimension at least two in Z . The morphism $p_X: U \rightarrow X \cong U//S$ is called the *universal torsor* over X .

Let a connected affine algebraic group G act on a normal variety X . Passing to a finite covering we may assume that $\mathrm{Cl}(G) = 0$ [12, Proposition 4.6]. Then the

action of G on X can be lifted to an action of G on the total coordinate space Z that commutes with the S -action, see [3, Section 4]. It turns out that the set U is $(G \times S)$ -invariant and $p_X: U \rightarrow X$ is a G -equivariant morphism.

Lemma 2.1. *The following conditions are equivalent.*

- (i) *The action of the group G on X is generically transitive.*
- (ii) *The action of the group $G \times S$ on Z is generically transitive.*

Proof. Let $X_0 \subseteq X$ be an open G -orbit. Each point $x \in X_0$ is smooth on X , and thus the fiber $p_X^{-1}(x)$ is isomorphic to the quasitorus S [10, Proposition 2.2, (iii)]. It shows that the group $G \times S$ acts on $p_X^{-1}(X_0)$ transitively.

Conversely, if $Z_0 \subseteq Z$ is an open $(G \times S)$ -orbit, then $Z_0 \subseteq U$ and the action of G on the quotient space $U//S$ is generically transitive. ■

Assume that the group G has trivial group of characters. Then the lifting of the action of the group G to Z is unique, compare [3, Remark 4.1] and [9, Proposition 1.8]. Let H be a closed subgroup of G . Every invertible regular function on the homogeneous space G/H is constant, see [13, Proposition 1.2].

Proposition 2.2. *The following conditions are equivalent.*

- (i) *The action of the group G on X is generically transitive and the complement of an open G -orbit has codimension at least two in X .*
- (ii) *The action of the group G on the total coordinate space Z is generically transitive.*
- (iii) *The action of the group G on the total coordinate space Z is generically transitive and the complement of an open G -orbit has codimension at least two in Z .*

Proof. We check "(i) \Rightarrow (iii)". Let $X_0 \subseteq X$ be an open G -orbit. The condition $\text{codim}_X(X \setminus X_0) \geq 2$ implies that $p_X: p_X^{-1}(X_0) \rightarrow X_0$ is the universal torsor over X_0 and that the complement to $p_X^{-1}(X_0)$ in Z does not contain divisors, see [2, Section 2]. By [2, Lemma 3.14] (see also [1, Theorem 4.1]), the universal torsor over a homogeneous space G/H is the projection $G/H_1 \rightarrow G/H$, where H_1 is the intersection of kernels of all characters of the subgroup H . This shows that the group G acts on $p_X^{-1}(X_0)$ transitively.

In order to obtain "(iii) \Rightarrow (i)" note that $p_X(Z_0)$, where Z_0 is the open G -orbit in Z , is an open G -orbit in X whose complement does not contain divisors. The implication "(iii) \Rightarrow (ii)" is obvious.

To verify "(ii) \Rightarrow (iii)" let $Z_0 \subseteq Z$ be an open G -orbit. Since the subset Z_0 is S -invariant, for every prime divisor $D \subset Z$ in the complement to Z_0 the set $S \cdot D$ is an S -invariant Weil divisor. Each S -invariant Weil divisor on Z is a principal divisor $\text{div}(f)$ of a regular function $f \in \mathbb{K}[Z]$, see [10, Proposition 2.2, (iv)]. Then the non-constant function f is invertible on Z_0 , a contradiction. ■

The same arguments lead to the following result.

Proposition 2.3. *The action of the group G on X is transitive if and only if the open subset $U \subseteq Z$ is a G -orbit.*

3. Classification of homogeneous toric varieties

In this section we prove Theorem 1.1. Since the variety X is toric, its total coordinate space Z is an affine space.

Lemma 3.1. *Let a semisimple group G act on a toric variety X with an open orbit. Then X is non-degenerate and the action of the group $G \times S$ on the affine space Z is equivalent to a linear one.*

Proof. Since any invertible function of the open G -orbit is constant, the variety X is non-degenerate. By Lemma 2.1, the action of the group $G \times S$ on the space Z is generically transitive, and the second statement follows from [14, Proposition 5.1]. ■

Later on we assume that $G = G_1 \times \dots \times G_m$ acts on X transitively. Denote by V the total coordinate space Z of the variety X regarded as the $(G \times S)$ -module. We proceed with a description of the G -module structure on V .

