Journal of Lie Theory
Volume **20** (2010) 393–407

Irreducible SL$_{n+1}$–Representations Remain Indecomposable Restricted to Some Abelian Subalgebras

Paolo Casati

Communicated by E. B. Vinberg

Abstract. In this paper we show that any irreducible finite dimensional representation of SL_{n+1} remains indecomposable if restricted to n–dimensional abelian subalgebras spanned by simple root vectors.
Mathematics Subject Classification 2000: 22E47.
Key Words and Phrases: Simple Lie algebras, indecomponsable representations.

1. Introduction

Surely the finite dimensional irreducible representations of complex simple Lie algebras are one of the most fascinating and studied subjects in the theory of representations. Their beautiful and complicate structure still presents unknown aspects worth to be studied (see [1] and [4] for recent examples). This paper concerns with one of these, namely the restriction of such representations to some subalgebras. More precisely we shall show that any finite dimensional irreducible representation of a complex simple Lie algebra of type A remains indecomposable if restricted to some abelian subalgebras (Theorem 3.9). Such abelian subalgebra $\mathfrak{a}$ can be constructed as follows. Let $\mathfrak{g}$ be the complex simple Lie algebra A_n, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the corresponding set of roots. Further for any $\alpha \in \Delta$ let X_α be a basis of $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}|\ [H, X] = \alpha(H)X\ \forall H \in \mathfrak{h}\}$, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ a set of simple roots in Δ and set $Y_{\alpha_i} = X_{-\alpha_i}$, then $\mathfrak{a}$ is the abelian subalgebra of $\mathfrak{g}$ spanned by the vectors $\{Y_{\alpha_{2i+1}}\}$ ($i = 0, \ldots, \left[\frac{n}{2}\right]$) and $\{X_{\alpha_{2j}}\}$ ($j = 1, \ldots, \left[\frac{n}{2}\right]$), where $[x]$ denotes the integer part of x.

Theorem 3.9 is almost trivial for the Lie algebra A_1, while for the Lie algebra A_2 was proved by Douglas and Premat in [5], and for the remaining simple Lie algebras of rank two B_2 and G_2 by Premat in [13]. These two papers have played an inspiring role in our work. As far as we know Theorem 3.9 is still unknown for A_n with $n \geq 3$.

The paper is organized as follows. In section 2 we recall some known facts about the simple Lie algebras of type A and their finite dimensional modules, and describe the abelian Lie algebra $\mathfrak{a}$. This section also devoted to present basis of the finite dimensional irreducible A_n–modules found by Littelmann in [10]. In section 3 we find a minimal set of generators for the restriction to the abelian subalgebra $\mathfrak{a}_n$ of the

ISSN 0949–5932 / \$2.50

finite dimensional representations of the Lie algebra A_n, and prove the main result of this paper: the indecomposableness of such restricted representations.

The author wishes to thank Alejandra Premat for sending her preprint [13], which plays a crucial role in the present work, and Veronica Magenes for discussions about the case concerning the Lie algebra $\mathfrak{sl}(4,\mathbb{C})$. The author is also grateful to the referee for carefully reading the paper, the welcomed suggestions and for pointing out the reference [11].

2. Irreducible finite dimensional $\mathfrak{sl}(n+1,\mathbb{C})$–modules

In this section we recall some basic facts on $\mathfrak{sl}(n+1,\mathbb{C})$ and its irreducible finite dimensional representations, and describe the basis of such representations constructed by Littelmann in [10]. It is worth to mention that a similar basis for such modules was alraedy considered by Sai-Ping Li, R.V.Moody, M.Nocolescu, J.Patera in [11]. Good references on the structure and representation theory of the complex simple Lie groups and Lie algebras are, for instance, the books [6, 7].

Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ be the simple Lie algebra of all $(n+1)\times(n+1)$ complex matrices of zero trace, let $\mathfrak{h}$ be its Cartan subalgebra given by all diagonal matrices in $\mathfrak{sl}(n+1,\mathbb{C})$, $\mathfrak{h}^*$ its complex dual, and $\Delta = \Delta(\mathfrak{sl}(n+1,\mathbb{C}),\mathfrak{h}) \subset \mathfrak{h}^*$ the corresponding set of roots. Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be its decomposition into the direct sum of strictly upper triangular, diagonal, and strictly lower triangular matrices, and $\Delta = \Delta^+ \cup -\Delta^+$ the decomposition of the set of root such that

$$\mathfrak{n}^+ = \sum_{\beta\in\Delta^+} \mathfrak{g}_\beta, \quad \mathfrak{n}^- = \sum_{\beta\in-\Delta^+} \mathfrak{g}_\beta$$

where $\mathfrak{g}_\beta = \{X \in \mathfrak{g} |\ [H,X] = \beta(H)X\ \forall H \in \mathfrak{h}\}$. We denote by $\Pi = \{\alpha_1,\dots,\alpha_n\}$ the corresponding set of simple roots and accordingly we fix a Chevalley basis of $\mathfrak{g}$: $X_\beta \in \mathfrak{g}_\beta$ and $Y_\beta \in \mathfrak{g}_{-\beta}$ for $\beta \in \Delta^+$, and $H_\alpha \in \mathfrak{h}$ for α simple, in such a way that $[X_\alpha, Y_\alpha] = H_\alpha$. The Weyl group of $\mathfrak{sl}(n+1,\mathbb{C})$ is denoted by W, as subgroup of $GL(\mathfrak{h}^*)$ it is generated by the hyperplane reflections $s_\alpha : \lambda \mapsto \lambda - \lambda(H_\alpha)\alpha$ for any $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta$.

Denote by $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{n}^+)$, $\mathcal{U}(\mathfrak{n}^-)$ the universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{n}^+$, $\mathfrak{n}^-$ respectively. (More in general $\mathcal{U}(\mathfrak{a})$ will denote the universal enveloping algebra of a given subalgebra $\mathfrak{a}$ of $\mathfrak{g}$.) Following Littelmann [10] we use the following abbreviations:

$$Y_\beta^{(k)} := \frac{Y_\beta^k}{k!} \quad X_\beta^{(k)} := \frac{X_\beta^k}{k!} \quad \binom{H_\alpha}{k} := \frac{H_\alpha(H_\alpha-1)\cdots(H_\alpha-k+1)}{k!}.$$

Fix an ordering $\{\gamma_1,\dots,\gamma_N\}$ of the positive roots ($N = n(n+1)/2$). For $(\mathbf{n}) \in \mathbb{N}^N$ we set:

$$X^{(\mathbf{N})} := X_{\gamma_1}^{(n_1)}\cdots X_{\gamma_N}^{(n_N)}, \quad Y^{(\mathbf{N})} := Y_{\gamma_1}^{(n_1)}\cdots Y_{\gamma_N}^{(n_N)}.$$

Fix an ordering $\{\alpha_1,\dots,\alpha_n\}$ of the simple roots. For $(\mathbf{k}) \in \mathbb{N}^n$ we set:

$$H^{(\mathbf{k})} := \binom{H_{\alpha_1}}{k_1}\cdots\binom{H_{\alpha_n}}{k_n};$$

(we shall sometime write X_i, Y_i, H_i respectively for $X_{\alpha_i}, Y_{\alpha_i}, H_{\alpha_i}$, for a simple root α_i).

Recall that the monomials $Y^{(\mathbf{m})}H^{(\mathbf{k})}X^{(\mathbf{n})}$ form a Poincaré–Birkhoff–Witt basis of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, and the monomials $X^{(\mathbf{n})}$ and $Y^{(\mathbf{m})}$ form a P–B–W basis of $\mathcal{U}^+ = \mathcal{U}(\mathfrak{n}^+)$ respectively $\mathcal{U}^- = \mathcal{U}(\mathfrak{n}^-)$.

An element of $\mathfrak{h}^*$ is called a weight. The set $P = \{\lambda \in \mathfrak{h}^* | \ \lambda(h_\alpha) \in \mathbb{Z}, \ \forall \alpha \in \Delta\}$ is said the set of integral weights of $\mathfrak{g}$. A weight λ of P is said dominant if $\lambda(H_\alpha) \geq 0$ for any simple root α. The complex finite dimensional irreducible representations of $\mathfrak{sl}(n+1, \mathbb{C})$ are parameterized by the dominant integral weights. We denote by $V(\lambda)$ the finite dimensional irreducible $\mathfrak{sl}(n+1, \mathbb{C})$–module corresponding to the integral dominant weight λ. A element μ of $\mathfrak{h}^*$ is said a weight of an irreducible finite dimensional module $V(\lambda)$ if the weight space $V_\mu = \{v \in V(\lambda) | \ Hv = \mu(H)v \ \forall H \in \mathfrak{h}\}$ is different from zero. Denote by $P(\lambda)$ the set of all weights of $V(\lambda)$ then $V(\lambda)$ may be decomposed as the direct sum of its weight spaces:

$$V(\lambda) = \bigoplus_{\mu \in P(\lambda)} V_\mu. \tag{2.1}$$

Let $\Pi = \Pi_Y \cup \Pi_X$ a decomposition of the set of simple roots Π such that the n–dimensional subalgebra spanned by the elements $\{X_\alpha, Y_\beta\}_{\alpha \in \Pi_X, \beta \in \Pi_Y}$ is an abelian subalgebra. If $\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_n = \varepsilon_n - \varepsilon_{n+1}$ is the usual ordering of the simple roots of $\mathfrak{sl}(n+1, \mathbb{C})$, where $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ denotes the projection of a diagonal matrix onto its i–th entry, then it easy to see that this decomposition of the set of simple roots Π can be achieved in two ways: either $\Pi_Y = \{\alpha_{2i+1}\}_{i=0,\ldots,[\frac{n}{2}]}$ and $\Pi_X = \{\alpha_{2i}\}_{i=1,\ldots,[\frac{n}{2}]}$, where $[x]$ denote the integer part of x or the converse case. Since the two choices are equivalent, let us for the sake of concreteness choose in this paper the first one and give the

Definition 2.1. Let $\mathfrak{a}_n$ be the abelian subalgebra of $\mathfrak{sl}(n+1, \mathbb{C})$ spanned by the simple root–vectors $\{X_{\alpha_{2i}}, Y_{\alpha_{2j+1}}\}$, $1 \leq i \leq \left[\frac{n}{2}\right]$, $0 \leq j \leq \left[\frac{n}{2}\right]$.

