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Conformal Actions on Homogeneous Lorentzian Manifolds

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Abstract. A conformal group action on a pseudo-Riemannian manifold is essential if the action is not an isometric action with respect to a conformally equivalent metric. We classify all essential actions of simple Lie groups on Riemannian, Lorentzian and sub-Lorentzian manifolds.
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1. Introduction

The purpose of the following paper is to find all Lorentzian manifold, endowed with a transitive simple Lie-group of conformal transformations which acts essentially.

We have in mind an application, which we are going to publish in sequel paper, [1]. Towards that purpose we establish here a little more. In fact we classify all possible *sub-Lorentzian* manifolds endowed with a transitive simple Lie-group of conformal transformations which acts *pre-essentially.* Exact definitions of these terms will be given in the next section. The main theorem will also be formulated there. Section 3 will be devoted to introducing some notations, and stating some preliminary Lemmas. Section 4, which is the heart of this manuscript, is a case by case classification. In section 5, we state some corollaries which are application oriented, and are going to be used in [1].

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2. Definitions and statement of results

Let G be a Lie-group. Let (X,t) be a manifold with a bilinear structure (see [2]). We assume that G acts conformally on X.

Definition 2.1. We say that G acts *essentially* on (X,t) if there is no conformally equivalent bilinear structure s, such that G acts isometrically on (X,s).

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When the action of G on X is transitive, there is an infinitesimal criterion for essentiality. Pick a point $x \in X$. Consider the action of $\mathrm{Stab}(x)$ on the vector space T_xX. It is conformal with respect to t_x. If it is orthogonal, then the G action on (X,t) is not essential (see [2, Lemma 2.6]). We wish to weaken the definition of essentiality. Let $\mathfrak{g}$ be the Lie algebra of G. Consider the form v, induced on $\mathfrak{g}$ by t_x, via the map $\mathfrak{g} \to \mathfrak{g}/\mathrm{Lie}(\mathrm{Stab}(x)) \simeq T_xX$. Denote by $\mathrm{CO}_G(v)$ the conformal group of v in G (that is, the pre-image of $\mathrm{CO}(v) < \mathrm{GL}(\mathfrak{g})$ under the adjoint map of G).

Definition 2.2. We call (X,t) a *G-pre-essential space* if X is a G-homogeneous space and $\mathrm{CO}_G(v)$ does not act orthogonally on $(\mathfrak{g}, v)$.

If G acts essentially on the homogeneous space (X,t), then (X,t) must be a pre-essential G-space (because $\mathrm{Stab}(x) < \mathrm{CO}_G(v)$). It will turn out to be that when G is an almost-simple Lie-group, a partial converse is also true. That is, every Riemannian or Lorentzian space, which is pre-essential with respect to a transitive, conformal action of some almost simple Lie-group, is in fact essential. This will follow from a classification of such spaces. In fact we classify a slightly wider class of spaces.

Definition 2.3. Let (V,B) a vector space endowed with a symmetric bilinear form. (V,B) is called *sub-Lorentzian* if one of the following holds:

Lorentzian case: B is of signature $(\dim(V)-1,1)$.

Riemannian case: B is positive definite.

Degenerate case: B is semi-positive definite, and has one dimensional radical.

Definition 2.4. (X,t) is called a *sub-Lorentzian pre-essential G-space* if it is G-pre-essential, and for some $x \in X$, (T_xX, t_x) is sub-Lorentzian.

It is well known that the standard sphere S^n is a Riemannian $\mathrm{SO}(n+1,1)$-essential (and pre-essential) space. Recall the definition of $C^{n,1}$ from [2]. This is an example of a Lorentzian $\mathrm{SO}(n+1,2)$-essential (and pre-essential) space. If n is odd then it is also $\mathrm{SU}(\frac{n+1}{2},1)$ essential space, via the natural map $\mathrm{SU}(\frac{n+1}{2},1) \to \mathrm{SO}(n+1,2)$.

Theorem 2.5. *Let G be a connected almost simple Lie-group with a finite center. Assume G is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Let X be a sub-Lorentzian G-pre-essential space. Then we have*

- *If X is Riemannian then G is locally isomorphic to $\mathrm{SO}(n,1)^o$ for some $n>2$, and $X \simeq S^{n-1}$. G acts on X by Möbius transformations.*

- *If X is Lorentzian then either G is locally isomorphic to $\mathrm{SO}(n,2)^o$ and X is a finite cover of the standard model space $C^{n-1,1}$ or G is locally isomorphic to $\mathrm{SU}(\frac{1}{2}n,1)$ and X is commensurable to $C^{n-1,1}$ - that is, G and $\mathrm{SU}(n,1)$*

have a common finite cover group which acts on a common finite cover of X and $C^{n-1,1}$, covering the actions of G and of $\mathrm{SU}(n,1)$.

- *If X is degenerate then G is locally isomorphic to* $\mathrm{SO}(n,1)^o$, *and X is a fiber bundle over S^{n-1} with one dimensional fibers.*

3. Preliminaries and notations

In the following we are using some basic algebraic group theory, in particular, the relations between algebraic groups and Lie group theory. A standard reference for algebraic group theory is [3]. Some preliminaries concerning real algebraic groups and the relation with Lie theory could be found in [5, Chapter 3]. Abusing the notation (in a way that already became standard), we refer to the Lie-group H as an "algebraic group" not only if H is isomorphic as a Lie-group to a group $\mathbf{G}(\mathbb{R})$, the group of real points of a honest algebraic group $\mathbf{G}$ which is defined over $\mathbb{R}$, but also if H^o, the connected component of the identity, is of finite index in H and it admits a Lie-group homomorphism with a finite kernel onto $\mathbf{G}(\mathbb{R})^o$, the (topological) connected component of the identity in $\mathbf{G}(\mathbb{R})$. A Lie group morphism between the algebraic groups H, H' is called algebraic if there exists a honest algebraic group morphism, defined over $\mathbb{R}$, between the corresponding algebraic groups $\mathbf{G}, \mathbf{G}'$ such that the obvious commutative diagram is satisfied.

Recall that a semisimple Lie-group with a finite center is algebraic, and any Lie-group morphism between semisimple Lie-groups with finite centers is algebraic. A parabolic subgroup of a semisimple group with a finite center is algebraic. The intersection of two algebraic subgroups of an algebraic group is algebraic. We will constantly use the notion of $\mathbb{R}$-split rank, denoted $\mathrm{rk}_{\mathbb{R}}$, which is defined for every algebraic group. Recall that, given an injective algebraic morphism, $G \to H$, we must have $\mathrm{rk}_{\mathbb{R}}(G) \leq \mathrm{rk}_{\mathbb{R}}(H)$.