Proposition 3.2. *Let $V = V_1 \oplus \dots \oplus V_s$ be a decomposition into irreducible summands. Then every simple component G_i acts not identically only on one summand V_i (up to renumbering), and thus $m = s$. Moreover, every G_i acts on the set of nonzero vectors in V_i transitively.*

Proof. By Proposition 2.3, the complement of the open G -orbit U in V is a union of coordinate subspaces (in some, possibly nonlinear, coordinate system). Thus each irreducible component of the complement is a smooth variety. The linear action of the group G on V commutes with the group \mathbb{K}^\times of scalar operators, and the open orbit U as well as any component of the complement $V \setminus U$ is $(G \times \mathbb{K}^\times)$ -invariant. But a cone is a smooth variety if and only if it is a subspace. This shows that each component of $V \setminus U$ is a maximal proper submodule of V . In particular, the number of maximal proper submodules is finite and thus the G -modules V_1, \dots, V_s are pairwise non-isomorphic. The orbit U is the set of vectors $v \in V$ whose projection on each V_i is nonzero. This implies that the group G acts on the set of nonzero vectors of each submodule V_i transitively.

If several components of G act on some V_i not identically, then V_i is isomorphic to the tensor product of simple modules of these components. Then the cone of decomposable tensors in V_i is G -invariant, a contradiction.

Suppose that a simple component G_l acts on both V_i and V_j not identically. Then G_l acts transitively on the set of pairs (v_i, v_j) with nonzero v_i and v_j . In particular, any such pair is an eigenvector of a Borel subgroup of G_l . Fix a Borel subgroup $B \subset G_l$ and a highest vector for B in V_i as v_i and a lowest vector for B in V_j as v_j . Since the intersection of two opposite parabolic subgroups of G_l

does not contain a Borel subgroup, we get a contradiction. ■

The following lemma is well known. We give a short self-contained proof suggested by the referee.

Lemma 3.3. *Finite-dimensional rational modules of a simple group G such that G acts on the set of nonzero vectors transitively are*

1. *the tautological $SL(n)$ -module \mathbb{K}^n and $Sp(2n)$ -module \mathbb{K}^{2n} ;*
2. *the dual $SL(n)$ -module $(\mathbb{K}^n)^*$.*

Proof. Since G acts on $V \setminus \{0\}$ transitively, V is a simple G -module of highest weight λ and $V = \mathfrak{g}v_\lambda$, where \mathfrak{g} is the tangent algebra of the group G and v_λ is a highest weight vector. In particular, a lowest weight vector is $v_{-\lambda^*} = e_{-\alpha}v_\lambda$, where α is a positive root, whence $\alpha = \lambda + \lambda^*$ is the highest root. This occurs only for $G = SL(n)$ with fundamental weights $\lambda = \omega_1, \omega_{n-1}$, and $G = Sp(2n)$ with $\lambda = \omega_1$. ■

Applying an outer automorphism of G , we may assume that

$$G = G_1 \times \dots \times G_m \text{ and } V = V_1 \oplus \dots \oplus V_m,$$

where every component G_i is either $SL(n_i)$ or $Sp(n_i)$, and V_i is the tautological G_i -module with identical action of other components. The open G -orbit U in V coincides with the subvariety $\mathcal{X} = \mathcal{X}(n_1, \dots, n_m)$. Therefore the variety X is obtained from \mathcal{X} by central factorization.

Let $\mathbb{S} = (\mathbb{K}^\times)^m$ be an algebraic torus acting on $V = V_1 \oplus \dots \oplus V_m$ by component-wise scalar multiplication. It remains to explain why for any subgroup $S \subseteq \mathbb{S}$ there exists a geometric quotient $\mathcal{X} \rightarrow \mathcal{X}/S$. This follows from the fact that \mathcal{X} is a homogeneous space of the group $\overline{G} := GL(n_1) \times \dots \times GL(n_m)$, and S is a central subgroup of \overline{G} . The proof of Theorem 1.1 is completed.

Remark 3.4. The collection (n_1, \dots, n_m) is determined by a homogeneous toric variety X uniquely. Indeed, if $\mathbb{K}^d \supset U \rightarrow X$ is the Cox realization of X and C_1, \dots, C_m are irreducible components of the complement $\mathbb{K}^d \setminus U$, then $n_i = d - \dim C_i$.