The aim of this paper is to show how any irreducible $\mathfrak{sl}(n+1, \mathbb{C})$–module $V(\lambda)$ restricted to the abelian subalgebra $\mathfrak{a}_n$ remains indecomposable.

Further since any of such abelian algebra $\mathfrak{a}_n$ may be imbedded in a solvable Lie algebra endowed with a non singular ad–invariant bilinear form [12] [3] which is still a subalgebra of $\mathfrak{sl}(n+1, \mathbb{C})$, this result provides a way to construct a fairly wide class of indecomposable (and therefore not trivial) finite dimensional modules of solvable quadratic Lie algebras [2]. In order to achieve such result we need to consider the basis of the irreducible $\mathfrak{sl}(n+1, \mathbb{C})$–modules discovered by Littelmann in [10] (but see also [9] [8]). First we introduce the following concepts

Definition 2.2. A monomial in the Y_{α_i} is called semi-standard if it is of the form:

$$Y^{(\mathbf{a})} = Y_1^{(a_1^1)} \left(Y_2^{(a_2^2)} Y_1^{(a_1^2)}\right) \left(\cdots\right) \left(Y_i^{(a_i^i)} Y_{i-1}^{(a_{i-1}^i)} \cdots Y_1^{(a_1^i)}\right) \left(\cdots\right) \left(Y_n^{(a_n^n)} \cdots Y_2^{(a_2^n)} Y_1^{(a_1^n)}\right)$$

where $\mathbf{a} = (a_1^1, a_2^2, a_1^2, \ldots, a_n^n, \ldots, a_1^n) \in \mathbb{N}^n$. The tuple $\mathbf{a}$ and the monomial $Y^{(\mathbf{a})}$ are called standard if:

$$\mathbf{a} \in \mathcal{S} = \{(\mathbf{a}) \in \mathbb{N}^n | \ a_2^2 \geq a_1^2, a_3^3 \geq a_2^3 \geq a_1^3, \ldots, a_i^i \geq a_{i-1}^i \geq \cdots \geq a_1^i, \ldots a_n^n \geq a_{n-1}^i \geq \cdots \geq a_1^n\}.$$

Then we can formulate the following important result due to Littelman.

Theorem 2.3. *[10] For a dominant weight λ of $\mathfrak{g}$, let $V(\lambda)$ be the corresponding irreducible finite dimensional $\mathfrak{g}$–module of highest weight λ and $u_\lambda \in V(\lambda)$ be a highest weight vector.*
Denote by λ_i^j the weight of

$$\left(Y_i^{(a_i^j)} \cdots Y_1^{(a_1^j)}\right)\left(\cdots\right)\left(Y_n^{(a_n^n)} \cdots Y_2^{(a_2^n)} Y_1^{(a_1^n)}\right) u_\lambda$$

and set
$\lambda_0^n := \lambda$, and $\lambda_0^{j-1} := \lambda_j^j$ for $1 \le j \le n$. Then the elements of $V(\lambda)$

$$Y^{(\mathbf{a})} u_\lambda = Y_1^{(a_1^1)}\left(Y_2^{(a_2^2)} Y_1^{(a_1^2)}\right)\left(\cdots\right)\left(Y_i^{(a_i^i)} Y_{i-1}^{a_{(i-1)}^i} \cdots Y_1^{(a_1^i)}\right)\left(\cdots\right)\left(Y_n^{(a_n^n)} \cdots Y_2^{a_2^n} Y_1^{a_1^n}\right) u_\lambda$$

with $\mathbf{a} \in \mathcal{S}$ such that

$$\begin{array}{llllll}
\lambda_0^n(H_1) \ge a_1^n & \lambda_1^n(H_2) \ge a_2^n & \lambda_2^n(H_3) \ge a_3^n & \ldots & \lambda_{i-1}^n(H_i) \ge a_i^n & \ldots \quad \lambda_{n-1}^n(H_n) \ge a_n^n \\
\quad\ldots & \quad\ldots & \quad\ldots & \ldots & \quad\ldots & \ldots \\
\lambda_0^j(H_1) \ge a_1^j & \quad\ldots & \quad\ldots & \ldots & \lambda_{j-1}^j(H_j) \ge a_j^j & \\
\quad\ldots & \quad\ldots & \quad\ldots & \ldots & & \\
\lambda_0^2(H_1) \ge a_1^2 & \lambda_1^2(H_2) \ge a_2^2 & & & & \\
\lambda_0^1(H_1) \ge a_1^1 & & & & &
\end{array}$$

form a basis $\mathfrak{L}_\lambda$ of $V(\lambda)$.

Remark 2.4. Let λ_i $i = 1, \ldots n$ be the elements of $\mathfrak{h}^*$ defined by the relations $\Lambda_i(\alpha_j) = \delta_{ij}$ where δ_{ij} is the usual Kronecker delta. Then if we write the dominant weight λ in the form: $\lambda = \sum_{i=1}^m m_i \lambda_i$ (with $m_i \in \mathbb{N}$, $i = 1, \ldots n$), the conditions (2.3) become:

$$\begin{array}{ll}
0 \le a_1^i \le m_1 - 2\sum_{j=i+1}^n a_1^j + \sum_{j=i+1}^n a_2^j & i = 1, \ldots, n \\
a_{k-1}^i \le a_k^i \le m_k - 2\sum_{j=i+1}^n a_k^j + \sum_{j=i}^n a_{k-1}^j + \sum_{j=i+1}^n a_{k+1}^i & i = 1, \ldots, n-k+1 \quad 2 \le k \le n-1 \\
a_{n-1}^n \le a_n^n \le m_n + a_{n-1}^n. &
\end{array}$$

Finally observe that we can not find for any complex simple Lie algebra $\mathfrak{sl}(n+1, \mathbb{C})$ a subalgebra of dimension strictly less then n such that any irreducible finite dimensional $\mathfrak{sl}(n+1, \mathbb{C})$–module remanins indecomposable if restricted to it.
Let us indeed consider the first non trivial case, namely the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. In this case it is easy to show that there is no an one dimensional subalgebra such that the restriction on it of any irreducible finite dimensional representation of $\mathfrak{sl}(3, \mathbb{C})$ remains indecomposable. Let X be indeed a basis for such algebra. Then X must act as a single Jordan block in any irreducible finite dimensional representation of $\mathfrak{sl}(3, \mathbb{C})$. In particular if $\pi : \mathfrak{sl}(3, \mathbb{C}) \to \mathrm{End}(\mathbb{C}^3)$ is the irreducible representation with $V(\lambda) = V(\lambda_1)$

(so that $\dim_{\mathbb{C}}(V(\lambda)) = 3$) then, since the trace of $\pi(X)$ is zero, it must exist a $\xi \in \mathrm{Aut}(\mathbb{C}^3)$ such that

$$\xi\pi(X)\xi^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{or eq.} \quad \xi\pi(X)\xi^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

I.e., X can be take equal to $Y_1 + Y_2$ (or eq. to $X_1 + X_2$). But the restriction to these one-dimensional subalgebras of the module $V(\lambda) = V(\lambda_1 + \lambda_2)$ is not indecomposable because on it in both cases $X^5 = 0$ while $\dim(V(\lambda)) = 8$.

3. $V(\lambda)$ as indecomposable $\mathfrak{a}_n$–module

Let us fix a dominant integral weight λ of $\mathfrak{sl}(n+1, \mathbb{C})$. We shall show in this section that the $\mathfrak{sl}(n+1, \mathbb{C})$–module $V(\lambda)$ viewed as $\mathfrak{a}_n$–modules is indecomposable.

We first need to find a (minimal) set of generators for the $\mathfrak{a}_n$–modules $V(\lambda)$.

Definition 3.1. Let V be a $\mathfrak{a}_n$–module, a subset of elements $\{v_1, \dots v_m\}$ in V is said to be a set of generators of V if $V = \mathcal{U}(\mathfrak{a}_n)\{v_1, \dots v_m\}$. The set is called a minimal set of generators if fewer than m vectors will not generate V. In the case of the $\mathfrak{a}_n$–modules $V(\lambda)$ a set of generators $\mathfrak{W}$ is a set of homogeneous generators if any element in $\mathfrak{W}$ is a $\mathfrak{sl}(n+1, \mathbb{C})$–weight vector.

Theorem 3.2. *Let $\mathcal{G}_\lambda$ be the subset of $\mathcal{L}_\lambda = \{\mathbf{a} \in \mathcal{S} | \ Y^{(\mathbf{a})} u_\lambda \in \mathfrak{L}_\lambda\}$ given by:* $\mathcal{G}_\lambda =$

$$\left\{ \mathbf{g} \in \mathcal{L}_\lambda \;\middle|\; \begin{array}{ll} a^{2j}_{2j} = \lambda^{2j}_{2j-1}(H_{2j}) & j = 1, \dots \left[\frac{n}{2}\right] \\ a^{2j+1}_{2j} \neq 0 \Rightarrow a^{2j}_{2j-1} \neq 0 & j = 1, \dots \left[\frac{n-1}{2}\right] \\ a^{2j+1}_{1} = 0 & j = 0, \dots \left[\frac{n-1}{2}\right] \\ \lambda^{2j+1}_{2i-1}(H_{2i}) \neq 0 \textit{ and } \lambda^{2j+1}_{0}(H_1) = 0,\ \lambda^{2j+1}_{2r-1}(H_{2r}) = a^{2j+1}_{2r}, & 1 \leq r < i \\ \Rightarrow a^{2j}_{2i-1} = \lambda^{2j}_{2i-2}(H_{2i-1}) \quad i = 1, \dots 2j-1 & j = 1, \dots, \left[\frac{n+1}{2}\right] \end{array} \right\}$$

then the corresponding subset $\mathfrak{G}_\lambda = \{Y^{(\mathbf{a})} u_\lambda | \ \mathbf{a} \in \mathcal{G}_\lambda\}$ of the Littelmann basis $\mathfrak{L}_\lambda$ is a set of homogeneous $\mathfrak{a}_n$–generators of $V(\lambda)$.

