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Degenerate Principal Series for the Exceptional p-adic Groups of type $\mathbf{F}_4$

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Abstract. We determine reducibility points of degenerate principal series for exceptional p-adic groups of type F_4 via Jacquet module techniques. The number of irreducible subrepresentations and quotients is also determined.
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1. Introduction

In this paper, we analyze degenerate principal series for p-adic F_4. More precisely, let F be a p-adic field with $\mathrm{char} F \neq 2$. We let G be the F-points of a split simply connected p-adic group of type F_4. For degenerate principal series, we take a maximal proper parabolic subgroup $P = MU$ and a one-dimensional representation χ of M, extended trivially to P. The induced representation $\mathrm{Ind}_P^G(\chi)$ is then a degenerate principal series. In this paper, we determine when $\mathrm{Ind}_P^G(\chi)$ is reducible, and when reducible, determine the number of irreducible subrepresentations and quotients which occur, as well as showing they are distinct. We note that for p-adic F_4, the reducibility points for the case where χ is unramified with real infinitesimal character were known by [8], but done using different methods.

An understanding of degenerate principal series is helpful in understanding the representation theory of a group, as well as having applications to automorphic forms (e.g., via the associated Eisenstein series). The reducibility points and composition series of degenerate principal series are already known for p-adic general linear groups ([21]), symplectic and orthogonal groups ([11],[3]), and the exceptional group of type G_2 ([14]).

Our analysis uses Jacquet module methods. The systematic use of Jacquet modules to study induced representations has its roots in the work of Tadić ([18],[16],[20]), which in turn have their roots in the work of Zelevinsky ([21],[22]). We remark that the situation at hand is more difficult in one important respect:

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unlike the case of classical or general linear groups, the structure of F_4 does not allow for a convenient explicit realization, so we must work via root data. The approach used is therefore more general, and could be applied to other exceptional groups. In practical terms, to deal with larger exceptional groups, it seems likely that either a greater use of computer calculations or a shift in emphasis to Jacquet modules with respect to maximal parabolic subgroups would be needed to efficiently carry out the analysis.

We close by briefly describing how this paper is laid out. The next section reviews the structure of F_4, as well as recalling some results on Jacquet modules which are needed later. The third section contains the main result on reducibility of degenerate principal series for F_4 (Theorem 3.1). The fourth and fifth sections contain the proof of this result; the fourth section covers the regular case, while the fifth section covers the non-regular case. We note that the nonregular cases are subtler and done on a case-by-case basis. In the last section, we give some results on the structure of composition series for reducible degenerate principal series. In particular, Theorem 6.1 gives the number of irreducible subrepresentations and quotients for the reducible degenerate principal series. We close with an appendix giving double-coset representatives used in the Jacquet module arguments.

Some of the material from this paper is based on the first author's thesis ([7]), and he would like to thank Allen Moy for his guidance. The second author would like to thank Salman Abdulali and Mahdi Asgari for answering questions which arose during this work. Finally, the authors would like to thank the referees for comments and corrections which were helpful in the revision of this paper.

2. Preliminaries and Methodology

We begin this section with a discussion of F_4. We then review results needed to do the Jacquet module arguments in this paper.

Let F be a p-adic field with $\mathrm{char} F \neq 2$, ϖ a uniformizer. Let G be the F-points of a split group of type F_4 defined over F. Fix a minimal parabolic subgroup $B = AU_{min}$ for G. We denote the simple roots by $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where α_1 and α_2 are the short roots while α_3 and α_4 are the long roots. Since G is simply connected, every element $h \in A$ can be written uniquely as

$$h(t_1, t_2, t_3, t_4) = \alpha_1^\vee(t_1)\alpha_2^\vee(t_2)\alpha_3^\vee(t_3)\alpha_4^\vee(t_4)$$

where $\alpha_i^\vee$ is the coroot associated with α_i. Note that this allows us to write a character of A in the form $\chi = \chi_1 \otimes \chi_2 \otimes \chi_3 \otimes \chi_4$, where χ_i is a character of $F^\times$ and

$$\chi(h(t_1, t_2, t_3, t_4)) = \chi_1(t_1)\chi_2(t_2)\chi_3(t_3)\chi_4(t_4).$$

Let W denote the Weyl group for G. To describe the action of W on characters, it suffices to describe the action of simple reflections. For $\alpha_i \in \Delta$, let s_i denote the corresponding simple reflection. The action of the simple reflections on characters is summarized below:

$$\begin{aligned}
s_1(\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4) &= \chi_1^{-1}\otimes\chi_1\chi_2\otimes\chi_3\otimes\chi_4\\
s_2(\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4) &= \chi_1\chi_2\otimes\chi_2^{-1}\otimes\chi_2\chi_3\otimes\chi_4\\
s_3(\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4) &= \chi_1\otimes\chi_2\chi_3^2\otimes\chi_3^{-1}\otimes\chi_3\chi_4\\
s_4(\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4) &= \chi_1\otimes\chi_2\otimes\chi_3\chi_4\otimes\chi_4^{-1}.
\end{aligned}$$

Note that we often write a character χ of $F^\times$ in the form $\chi=\nu^s\chi_0$, where $\nu=|\cdot|$, $s\in\mathbb{R}$ and χ_0 unitary.