4. Properties of homogeneous toric varieties

In this section we use standard notation of toric geometry, see [7]. Let \mathcal{N} be the lattice of one-parameter subgroups of a d -dimensional torus \mathbb{T} and \mathcal{M} be the lattice of characters of \mathbb{T} . The torus \mathbb{T} acts diagonally on the space $\mathbb{K}^d = V = V_1 \oplus \dots \oplus V_m$, and $\mathbb{S} \subset \mathbb{T}$ is the m -dimensional subtorus acting on every V_i by scalar multiplication. Identification of \mathbb{T} with $(\mathbb{K}^\times)^d$ defines standard bases in \mathcal{N} and \mathcal{M} . Moreover, the decomposition $V = V_1 \oplus \dots \oplus V_m$ divides the standard basis of \mathcal{N} into m groups I_1, \dots, I_m , where each group I_j contains n_j basis vectors and $n_j := \dim V_j$. The open subvariety $\mathcal{X}(n_1, \dots, n_m) = U \subset V$ is a toric \mathbb{T} -variety. Its fan $\mathcal{C} = \mathcal{C}(n_1, \dots, n_m)$ in the lattice \mathcal{N} consists of the cones generated by all collections of standard basis vectors that do not contain any subset I_j .

Let $S \subseteq \mathbb{S}$ be a closed subgroup. There is a sequence of lattices of one-parameter subgroups $\mathcal{N}_S \subseteq \mathcal{N}_{\mathbb{S}} \subset \mathcal{N}$, where the lattice \mathcal{N}_S is determined by the connected component S^0 of the quasitorus S . The fan \mathcal{C}_{S^0} of the quotient space \mathcal{X}/S^0 is the image of the fan \mathcal{C} under the projection

$$\mathcal{N}_{\mathbb{Q}} \rightarrow (\mathcal{N}/\mathcal{N}_S)_{\mathbb{Q}}.$$

The fan \mathcal{C}_S of the variety \mathcal{X}/S coincides with the fan \mathcal{C}_{S^0} considered with regard to an overlattice of $\mathcal{N}/\mathcal{N}_S$ of finite index, see [7, Section 2.2]. In particular, the fan $\mathcal{C}_{\mathbb{S}}$ coincides with the fan \mathcal{P} of the product of projective spaces $\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_m-1}$, and \mathcal{C}_S may be considered as an intermediate step of the projection:

$$\mathcal{C} \rightarrow \mathcal{C}_S \rightarrow \mathcal{P}.$$

Let us define a sublattice $\mathcal{M}_S \subseteq \mathcal{M}$ as the set of characters of the torus \mathbb{T} containing S in the kernel. Elements of \mathcal{M}_S are linear functions on the space $(\mathcal{N}/\mathcal{N}_S)_{\mathbb{Q}}$.

Proposition 4.1. *Let $X = \mathcal{X}/S$ be a homogeneous toric variety. Then*

1. *the variety X is quasiprojective;*
2. *the variety X is not affine;*
3. *the variety X is projective if and only if it coincides with $\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_m-1}$;*
4. *the variety X is quas affine if and only if the lattice \mathcal{M}_S contains a vector with positive coordinates;*
5. *the variety X has a nonconstant regular function if and only if the lattice \mathcal{M}_S contains a nonzero vector with nonnegative coordinates.*

Proof. (1) By Chevalley's Theorem, any homogeneous space of an affine algebraic group is a quasiprojective variety.

(2) A toric variety obtained via Cox construction is affine if and only if $U = V$. In our situation this is not the case.

(3) Maximal dimension of a cone in the fan \mathcal{C} equals $n_1 + \dots + n_m - m$. Therefore the fan \mathcal{C}_S is complete if and only if it is obtained from \mathcal{C} by projection to $(\mathcal{N}/\mathcal{N}_{\mathbb{S}})_{\mathbb{Q}}$, and thus \mathcal{C}_S coincides with \mathcal{P} .

(4) A toric variety is quas affine if and only if its fan is a collection of faces of a strongly convex polyhedral cone. In our case, this condition implies that the projection K of the support of the fan \mathcal{C} to $(\mathcal{N}/\mathcal{N}_S)_{\mathbb{Q}}$ is a strongly convex cone. The latter is equivalent to existence of a linear function on the space $(\mathcal{N}/\mathcal{N}_S)_{\mathbb{Q}}$ that is positive on $K \setminus \{0\}$. This gives the desired element of the lattice \mathcal{M}_S .

Conversely, assume that the lattice \mathcal{M}_S contains a vector v with positive coordinates. We have to show that the projection of each cone of the fan \mathcal{C} is a face of K . Fix proper subsets $J_1 \subset I_1, \dots, J_m \subset I_m$ of the sets of standard basis vectors of the lattice \mathcal{N} . We claim that there is an element of the lattice \mathcal{M}_S , which vanishes on the vectors of $J_1 \cup \dots \cup J_m$ and is positive on other standard basis

vectors. Indeed, the sublattice \mathcal{M}_S is defined in terms of the sums of coordinates of a character over all m groups of its coordinates. The desired vector should have the same sums of coordinates over the groups as the vector v .