Proof. Set $\mathcal{L}^0_\lambda = \{\mathbf{a} \in \mathcal{L}_\lambda | \ a^1_1 = 0\}$. Since X_1 belongs to $\mathfrak{a}_n$, we have of course only to prove that acting with $\mathfrak{a}_n$ we may construct the subset $\mathfrak{L}^0_\lambda = \{Y^{(\mathbf{a})} u_\lambda | \mathbf{a} \in \mathcal{L}^0_\lambda\}$ of $\mathfrak{L}_\lambda$. We divide the proof in four steps.

1. First, if we define $\mathcal{L}^1_\lambda =$

$$\left\{ \mathbf{a} \in \mathcal{L}^0_\lambda \;\middle|\; \begin{array}{ll} a^{2h+1}_1 = 0,\ a^{2h+1}_{2h} \neq 0 \Rightarrow a^{2h}_{2h-1} \neq 0 & h = 1, \dots \left[\frac{n-1}{2}\right] \\ \lambda^{2j+1}_{2i-1}(H_{2i}) \neq 0 \text{ and } \lambda^{2j+1}_{0}(H_1) = 0,\ \lambda^{2j+1}_{2r-1}(H_{2r}) = a^{2j+1}_{2r}, & 1 \leq r < i \\ \Rightarrow a^{2j}_{2i-1} = \lambda^{2j}_{2i-2}(H_{2i-1}) \quad i = 1, \dots 2j-1 & j = 1, \dots, \left[\frac{n+1}{2}\right] \end{array} \right\}$$

and set $\mathfrak{L}^1_\lambda = \{Y^{(\mathbf{a})} | \, \mathbf{a} \in \mathcal{L}^1_\lambda\}$, then $\mathfrak{L}^1_\lambda \subset \mathcal{U}(\mathfrak{a}_n)(\mathfrak{G}_\lambda)$.

Let us consider indeed for any $1 \le j \le \left[\frac{n}{2}\right]$ the subsets $\mathfrak{L}^1_{2j}$ of $\mathfrak{L}^1_\lambda$ given by:

$$\mathfrak{L}^1_{2j} = \left\{ Y^{(\mathbf{a})} u_\lambda \in \mathfrak{L}^1_\lambda | \, \exists \, Y^{(\mathbf{a}_\mathcal{G})} u_\lambda \in \mathfrak{G} \;\middle|\; \begin{array}{lll} (a_\mathcal{G})^h_{2l+1} = a^h_{2l+1}, & l = 0, \dots, \left[\frac{n-1}{2}\right], & 2l+1 \le h \le n \\ (a_\mathcal{G})^h_{2k} = a^h_{2k}, & k = j, \dots, \left[\frac{n}{2}\right] & 2l \le h \le n, \end{array} \right\}$$

and the corresponding filtration of $\mathfrak{L}^1_\lambda$:

$$\mathfrak{G}_\lambda = \mathfrak{L}^1_2 \subset \dots \subset \mathfrak{L}^1_{2j} \dots \subset \mathfrak{L}^1_{2[\frac{n}{2}]} \subset \mathfrak{L}^1_{2[\frac{n}{2}]+2} = \mathfrak{L}^1_\lambda.$$

Obviously it suffices to show that $\mathfrak{L}^1_{2j} \subset \mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^1_{2j-2})$ for any $1 \le j \le \left[\frac{n}{2}\right]$. We shall do it (for a fixed index j) by induction over the partial ordering "$\le_j$" of $\mathcal{L}^0_\lambda$ (and of $\mathfrak{L}^0_\lambda$ as well) given by the relations

$$\mathbf{a} \le_j \mathbf{b} \Leftrightarrow a^i_{2j} - a^i_{2j+1} \le b^i_{2j} - b^i_{2j+1} \quad i = 2j+1, \dots, n.$$

With respect to this ordering the minimal elements in $\mathfrak{L}^1_{2j}$ are those $Y^{(\mathbf{a})} u_\lambda$ with $a^i_{2j} - a^i_{2j+1} = -\lambda^i_{2j}(H_{2j+1})$, $2j+1 \le i \le n$. For any of this element there exists a positive integer number k (namely $k = a^{2j}_{2j} - \lambda^{2j}_{2j+1}(H_{2j})$) such that the element $Y^{(\mathbf{g}(\mathbf{a}))} u_\lambda$ with $g(a)^{2j}_{2j} = a^{2j}_{2j} + k$ and $g(a)^i_l = a^i_l$ if $(i,l) \ne (2j, 2j)$ belongs to $\mathfrak{G}_\lambda$ and (using the $\mathfrak{sl}(2, \mathbb{C})$–representation theory and [10])

$$X^k_{2j} \cdot Y^{(\mathbf{g}(\mathbf{a}))} u_\lambda = c^{2j}_k(\mathbf{a}) Y^{(\mathbf{a})} u_\lambda = \prod_{i=1}^{k-1} \left(\sum_{h=1}^{n-2j} a^{2j+h}_{2j+1} - 2 \sum_{h=1}^{n-2j} a^{2j+h}_{2j} + \sum_{h=0}^{n-2j} a^{2j+h}_{2j-1} - a^{2j}_{2j} + k + i \right) Y^{(\mathbf{a})} u_\lambda$$

hence, since in our Hypothesis the coefficients $c^{2j}_k(\mathbf{a})$ are always different from zero, any minimal element of $\mathfrak{L}^1_{2j}$ belongs to $\mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^1_{2j-2})$. Suppose now by induction hypothesis that we have constructed any elements $Y^{(\mathbf{b})} u_\lambda \in \mathfrak{L}^1_{2j}$ for any $\mathbf{b} <_j \mathbf{a}$. Since there exists a tuple $\mathbf{a}_{\mathcal{L}^1_{2j-2}}$ such that $(a_{\mathcal{L}^1_{2j-2}})^i_l = a^i_l$, if $(i,l) \ne (2j, 2j)$ and $Y^{(\mathbf{a}_{\mathcal{L}^1_{2j-2}})} u_\lambda \in \mathfrak{L}^1_{2j-2}$, we have

$$X^k_{2j} \cdot Y^{(\mathbf{a}_{\mathcal{L}^1_{2j-2}})} u_\lambda = c^{2j}_k(\mathbf{a}_{\mathcal{L}^1_{2j-2}}) Y^{(\mathbf{a})} u_\lambda + \sum_{\mathbf{b} <_j \mathbf{a}} c_\mathbf{b} Y^{(\mathbf{b})} u_\lambda$$

which shows (again being $c^{2j}_k(\mathbf{a}_{\mathcal{L}^1_{2j-2}}) \ne 0$) that also $Y^{(\mathbf{a})} u_\lambda$ belongs to $\mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^1_{2j-2})$.

2. Define now $\mathcal{L}^2_\lambda =$

$$\left\{ \mathbf{a} \in \mathcal{L}^0_\lambda \;\middle|\; \begin{array}{ll} a^{2h+1}_1 = 0, & h = 1, \dots \left[\frac{n-1}{2}\right] \\ \lambda^{2j+1}_{2i-1}(H_{2i}) \ne 0 \text{ and } \lambda^{2j+1}_0(H_1) = 0, \; \lambda^{2j+1}_{2r-1}(H_{2r}) = a^{2j+1}_{2r}, & 1 \le r < i \\ \Rightarrow a^{2j}_{2i-1} = \lambda^{2j}_{2i-2}(H_{2i-1}) \quad i = 1, \dots 2j-1 & j = 1, \dots, \left[\frac{n+1}{2}\right] \end{array} \right\}$$

and set $\mathfrak{L}^2_\lambda = \left\{ Y^{(\mathbf{a})} u_\lambda | \, \mathbf{a} \in \mathcal{L}^2_\lambda \right\})$ then $\mathfrak{L}^2_\lambda \subset \mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^1_\lambda)$.

For any $1 \le s \le \left[\frac{n-1}{2}\right]$ let $\mathfrak{L}^2_s$ be the set

$$\mathfrak{L}^2_s = \left\{ Y^{(\mathbf{a})} u_\lambda \in \mathfrak{L}^2_\lambda | \, a^{2k+1}_{2k} \ne 0 \Rightarrow a^{2k}_{2k-1} \ne 0 \quad k = s, \dots \left[\frac{n-1}{2}\right] \right\}$$

and consider the corresponding filtration of $\mathfrak{L}^2_\lambda$:

$$\mathfrak{L}^1_\lambda = \mathfrak{L}^2_1 \subset \cdots \subset \mathfrak{L}^2_s \subset \cdots \subset \mathfrak{L}^2_{\left[\frac{n-1}{2}\right]} \subset \mathfrak{L}^2_{\left[\frac{n-1}{2}\right]+1} = \mathfrak{L}^2_\lambda.$$