A sub-Lie-algebra of the Lie-algebra of an algebraic group is called algebraic, if it is the Lie algebra of an algebraic subgroup. Similarly, a sub-Lie-algebra is called compact/unipotent/torus, if it is the Lie algebra of a compact/unipotent/torus subgroup. Given an algebraic sub-algebra $\mathfrak{h}$, which is the Lie-algebra of an algebraic subgroup H, we will often abuse the notation using the term $\mathrm{rk}_{\mathbb{R}}(\mathfrak{h})$ instead of $\mathrm{rk}_{\mathbb{R}}(H)$.

Fix an almost simple Lie-group G. Let (X,t) be a G-pre-essential manifold. Assume the form t does not vanish identically. Fixing once and for all a point $x \in X$, we will denote in the sequel

$$H = \mathrm{Stab}_G(x), \ \mathfrak{h} = \mathrm{Lie}(H).$$

t_x is considered as a symmetric bilinear form defined on $\mathfrak{g}/\mathfrak{h}$. Pulling this form to $\mathfrak{g}$ by the map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ we obtain a bilinear form v on $\mathfrak{g}$, that is an element of $\mathrm{Sym}^2(\mathfrak{g})$. By our non-vanishing assumption on t, this element is non-zero, hence projects to the G-space $\mathbb{P}^+(\mathrm{Sym}^2(\mathfrak{g}))$, the space of symmetric bilinear forms on $\mathfrak{g}$ modulo multiplication by positive scalars. We denote the resulting element by $\bar{v} \in \mathbb{P}^+(\mathrm{Sym}^2(\mathfrak{g}))$. We denote

$$S = \mathrm{Stab}_G(\bar{v}), \ \mathfrak{s} = \mathrm{Lie}(S), \ \mathfrak{r} = \mathrm{rad}(\bar{v}).$$

S is an algebraic group. Observe that $H < S$, and that $\mathfrak{h} < \mathfrak{s} \cap \mathfrak{r}$. Consider the line $V = \mathbb{R}v < \mathrm{Sym}^2(\mathfrak{g})$ which sits above $\bar{v}$. There is a natural algebraic morphism $S \to \mathrm{GL}(V)$. By the definition of a pre-essential space, this map is not trivial. In particular we see that S has a real character, hence its real rank is non-zero. Let $\mathfrak{a}'$ be a maximal $\mathbb{R}$-split torus of $\mathfrak{s}$. Let 2α be the corresponding weight. Keep in mind that $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) \geq \mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) \geq 1$.

As an $\mathfrak{a}'$-module, $\mathfrak{g}$ splits into a direct sum $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{d}$, where $\mathfrak{d} \simeq \mathfrak{g}/\mathfrak{r}$. The following is an immediate corollary of [2, Lemma 3.3].

Lemma 3.1. *$\mathfrak{d}$ is a faithful $\mathfrak{a}'$-module.*

$\mathfrak{d}$ itself splits into weight-spaces $\mathfrak{d}_\lambda$, and in particular we have $\mathfrak{d}_\alpha < \mathfrak{d}$ ($\mathfrak{d}_\alpha$ might be 0). Observe that $\mathfrak{d}$ carries a natural bilinear form - induced from the form v on $\mathfrak{g}$, as $\mathfrak{r}$ is radical - and that $\mathfrak{a}'$ imbeds in $\mathfrak{co}(\mathfrak{d})$. By choosing a maximal $\mathbb{R}$-split torus $\tilde{\mathfrak{a}}' < \mathfrak{co}(\mathfrak{d})$ containing (the image of) $\mathfrak{a}'$ we obtain finer weight spaces, to be denoted $\mathfrak{d}_{\tilde{\lambda}}$. We will indeed use the suggestive notation, denoting the restriction of $\tilde{\lambda}$ to $\mathfrak{a}'$ by λ, thus $\mathfrak{d}_{\tilde{\lambda}} < \mathfrak{d}_\lambda$ (note that given a weight λ of $\mathfrak{a}'$ there is no "canonical" lift $\tilde{\lambda}$ - it might be that for non-equal weights of $\tilde{\mathfrak{a}}'$, $\tilde{\lambda}, \tilde{\mu}$ we have $\lambda = \mu$).

Lemma 3.2. *Consider the bilinear form induced by v on $\mathfrak{d} \simeq \mathfrak{g}/\mathfrak{r}$. If the signature is denoted by (p, q) then*

$$\mathrm{rk}_{\mathbb{R}}(S) \leq \min\{p, q\} + 1.$$

Proof. By Lemma 3.1, $\mathfrak{g}/\mathfrak{r}$ is a faithful $\mathfrak{a}'$-module. We get

$$\mathrm{rk}_{\mathbb{R}}(S) = \dim(\mathfrak{a}') \leq \dim(\tilde{\mathfrak{a}}') = \mathrm{rk}_{\mathbb{R}}(\mathrm{CO}(\mathfrak{g}/\mathfrak{r})) = \min\{p, q\} + 1. \qquad \blacksquare$$

$\mathfrak{a}'$ is contained in a maximal torus of $\mathfrak{g}$ which we will denote by $\mathfrak{a}$ (do not confuse $\mathfrak{a}$ with $\tilde{\mathfrak{a}}'$ - both are maximal tori containing $\mathfrak{a}'$, but in different ambient algebras). Denote the root system of $\mathfrak{g}$ associated to $\mathfrak{a}$ by Σ. Let $A \in \mathfrak{a}' < \mathfrak{a}$ be an element such that $\alpha(A) > 0$. We define the parabolic associated to A, $\mathfrak{p}_A$, written as a sum of root-spaces in the following form

$$\mathfrak{p}_A = \mathfrak{g}_0 \oplus \bigoplus_{\xi \in \Sigma,\ \xi(A) \leq 0} \mathfrak{g}_\xi = \mathfrak{m}_A \oplus \mathfrak{a}_A \oplus \mathfrak{n}_A^-$$

(the right hand side is the Langlands decomposition of $\mathfrak{p}_A$). Denote its direct complement by $\mathfrak{n}_A^+$ or just $\mathfrak{n}_A$. Observe that $\mathfrak{d}_\alpha < \mathfrak{g}_\alpha < \mathfrak{n}_A$ (notice that $\mathfrak{g}_\alpha$ is not a root-space, but a weight-space for $\mathfrak{a}'$).

The following are useful Lemmas.