(5) Since regular functions on X form a rational \mathbb{T} -module, one may consider only \mathbb{T} -semiinvariant regular functions. Further, regular \mathbb{T} -semiinvariants on X correspond to characters from \mathcal{M}_S that are nonnegative on the rays of the fan \mathcal{C} , see [7, Section 3.3]. ■

Remark 4.2. Let X be a homogeneous toric variety. Then X is projective if and only if X contains a \mathbb{T} -fixed point. Indeed, the latter condition means that the fan \mathcal{C}_S contains a cone of full dimension, thus $\mathcal{N}_S = \mathcal{N}_{\mathbb{S}}$ and $S = \mathbb{S}$.

Example 4.3. Let $m = 2$ and $n_1 = n_2 = 2$. Then $\mathcal{X} = (\mathbb{K}^2 \setminus \{0\}) \times (\mathbb{K}^2 \setminus \{0\})$. Set $S = \{(s, s, s, s) : s \in \mathbb{K}^\times\}$. Then

$$\mathcal{M}_S = \{(x_1, x_2, x_3, x_4) ; x_i \in \mathbb{Z}, x_1 + x_2 + x_3 + x_4 = 0\},$$

and the variety X is $\mathbb{P}^3 \setminus (D_1 \cup D_2)$, where $D_i \cong \mathbb{P}^1$. If we set

$$S = \{(s, s, s^{-1}, s^{-1}) : s \in \mathbb{K}^\times\},$$

then

$$\mathcal{M}_S = \{(x_1, x_2, x_3, x_4) ; x_i \in \mathbb{Z}, x_1 + x_2 = x_3 + x_4\},$$

and X is a three-dimensional quadratic cone with the apex removed.

Let us characterize the fans of homogeneous toric varieties. Let N be a lattice, Δ be a fan in $N_{\mathbb{Q}}$ and P be the set of primitive vectors on the rays of Δ . Denote by N_0 a sublattice of N generated by P . Fix a positive integer m .

Definition 4.4. A fan Δ is called m -partite if

- the set P spans the vector space $N_{\mathbb{Q}}$;
- the set P can be decomposed into m subsets $P = I_1 \sqcup \dots \sqcup I_m$, where each I_j contains at least two elements, and the cones of Δ are exactly the cones generated by subsets $J \subset P$ that do not contain any I_j .

Set $I_j = \{e_1^j, \dots, e_{n_j}^j\}$ and $q_j = e_1^j + \dots + e_{n_j}^j$. Let Q be a sublattice of N generated by q_1, \dots, q_m , and $Q_{\mathbb{Q}} = Q \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proposition 4.5. A fan Δ is the fan of a homogeneous toric variety if and only if

1. Δ is m -partite for some $m \geq 1$;
2. every linear relation among elements of P has the form $\lambda_1 q_1 + \dots + \lambda_m q_m = 0$ for some rational λ_i ;
3. $N \subset N_0 + Q_{\mathbb{Q}}$.

Proof. A fan is m -partite if and only if it is a projection of the fan $\mathcal{C}(n_1, \dots, n_m)$ with some $n_i \geq 2$. Condition 2 means that the kernel of the projection is of the form $(\mathcal{N}_{S^0})_{\mathbb{Q}}$, where $S \subseteq \mathbb{S}$. Finally, condition 3 means that N is generated by P and some elements

$$\frac{r_{1i}}{R_i}q_1 + \dots + \frac{r_{mi}}{R_i}q_m, \quad \text{where } r_{ji} \in \mathbb{Z}_{\geq 0}, \quad R_i \in \mathbb{Z}_{>0}, \quad r_{ji} < R_i, \quad \text{and } i = 1, \dots, l.$$

Equivalently, the corresponding toric variety is obtained as the quotient of the variety $\mathcal{X}(n_1, \dots, n_m)/S^0$ by an action of the group $\Gamma = \Gamma_1 \times \dots \times \Gamma_l$, where Γ_i is the cyclic group of R_i -th roots of unity and an element $\epsilon \in \Gamma_i$ multiplies the j -th factor of $\mathcal{X}(n_1, \dots, n_m)$ by $\epsilon^{r_{ji}}$. ■

5. Some generalizations

Let a connected reductive group G act on a toric variety X transitively. One may assume that $G = G^s \times L$, where G^s is a simply connected semisimple group, L is a central torus, and the G -action on X is locally effective. It is well known that any toric variety X is isomorphic to a direct product $X_0 \times X_1$, where X_0 is a non-degenerate toric variety and X_1 is an algebraic torus.