Again it will suffice to show that $\mathfrak{L}^2_s \subset \mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^1_{s-1})$ for any $2 \leq s \leq \left[\frac{n-1}{2}\right] + 1$. We shall still do it by induction. Indeed consider first an element in $\mathfrak{L}^2_s$ of the form $Y^{(\mathbf{a})}u_\lambda = \left(Y_2^{(a_2^2)}\cdots\right)\left(\cdots\right)\left(Y_{2s}^{(a_{2s}^{2s})}Y_{2s-2}^{(a_{2s-2}^{2s})}\cdots\right)\left(Y_{2s+1}^{(a_{2s+1}^{2s+1})}Y_{2s}\cdots\right)\left(\cdots\right)u_\lambda$ then the element $Y^{(\mathbf{b}(\mathbf{a}))}u_\lambda$ with the tuple $\mathbf{b}(\mathbf{a})$ given by the relations $b(a)^{2s}_{2s} = a^{2s}_{2s} + 1$, $b(a)^{2s+1}_{2s+1} = a^{2s+1}_{2s+1} - 1$, $b(a)^{2s+1}_{2s} = 0$ and $b(a)^i_j = a^i_j$ otherwise, belongs to $\mathfrak{L}^2_{s-1}$. Further from the relation [10]

$$\begin{aligned}
&Y_{2s+1}\cdot Y^{(\mathbf{b}(\mathbf{a}))}u_\lambda\\
&= p(1, a^{2s}_{2s}+1, a^{2s+1}_{2s+1}, 0)\left(\cdots\right)\left(Y_{2s}^{(a_{2s}^{2s}+1)}Y_{2s-2}^{(a_{2s-2}^{2s})}\cdots\right)\left(Y_{2s+1}^{(a_{2s+1}^{2s+1})}Y_{2s-1}^{(a_{2s+2}^{2s+1})}\cdots\right)\left(Y_{2s+2}^{(a_{2s+2}^{2s+2})}\cdots\right)\left(\cdots\right)u_\lambda\\
&+p(1, a^{2s}_{2s}+1, a^{2s+1}_{2s+1}, 1)\left(\cdots\right)\left(Y_{2s}^{(a_{2s}^{2s})}Y_{2s-2}^{(a_{2s-2}^{2s})}\cdots\right)\left(Y_{2s+1}^{(a_{2s+1}^{2s+1})}Y_{2s}\cdots\right)\left(Y_{2s+2}^{(a_{2s+2}^{2s+2})}\cdots\right)\left(\cdots\right)u_\lambda,
\end{aligned}$$

where

$$p(a,b,c,d) = \binom{a+c-b}{a-d} \quad a,b,c,d \in \mathbb{N} \quad d \leq b$$

and to have binomial coefficients also available for negative integers, following Littelmann we used the definition:

$$\binom{a}{b} = \lim_{t\to 0}\frac{\Gamma(a+1+t)}{\Gamma(b-a+1+t)\Gamma(b+1+t)};$$

it follows that $Y^{(\mathbf{a})}u_\lambda$ belongs to $\mathcal{U}(\mathfrak{a}_n)\left(\mathfrak{L}^2_{s-1}\right)$ because $p(1, a^{2s}_{2s}+1, a^{2s+1}_{2s+1}, 1) = 1$ (but also see [10] remark 7) and both $Y^{(\mathbf{b}(\mathbf{a}))}u_\lambda$ and
$\left(\cdots\right)\left(Y_{2s}^{(a_{2s}^{2s})}\cdots\right)\left(Y_{2s+1}^{(a_{2s+1}^{2s+1})}Y_{2s-1}^{(a_{2s+2}^{2s+1})}\cdots\right)\left(Y_{2s+2}^{(a_{2s+2}^{2s+2})}\cdots\right)\left(\cdots\right)u_\lambda$ are in $\mathcal{U}(\mathfrak{a}_n)\left(\mathfrak{L}^2_{s-1}\right)$.
Let us now consider an element in $\mathcal{L}^2_s$ of the type
$\left(Y_2^{(a_2^2)}\cdots\right)\left(\cdots\right)\left(Y_{2s}^{(a_{2s}^{2s})}Y_{2s-2}^{(a_{2s-2}^{2s})}\cdots\right)\left(Y_{2s+1}^{(a_{2s+1}^{2s+1})}Y_{2s}^{k+1}\cdots\right)\left(Y_{2s+2}^{(a_{2s+2}^{2s+2})}\right)\left(\cdots\right)u_\lambda$, since by induction Hypothesis $Y^{(\mathbf{b})}u_\lambda$, with $b^{2s}_{2s} = a^{2s}_{2s}+1$, $b^{2s+1}_{2s+1} = a^{2s+1}_{2s+1}-1$, $b^{2s+1}_{2s} = k$ and $b^i_j = a^i_j$ otherwise, belongs to $\mathcal{U}(\mathfrak{a}_n)\left(\mathcal{L}^2_{s-1}\right)$ from $Y_{2s+1}\cdot Y^{(\mathbf{b})}u_\lambda$

$$\begin{aligned}
&= p(1, a^{2s}_{2s}+1, a^{2s+1}_{2s+1}, 0)\left(\cdots\right)\left(Y_{2s}^{(a_{2s}^{2s}+1)}\cdots\right)\left(Y_{2s+1}^{(a_{2s+1}^{2s+1})}Y_{2s-1}^{(a_{2s+2}^{2s+1})}Y_{2s}^{k}\cdots\right)\left(Y_{2s+2}^{(a_{2s+2}^{2s+2})}\cdots\right)\left(\cdots\right)u_\lambda\\
&+p(1, a^{2s}_{2s}+1, a^{2s+1}_{2s+1}, 1)\left(\cdots\right)\left(Y_{2s}^{(a_{2s}^{2s})}Y_{2s-2}^{(a_{2s-2}^{2s})}\cdots\right)\left(Y_{2s+1}^{(a_{2s+1}^{2s+1})}Y_{2s}^{k+1}\cdots\right)\left(Y_{2s+2}^{(a_{2s+2}^{2s+2})}\cdots\right)\left(\cdots\right)u_\lambda
\end{aligned}$$

it follows that also

$$\left(Y_2^{(a_2^2)}\cdots\right)\left(\cdots\right)\left(Y_{2s}^{(a_{2s}^{2s})}Y_{2s-2}^{(a_{2s-2}^{2s})}\cdots\right)\left(Y_{2s+1}^{(a_{2s+1}^{2s+1})}Y_{2s}^{k+1}\cdots\right)\left(Y_{2s+2}^{(a_{2i+2}^{2i+2})}\right)\left(\cdots\right)u_\lambda$$

belongs to $\mathcal{U}(\mathfrak{a}_n)\left(\mathcal{L}^2_{s-1}\right)$.
3. Let us now define $\mathcal{L}^3_\lambda$ as: $\mathfrak{L}^0_s =$

$$\left\{\mathbf{a}\in\mathcal{L}_0 \;\middle|\; \begin{array}{lll} \lambda^{2j+1}_{2i-1}(H_{2i}) \neq 0 \text{ and } \lambda^{2j+1}_0(H_1) = 0,\ \lambda^{2j+1}_{2r-1}(H_{2r}) = a^{2j+1}_{2r}, & 1 \leq r < i & \\ \Rightarrow a^{2j}_{2i-1} = \lambda^{2j}_{2i-2}(H_{2i-1}) \quad i = 1, \ldots 2j-1 & & j = 1, \ldots, \left[\frac{n+1}{2}\right] \end{array}\right\}$$

and set $\mathfrak{L}^3_\lambda = \{Y^{(\mathbf{a})}u_\lambda |\, \mathbf{a} \in \mathcal{L}^3_\lambda\}$ then $\mathfrak{L}^3_\lambda \subset \mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^2_\lambda)$.
Defining for any $1 \le s \le \left[\frac{n-1}{2}\right]$ the sets

$$\mathfrak{L}^3_s = \{Y^{(\mathbf{a})}u_\lambda \in \mathfrak{L}^3_\lambda |\, a_1^{2j+1} = 0 \quad j = s, \dots \left[\frac{n-1}{2}\right]\}$$

we have the filtration of $\mathcal{L}^3$:

$$\mathfrak{L}^2_\lambda = \mathfrak{L}^3_1 \subset \cdots \subset \mathfrak{L}^3_s \subset \cdots \subset \mathfrak{L}^3_{[\frac{n-1}{2}]} \subset \mathfrak{L}^3_{[\frac{n-1}{2}]+1} = \mathfrak{L}^3_\lambda.$$

Once again it suffices to prove that $\mathfrak{L}^3_s \subset \mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^3_{s-1})$ for any fixed s. We proceed by induction. Let us first consider an element of $\mathfrak{L}^3_s$ of the type
$Y^{(\mathbf{a})}u_\lambda = \left(Y_2^{(a_2^2)}\cdots\right)\left(\cdots\right)\left(Y_{2s}^{(a_{2s}^{2s})}\cdots Y_1^{(a_1^{2s})}\right)\left(Y_{2s+1}^{(a_{2s+1}^{2s+1})}\dots Y_2^{(a_2^{2s+1})}Y_1\right)\left(\cdots\right)u_\lambda$ then the element $Y^{(\mathbf{b}(\mathbf{a}))}u_\lambda$ with $b(a)^{2s}_{2s-l} = a^{2s}_{2s-l}+1$, $0 \le l \le 2s-1$, $b(a)^{2s+1}_{2s-l} = a^{2s+1}_{2s-l}-1$, $-1 \le l \le 2s-1$, $b(a)^h_l = a^h_l$ otherwise, belongs to $\mathfrak{L}^3_{s-1}$, and we have