To each $\Theta\subset\Delta$, we associate the standard parabolic subgroup $P_\Theta = M_\Theta U_\Theta=\langle B, s_i \mid \alpha_i\in\Theta\rangle$ (identifying s_i with a representative in G). We let M_Θ denote the Levi factor of P_Θ. There are four maximal standard Levi subgroups in G. For each $i=1,2,3,4$, we define $P_i=M_iU_i=P_{\Delta-\{\alpha_i\}}$. We note that the following isomorphisms are used later (cf. [1]):

$$M_{\alpha_i}\cong GL(2,F)\times F^\times\times F^\times$$

for $i=1,2,3,4$, and

$$M_1\cong GSpin(7,F),\qquad M_4\cong GSp(6,F).$$

We review results need for our Jacquet module arguments in a more general setting. Let M,N be Levi factors of standard parabolic subgroups of G. We let i_M^G and r_M^G denote the functors for (normalized) parabolic induction and the (normalized) Jacquet module, respectively (cf. [4]). Set

$$W^{MN}=\{w\in W\mid w(\Phi_M^+)\subset\Phi^+, w^{-1}(\Phi_N^+)\subset\Phi^+\},$$

where Φ^+ denotes the set of positive roots. The following is a theorem of Bernstein-Zelevinsky [4], Casselman [6]:

Theorem 2.1. *Let Ω be an admissible representation of M. Then, $r_N^G\circ i_M^G(\Omega)$ has a composition series with factors $i_{N'}^N\circ w\circ r_{M'}^M(\Omega)$, $w\in W^{MN}$ where $M'=M\cap w^{-1}(N)$, $N'=w(M)\cap N$.*

Let $s_1,\ldots,s_n$ be the simple reflections in W. Let $B=AU_{min}$ be the minimal parabolic, and N_i the Levi factor of $P_{\{s_i\}}$. For $\pi=i_M^G(\Omega)$, we denote by $BZ_N(\pi)$ the collection of representations $i_{N'}^N\circ w\circ\tau$ as τ runs over the components of $r_{M'}^M(\Omega)$ and w runs over W^{MN}. We use the following theorem to determine reducibility points in regular cases. This theorem requires three hypotheses. First, we want Ω to be irreducible. Second, we want $r_A^M(\Omega)\neq 0$. Third, we require a regularity condition on Ω: if ψ is a character in $r_A^M(\Omega)$, we require that ψ be regular with respect to W. The following is Theorem 3.1.2 of [9]:

Theorem 2.2. *Under the three conditions above, the following are equivalent :*
1. π is irreducible
2. σ is irreducible for any i and $\sigma\in BZ_{N_i}(\pi)$.

Theorem 2.2 provides a criterion to determine the reducibility points in regular cases.

We remark that in much of this paper, we work in the Grothendieck group setting. In particular, we have write $\pi = \pi_1 + \pi_2$ if $\text{mult}(\rho, \pi) = \text{mult}(\rho, \pi_1) + \text{mult}(\rho, \pi_2)$ for any irreducible ρ, where $\text{mult}(\rho, \pi)$ denotes the multiplicity of ρ in π. Similarly, $\pi \leq \pi'$ if $\text{mult}(\rho, \pi) \leq \text{mult}(\rho, \pi')$ for every irreducible ρ. When we mean an equivalence rather than just an equality in the Grothendieck group, we write $\pi \cong \pi'$ instead of $\pi = \pi'$.

The Langlands classification is used to deal with some of the nonregular cases; we take a moment to note some relevant facts ([5], [17], [13]). We note that we use the subrepresentation version of the Langlands classification as it is a bit more amenable to Jacquet module arguments (by Frobenius reciprocity, the inducing representation appears in the corresponding Jacquet module). In this setting, the Langlands subrepresentation is the unique irreducible subrepresentation of some $i_M^G(\exp \mu \otimes \tau)$, with τ an irreducible tempered representation of M and $\mu \in (\mathfrak{a}_M^*)_-$ (cf. [5], [17], [13] for notation). Note that if $M = A$, $\exp \mu \otimes \tau$ has the form $\nu^{x_1}\chi_1 \otimes \nu^{x_2}\chi_2 \otimes \nu^{x_3}\chi_3 \otimes \nu^{x_4}\chi_4$, with $x_i < 0$ and χ_i unitary for all i. In our applications, we have either $M = A$ and $\exp \mu \otimes \tau$ of this form, or there is some $I \subset \{1, 2, 3, 4\}$ such that $x_i = 0$ for $i \in I$, $M = M_{\{\alpha_i \,|\, i \in I\}}$ and $\exp \mu \otimes \tau = i_A^M(\nu^{x_1}\chi_1 \otimes \nu^{x_2}\chi_2 \otimes \nu^{x_3}\chi_3 \otimes \nu^{x_4}\chi_4)$ (with the irreducibility of $\exp \mu \otimes \tau$ known).