Let us give a construction of a transitive G -action on a toric variety X . Take a G^s -homogeneous toric variety X_0 with a locally effective and G^s -equivariant action of a quasitorus L' . Fix an inclusion $L' \subseteq L$ into an algebraic torus L as a closed subgroup. The group $G = G^s \times L$ acts on $X_0 \times L$, where G^s acts on the first factor and L acts on the second one by multiplication. Consider the G -equivariant action of L' on $X_0 \times L$ given by $(x_0, l) \mapsto (sx_0, s^{-1}l)$ for every $s \in L'$. Then

$$X(X_0, G^s, L', L) := (X_0 \times L)/L'$$

is a G -homogeneous toric variety.

Proposition 5.1. *Let X be a toric variety endowed with a transitive and locally effective action of a connected reductive group $G = G^s \times L$. Then the non-degenerate factor X_0 of X is a G^s -homogeneous toric variety. Moreover, if L' is the stabilizer of a G^s -orbit on X in the torus L , then X is G -equivariantly isomorphic to $X(X_0, G^s, L', L)$.*

Proof. Since the G^s - and L -actions on X commute, all G^s -orbits are of the same dimension. Let Y be one of these orbits. Any invertible function on Y is constant. Consider the above decomposition $X = X_0 \times X_1$. Since points on X_1 are separated by invertible functions, Y is contained in a subvariety $X_0 \times \{x_1\}$, where $x_1 \in X_1$. Let L' be the stabilizer of the subvariety Y in the torus L . Then the stabilizer H of a point $x \in Y$ is contained in the subgroup $G^s \times L'$ and the homogeneous space G/H projects onto $G/(G^s \times L') \cong L/L'$. Points on L/L' are separated by invertible functions, hence $X_0 \times \{x_1\}$ is contained in a fiber of the projection. But the fibers coincide with G^s -orbits on X . This implies $Y = X_0 \times \{x_1\}$.

Let us identify the variety X_0 with the subvariety $Y \subseteq X$. Consider the

morphism

$$\varphi: X_0 \times L \rightarrow X, \quad (x_0, l) \mapsto lx_0.$$

Two pairs (x_0, l) and (\tilde{x}_0, \tilde{l}) are in the same fiber of φ if and only if $(\tilde{x}_0, \tilde{l}) = (sx_0, s^{-1}l)$ with $s = \tilde{l}^{-1}l$. This shows that φ induces a bijective morphism $X(X_0, G^s, L', L) \rightarrow X$. Clearly, this is an isomorphism of G -homogeneous spaces. ■

If the subgroup L' is connected, then $L \cong L' \times L''$ with some complementary subtorus L'' , and $X \cong X_0 \times L''$. But unlike the case of algebraic transformation groups of maximal rank [17, Theorem 2], this situation does not always occur. Indeed, one may consider a toric variety $(\mathbb{K}^2 \setminus \{0\}) \times \mathbb{K}^\times$ with a transitive locally effective action of the group $\mathrm{SL}(2) \times \mathbb{K}^\times$ given as $(g, t) \cdot (v, a) = (g(tv), t^2a)$.

Remark 5.2. It would be interesting to generalize [17, Theorem 3] and to describe toric varieties with transitive actions of non-reductive affine algebraic groups.

Besides homogeneous toric varieties, our method allows to describe toric varieties with a generically transitive action of a semisimple group G . By Lemma 2.1, they are quasitorus quotients of open subsets of generically transitive $(G \times \mathbb{S})$ -modules. Such modules are known as $(G \times \mathbb{S})$ -prehomogeneous vector spaces. For an explicit description, one needs a list of prehomogeneous vector spaces. The classification results here are known only under some restrictions on the group and on the module. For example, if G is simple and the number of irreducible summands of the module does not exceed three, the classification is given in a series of papers of M.Sato, T.Kimura, K.Ueda, T.Yoshigiaki and others.

If the complement of an open G -orbit on a toric variety X has codimension at least two in X , then X comes from a G -prehomogeneous vector space (Proposition 2.2). When the group G is simple, the list of G -prehomogeneous vector spaces is obtained in [16, Theorems 7-8], and the corresponding toric varieties are described in [2, Proposition 4.7]. In contrast to the homogeneous case, here appear singular [2, Example 5.8] and non-quasiprojective [2, Example 5.9] varieties.

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