Lemma 3.3. *Assume $\mathfrak{u} < \mathfrak{s}$ is an algebraic subalgebra which has no split semisimple element (i.e, it is the Lie algebra associated to an algebraic subgroup of S which has no split semisimple element). Then $\mathfrak{u}$ acts orthogonally on $\mathfrak{g}/\mathfrak{r}$.*

Proof. This follows immediately from the fact that, denoting by U an algebraic group with $\mathrm{Lie}(U) = \mathfrak{u}$, the maps $U \to \mathrm{CO}(\mathfrak{g}/\mathfrak{r}) \simeq \mathrm{O}(\mathfrak{g}/\mathfrak{r}) \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ are algebraic, hence the image of U in $\mathrm{CO}(\mathfrak{g}/\mathfrak{r})$ must be contained in the kernel of $\mathrm{CO}(\mathfrak{g}/\mathfrak{r}) \to \mathbb{R}_+^*$, namely $\mathrm{O}(\mathfrak{g}/\mathfrak{r})$. ∎

Lemma 3.4. *Let (X,t) be a sub-Lorentzian G-homogeneous manifold (not necessarily pre-essential). Then $\mathfrak{r}$ is a sub-algebra of $\mathfrak{g}$.*

Proof. If $\mathfrak{r} = \mathfrak{h}$ then this is trivial. Assume not. Then $\mathfrak{h}$ is a codimension one sub-algebra of $\mathfrak{r}$ which normalizes it (because $\mathfrak{r} = \mathrm{rad}(v)$, and v is $\mathfrak{h}$-invariant). Fix an element w of $\mathfrak{r} - \mathfrak{h}$. For $i = 1,2$, let $\alpha_i w + y_i$ be two elements of $\mathfrak{r}$, where $\alpha_i \in \mathbb{R}$ and $y_i \in \mathfrak{h}$. Then

$$[\alpha_1 w + y_1, \alpha_2 w + y_2] = [w, \alpha_1 y_2 - \alpha_2 y_1] + [y_1, y_2]$$

The two terms in the right hand side are in $\mathfrak{r}$, because $\mathfrak{h}$ normalizes $\mathfrak{r}$. The Lemma follows. ∎

4. The classification

In this section G is assumed to be an almost simple Lie-group with a finite center, not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. (X,t) is assumed to be a sub-Lorentzian pre-essential G-manifold. The division into subsections will correspond to the various cases of the form t.

In the classification that follows we will use in an essential way the fact that a simple real Lie-algebra is determined uniquely by the type of its real root system, together with the multiplicities of the long and the short roots (see [4, p. 535, Ex. 9]). Whenever this datum is given, one can use table VI in [4] in order to actually determine the Lie-algebra. We will do it over and over with out any further comment.

4.1. Riemannian case.

Here $\mathfrak{r} = \mathfrak{h}$, and the induced form on $\mathfrak{d} \simeq \mathfrak{g}/\mathfrak{r}$ is positive definite. $\mathfrak{a}'$ is the Lie-algebra of the group of homotheties of $\mathfrak{g}/\mathfrak{r}$, hence $\mathfrak{d} = \mathfrak{d}_\alpha$, and hence $\mathfrak{d} < \mathfrak{n}_A$. It follows that $\mathfrak{r} > \mathfrak{p}_A$, hence $\mathfrak{p}_A < \mathfrak{s}$. By Lemma 3.2, $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) \leq 1$, hence $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = \mathrm{rk}_{\mathbb{R}}(\mathfrak{p}_A) \leq 1$, and we get $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = 1$. It follows that $\mathfrak{r} = \mathfrak{p}_A$ and $\mathfrak{d} = \mathfrak{n}_A$.

By the assumption that the form is Riemannian, $\mathfrak{d}$ consists of a single $\tilde{\mathfrak{a}}'$ weight space, hence a single $\mathfrak{a}'$ weight space. As $\mathfrak{d} = \mathfrak{n}_A = \mathfrak{n}$ and $\mathfrak{a} = \mathfrak{a}'$ (because the rank is 1), $\mathfrak{n}$ consists of a single root space, and we deduce that the root system is reduced. From the classification, $\mathfrak{g} = \mathfrak{so}(n,1)$. As $\mathfrak{so}(n-1) \simeq \mathfrak{m} < \mathfrak{p}$ acts irreducibly on $\mathbb{R}^{n-1} \simeq \mathfrak{n}$, the Riemannian form on $\mathfrak{n}$ is uniquely determined (up to a scalar multiple). It follows that G is locally isomorphic to $\mathrm{SO}(n,1)^o$ (for some $n > 2$, as G is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$). We deduce that X is a finite cover of S^{n-1}. As the latter is simply connected, X coincides with S^{n-1}.

4.2. Lorentzian case. Here $\mathfrak{r} = \mathfrak{h}$, and the induced form on $\mathfrak{d} \simeq \mathfrak{g}/\mathfrak{r}$ is Lorentzian. In this case, where $\mathfrak{co}(\mathfrak{d}) \simeq \mathfrak{so}(p,1) \oplus \mathbb{R}$ and $\mathfrak{d} \simeq \mathbb{R}^{p,1}$, the weight-space decomposition of the maximal torus $\tilde{\mathfrak{a}}' < \mathfrak{co}(\mathfrak{d})$ is well known: it has three weights $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$, where $\tilde{\beta} + \tilde{\gamma} = 2\tilde{\alpha}$ and $\dim(\mathfrak{d}_{\tilde{\beta}}) = \dim(\mathfrak{d}_{\tilde{\gamma}}) = 1$. $2\tilde{\alpha}$ is the conformal weight, thus its restriction to $\mathfrak{a}'$ is 2α, as the notation suggests. Thus, as an ($\tilde{\mathfrak{a}}'$-module, hence also as an) $\mathfrak{a}'$-module, $\mathfrak{d}$ splits into a direct sum $\mathfrak{d} = \mathfrak{d}_{\tilde{\alpha}} \oplus \mathfrak{d}_{\tilde{\beta}} \oplus \mathfrak{d}_{\tilde{\gamma}}$.

Notice that β and γ might equal α (for these are restrictions to $\mathfrak{a}'$), and that $\mathfrak{d}_\alpha$ might vanish (if $p = 1$), but anyway $\alpha \neq 0$ (by our assumption of essentiality). We will always assume in the sequel that $\beta(A) \leq \gamma(A)$ (upon interchanging the role of $\tilde{\beta}$ and $\tilde{\gamma}$ if needed).

We split the discussion into three cases (recall that, by Lemma 3.2 $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) \leq 2$):

1) $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) = 2$**:** In that case, $\mathfrak{a}' = \tilde{\mathfrak{a}}'$. By picking the element $A \in \mathfrak{a}'$ in the kernel of $\tilde{\gamma} - \tilde{\alpha}$ (but satisfying $\tilde{\alpha}(A) > 0$) we can assume that $\alpha(A) = \beta(A) = \gamma(A)$. As before we get $\mathfrak{d} < \mathfrak{n}_A$, hence $\mathfrak{r} > \mathfrak{p}_A$. We then have $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = \mathrm{rk}_{\mathbb{R}}(\mathfrak{p}_A) \leq \mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) \leq \mathrm{rk}_{\mathbb{R}}(\mathfrak{g})$, hence $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = \mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) = 2$.