3. Main Results

The following contains the main results on reducibility. The proof is done in sections 4 (regular cases) and 5 (nonregular cases). We note that in the case of real infinitesimal character–$\chi_0 = 1$ in the theorem below–this recovers the results of [8] using a different approach.

Theorem 3.1. *Let G be a split group of type F_4 and Ω a character of the Levi factor M of a maximal parabolic subgroup of G, which is defined by a character χ of $F^\times$ as described below. We write $\chi = \nu^s \chi_0$, where $\nu = |\cdot|$, $s \in \mathbb{R}$ and χ_0 is a unitary character.*
(1) Let $M = M_1$ and Ω be defined by $\Omega(h(t_1, t_2, t_3, t_4)) = \chi(t_1)$. Then $i_M^G(\Omega)$ is reducible if and only if $s = \pm 11/2, \pm 5/2, \pm 1/2$ with $\chi_0 = 1$, or $s = \pm 1/2$, χ_0 with of order two.
(2) Let $M = M_2$ and Ω be defined by $\Omega(h(t_1, t_2, t_3, t_4)) = \chi(t_2)$. Then $i_M^G(\Omega)$ is reducible if and only if $s = \pm 7/2, \pm 5/2, \pm 3/2, \pm 1/2$ with $\chi_0 = 1$, $s = \pm 3/2, \pm 1/2$ with χ_0 of order two, or $s = \pm 1/2$ with χ_0 of order three.
(3) Let $M = M_3$ and Ω be defined by $\Omega(h(t_1, t_2, t_3, t_4)) = \chi(t_3)$. Then $i_M^G(\Omega)$ is reducible if and only if $s = \pm 5/2, \pm 3/2, \pm 1, \pm 1/2$ with $\chi_0 = 1$, $s = \pm 3/2, \pm 1, \pm 1/2$ with χ_0 of order two, $s = \pm 1/2$ with χ_0 of order three, or $s = \pm 1/2$, χ_0 of order four.
(4) Let $M = M_4$ and Ω be defined by $\Omega(h(t_1, t_2, t_3, t_4)) = \chi(t_4)$. Then $i_M^G(\Omega)$ is reducible if and only if $s = \pm 4, \pm 2, \pm 1$ with $\chi_0 = 1$, or $s = \pm 2$ with χ_0 of order two.

4. Regular Cases

For each maximal standard Levi subgroup of G, we determine the regularity conditions on $r_A^M(\Omega)$ and the reducibility points of $BZ_{N_i}(\pi)$ (i=1,2,3,4). We apply Theorem 2.2 with this computation to find the reducibility points for the regular cases.

We use the following result for $G_0 = GL(2,F)$ in [21]: $i_{A_0}^{G_0}(\chi_0)$ is reducible if and if $\chi_0/\chi_0^{s_0}(diag(a,b)) = \nu^{\pm 1}(a/b)$ where $diag(a,b) \in A_0$ and s_0 is a simple reflection in G_0. To use this result for $GL(2,F)$, we need to describe $diag(p,q)$ inside $GL(2,F)$ of N_1, N_2, N_3, and N_4 which are isomorphic to $GL(1,F)\times GL(1,F)\times GL(2,F)$ [1]: $diag(p,q)$ in N_1 is expressed as $h(p,pq,r,s)$; $diag(q,r)$ in N_2 is expressed as $h(p,q,qr/p,s)$; $diag(r,s)$ in N_3 is expressed as $h(p,q,r,rs/q^2)$; and $diag(r,s)$ in N_4 is expressed as $h(p,q,rs,s)$. Combining these observations, we see that $i_A^{N_i}(\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4)$ is reducible if and only if $\chi_i = \nu^{\pm 1}$.