$$\begin{aligned}
&Y_{2s+1}\cdot Y^{(\mathbf{b}(\mathbf{a}))}u_\lambda = p(1,a^{2s}_{2s}+1,a^{2s+1}_{2s+1}-1,0)\left(\cdots\right)\left(Y_{2s}^{(a^{2s}_{2s}+1)}\cdots\right)\left(Y_{2s+1}^{(a^{2s+1}_{2s+1})}Y_{2s}^{(a^{2s+1}_{2s}-1)}\cdots\right)\left(\cdots\right)u_\lambda\\
&+p(1,a^{2s}_{2s}+1,a^{2s+1}_{2s+1}-1,1)\left(\cdots\right)\left(Y_{2s}^{(a^{2s}_{2s})}Y_{2s+1}^{(a^{2s+1}_{2s+1})}Y_{2s}Y_{2s-1}^{(a^{2s}_{2s-1}+1)}\cdots\right)\left(Y_{2s}^{(a^{2s+1}_{2s}-1)}\cdots\right)\left(\cdots\right)u_\lambda\\
&= \dots\\
&= \textstyle\sum_{k=1}^h p^s_k(\mathbf{a})\left(\cdots\right)\left(\prod_{l=0}^{k-2}Y_{2s-l}^{(a^{2s}_{2s-l})}\right)\left(\prod_{l=k-1}^{2s-1}Y_{2s-l}^{(a^{2s}_{2s-l}+1)}\right)\left(\prod_{l=0}^{k-1}Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l})}\right)\left(\prod_{l=k}^{2s-1}Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l}-1)}\right)\left(\cdots\right)u_\lambda\\
&+q^s_h(\mathbf{a})\left(\cdots\right)\left(\prod_{l=0}^{h-1}Y_{2s-l}^{(a^{2s}_{2s-l})}\right)\left(\prod_{l=0}^{h-1}Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l})}\right)\left(Y_{2s-h+1}Y_{2s-h}^{(a^{2s}_{2s-h}+1)}Y_{2s-h+1}^{(a^{2s+1}_{2s-h+1}-1)}\right)\\
&\left(\prod_{l=h+1}^{2s-1}Y_{2s-l}^{(a^{2s}_{2s-l}+1)}\right)\left(\prod_{l=h+1}^{2s-1}Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l}-1)}\right)\left(\cdots\right)u_\lambda = \dots\\
&= \textstyle\sum_{k=1}^{2s} p^s_k(\mathbf{a})\,(\cdots)\left(\prod_{l=0}^{k-2}Y_{2s-l}^{(a^{2s}_{2s-l})}\right)\left(\prod_{l=k-1}^{2s-1}Y_{2s-l}^{(a^{2s}_{2s-l}+1)}\right)\left(\prod_{l=0}^{k-1}Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l})}\right)\left(\prod_{l=k}^{2s-1}Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l}-1)}\right)\left(\cdots\right)u_\lambda\\
&+q^s_{2s}(\mathbf{a})Y^{(\mathbf{a})}u_\lambda
\end{aligned}$$

where

$$\begin{aligned}
p^s_k(\mathbf{a}) &= p(1,a^{2s}_{2s+1-k}+1,a^{2s+1}_{2s-k+2}-1,0)\textstyle\prod_{l=1}^{k-1}p(1,a^{2s}_{2s+1-l}+1,a^{2s+1}_{2s-l+2}-1,1)\\
q^s_k(\mathbf{a}) &= \textstyle\prod_{l=1}^{k}p(1,a^{2s}_{2s+1-l}+1,a^{2s+1}_{2s-l+2}-1,1),
\end{aligned}$$

which show that $Y^{(\mathbf{a})}u_\lambda \in \mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^3_{s-1})$ because $q^s_{2s}(\mathbf{a}) \neq 0$ and all other elements belong to $\mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^3_{s-1})$. If we now consider an element $Y^{(\mathbf{a})}u_\lambda$ of the type
$Y^{(\mathbf{a})}u_\lambda = \left(Y_2^{(a_2^2)}\cdots\right)\left(\cdots\right)\left(Y_{2s}^{(a_{2s}^{2s})}\cdots\cdots Y_1^{(a_1^{2s})}\right)\left(Y_{2s+1}^{(a_{2s+1}^{2s+1})}\dots Y_2^{(a_2^{2s+1})}Y_1^k\right)\left(\cdots\right)u_\lambda$ then by induction Hypothesis the element $Y^{(\mathbf{b}(\mathbf{a}))}u_\lambda$ with $b(a)^{2s}_{2s-l} = a^{2s}_{2s-l}+1$, $0 \le l \le 2s-1$, $b(a)^{2s+1}_{2s-l} = a^{2s+1}_{2s-l}-1$, $-1 \le l \le 2s-1$, $b(a)^h_l = a^h_l$ otherwise, belongs to $\mathfrak{L}^3_{s-1}$, and with the same computations done before we have

$$\begin{aligned}
&Y_{2s+1}\cdot Y^{(\mathbf{b}(\mathbf{a}))}u_\lambda\\
&= \textstyle\sum_{j=1}^{2s} p^s_j(\mathbf{a})\left(\cdots\right)\left(\Pi_{l=0}^{j-2}Y_{2s-l}^{(a^{2s}_{2s-l})}\right)\left(\prod_{l=j-1}^{2s-1}Y_{2s-l}^{(a^{2s}_{2s-l}+1)}\right)\left(\prod_{l=0}^{j-1}Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l})}\right)\left(\prod_{l=j}^{2s-1}Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l}-1)}\right)\left(\cdots\right)u_\lambda\\
&+q^s_{2s}(\mathbf{a})Y^{(\mathbf{a})}u_\lambda,
\end{aligned}$$

which implies (by induction) $Y^{(\mathbf{a})}u_\lambda \in \mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^3_{s-1})$.
4. Finally we can show that $\mathfrak{L}^0_\lambda \subset \mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^3_\lambda)$. The computations are similar to those done in the previous step. We define indeed for any $1 \leq s \leq \left[\frac{n-1}{2}\right]$ the sets $\mathfrak{L}^0_s =$

$$\left\{ Y^{(\mathbf{a})}u_\lambda \in \mathfrak{L}^0_\lambda \;\middle|\; \begin{array}{l} \lambda^{2j+1}_{2i-1}(H_{2i}) \neq 0 \text{ and } \lambda^{2j+1}_0(H_1) = 0,\ \lambda^{2j+1}_{2r-1}(H_{2r}) = a^{2j+1}_{2r}, \quad 1 \leq r < i \\ \Rightarrow a^{2j}_{2i+1} = \lambda^{2j}_{2i}(H_{2i+1}), \quad i = 1, \dots 2j-1, \qquad\qquad j = s, \dots, \left[\frac{n+1}{2}\right] \end{array} \right\}$$

and consider the corresponding filtration of $\mathfrak{L}^0_\lambda$

$$\mathfrak{L}^3_\lambda = \mathfrak{L}^0_1 \subset \cdots \subset \mathfrak{L}^0_s \subset \cdots \subset \mathfrak{L}^0_{\left[\frac{n+1}{2}\right]} \subset \mathfrak{L}^0_{\left[\frac{n+1}{2}\right]+1} = \mathfrak{L}^0_\lambda.$$

Again, we need only to prove (always by induction) that $\mathfrak{L}^0_s \subset \mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^0_{s-1})$ for any $1 \leq s \leq \left[\frac{n-1}{2}\right]$.
For a fixed i, $1 \leq i \leq 2j-1$, let $Y^{(\mathbf{a})}u_\lambda \in \mathfrak{L}^0_s$ be of the type

$$Y^{(\mathbf{a})}u_\lambda = \Big(Y_2^{(a^2_2)} \cdots\Big)\Big(\cdots\Big)\Big(Y_{2s}^{(a^{2s}_{2s})} \cdots Y_1^{(a^{2s}_1)}\Big)\Big(Y_{2s+1}^{(a^{2s+1}_{2s+1})} \cdots Y_{2i}^{(\lambda^{2s+1}_{2i-2}(H_{2i-1})+a^{2s}_{2i-1}+1)} \cdots\Big)\Big(\cdots\Big)u_\lambda$$

then the element $Y^{(\mathbf{b(a)})}u_\lambda$ with $b(a)^{2s}_{2s-l} = a^{2s}_{2s-l} + 1$, $0 \leq l \leq 2s-2i$, $b(a)^{2s+1}_{2s+1-l} = a^{2s+1}_{2s+1-l} - 1$, $0 \leq l \leq 2s-2i+1$, $b(a)^h_l = a^h_l$ otherwise, belongs to $\mathfrak{L}^0_{s-1}$ and we have with the same computations of the previous step and the results of [10]:

$$\begin{aligned}
&Y_{2s+1} \cdot Y^{(\mathbf{b(a)})}u_\lambda \\
&= \textstyle\sum_{k=1}^{2s-2i+1} p^s_k(\mathbf{a})\Big(\cdots\Big)\Big(\prod_{l=0}^{k-2} Y_{2s-l}^{(a^{2s}_{2s-l})}\Big)\Big(\prod_{l=k-1}^{2s-1} Y_{2s-l}^{(a^{2s}_{2s-l}+1)}\Big)\Big(\prod_{l=0}^{k-1} Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l})}\Big)\Big(\prod_{l=k}^{2s-1} Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l}-1)}\Big)\Big(\cdots\Big)u_\lambda \\
&+ p^s_{2s-2i+2}(\mathbf{a})Y^{(\mathbf{a})}u_\lambda \\
&+ \textstyle\sum_{k=s-i+2}^{s} p^s_{2k-1}(\mathbf{a})\Big(\cdots\Big)\Big(\prod_{l=0}^{2k-3} Y_{2s-l}^{(a^{2s}_{2s-l})}\Big)\Big(\prod_{l=2k-1}^{2s-1} Y_{2s-l}^{(a^{2s}_{2s-l}+1)}\Big)\Big(\prod_{l=0}^{2k-2} Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l})}\Big) \\
&\Big(\prod_{l=2k-1}^{2s-1} Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l}-1)}\Big)\Big(\cdots\Big)u_\lambda
\end{aligned}$$

which implies that $Y^{(\mathbf{a})}u_\lambda$ belongs to $\mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^0_{s-1})$. Now suppose by induction Hypothesis that we have already constructed all the elements of $\mathfrak{L}^0_s$ with $a^{2s+1}_{2i} = \lambda^{2s+1}_{2i-2}(H_{2i-1}) + a^{2s}_{2i-1} + k$. Then for any element $Y^{(\mathbf{a})}u_\lambda \in \mathfrak{L}^0_s$ of the type