The question of finding $\mathfrak{p}$ conformally invariant Lorentzian form on $\mathfrak{g}/\mathfrak{p}$ for rank two groups was solved in [2], and the only solution is $\mathfrak{g} = \mathfrak{so}(n, 2)$. It follows that G is locally isomorphic to $\mathrm{SO}(n, 2)^o$, and X is a finite cover of $C^{n-1,1}$.

2) $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) = 1$**,** $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = 1$**:** In that case, $\mathfrak{a}' = \mathfrak{a}$. If $\alpha = \beta = \gamma$ we get, as in the discussion in the Riemannian case, that $\mathfrak{g} = \mathfrak{so}(n, 1)$ and $\mathfrak{r} = \mathfrak{p}$, which lead to a contradiction (because the unique $\mathfrak{p}$ conformally invariant form on $\mathfrak{g}/\mathfrak{p}$ is Riemannian).

We get that $\mathfrak{a}'$ has more than one weight in $\mathfrak{d} < \mathfrak{g}$. $\mathfrak{a}'$ being a maximal torus in the rank 1 algebra $\mathfrak{g}$, we deduce that the root system of $\mathfrak{g}$ is not reduced: it has two positive roots $\xi, 2\xi$, given a choice of positivity. Choosing the positivity so that these positive roots are positive on A, and using $\beta(A) \leq \gamma(A)$ and $\beta + \gamma = 2\alpha$, we obtain the only possibility: $\beta = 0$, $\gamma = 2\xi$ and $\alpha = \xi$. It follows, in particular, that $\mathfrak{r} \oplus \mathfrak{d}_0$ contains the parabolic $\mathfrak{p}_A$.

We next show that $\mathfrak{a} < \mathfrak{r}$. Otherwise, there exists some $X_{2\xi} \in \mathfrak{d}_\gamma < \mathfrak{g}_{2\xi}$ with $\langle A, X_{2\xi} \rangle \neq 0$ (A projects to $\mathfrak{d}_0 = \mathfrak{d}_\beta$ modulo $\mathfrak{r}$, which is orthogonal to $\mathfrak{d}_\beta \oplus \mathfrak{d}_\alpha$). By Jacobson Morozov Theorem (and the rank 1 assumption on $\mathfrak{g}$), there exists $X_{-2\xi} \in \mathfrak{g}_{-2\xi}$ satisfying $[X_{-2\xi}, X_{2\xi}] = A$. Now, $X_{-2\xi} \in \mathfrak{g}_{-2\xi} < \mathfrak{p}_A < \mathfrak{r} \oplus \mathfrak{d}_0 < \mathfrak{r} \oplus \mathfrak{g}_0$ projects trivially to $\mathfrak{g}_0$. Hence $X_{-2\xi} \in \mathfrak{r} = \mathfrak{h} < \mathfrak{s}$. Being unipotent, by Lemma 3.3, it acts orthogonally on $\mathfrak{d}$. We get

$$\langle [X_{-2\xi}, X_{2\xi}], X_{2\xi} \rangle = -\langle X_{2\xi}, [X_{-2\xi}, X_{2\xi}] \rangle,$$

and conclude $\langle A, X_{2\xi} \rangle = 0$ - a contradiction.

Writing $\mathfrak{p}_A = \mathfrak{a} \oplus \mathfrak{m}_A \oplus \mathfrak{n}_A^-$, we conclude that $\mathfrak{m}_A$ projects modulo $\mathfrak{r}$ onto $\mathfrak{d}_0$ (as $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}_A$ does, and $\mathfrak{a}$ is in the kernel). Thus, $\mathfrak{r} \cap \mathfrak{m}_A$ is of codimension one subalgebra (Lemma 3.4) in $\mathfrak{m}_A$. From the classification, $\mathfrak{g}$, being a rank one simple algebra with a non-reduced root system, is isomorphic to either $\mathfrak{su}(n, 1), \mathfrak{sp}(n, 1)$ or $\mathfrak{f}_{4,-20}$. From all of the above cases, the only one in which the algebra $\mathfrak{m}$ in the Langlands decomposition of a parabolic subalgebra has a codimension one subalgebra is $\mathfrak{su}(n, 1)$ for which $\mathfrak{m} = \mathfrak{su}(n-1) \oplus \mathfrak{u}(1)$. Since $\mathfrak{su}(n-1)$ has no codimension one subalgebra as well, we conclude that it (or rather, the corresponding isomorphic subalgebra in $\mathfrak{m}_A$ which from now on we identify with $\mathfrak{su}(n-1)$) is contained in $\mathfrak{r}$.

Thus, $\mathfrak{g} \simeq \mathfrak{su}(n,1)$, $\mathfrak{m}_A \simeq \mathfrak{su}(n-1) \oplus \mathfrak{u}(1)$ and we have the following inclusions of subalgebras:

$$\mathfrak{n}_A^- \oplus \mathfrak{a} \oplus \mathfrak{su}(n-1) < \mathfrak{r} \lneq \mathfrak{g} \simeq \mathfrak{su}(n,1).$$

We claim that $\mathfrak{p}_A$ is the only subalgebra $\mathfrak{q} < \mathfrak{g}$ which satisfies

$$\mathfrak{n}_A^- \oplus \mathfrak{a} \oplus \mathfrak{su}(n-1) \lneq \mathfrak{q} \lneq \mathfrak{g} \simeq \mathfrak{su}(n,1).$$

Indeed, it is known that a proper maximal dimensional subalgebra is parabolic, and $\mathfrak{n}_A^- \oplus \mathfrak{a} \oplus \mathfrak{su}(n-1)$ is of codimension one in $\mathfrak{p}_A$, so every such subalgebra $\mathfrak{q}$ must be parabolic. A parabolic $\mathfrak{q}$ that contains $\mathfrak{a}$ contains also its centralizer, $\mathfrak{m}$, thus $\mathfrak{q} > \mathfrak{n}_A^- \oplus \mathfrak{a} \oplus \mathfrak{m} = \mathfrak{p}_A$, and by the rank one assumption, $\mathfrak{q} = \mathfrak{p}_A$. From the fact that $\mathfrak{r} \neq \mathfrak{p}_A$, we finally obtain that $\mathfrak{r} = \mathfrak{n}_A^- \oplus \mathfrak{a} \oplus \mathfrak{su}(n-1)$.