Lemma 4.1. *Let χ be a character of $F^\times$.*

1. *Let Ω be a character of M_1 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_1)$. Then $r_A^{M_1}(\Omega)$ is nonregular if and only if $\chi = \nu^{\pm 9/2,\pm 7/2,\pm 5/2,\pm 3/2,\pm 1/2}$ or $\chi^2 = 1$.*

2. *Let Ω be a character of M_2 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_2)$. Then $r_A^{M_2}(\Omega)$ is nonregular if and only if $\chi = \nu^{\pm 5/2,\pm 3/2,\pm 1/2}$, $\chi^2 = \nu^{\pm 2,\pm 1,0}$, or $\chi^3 = \nu^{\pm 1/2}$.*

3. *Let Ω be a character of M_3 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_3)$. Then $r_A^{M_3}(\Omega)$ is nonregular if and only if $\chi = \nu^{\pm 3/2,\pm 1/2}$, $\chi^2 = \nu^{\pm 2,\pm 1,0}$, $\chi^3 = \nu^{\pm 1/2}$, or $\chi^4 = \nu^{\pm 1,0}$.*

4. *Let Ω be a character of M_4 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_4)$. Then $r_A^{M_4}(\Omega)$ is nonregular if and only if $\chi = \nu^{\pm 3,\pm 2,\pm 1,0}$ or $\chi^2 = \nu^{\pm 3,\pm 2,\pm 1,0}$.*

Proof. We start with (1). Set $\psi = r_A^{M_1}(\Omega)$, so that $\psi = \nu^{9/2}\chi\otimes\nu^{-1}\otimes\nu^{-1}\otimes\nu^{-1}$. For $w = s_1$,

$$w\cdot\psi = \nu^{-9/2}\chi^{-1}\otimes\nu^{7/2}\chi\otimes\nu^{-1}\otimes\nu^{-1}$$

(see section 2). Then, $\psi = w\cdot\psi$ implies that $\chi = \nu^{-9/2}$. Similar calculations for the other elements of W check the regularity of ψ and give us the regularity condition on $r_A^M(\Omega)$.

For (2), we use $\psi = \nu^{-1}\otimes\nu^{5/2}\chi\otimes\nu^{-1}\otimes\nu^{-1} = r_A^{M_2}(\Omega)$, and argue as above. For (3), we have $\psi = \nu^{-1}\otimes\nu^{-1}\otimes\nu^{3/2}\chi\otimes\nu^{-1} = r_A^{M_3}(\Omega)$. For (4), we use $\psi = \nu^{-1}\otimes\nu^{-1}\otimes\nu^{-1}\otimes\nu^{3}\chi = r_A^{M_4}(\Omega)$. ■

Lemma 4.2. *Let χ be a character of $F^\times$.*

1. *Let Ω be a character of M_1 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_1)$. All the BZ composition factors of $r_{N_i}^G(\pi)$ (i=1,2,3,4) are irreducible except when $\chi = \nu^{\pm 11/2,\pm 9/2,\pm 7/2,\pm 5/2,\pm 3/2,\pm 1/2}$ or $\chi^2 = \nu^{\pm 1}$.*

2. *Let Ω be a character of M_2 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_2)$. All the BZ composition factors of $r^G_{N_i}(\pi)$ (i=1,2,3,4) are irreducible except when $\chi = \nu^{\pm 7/2,\pm 5/2,\pm 3/2,\pm 1/2}$, $\chi^2 = \nu^{\pm 3,\pm 2,\pm 1,0}$, or $\chi^3 = \nu^{\pm 3/2,\pm 1/2}$.*

3. *Let Ω be a character of M_3 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_3)$. All the BZ composition factors of $r^G_{N_i}(\pi)$ (i=1,2,3,4) are irreducible except when $\chi = \nu^{\pm 5/2,\pm 3/2,\pm 1/2}$, $\chi^2 = \nu^{\pm 3,\pm 2,\pm 1,0}$, $\chi^3 = \nu^{\pm 3/2,\pm 1/2}$, or $\chi^4 = \nu^{\pm 2,\pm 1,0}$.*

4. *Let Ω be a character of M_4 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_4)$. All the BZ composition factors of $r^G_{N_i}(\pi)$ (i=1,2,3,4) are irreducible except when $\chi = \nu^{\pm 4,\pm 3,\pm 2,\pm 1,0}$ or $\chi^2 = \nu^{\pm 4,\pm 3,\pm 2,\pm 1,0}$.*

Proof. We start with (1). In case of $N = N_1$,

$$\begin{aligned} W^{M_1N_1} = \{&1, s_2s_1, s_3s_2s_1, s_2s_3s_2s_1, s_4s_3s_2s_1, s_4s_2s_3s_2s_1, s_3s_4s_2s_3s_2s_1,\\ &s_2s_3s_4s_2s_3s_2s_1, s_2s_3s_4s_1s_2s_3s_2s_1, s_3s_2s_3s_4s_1s_2s_3s_2s_1,\\ &s_4s_3s_2s_3s_4s_1s_2s_3s_2s_1, s_2s_3s_1s_2s_3s_4s_1s_2s_3s_2s_1, s_4s_2s_3s_1s_2s_3s_4s_1s_2s_3s_2s_1,\\ &s_3s_4s_2s_3s_1s_2s_3s_4s_1s_2s_3s_2s_1, s_2s_3s_4s_2s_3s_1s_2s_3s_4s_1s_2s_3s_2s_1\} \end{aligned}$$