$$Y^{(\mathbf{a})}u_\lambda = \Big(Y_2^{(a^2_2)} \cdots\Big)\Big(\cdots\Big)\Big(Y_{2s}^{(a^{2s}_{2s})} \cdots Y_1^{(a^{2s}_1)}\Big)\Big(Y_{2s+1}^{(a^{2s+1}_{2s+1})} \cdots Y_{2i}^{(\lambda^{2s+1}_{2i-2}(H_{2i-1})+a^{2s-1}_{2i-1}+k+1)} \cdots\Big)\Big(\cdots\Big)u_\lambda$$

the element $Y^{(\mathbf{b(a)})}u_\lambda$ with $b(a)^{2s}_{2s-l} = a^{2s}_{2s-l} + 1$, $0 \leq l \leq 2s-2i$, $b(a)^{2s+1}_{2s+1-l} = a^{2s+1}_{2s+1-l} - 1$, $0 \leq l \leq 2s-2i+1$, $b(a)^h_l = a^h_l$ otherwise, belongs to $\mathfrak{L}^0_{s-1}$ and we have

$$\begin{aligned}
&Y_{2s+1} \cdot Y^{(\mathbf{b(a)})}u_\lambda \\
&= \textstyle\sum_{k=1}^{2s-2i+1} p^s_k(\mathbf{a})\Big(\cdots\Big)\Big(\prod_{l=0}^{k-2} Y_{2s-l}^{(a^{2s}_{2s-l})}\Big)\Big(\prod_{l=k-1}^{2s-1} Y_{2s-l}^{(a^{2s}_{2s-l}+1)}\Big)\Big(\prod_{l=0}^{k} Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l})}\Big)\Big(\prod_{l=k+1}^{2s-1} Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l}-1)}\Big)\Big(\cdots\Big)u_\lambda \\
&+ p^s_{2s-2i+2}(\mathbf{a})Y^{(\mathbf{a})}u_\lambda \\
&+ \textstyle\sum_{k=s-i+2}^{s} p^s_{2k-1}(\mathbf{a})\Big(\cdots\Big)\Big(\prod_{l=0}^{2k-3} Y_{2s-l}^{(a^{2s}_{2s-l})}\Big)\Big(\prod_{l=2k-1}^{2s-1} Y_{2s-l}^{(a^{2s}_{2s-l}+1)}\Big)\Big(\prod_{l=0}^{2k-2} Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l})}\Big) \\
&\Big(\prod_{l=2k-1}^{2s-1} Y_{2s+1-l}^{(a^{2s+1}_{2s+1-l}-1)}\Big)\Big(\cdots\Big)u_\lambda
\end{aligned}$$

which once again implies that $Y^{(\mathbf{b}(\mathbf{a}))}u_\lambda \in \mathcal{U}(\mathfrak{a}_n)(\mathfrak{L}^0_{s-1})$. This closes the proof of the Theorem. ■

In the first non trivial case beyond that treated by Douglas and Premat [5] namely the restriction of irreducible finite dimensional $\mathfrak{sl}(4,\mathbb{C})$–module $V(\lambda)$, $\lambda = n\Lambda_1 + m\Lambda_2 + p\Lambda_3$ to the abelian three dimensional Lie algebra spanned by the element Y_1, X_2, Y_3 the set of generators $\mathfrak{G}_\lambda$ is:

$$\mathfrak{G}_\lambda = \left\{ Y_2^{m-j+i+h} Y_1^i Y_3^{j+h} Y_2^j u_\lambda \quad 0 \le j \le m \quad 0 \le h \le p \quad 0 \le i \le j+n \quad j \ne 0 \Rightarrow i \ne 0 \right\}$$

if $\lambda = n\Lambda_1 + m\Lambda_2 + p\Lambda_3$ with $n > 0$ and:
$\mathfrak{G}_\lambda = \left\{ Y_2^{m+h} Y_1 Y_3^{1+h} Y_2 u_\lambda, Y_2^{m+h} Y_3^h Y_2 u_\lambda \quad 0 \le h \le p \right\}$ if $\lambda = m\Lambda_2 + p\Lambda_3$.
Although we do not need this fact in order to prove that the $\mathfrak{a}_n$–module $V(\lambda)$ are indecomposable, let us first show that the set of generators $\mathfrak{G}$ is a minimal set of generators. We begin with

Lemma 3.3. *No proper subset $\mathfrak{G}'_\lambda$ of $\mathfrak{G}_\lambda$ ($\mathfrak{G}'_\lambda \subsetneq \mathfrak{G}_\lambda$) generates $\mathfrak{G}_\lambda$.*

Proof. It suffices to show that any expression of the form

$$\sum_{g \in \mathfrak{G}_\lambda} P_g(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_g})} u_\lambda \qquad P_g(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_g})} u_\lambda \ne 0 \quad \forall g \in \mathfrak{G}_\lambda \tag{3.1}$$

where $P_g(X_{2j}, Y_{2j+1})$ are non trivial polynomials in the operators X_{2j}, $j = 1, \cdots, \left[\frac{n}{2}\right]$, Y_{2i+1}, $i = 1, \ldots \left[\frac{n-1}{2}\right]$, does not belong to the linear span $\langle \mathfrak{G}_\lambda \rangle$ of $\mathfrak{G}_\lambda$.
Let us denote by $V(\lambda)^-$ the linear span of all element of the Littelmann basis with $(a_k)^{2j}_{2j} < \lambda^{2j}_{2j-1}(H_{2j})$, since for every element of $\mathfrak{G}_\lambda$ yields $a^{2j}_{2j} = \lambda^{2j}_{2j-1}(H_{2j})$, we have $V(\lambda)^- \cap \langle \mathfrak{G}_\lambda \rangle = \{0\}$.
Now from the proof of Theorem 3.2 point 1. for any element $Y^{(\mathbf{a})}u_\lambda$ in $\mathfrak{L}_\lambda$ and any operator X_{2j}, we have $X_{2j} Y^{(\mathbf{a})} u_\lambda = \sum_k c_k Y^{(\mathbf{a_k})} u_\lambda \in V(\lambda)^-$. Therefore it remains only to consider those combinations of the type (3.1) where there exists at least a monomial which contains only operators of odd index. For any such monomial P if $V(\lambda)^+$ is a subspace of $V(\lambda)$ such that $V(\lambda) = V(\lambda)^+ \oplus (V(\lambda)^- \oplus \langle \mathfrak{G}_\lambda \rangle)$, then from the proof of Theorem 3.2 points 3. and 4, it follows that for any $g \in \mathfrak{G}_\lambda$ $Pg = v^P_g + w^P_g$ with $v^P_g \in V(\lambda)^+$, $w^P_g \in (V(\lambda)^- \oplus \langle \mathfrak{G}_\lambda \rangle)$ and $v^P_g \ne 0$, moreover if $g' \ne g$, $g, g' \in \mathfrak{G}_\lambda$, or $P \ne Q$ then v^P_g is linear independent from $v^Q_{g'}$. But then for any expression of type (3.1) where there exist at least a monomial which is a product of only the operators Y_{2j+1} ($i = 1, \ldots \left[\frac{n-1}{2}\right]$) we have

$$\sum_{g \in G} P_g(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_g})} u_\lambda \notin (V(\lambda)^- \oplus \langle \mathfrak{G}_\lambda \rangle)$$

■

Theorem 3.4. *The set $\mathfrak{G}_\lambda$ is a minimal set of $\mathfrak{a}_n$–generators.*

Proof. Let $\{w_1, \ldots w_k\}$ be another set of generators, then for all $1 \le l \le k$, choosing any ordering $G_\lambda = \{1, \ldots, \#(\mathcal{G}_\lambda)\}$ (where $\#(S)$ denotes the number of elements in the set S) of the set $\mathcal{G}_\lambda$, we have:

$$w_l = \sum_{g\in G_\lambda} a_{lg} Y^{(\mathbf{a_g})} u_\lambda + \sum_{g\in G_\lambda} P_{lg}(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_g})} u_\lambda$$

where $a_{lg} \in \mathbb{C}$ and $P_{lg}(X_{2j}, Y_{2j+1})$ are polynomials in the operators X_{2j}, $j = 1, \cdots, \left[\frac{n}{2}\right]$, Y_{2i+1}, $i = 1, \ldots \left[\frac{n-1}{2}\right]$ without constant term. Since the set $\{w_1, \ldots w_k\}$ generates $V(\Lambda)$ we may obtain acting on it the elements of $\mathfrak{G}_\lambda$. Let $\mathcal{T} = \{w_r | \, a_{rg} \neq 0 \text{ for some } g \in G_\lambda\}$ and $T = \{j \mid 1 \leq j \leq k \mid w_r \in S\}$. Let $g \in G_\lambda$. Then

$$Y^{(\mathbf{a_g})} u_\lambda = \sum_{l\in T} b_{gl} \left(\sum_{g'\in G_\lambda} a_{lg'} Y^{(\mathbf{a_{g'}})} u_\lambda \right) + \sum_{g'\in G_\lambda} P'_{tg'}(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_{g'}})} u_\lambda$$

with polynomials $P'_{tg'}$ in the variables X_{2j}, $j = 1, \cdots, \left[\frac{n}{2}\right]$, Y_{2i+1}, $i = 1, \ldots, \left[\frac{n-1}{2}\right]$ without constant term. From the proof of Lemma 3.3 it follows that the sum $\sum_{g'\in G_\lambda} P'_{tg'}(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_{g'}})} u_\lambda$ can not be equal to any combination of elements of $\mathfrak{G}_\lambda$.

$$\text{Hence } Y^{(\mathbf{a_g})} u_\lambda = \sum_{l\in T} b_{gl} \left(\sum_{g'\in G_\lambda} a_{lg'} Y^{(\mathbf{a_{g'}})} u_\lambda \right).$$

This implies that if we put $B = \left(b_{gi}\right)_{\substack{g\in G_\lambda \\ i\in T}}$ and $A = \left(a_{ig}\right)_{\substack{i\in T \\ g\in G_\lambda}}$ then BA is the identity matrix. Hence $k \geq \#(T) \geq \operatorname{rank}(B) \geq \#(G_\lambda)$, so $\mathfrak{G}_\lambda$ is a minimal set of generators. The argument of this proof is due to Premat [13]. ■

Corollary 3.5. *Let $\mathfrak{W} = \{w_1, \ldots w_k\}$ be a set (non necessarily minimal) of $\mathfrak{a}_n$-generators, then there exist a injective map $\phi_{\mathfrak{W}} : \mathfrak{G}_\lambda \to \mathfrak{W}$, such that for every $Y^{(\mathbf{a_g})} u_\lambda \in \mathfrak{G}_\lambda$:*

$$Y^{(\mathbf{a_g})} u_\lambda \mapsto w_{Y^{(\mathbf{a_g})} u_\lambda} = \phi_{\mathfrak{W}}(Y^{(\mathbf{a_g})} u_\lambda) = a_g Y^{(\mathbf{a_g})} u_\lambda + \sum_{g'\in G_\lambda} P_{gg'}(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_{g'}})} u_\lambda \tag{3.2}$$

for some $a_g \in \mathbb{C}$, $a_g \neq 0$, where $P_{gg'}$ are polynomials in the variables X_{2j}, $j = 1, \cdots, \left[\frac{n}{2}\right]$, Y_{2i+1}, $i = 1, \ldots \left[\frac{n-1}{2}\right]$, and the polynomial P_{gg} has no constant term.