As the $\mathfrak{a}'$-module $\mathfrak{d}$ was chosen arbitrarily as a direct complement of $\mathfrak{r}$, we may (and will) change it to be

$$\mathfrak{d} = \mathfrak{u}(1) \oplus \mathfrak{n}_A^+ = \mathfrak{u}(1) \oplus \mathfrak{g}_\xi \oplus \mathfrak{g}_{2\xi}$$

We proceed to show that there is a unique $\mathfrak{r}$ conformally invariant Lorentzian form on $\mathfrak{d}$. Indeed, by Lemma 3.3, the action of $\mathfrak{su}(n-1)$ on $\mathfrak{d}$ is orthogonal, hence preserves the Riemannian form on $\mathfrak{g}_\xi$. As the action of $\mathfrak{su}(n-1)$ on $\mathfrak{g}_\xi$ is a conjugate of the standard $\mathfrak{su}(n-1)$ action on $\mathbb{R}^{2n-2}$, the form on $\mathfrak{g}_\xi$ must be a conjugate of the standard inner product on $\mathbb{R}^{2n-2}$, up to homothety, as every form on $\mathbb{R}^{2n-2}$ which is invariant with respect to the standard $\mathfrak{su}(n-1)$ action is a scalar multiple of the standard inner product (this follows from the irreducibility of the action). The $(1,1)$-form on $\mathfrak{u}(1) \oplus \mathfrak{g}_{2\xi}$ is also determined, up to homothety, by the fact that the lines $\mathfrak{u}(1)$ and $\mathfrak{g}_{2\xi}$ are isotropic. We are left to show that the ratio between the two homothety constants is determined as well. This follows from the fact that $[\mathfrak{g}_{-\xi}, \mathfrak{g}_{2\xi}] = \mathfrak{g}_\xi$. Indeed, fixing elements $X_{-\xi} \in \mathfrak{g}_{-\xi}$ and $X_{2\xi} \in \mathfrak{g}_{2\xi}$ such that $0 \neq X_\xi = [X_{-\xi}, X_{2\xi}] \in \mathfrak{g}_\xi$, we know by Lemma 3.3 that $X_{-\xi}$ acts orthogonally, hence

$$0 \neq \langle X_\xi, X_\xi \rangle = \langle [X_{-\xi}, X_{2\xi}], X_\xi \rangle = -\langle X_{2\xi}, [X_{-\xi}, X_\xi] \rangle.$$

Letting $X_0 \in \mathfrak{u}(1)$ be the projection modulo $\mathfrak{r}$ of $[X_{-\xi}, X_\xi]$, we get that the number $\langle X_0, X_{2\xi} \rangle$ (which completely determines the form on $\mathfrak{u}(1) \oplus \mathfrak{g}_{2\xi}$) is determined by the form on $\mathfrak{g}_\xi$.

Now, there exists a familiar Lorentzian action of $\mathrm{SU}(n,1)$, obtained by imbedding $\mathrm{SU}(n,1)$ in $\mathrm{SO}(2n,2)$ and letting it act on the model space $C^{2n-1,1}$ (which can easily checked to be pre-essential). We have seen above that, assuming $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = \mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) = 1$, G has to be locally isomorphic to $\mathrm{SU}(n,1)$, and X is completely determined (locally). It follows that the action of G on X is commensurable to this standard action of $\mathrm{SU}(n,1)$: G has a finite cover which is also a finite cover of $\mathrm{SU}(n,1)$ and X has a finite cover which is also a finite cover of $C^{2n-1,1}$.

3) $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) = 1$**,** $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) \geq 2$**:** We will show that there are no more examples of pre-essential spaces.

In this case $\mathfrak{a}' \lneq \mathfrak{a}$. As $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{s}$, we get that $\mathfrak{a}$ is not contained in $\mathfrak{s}$, and in particular $\mathfrak{a} \not< \mathfrak{r} = \mathfrak{h} < \mathfrak{s}$. We conclude that $\mathfrak{a}$ projects non-trivially to $\mathfrak{d}_0 < \mathfrak{d}$. In particular $\mathfrak{d}_0 \neq 0$. We know that $\mathfrak{d} = \mathfrak{d}_\beta \oplus \mathfrak{d}_\alpha \oplus \mathfrak{d}_\gamma$, and that α, γ are positive weights of $\mathfrak{a}'$ (by our assumptions $2\alpha(A) > 0$ and $\beta + \gamma = 2\alpha$, $\gamma(A) \geq \beta(A)$). Also we know that $\mathfrak{d}_\beta$ is one dimensional. We conclude that $\beta = 0$ (hence $\gamma = 2\alpha$) and that $\mathfrak{d}_0$ is one dimensional. Since the kernel of the projection of $\mathfrak{a}$ to $\mathfrak{d}_0$ is at most one-dimensional (being contained in $\mathfrak{a}'$, by the rank one assumption on $\mathfrak{s}$), we conclude that $\mathfrak{a}' < \mathfrak{r}$, $\dim(\mathfrak{a}) = 2$, and in particular $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = 2$.

Next we show that the parabolic subalgebra $\mathfrak{p}_A$ is a minimal parabolic. This is well known to be the case if and only if the subalgebra $\mathfrak{m}_A$ contains no copy of $\mathfrak{sl}(2,\mathbb{R})$. Assume, by negation, the existence of a subalgebra $\mathfrak{c} < \mathfrak{m}_A$, $\mathfrak{c} \simeq \mathfrak{sl}(2,\mathbb{R})$). Observe that by its very definition, $\mathfrak{p}_A < \mathfrak{r} + \mathfrak{a} = \mathfrak{r} \oplus \mathfrak{d}_0$, hence the codimension of $\mathfrak{p}_A \cap \mathfrak{r}$ in $\mathfrak{p}_A$ is at most one, and we get that the codimension of $\mathfrak{c} \cap \mathfrak{r}$ in $\mathfrak{c}$ is at most one as well. It follows that $\mathfrak{c} \cap \mathfrak{r}$ contains an $\mathbb{R}$-split semisimple element. Observe that $\mathfrak{a}' < \mathfrak{a}_A$ intersects $\mathfrak{m}_A$ trivially. We conclude that the algebra $\mathfrak{a}' \oplus (\mathfrak{m}_A \cap \mathfrak{r}) < \mathfrak{r} < \mathfrak{s}$ contains a two dimensional $\mathbb{R}$-split torus, contradicting our assumption $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) = 1$.

Thus, $A < \mathfrak{a}$ is a regular element, as $\mathfrak{p}_A$ is a minimal parabolic. Therefore, every A-submodule of $\mathfrak{g}$ is automatically an $\mathfrak{a}$ submodule. In particular, we get that $\mathfrak{a}$ normalizes $\mathfrak{r}$. We conclude that $\mathfrak{r} + \mathfrak{a} = \mathfrak{r} \oplus \mathfrak{d}_0$ is a subalgebra, to be denoted $\mathfrak{p}_0$. From $\mathfrak{p}_A < \mathfrak{r} + \mathfrak{a}$ we get that $\mathfrak{p}_0$ is parabolic, and in particular, algebraic.