(see the appendix). For $w = s_2s_1$, $s_3s_2s_1$, $s_4s_3s_2s_1$, $s_2s_3s_1s_2s_3s_4s_1s_2s_3s_2s_1$, $s_4s_2s_3s_1s_2s_3s_4s_1s_2s_3s_2s_1$, $s_3s_4s_2s_3s_1s_2s_3s_4s_1s_2s_3s_2s_1$ in W^{M_1N}, $N' = N_1$ and $M' = w^{-1}(N_1)$ imply that $i^{N_1}_{N'} \circ w \circ \psi$ is irreducible. In the remaining cases of $w \in W^{M_1N_1}$, $N' = M' = A$ and $i^{N_1}_{N'} \circ w \circ \psi$ is irreducible except when $\chi = \nu^{\pm 11/2,\pm 7/2,\pm 3/2,\pm 1/2}$ or $\chi^2 = \nu^{\pm 1}$. For example, consider $w = 1$. Then $i^{N_1}_A \circ w \circ \psi = i^{N_1}_A(\nu^{9/2}\chi \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1})$ is reducible if and only if $\nu^{9/2}\chi = \nu^{\pm 1}$, i.e., $\chi = \nu^{-7/2,-11/2}$. Similarly, we check the reducibility of the remaining BZ composition factors of $r^G_{N_i}(\pi)$, i=1,2,3,4.

Cases (2),(3), and (4) are similar. ∎

Corollary 4.3. *Let χ be a character of $F^\times$.*

1. *Let Ω be a character of M_1 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_1)$. Under the regularity condition on $r^{M_1}_A(\Omega)$, $\pi = i^G_M(\Omega)$ is reducible if and only if $\chi = \nu^{\pm 11/2}$ or $\chi = \nu^{\pm 1/2}\chi_0$ with χ_0 of order two.*

2. *Let Ω be a character of M_2 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_2)$. Under the regularity condition on $r^{M_2}_A(\Omega)$, $\pi = i^G_M(\Omega)$ is reducible if and only if $\chi = \nu^{\pm 7/2}$, $\chi = \nu^{\pm 3/2}\chi_0$ with χ_0 of order two, or $\chi = \nu^{\pm 1/2}\chi_0$ with χ_0 of order three.*

3. *Let Ω be a character of M_3 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_3)$. Under the regularity condition on $r^{M_3}_A(\Omega)$, $\pi = i^G_M(\Omega)$ is reducible if and only if $\chi = \nu^{\pm 5/2}$, $\chi = \nu^{\pm 3/2}\chi_0$ with χ_0 of order two, $\chi = \nu^{\pm 1/2}\chi_0$ with χ_0 of order three, or $\chi = \nu^{\pm 1/2}\chi_0$ with χ_0 of order four.*

4. *Let Ω be a character of M_4 defined by $\Omega(h(t_1,t_2,t_3,t_4)) = \chi(t_4)$. Under the regularity condition on $r^{M_4}_A(\Omega)$, $\pi = i^G_M(\Omega)$ is reducible if and only if $\chi = \nu^{\pm 4}$ or $\chi = \nu^{\pm 1}\chi_0$ with χ_0 of order two.*

Proof. This follows from Theorem 2.2 and the two preceding lemmas. ∎

5. Nonregular cases

In this section, we use Jacquet module methods to determine reducibility in the nonregular cases. We remark that in [7], this is done for those cases not having real infinitesimal character–the case of real infinitesimal character being known from [8]–using the Hecke algebra results of [15] to reduce the problem to simpler groups. We take a different approach here to correct certain errors in [7], give a more stylistically consistent approach, and drop certain restrictions on F required by [15].

In the following lemma, we use the shorthand of [21] and [19] (and its obvious extension to $GSpin$ groups, cf. [2]) for induced representations. The realizations of $GSp(6,F)$ and $GSpin(7,F)$ are the usual ones (e.g., [19],[2]). We note that some of these results are not used until the next section.