Proof. In the proof of Theorem 3.4 we have shown that the elements of $\mathcal{W}$ can be written in the form

$$w_l = \sum_{g\in G_\lambda} a_{lg} Y^{(\mathbf{a_g})} u_\lambda + \sum_{g'\in G_\lambda} P_{lg}(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_{g'}})} u_\lambda$$

where $A = (a_{lt})_{\substack{l=1,\ldots,\#(\mathfrak{W}) \\ t=1,\ldots,\#(G_\lambda)}}$ is a matrix of rank at least $\#(\mathcal{G}_\lambda)$ (recall that $\#(\mathfrak{W}) \leq \#(\mathcal{G}_\lambda)$). This implies that for any $g \in \{1, \ldots \#(\mathcal{G}_\lambda)\}$ we can construct a map $\phi : \{1, \ldots \#(\mathcal{G}_\lambda)\} \to \{1, \ldots \#(\mathcal{G}_\lambda)\}$ such that for any $g \in \{1, \ldots \#(\mathcal{G}_\lambda)\}$, $a_{\phi(g),g}$ is different from zero and $g \neq g'$ implies $\phi(g) \neq \phi(g')$. Then the map $\mathfrak{G}_\lambda \to \mathfrak{W}$

$$\begin{aligned} Y^{(\mathbf{a_g})} u_\lambda \mapsto \quad & \phi_{\mathfrak{W}}(Y^{(\mathbf{a_g})} u_\lambda) = w_{\phi(g)} = a_{\phi(g)g} Y^{(\mathbf{a_g})} u_\lambda + \textstyle\sum_{\substack{g'\in G_\lambda \\ g\neq g'}} a_{\phi(g)g'} Y^{(\mathbf{a_{g'}})} u_\lambda \\ & + \textstyle\sum_{g'\in G_\lambda} P_{\phi(g)g'}(X_{2j}, Y_{2j+1})(Y^{(\mathbf{a_{g'}})} u_\lambda) \end{aligned}$$

is the wanted map. ■

Proposition 3.6. *Any set $\mathfrak{W} = \{w_1, \dots w_k\}$ of homogeneous $\mathfrak{a}_n$–generators contains an element $w_{\overline{g}}$ such that:*

$$w_{\overline{g}} = a_{\overline{g}}\overline{g} \qquad a_{\overline{g}} \neq 0 \in \mathbb{C}.$$

where $\overline{g}$ is the element of the set of $\mathfrak{a}_n$–generators $\mathfrak{G}_\lambda$ given by

$$\overline{g} = Y_{2[\frac{n}{2}]}^{\left(\lambda^{2[\frac{n}{2}]}_{2[\frac{n}{2}]-1}\left(H_{[\frac{n}{2}]}\right)\right)} \cdots Y_{2j}^{(\lambda^{2j}_{2j-1}(H_{2j}))} \cdots Y_2^{(\lambda^2_1(H_2))} u_\lambda.$$

Proof. The $\mathfrak{sl}(n+1, \mathbb{C})$–weight $\mu_{\overline{g}}$ of the element $\overline{g}$ is $\mu_{\overline{g}} = \lambda - \sum_{i=1}^{[\frac{n}{2}]} \lambda^{2i}_{2i-1}(H_{2i})\alpha_{2i}$ since

$$\mu_{\overline{g}} = s_{\alpha_{2[\frac{n}{2}]}} \cdots s_{\alpha_2}(\lambda)$$

we have $\dim(V_{\mu_{\overline{g}}}) = 1$, but then the claim follows from Corollary 3.5 and the first part of this proposition. ∎

Observe that we do not need the fact that $\mathfrak{G}_\lambda$ is a minimal set of generators in order to prove Proposition 3.6. The simple fact that $\mathfrak{G}$ is a set of generators implies

Lemma 3.7. *Let $\mathfrak{W} = \{w_1, \dots w_k\}$ be a set (non necessarily minimal) of $\mathfrak{a}_n$–generators then there exists a $w_{\overline{k}}$ in $\mathfrak{W}$ such that*

$$w_{\overline{k}} = \overline{a}\,\overline{g} + \sum_{g \in \mathfrak{G}_\lambda, g \neq \overline{g}} P_{lg}(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_g})} u_\lambda$$

with $\overline{a}$ complex number different from zero.

Proof. Since:

$$\overline{g} \notin X_{2j}(V(\lambda)) \quad j = 1, \cdots, \left[\frac{n}{2}\right] \qquad \overline{g} \notin Y_{2j+1}(V(\lambda)) \quad i = 1, \dots \left[\frac{n-1}{2}\right],$$

the set $\mathfrak{W}$ is a set of generators of $V(\lambda)$ only if it contains an element w of the form

$$w = \overline{a}\,\overline{g} + \sum_{g \in \mathfrak{G}_\lambda, g \neq \overline{g}} P_{lg}(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_g})} u_\lambda$$

with $\overline{a}$ complex number different from zero. ∎

Using Lemma 3.7 is obviously possible to prove directly Proposition 3.6.

Let $\mathfrak{s}_n = \mathfrak{h} \rtimes \mathfrak{a}_n$ be the subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ given by the semidirect product of the Cartan subalgebra $\mathfrak{h}$ and the subalgebra $\mathfrak{a}_n$. The $\mathfrak{sl}(n+1, \mathbb{C})$–module ($\mathfrak{a}_n$–module) $V(\lambda)$ is also a $\mathfrak{s}_n$–module, on which the subalgebra $\mathfrak{h}$ acts diagonally. Obviously any set of generators of the $\mathfrak{a}_n$–module $V(\lambda)$ is also a set of generators of the $\mathfrak{s}_n$–module $V(\lambda)$. Moreover for what said above any $\mathfrak{s}_n$–submodule of $V(\lambda)$ is a $\mathfrak{sl}(n+1, \mathbb{C})$–weight module, i.e., it can decomposed as a direct sum of $\mathfrak{sl}(n+1, \mathbb{C})$–weight spaces. From these facts it follows the

Proposition 3.8. *If the $\mathfrak{s}_n$–module $V(\lambda)$ decomposes in a direct sum of two subsmodules: $V(\lambda) = U \oplus T$, then $\overline{g}$ belongs either to U or to T.*

Proof. Let $\mathfrak{W}_U = \{w_1, \dots w_j\}$ and $\mathfrak{W}_T = \{w_{j+1}, \dots w_h\}$ be respectively a set of generators of U and of W. Since U and T are $\mathfrak{sl}(n+1, \mathbb{C})$–weight modules we may suppose that both $\mathfrak{W}_U$ and $\mathfrak{W}_T$ are made by homogeneous elements, and therefore $\mathfrak{W} = \mathfrak{W}_U \cup \mathfrak{W}_T = \{w_1, \dots w_j, w_{j+1}, \dots w_h\}$ is a set of homogeneous generators of $V(\lambda)$. Then form proposition 3.6 it follows that there exists an index $\bar{l}$, $1 \le \bar{l} \le h$ such that $\overline{g} = c_{\overline{g}} w_{\bar{l}}$. Hence $\overline{g}$ belongs either to U or to T. ■

Theorem 3.9. *The $\mathfrak{a}_n$–module $V(\lambda)$ is indecomposable.*

Proof. Let us first show that the $\mathfrak{s}_n$–module $V(\lambda)$ is indecomposable. Let us suppose that $V(\lambda)$ is the direct sum $V(\lambda) = U \oplus T$ of two $\mathfrak{s}_n$–modules U and T and let $\mathfrak{W}_U = \{w_1, \dots w_i\}$ (res. $\mathfrak{W}_T = \{w_{i+1}, \dots w_h\}$) be a set of homogeneous generators of U (res. of T). We know from Proposition 3.8 that either $\overline{g}$ belongs to U or to T. Say $\overline{g} \in U$, then we shall show that $V(\lambda) = U$.

We say that an element $Y^{(\mathbf{a})}u_\lambda$ of the Littelmann basis is of level l if l is the minimal nonnegative integer such that $Y^{(\mathbf{a})}u_\lambda = P_l \cdots P_1 u_\lambda$ and any monomial P_j $1 \le j \le l$ is a product of elements Y_i of index either odd or even.