We remark that $\mathfrak{r} = \mathfrak{s} \cap \mathfrak{p}_0$. This is because $\mathfrak{r}$ is of codimension one in $\mathfrak{p}_0$, hence maximal among proper subspaces, and is contained in (the proper subspace by rank consideration) $\mathfrak{s} \cap \mathfrak{p}_0$. In particular $\mathfrak{r}$ is algebraic.

We proceed to show that $\mathfrak{p}_A = \mathfrak{p}_0$. As $\mathfrak{p}_A < \mathfrak{p}_0$ it is enough to show that $\mathfrak{p}_0$ is a minimal parabolic. As before, we will see that the semisimple part of its Langlands decomposition does not contain a copy of $\mathfrak{sl}(2,\mathbb{R})$. Let $\mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0^-$ be the Langlands decomposition of $\mathfrak{p}_0$. Denote by $\mathfrak{m}_0'$ the semisimple component of $\mathfrak{m}_0$ (indeed, its commutator subalgebra). We need to show that $\mathfrak{m}_0'$ contains no copy of $\mathfrak{sl}(2,\mathbb{R})$. Denote the complementary subalgebra of $\mathfrak{p}_0$ by $\mathfrak{n}_0 = \mathfrak{n}_0^+$. As an $\mathfrak{a}'$ module, $\mathfrak{n}_0 \simeq \mathfrak{d}_\alpha \oplus \mathfrak{d}_{2\alpha}$. Observe that all the eigenvalues of A on $\mathfrak{n}_0$ are positive - these are $\alpha(A)$ and $2\alpha(A)$. $\mathfrak{m}_0'$ is semisimple, hence its action on $\mathfrak{n}_0$ is via $\mathfrak{sl}(\mathfrak{n}_0)$, hence $\mathfrak{a}' \cap \mathfrak{m}_0' = \emptyset$ (A spans $\mathfrak{a}'$). It follows that there is no subalgebra $\mathfrak{c} < \mathfrak{m}_0'$, $\mathfrak{c} \simeq \mathfrak{sl}(2,\mathbb{R})$). Indeed, as before, for any such $\mathfrak{c}$, the codimension of $\mathfrak{c} \cap \mathfrak{r}$ is at most one, hence $\mathfrak{c} \cap \mathfrak{r}$ contains an $\mathbb{R}$-split semisimple element, which forms, together with $\mathfrak{a}'$ a two dimensional $\mathbb{R}$-split torus, contradicting our assumption $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) = 1$.

By now we have shown that $\mathfrak{p}_A = \mathfrak{p}_0$ is a minimal parabolic, and its complement subalgebra $\mathfrak{n}_A$ has the two $\mathfrak{a}'$ weights, α and 2α, where the 2α weight space is one dimensional. We have enough information in order to determine the root system. Denote the simple roots of Σ by $\Delta = \{\sigma, \tau\}$. $\mathfrak{g}$ is simple, hence σ and τ are connected in the Dynkin diagram, and hence $\sigma+\tau$ is a (positive) root too (recall that if the angle between two roots is acute then their sum is also a root). We know that $\sigma(A), \tau(A)$ and $(\sigma+\tau)(A)$ are all taking the values $\alpha(A)$ or $2\alpha(A)$. It follows that $\sigma(A)=\tau(A)=\alpha(A)$.

We claim that that the set of positive roots, Σ_+, is exactly $\{\sigma,\tau,\sigma+\tau\}$. Assume, for negation, there exists a root $\xi\in\Sigma_+-\{\sigma,\tau,\sigma+\tau\}$. Recall that ξ is a linear combination of σ and τ with integer coefficients. The sum of these coefficients must be 2 (since $\xi(A)=2\alpha(A)$), hence we must have $\xi=2\sigma$ (with out loss of generality). But then $\xi+\tau$ is also a root and $(\xi+\tau)(A)=3\alpha(A)$, a contradiction.

It follows that Σ is of type A_3. As $\mathfrak{g}_{\sigma+\tau}$ is one dimensional (having the $\mathfrak{a}'$ weight 2α), all roots-spaces are one dimensional (all the roots are of the same length), and hence $\mathfrak{g}$ is the split form - $\mathfrak{sl}(3,\mathbb{R})$.

Fix non zero elements $X\in\mathfrak{g}_{-\tau}$, $Y\in\mathfrak{g}_{\tau+\sigma}$, and set $Z=[X,Y]$. Z is a generator of $\mathfrak{g}_\sigma$. $[X,Z]$ is in $\mathfrak{g}_{\sigma-\tau}=\{0\}$, hence $[X,Z]=0$. $\mathfrak{g}_\sigma$ is in the α $\mathfrak{a}'$ weight space, hence the form on it is Riemannian. It follows that

$$\langle[X,Y],Z\rangle+\langle Y,[X,Z]\rangle=\langle Z,Z\rangle\neq 0.$$

We see that the action of X is not orthogonal, contradicting Lemma 3.3, as X is a unipotent element of $\mathfrak{r}<\mathfrak{s}$.

4.3. The degenerate case.

Here we assume that $\mathfrak{h}$ is of codimension one in $\mathfrak{r}$, and that the form on $\mathfrak{g}/\mathfrak{r}$ is positive definite. By Lemma 3.4, $\mathfrak{r}$ is a proper subalgebra of $\mathfrak{g}$.

By Lemma 3.2, $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s})=1$. Recall that $\mathfrak{a}'$ denotes a maximal split torus in $\mathfrak{s}$ and A is an element that spans $\mathfrak{a}'$. We have the decomposition of $\mathfrak{g}$ as an $\mathfrak{a}'$-module, $\mathfrak{g}=\mathfrak{r}\oplus\mathfrak{d}$. The form v restricted to $\mathfrak{d}$ is positive definite, hence $\mathfrak{a}'$ acts by homotheties on $\mathfrak{d}$. It follows that $\mathfrak{d}$ is contained in the weight space corresponding to the weight α of $\mathfrak{a}'$. We assume $\alpha(A)>0$. It follows that $\mathfrak{r}>\mathfrak{p}_A$, hence $\mathfrak{r}$ is a (proper) parabolic sub-algebra of $\mathfrak{g}$.