Lemma 5.1. *For $GSp(2n,F)$ we let $triv_n(\chi)$ denote the one-dimensional subrepresentation of $\nu^{-n}\times\nu^{-n+1}\times\cdots\times\nu^{-1}\rtimes\chi$. Let χ_0 denote a character of order two. We have the following results on degenerate principal series for $GSp(6,F)$:*

1. $1_{GL(1)}\rtimes triv_2(\nu^{-1})\cong\sigma_1\oplus\sigma_2$ *with*

$$r_A^G(\sigma_1)=1\otimes\nu^{-2}\otimes\nu^{-1}\otimes\nu^{-1}+\nu^{-2}\otimes 1\otimes\nu^{-1}\otimes\nu^{-1}\ \textit{and}$$

$$r_A^G(\sigma_2)=1\otimes\nu^{-2}\otimes\nu^{-1}\otimes\nu^{-1}+\nu^{-2}\otimes 1\otimes\nu^{-1}\otimes\nu^{-1}+2\nu^{-2}\otimes\nu^{-1}\otimes 1\otimes\nu^{-1}.$$

2. $\nu^{\frac{1}{2}}\circ det_{GL(2)}\rtimes triv_1(\nu^{-1})$ *has a unique irreducible quotient* σ_1 *and unique irreducible subrepresentation* σ_2*. We have*

$$r_A^G(\sigma_1)=4\nu^{-1}\otimes\nu^{-1}\otimes 1\otimes 1+2\nu^{-1}\otimes 1\otimes\nu^{-1}\otimes 1+\nu^{-1}\otimes 1\otimes\nu\otimes\nu^{-1}\ \textit{and}$$

$$r_A^G(\sigma_2)=\nu^{-1}\otimes 1\otimes\nu^{-1}\otimes 1+2\cdot 1\otimes\nu^{-1}\otimes\nu^{-1}\otimes 1+1\otimes\nu\otimes\nu^{-1}\otimes\nu^{-1}+1\otimes\nu^{-1}\otimes\nu\otimes\nu^{-1}.$$

3. $\chi_0\circ det_{GL(2)}\rtimes triv_1(\nu^{-1})$ *is irreducible.*

4. $\chi_0\circ det_{GL(3)}\rtimes\nu^{-1}$ *is irreducible.*

5. $1_{GL(3)}\rtimes\nu^2\cong\sigma_1\oplus\sigma_2$, *with*

$$r_A^G(\sigma_1)=\nu^{-1}\otimes 1\otimes\nu\otimes\nu^{-2}+2\nu^{-1}\otimes 1\otimes\nu^{-1}\otimes\nu^{-1}+4\nu^{-1}\otimes\nu^{-1}\otimes 1\otimes\nu^{-1}$$

$$\textit{and}\ r_A^G(\sigma_2)=\nu^{-1}\otimes 1\otimes\nu\otimes\nu^{-2}.$$

For $GSpin(2n+1,F)$ we let $triv_n(\chi)$ denote the one-dimensional subrepresentation of $\nu^{\frac{-n+1}{2}}\chi\times\nu^{\frac{-n+3}{2}}\chi\times\cdots\times\nu^{-\frac{1}{2}}\chi\rtimes\chi^2$. Let χ_0 denote a character of order two. We have the following results on degenerate principal series for $GSpin(7,F)$:

1. $\nu^{-\frac{3}{2}}\circ det_{GL(2)}\rtimes triv_1(\nu^{-\frac{3}{2}})$ *is irreducible.*

2. $\nu^{-1}\circ det_{GL(3)}\rtimes\nu^{-2}\chi_0$ *is irreducible.*

3. $\nu^{-1}\chi_0 \circ det_{GL(3)} \rtimes \nu^{-1} = \sigma_1 + \sigma_2 + \sigma_3$.

We have $r_A^G(\sigma_1) = 2\nu^{-2}\chi_0 \otimes \nu^{-1}\chi_0 \otimes \nu^{-1}\chi_0 \otimes \nu^{-1}$

$$+\nu^{-2}\chi_0 \otimes \nu^{-1}\chi_0 \otimes \chi_0 \otimes \nu^{-1} + \nu^{-1}\chi_0 \otimes \nu^{-2}\chi_0 \otimes \nu^{-1}\chi_0 \otimes \nu^{-1},$$

$r_A^G(\sigma_2) = \nu^{-2}\chi_0 \otimes \nu^{-1}\chi_0 \otimes \chi_0 \otimes \nu^{-1}$, *and* $r_A^G(\sigma_3) = \nu^{-1}\chi_0 \otimes \nu^{-2}\chi_0 \otimes \chi_0 \otimes \nu^{-1}$

$$+\nu^{-1}\chi_0 \otimes \chi_0 \otimes \nu\chi_0 \otimes \nu^{-1} + \nu^{-1}\chi_0 \otimes \chi_0 \otimes \nu^{-2}\chi_0 \otimes \nu^{-1}.$$

Further, $\sigma_1 \oplus \sigma_2$ *appears as a subrepresentation and* σ_3 *as the unique irreducible quotient.*

Proof. The arguments are like those used in [10]; the composition series structure follows easily from the Jacquet modules and Frobenius reciprocity. Note that the results on reducibility for $GSp(4,F) \cong GSpin(5,F)$ (resp., $GL(3,F)$) needed to do these may be found in [16] (resp., [21]). ∎