It is immediate to see that all the elements of the Littelmann basis of length 1 and 0 are in $\mathcal{U}(\mathfrak{a})(\overline{g})$ and therefore in U. Let now us suppose by induction that any element in $\mathfrak{G}_\lambda$ of level less or equal l is in U. We need to show that any element in $\mathfrak{G}_\lambda$ of level $l+1$ also belongs to U. First, since any element $Y^{(\mathbf{a})}u_\lambda$ in $\mathfrak{G}_\lambda$ is of the type $Y^{(\mathbf{a})}u_\lambda = Y_{2h}^{a_{2h}^{2h}}\Big(\cdots\Big)u_\lambda$ with $a_{2h}^{2h} \neq 0$, $0 \le h \le \left[\frac{n}{2}\right]$, $\mathfrak{G}_\lambda$ decomposes as

$$\begin{aligned}
\mathfrak{G}_\lambda &= \bigcup\nolimits_{1 \le j_1 \le \cdots \le j_s \le \left[\frac{n}{2}\right]} \mathfrak{G}_{\lambda, j_1, \dots, j_s} \\
\mathfrak{G}_{\lambda, j_1, \dots, j_s} &= \left\{ g \in \mathfrak{G}_\lambda \;\middle|\; \begin{array}{l} g = Y^{(\mathbf{a})}u_\lambda = Y_{2j_1}^{a_{2j_1}^{2j_1}}(\cdots)Y_{2j_s}^{a_{2j_s}^{2j_s}} Y_{2k_1}^{a_{2k+1}^r} u_\lambda \\ \text{with } a_{2j_i}^{2j_i} > 0 \; i = 1, \dots s \; {a^r}_{2k+1} \neq 0, \, k < j_s, \, r > 2j_s. \end{array} \right\}
\end{aligned}$$

Therefore it is enough to show that for any fixed set $\{j_1, \dots j_s\}$ ($1 \le j_1 \le \cdots \le j_s \le \left[\frac{n}{2}\right]$) the elements of length $l+1$ in $\mathfrak{G}_{\lambda, j_1, \dots, j_s}$ belong to U. We shall do it by induction over the orderings $\{\le_{j_1}, \dots, \le_{j_s}\}$ defined in the proof of Theorem 3.2. If $g \in \mathfrak{G}_\lambda$ is minimal with respect all the ordering $\{\le_{j_1}, \dots, \le_{j_s}\}$ then

$$X_{2j_1}^{a_{2j_1}^{2j_1}}\Big(\cdots\Big)X_{2j_s}^{a_{2j_s}^{2j_s}} g = c_{2j_1, \dots, 2j_s}^{2j_1, \dots, 2j_s} Y^{(\mathbf{a})}u_\lambda = c_{2j_1, \dots, 2j_s}^{2j_1, \dots, 2j_s} Y_{2k+1}^{a_{2k+1}^r}\Big(\cdots\Big)u_\lambda$$

with $c_{2j_1, \dots, 2j_s}^{2j_1, \dots, 2j_s} \neq 0$, $a_{2k+1}^r \neq 0$ and $Y^{(\mathbf{a})}u_\lambda = Y_{2k+1}^{a_{2k+1}^r}\Big(\cdots\Big)u_\lambda$ element of $\mathfrak{L}$ of level l. Since $X_{2j_1}^{a_{2j_1}^{2j_1}}\Big(\cdots\Big)X_{2j_s}^{a_{2j_s}^{2j_s}} g$ has been obtained from an element of the set $\mathfrak{G}_\lambda$ of level $l+1$ by erasing the operators Y_{2j_h}, $1 \le h \le s$, and it is of the type $Y_{2k+1}^{a_{2k+1}^r}\Big(\cdots\Big)u_\lambda$ with $a_{2k+1}^r > 0$, $0 \le k \le \left[\frac{n-1}{2}\right]$, by the very definition of the set $\mathfrak{G}_\lambda$ it can be generated by an element of $\mathfrak{G}_\lambda$ of level $l-1$ and it is therefore by induction hypothesis a non trivial element in U. We can now decompose g as $g = g_U + g_T$ with $g_U = \sum_{k=1}^{i} P_k(X_{2s}, Y_{2s+1})w_k \in U$ and $g_T = \sum_{k=i+1}^{h} P_i(X_{2s}, Y_{2s+1})w_k \in T$ and since all the elements w_k are homogeneous, g_U

and g_T are of the same $\mathfrak{sl}(n+1,\mathbb{C})$–weight of g. Now

$$X_{2j_1}^{a_{2j_1}^{2j_1}}\Big(\cdots\Big)X_{2j_s}^{a_{2j_s}^{2j_s}}g_T = X_{2j_1}^{a_{2j_1}^{2j_1}}\Big(\cdots\Big)X_{2j_s}^{a_{2j_s}^{2j_s}}g - X_{2j_1}^{a_{2j_1}^{2j_1}}\Big(\cdots\Big)X_{2j_s}^{a_{2j_s}^{2j_s}}g_U \in U \Longrightarrow X_{2j_1}^{a_{2j_1}^{2j_1}}\Big(\cdots\Big)X_{2j_s}^{a_{2j_s}^{2j_s}}g_T = 0.$$

But the fact that g_U and g_T has the same $\mathfrak{sl}(n+1,\mathbb{C})$–weight of g implies that they have also the same weight of g with respect any subalgebra $\mathfrak{g}_{2j_r}$ spanned by the vector $H_{2j_r}, X_{2j_r}, Y_{2j_r}$ $1 \le r \le s$ and equivalent to the complex simple Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Since $H_{2j_r}g = -a_{2j_r}^{2j_r}g$ with $a_{2j_r}^{2j_r} > 0$ for $1 \le r \le s$ the theory of the $\mathfrak{sl}(2,\mathbb{C})$–finite dimensional modules implies that for $1 \le r \le s$, $X_{2j_r}^{a_{2j_r}^{2j_r}}g_T = 0$ if and only if $g_T = 0$. Hence $g = g_U \in U$. Now, since for any element $\widetilde{g}$ in $\mathfrak{G}_{\lambda,j_1,\dots,j_s}$ which is not a minimal element for at least one of the ordering $\le_{j_s}$ $(1 \le r \le s)$ we have

$$X_{2j_1}^{a_{2j_1}^{2j_1}}\Big(\cdots\Big)X_{2j_s}^{a_{2j_s}^{2j_s}}\widetilde{g} = \widetilde{c}_{2j_1,\dots,2j_s}^{2j_1,\dots,2j_s}Y_{2k+1}^{a_{2k+1}^{r}}\Big(\cdots\Big)u_\lambda + \sum_{\substack{(\mathbf{b})<_{j_r}(\mathbf{a})\\ s=1,\dots r}} \widetilde{c}_{(\mathbf{b})}Y^{(\mathbf{b})}u_\lambda$$

by induction over the orderings $\le_{j_s}$ $(1 \le r \le s)$ we have that $X_{2j_1}^{a_{2j_1}^{2j_1}}\Big(\cdots\Big)X_{2j_s}^{a_{2j_s}^{2j_s}}\widetilde{g} \in U$. Then from the same argument used above $\widetilde{g} \in U$. We have therefore proved that any element of $\mathfrak{G}_\lambda$ of length $l+1$ belong to U, if any element of $\mathfrak{G}_\lambda$ of length l does. Therefore by induction the set of generators $\mathfrak{G}_\lambda$ belongs to U. Since $\mathfrak{G}_\lambda$ generates $V(\lambda)$ under the action of $\mathfrak{a}_n$, we have $V(\lambda) = U$ also as $\mathfrak{a}_n$–module. Hence the $\mathfrak{a}_n$–module $V(\lambda)$ is indecomposable for any integer dominant weight λ. ■

References

[1] Bliem, T., *Weight Multiplicities for* $\mathfrak{so}5(\mathbb{C})$, Proceedings of the 2008 International Conference on Information Theory and Statistical Learning, CSREA Press, 2008, 80–86.

[2] Casati, P., S. Minniti, and V. Solari, *Indecomposable Representations of the Diamond Lie Algebra* J. of Mathematical Physics, to appear.

[3] Casati, P., and G. Ortenzi, *New Integrable Hierarchies from Vertex Operator Representations of Polynomial Lie Algebras*, J. of Geometry and Physics **56** (2006), 418–449.

[4] Cagliero, L., and P. Tirao, *A closed formula for weight multiplicities of representations of* $Sp_2(\mathbb{C})$, Manuscripta mathematica **115** (2004), 417–426.

[5] Douglas, A., and A. Premat, *A Class of Nonunitary, Finite Dimensional Representations of the Euclidian Lie Algebra* $\mathfrak{e}(2)$, Communication in Algebra **35** (2007), 1433–1448.

[6] Humphreys, J., "Introduction to Lie Algebras and Representation Theory," Springer Verlag, Berlin, Heidelberg, New York NY, 1972.

[7] Knapp, A., *Lie Groups Beyond an Introduction*, Progress in Mathematics **140**, Birkhäuser, Boston, 2002. 812+xviii pp.

[8] Littelmann, P., *Cones, crystals, and patterns*, Transform. Groups **3** (1998), 145–179.

[9] —, *Paths and Root Operators in Representation Theory*, Ann. of Math. **142** (1995), 499–525.

[10] —, *An algorithm to compute bases and representation matrices for* $\mathfrak{SL}_{n+1}$*–representations*, J. of Pure and Applied Algebra **117** & **118** (1997), 447–468.

[11] Sai-Ping Li, R. V.Moody, M. Nocolescu, and J. Patera, *Verma bases for representations of classical simple Lie algebras*, J. Math. Phys. **27** (1986), 668–677.

[12] Medina, A., and P. Revoy, *Algèbres de Lie et produit scalaire invariant.* Ann. Sci. École Norm. Sup. (4) **18** (1985), 553–561.

[13] Premat, A., *Indecomposable Representations of the Euclidean Algebra* Preprint, University of York Toronto 2007.

Paolo Casati
Dipartimento di Matematica e applicazioni
Università di Milano-Bicocca
Via Cozzi 53, I-20125 Milano, Italy
casati@matapp.unimib.it

Received February 2, 2010
and in final form May 15, 2010