We claim that $\mathfrak{r}=\mathfrak{p}_A$, and that this is a minimal parabolic in $\mathfrak{g}$. It is enough to show that $\mathfrak{r}$ is a minimal parabolic. That is, its semisimple part contains no copy of $\mathfrak{sl}(2,\mathbb{R})$. Let $\mathfrak{r}=\mathfrak{m}_\mathfrak{r}\oplus\mathfrak{a}_\mathfrak{r}\oplus\mathfrak{n}_\mathfrak{r}$ be the Langlands decomposition of $\mathfrak{r}$. Assume $\mathfrak{m}_\mathfrak{r}$ contains a copy of $\mathfrak{sl}(2,\mathbb{R})$. Denote this copy by $\mathfrak{c}$. Consider $\mathfrak{s}\cap(\mathfrak{c}\oplus\mathfrak{a}_\mathfrak{r})$. Its codimension inside $\mathfrak{c}\oplus\mathfrak{a}_\mathfrak{r}$ is at most one (because $\mathfrak{h}$ is a codimension one subalgebra of $\mathfrak{r}$ and is contained in $\mathfrak{s}$). On the other hand its real rank is at most one (as the real rank of $\mathfrak{s}$ is). Hence it must be equal $\mathfrak{c}$. It follows that $\mathfrak{c}$ is contained in $\mathfrak{s}$. Combining this with the fact that $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s})=1$, we get that S has no non-trivial algebraic morphism into $\mathbb{R}^*$. This is a contradiction because we already considered the non-trivial map $S\to\mathrm{GL}(V)$.

We conclude that $\mathfrak{r}$ is a minimal parabolic, and that A is a regular element in $\mathfrak{a}$. In particular we get that the weight space decomposition of $\mathfrak{g}$ as an $\mathfrak{a}'$-modules coincides with its root space decomposition with respect to $\mathfrak{a}$. Since $\mathfrak{d}$ is an $\mathfrak{a}'$-complement of $\mathfrak{r}=\mathfrak{p}_A$, we conclude that it is an $\mathfrak{a}$ complement as well, and that it consists of the sum of all positive roots (regarding the positivity given by A). Since $\mathfrak{d}$ consists of a single $\mathfrak{a}'$-weight, we conclude that the set of positive roots in the root system of $\mathfrak{g}$ consists of a single root. It follows that $\mathfrak{g}$ is a rank one simple algebra with a reduced root system. That is $\mathfrak{g} = \mathfrak{so}(n,1)$ for some n.

It follows that G is locally isomorphic to $\mathrm{SO}(n,1)^o$. The stabilizer of a point in X, H, has the Lie-algebra $\mathfrak{h}$ which is a codimension one subalgebra of the parabolic algebra $\mathfrak{r}$. Denoting by P the associated parabolic group, we get that H is a codimension one subgroup of P, and that X is a fiber bundle over $S^{n-1} \simeq G/P$ with one-dimensional fibers. The induced conformal structure on S^{n-1} must be the standard one - the only $SO(n,1)^o$ conformally invariant structure.

5. Corollaries and applications

Let G_1, G_2 be connected Lie groups and $G_1 \to G_2$ a surjection with a finite central kernel K. Observe that every G_1 homogenous space X gives rise to a G_2 homogenous space, namely X/K. Furthermore, observe that if G_1 preserves a conformal structure t on X, t descends canonically to a conformal structure $\bar{t}$ on X/K which is being conformally preserved by G_2. The property of being pre-essential is an infinitesimal one, thus the G_1 action on (X,t) posses it if and only if so does the G_2 action on $(X/K, \bar{t})$. Similarly, every G_2 homogenous pre-essential conformal space is clearly a G_1 homogenous pre-essential conformal space, just by inflating the action. These simple observations make it natural to make the following definition.

Definition 5.1. Let G be a connected Lie-group with a finite center. For every G-homogeneous manifold (X,t), on which the G action is conformal, and every $x \in X$, we consider the map $\mathfrak{g} \to \mathfrak{g}/\mathrm{Lie}(\mathrm{Stab}(x)) \simeq T_x X$. We define the subspace $\mathfrak{r}_x$ of $\mathfrak{g}$, which is the radical of the form induced by t_x on $\mathfrak{g}$. Denote by $V_{\mathfrak{g}}$ the subset of the full Grassmannian, $\mathrm{Gr}(\mathfrak{g})$ obtained when we vary over all possibilities of pre-essential sub-Lorentzian G-manifolds (X,t) and all points $x \in X$.

Note that by the remarks above the subset $V_{\mathfrak{g}} \subset \mathrm{Gr}(\mathfrak{g})$ depends indeed only on $\mathfrak{g}$ rather then G, so the notation is justified. The following Lemma is a corollary of Theorem 2.5, and of its proof.

Lemma 5.2. *Let G be a connected almost simple Lie-group with a finite center. The set $V_{\mathfrak{g}}$, if not empty, is a single compact G-orbit in* $\mathrm{Gr}(\mathfrak{g})$.

Obviously, if the group G does not appear in the list given in Theorem 2.5, then $V_{\mathfrak{g}}$ is empty. For the groups that do appear, we get from the theorem and from its proof that indeed, $V_{\mathfrak{g}}$ is a single orbit. The Lie-algebras in $V_{\mathfrak{g}}$ are seen to have cocompact normalizers, and the compactness assertion follows.

Let (X,t) be a Lorentzian manifold, and assume that G acts on it conformally (but not necessarily transitively). For every point $x \in X$, define the *orbit manifold of* x, $O_x = G/Stab(x)$. There is a natural injection map (though generally not an imbedding), $O_x \to X$, given by $g\mathrm{Stab}(x) \mapsto gx$. Using this map, we pull back the Lorentzian structure t from X, and obtain a bilinear structure, denoted t^*, on O_x. G acts conformally on (O_x, t^*). (O_x, t^*) is an homogeneous sub-Lorentzian manifold.

Definition 5.3. We say that x is a *pre-essential point* if (O_x, t^*) is a pre-essential manifold. The subset of X consisting of all pre-essential points is called *the pre-essential part of* X.

We define a map $r : X \to \mathrm{Gr}(\mathfrak{g})$ by mapping a point $x \in X$ to the radical, $\mathfrak{r}_x$, of the form induced on $\mathfrak{g}$, by a pull back of t_x via the map $\mathfrak{g} \to \mathfrak{g}/\mathrm{Lie}(\mathrm{Stab}(x)) \to T_xX$. This is the same form (and the same radical) as the form (and the radical) which is induced on $\mathfrak{g}$ via the map $\mathfrak{g} \to T_xO_x$. By definition, the pre-essential part of X is mapped into $V_{\mathfrak{g}} \subset \mathrm{Gr}(\mathfrak{g})$. This gives

Lemma 5.4. *The map* $r : X \to \mathrm{Gr}(\mathfrak{g})$ *maps the pre-essential part of* X *into* $V_{\mathfrak{g}}$.

The next lemma is less obvious.