Remark 5.2. Consider the character $\chi_1 \otimes \chi_2 \otimes \chi_3 \otimes \chi_4$ of $A \subset F_4$ (which is also the maximal split torus for M_1 and M_4). Now, we have $M_4 = M_{\alpha_1,\alpha_2,\alpha_3} \cong GSp(6,F)$ and $M_1 = M_{\alpha_2,\alpha_3,\alpha_4} \cong GSpin(7,F)$. With the usual realizations of $GSp(6,F)$ and $GSpin(7,F)$ (e.g., those used by Tadić and Asgari, resp.), $\chi_1 \otimes \chi_2 \otimes \chi_3 \otimes \chi_4$ may be identified with the following:

1. under $M_4 \cong GSp(6,F)$, it is identified with $\lambda_1 \otimes \lambda_2 \otimes \lambda_3 \otimes \lambda$, where

$$\lambda_1 = \chi_1\chi_2\chi_3, \quad \lambda_2 = \chi_2\chi_3, \quad \lambda_3 = \chi_3, \quad \lambda = \chi_4.$$

   Note that in this case, the simple roots $\alpha_1, \alpha_2, \alpha_3$ of M_4 correspond to the simple roots $\alpha'_1, \alpha'_2, \alpha'_3$, resp., of (the usual realization of) $GSp(6,F)$. In the opposite direction,

$$\chi_1 = \lambda_1\lambda_2^{-1}, \quad \chi_2 = \lambda_2\lambda_3^{-1}, \quad \chi_3 = \lambda_3, \quad \chi_4 = \lambda.$$

2. under $M_1 \cong GSpin(7,F)$, it is identified with $\lambda_1 \otimes \lambda_2 \otimes \lambda_3 \otimes \lambda$, where

$$\lambda_1 = \chi_1\chi_2^2\chi_3^3\chi_4^2, \quad \lambda_2 = \chi_1\chi_2^2\chi_3^3\chi_4, \quad \lambda_3 = \chi_1\chi_2^2\chi_3^2\chi_4, \quad \lambda = \chi_1^2\chi_2^3\chi_3^4\chi_4^2.$$

   Note that in this case, the simple roots $\alpha_2, \alpha_3, \alpha_4$ of M_1 correspond to the simple roots $\alpha'_3, \alpha'_2, \alpha'_1$, resp., of (the usual realization of) $GSpin(7,F)$. In the opposite direction,

$$\chi_1 = \lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1}\lambda^2, \quad \chi_2 = \lambda_3^2\lambda^{-1}, \quad \chi_3 = \lambda_2\lambda_3^{-1}, \quad \chi_4 = \lambda_1\lambda_2^{-1}.$$

Proposition 5.3. *With notation as in Theorem 3.1, the reducibility for* $i_M^G(\Omega)$ *for nonregular cases (cf. Lemma 4.1) is as follows:*

1. $M = M_1$

We have reducibility for $s = \pm 5/2, \pm 1/2$ *with* $\chi_0 = 1$. *We have irreducibility for* $s = \pm 9/2, \pm 7/2, \pm 3/2, 0$ *with* $\chi_0 = 1$, *and* $s = 0$ *with* χ_0 *of order two.*

2. $M = M_2$

We have reducibility for $s = \pm 5/2, \pm 3/2, \pm 1/2$ *with* $\chi_0 = 1$*, and* $s = \pm 1/2$ *with* χ_0 *of order two. We have irreducibility for* $s = \pm 1, \pm 1/6, 0$ *with* $\chi_0 = 1$*, and* $s = \pm 1, 0$ *with* χ_0 *of order two, and* $s = \pm 1/6$ *with* χ_0 *of order three.*

3. $M = M_3$

We have reducibility for $s = \pm 3/2, \pm 1, \pm 1/2$ *with* $\chi_0 = 1$*, and* $s = \pm 1, \pm 1/2$ *with* χ_0 *of order two. We have irreducibility for* $s = \pm 1/4, \pm 1/6, 0$ *with* $\chi_0 = 1$*;* $s = 0$ *with* χ_0 *of order two;* $s = \pm 1/6$ *with* χ_0 *of order three;* $s = \pm 1/4, 0$ *with* χ_0 *of order four.*

4. $M = M_4$

We have reducibility for $s = \pm 2, \pm 1$ *with* $\chi_0 = 1$*. We have irreducibility for* $s = \pm 3, \pm 3/2, \pm 1/2, 0$ *with* $\chi_0 = 1$*, and* $s = \pm 3/2, \pm 1, \pm 1/2, 0$ *with* χ_0 *of order two.*

Proof. By contragredience, it suffices to restrict our attention to $s \leq 0$.