Lemma 5.5. *Let* G *be a connected almost simple Lie-group with a finite center. Assume that* G *is not locally isomorphic to* $\mathrm{SL}_2(\mathbb{R})$. *Assume that* G *acts conformally on a Lorentzian manifold* (X, t). *Assume that there is no* G*-fixed point in* X. *Then* $r^{-1}(V_{\mathfrak{g}})$ *is a closed subset of* X.

Proof. In case $V_{\mathfrak{g}}$ is empty, there is nothing to prove, hence we assume that G is one that appears in the list given in Theorem 2.5.

The integer valued function $\dim(r(x))$ is easily seen to be upper semi-continuous on X, hence the pre-image under r of the subset of $r(X)$ consisting of maximal dimension spaces is closed in X. We will show that the spaces in $V_{\mathfrak{g}}$ are of maximal dimension inside $r(X)$. This will finish the proof, because, by Lemma 5.2, $V_{\mathfrak{g}}$ is compact, hence it consists of a closed subset of the closed subset of $r(X)$ consisting of maximal dimension spaces.

By Lemma 3.4, $r(X)$ consists of sub-algebras of $\mathfrak{g}$, hence it is enough to show that $V_{\mathfrak{g}}$ consists of maximal dimensional sub-algebras inside $r(X)$. This is what we proceed to show.

The first thing to show is that $r(X)$ contains only proper sub-algebras. We claim that this is indeed the case. G is a simple Lie-group not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$, hence it does not have a codimension one closed sub-group (see, for example, [2, Lemma 3.4 and its proof]). Therefore there are no (locally) one-dimensional orbits in X. There are no zero dimensional orbits in X (as G is connected any such an orbit is a fixed point). It follows that every G-orbit in X is at least (locally) two-dimensional. A Lorentzian form cannot vanish when restricted to a two-dimensional subspace, and the claim follows.

In case $\mathfrak{g} \simeq \mathfrak{so}(n, 2)$ or $\mathfrak{so}(n, 1)$, the sub-algebras in $V_{\mathfrak{g}}$ are maximum dimensional proper parabolic sub-algebras, hence maximum dimensional proper sub-algebras (see for example the proof of [2, Lemma 3.4]). Therefore we can and will assume from now on that $\mathfrak{g} \simeq \mathfrak{su}(n,1)$. In this case $\mathfrak{r}$ has codimension one in a proper parabolic sub-algebra, $\mathfrak{p}$. The parabolic sub-algebras are maximum dimensional algebras so we will be done if we show that $r(X)$ does not contain any parabolic sub-algebra.

We are left to show that, for G locally isomorphic to $\mathrm{SU}(n,1)$, there is no sub-Lorentzian G-homogeneous manifold Y with a parabolic radical, $\mathfrak{p}$. We deal separately with the case that Y is degenerate and the case it is not.

Assume first that Y is non-degenerate. Then $\mathfrak{p} = \mathfrak{r} < \mathfrak{s}$. Let $\mathfrak{a}$ be a maximal (one dimensional) $\mathbb{R}$-split torus in $\mathfrak{p}$, and fix an $\mathfrak{a}$-module decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{d}$. Clearly, $\mathfrak{d} = \mathfrak{n}^+ = \mathfrak{g}_\xi \oplus \mathfrak{g}_{2\xi}$ is the sum of all positive root spaces. Set $\tilde{\mathfrak{a}}$ to be a maximal torus in $\mathfrak{co}(\mathfrak{d})$. If $\mathfrak{d}$ is Riemannian than $\mathfrak{d}$ consists of a single $\tilde{\mathfrak{a}}$-weight space, hence a single $\mathfrak{a}$-weight space, which is a contradiction. Assume then that $\mathfrak{d}$ is Lorentzian. As an $\tilde{\mathfrak{a}}$-module, $\mathfrak{d} = \mathfrak{d}_\alpha \oplus \mathfrak{d}_\beta \oplus \mathfrak{d}_\gamma$ where $\dim(\mathfrak{d}_\beta) = \dim(\mathfrak{d}_\gamma) = 1$ and $\beta + \gamma = 2\alpha$. It follows that $\dim(\mathfrak{g}_\xi) = 1$, which contradicts the fact that $\dim(\mathfrak{g}_\xi) = 2n-2$ is even.

Assume now that Y is degenerate. Then the form on $\mathfrak{n}^+ = \mathfrak{g}_\xi \oplus \mathfrak{g}_{2\xi} \simeq \mathfrak{g}/\mathfrak{r}$ must be Riemannian. It follows that S contains no split semisimple elements (such an element must act faithfully on $\mathfrak{g}/\mathfrak{r}$ by [2, Lemma 3.3]) . Hence $\mathfrak{h} = \mathfrak{s} \cap \mathfrak{r} = \mathfrak{s} \cap \mathfrak{p}$ is a codimension one algebraic sub-algebra of $\mathfrak{p}$ which contains no split semisimple elements. We conclude that $\mathfrak{h}$ contains the unipotent radical of $\mathfrak{p}$, $\mathfrak{n}^-$. Pick elements $X_{-\xi} \in \mathfrak{g}_{-\xi}$ and $X_{2\xi} \in \mathfrak{g}_{2\xi}$ such that $0 \neq X_\xi = [X_{-\xi}, X_{2\xi}] \in \mathfrak{g}_\xi$. By Lemma 3.3, $X_{-\xi}$ must act orthogonally on $\mathfrak{g}/\mathfrak{r}$. On the other hand

$$\langle [X_{-\xi}, X_{2\xi}], X_\xi \rangle + \langle X_{2\xi}, [X_{-\xi}, X_\xi] \rangle = \langle X_\xi, X_\xi \rangle \neq 0$$

This is a contradiction. ■

The Lemmas above will be used in [1] in order to prove

Theorem 5.6. *Let G be a connected almost simple Lie-group with finite center which is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Let X be a connected compact Lorentzian manifold. Assume G acts conformally on X, with no fixed points. Then one of the following holds:*

- *There exist an open and dense G-invariant set $U \subset X$ such that the G-action on U is not essential.*
- *X is commensurable to $C^{n,1}$ for some $n \geq 2$, and G acts transitively on X.*

References

[1] U. Bader "Conformal actions on compact Lorentzian manifolds," Pre-print, 2001.

[2] Bader, U., and A. Nevo, *Conformal actions of simple Lie groups on compact pseudo-Riemannian manifolds*, Journal of Differential Geometry, **60** (2002), 355–387.

[3] Borel, A., "Linear Algebraic Groups," Springer-Verlag, 1991.

[4] Helgason, S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, 1978.

[5] Zimmer, R. J., "Ergodic Theory and Semisimple Groups," Birkhäuser, 1984.

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