We first address the reducibility cases. The basic strategy for proving reducibility is the same for each case. One finds an induced representation $i_L^G(\lambda)$ such that the following hold: (1) $i_M^G(\Omega)$ and $i_L^G(\lambda)$ have an irreducible subquotient π_1 in common, and (2) $r_A^G(i_M^G(\Omega))$ is not contained in $r_A^G(i_L^G(\lambda))$. This clearly suffices; were $i_M^G(\Omega)$ irreducible, (1) would force $i_M^G(\Omega) \leq i_L^G(\lambda)$, from which it would follow that $r_A^G(i_M^G(\Omega)) \leq r_A^G(i_L^G(\lambda))$, contradicting (2).

Consider the case $M = M_3$, $s = -1$, and χ_0 of order two. We note that $r_A^G(i_{M_3}^G(\Omega))$ may be calculated using Theorem 2.1 and the results in the appendix. Doing so, one sees that $\nu^{-1} \otimes 1 \otimes \nu^{-\frac{1}{2}}\chi_0 \otimes \nu^{-\frac{1}{2}}\chi_0 \leq r_A^G(i_{M_3}^G(\Omega))$. Let π_1 be an irreducible subquotient of $i_{M_4}^G(\Omega)$ which contains $1 \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi_0 \otimes \nu^{-\frac{1}{2}}\chi_0$ in its Jacquet module. By central character considerations,

$$\pi \hookrightarrow i_A^G(\nu^{-1} \otimes 1 \otimes \nu^{-\frac{1}{2}}\chi_0 \otimes \nu^{-\frac{1}{2}}\chi_0) \cong i_{M_{\alpha_2}}^G\left(i_A^{M_{\alpha_2}}(\nu^{-1} \otimes 1 \otimes \nu^{-\frac{1}{2}}\chi_0 \otimes \nu^{-\frac{1}{2}}\chi_0)\right).$$

Now, $i_A^{M_{\alpha_2}}(\nu^{-1} \otimes 1 \otimes \nu^{-\frac{1}{2}}\chi_0 \otimes \nu^{-\frac{1}{2}}\chi_0)$ is an irreducible representation of M_{α_2} and satisfies the requirements for Langlands data in the subrepresentation setting of the Langlands classification. In particular, $L(i_A^{M_{\alpha_2}}(\nu^{-1} \otimes 1 \otimes \nu^{-\frac{1}{2}}\chi_0 \otimes \nu^{-\frac{1}{2}}\chi_0))$ is the unique irreducible subrepresentation of $i_{M_{\alpha_2}}^G(i_A^{M_{\alpha_2}}(\nu^{-1} \otimes 1 \otimes \nu^{-\frac{1}{2}}\chi_0 \otimes \nu^{-\frac{1}{2}}\chi_0))$, hence $\pi_1 = L(i_A^{M_{\alpha_2}}(\nu^{-1} \otimes 1 \otimes \nu^{-\frac{1}{2}}\chi_0 \otimes \nu^{-\frac{1}{2}}\chi_0))$. A similar argument shows that π_1 also appears in $i_{M_4}^G(\lambda)$, where λ is the character of M_4 corresponding to Ω having $s = -\frac{1}{2}$ and χ_0 of order two. One can calculate $r_A^G(i_{M_4}^G(\lambda))$ similarly to see that it does not contain $r_A^G(i_{M_3}^G(\Omega))$ (or simply observe that this holds from cardinality considerations). Thus $i_{M_3}^G(\Omega)$ is reducible.

The arguments in the other cases of nontrivial χ_0 are similar; we just remark on the necessary changes. For $M = M_2$, $s = -\frac{1}{2}$ with χ_0 of order two, we compare with $i_{M_4}^G(\lambda)$, where $\lambda = \sigma_1$ is the irreducible subquotient of $\nu^{-\frac{1}{2}}\chi_0 \circ det_{GL(2)} \rtimes$

$triv_1(\chi_0)$ given in Lemma 5.1, (7). We remark that by Remark 5.2,

$$\begin{aligned} r_A^{M_4}(\lambda) &= \nu^{-1}\otimes\nu\chi_0\otimes\nu^{-1}\otimes\chi_0+\nu^{-1}\otimes\nu\chi\otimes\nu^{-1}\otimes 1+\chi_0\otimes\nu^{-1}\chi_0\otimes\chi_0\otimes\chi_0 \\ &+\chi_0\otimes\nu^{-1}\chi_0\otimes\chi_0\otimes 1+\chi_0\otimes\nu^{-1}\otimes\chi_0\otimes\chi_0+\chi_0\otimes\nu^{-1}\otimes\chi_0\otimes 1, \end{aligned}$$

which can then be used to calculate $r_A^G(i_{M_4}^G(\lambda))$. We take

$$\pi_1 = L(i_A^{M_2}(\chi_0\otimes\nu^{-1}\otimes\chi_0\otimes\chi_0))$$

for the common irreducible subquotient. In the case $M=M_3$, $s=-\frac{1}{2}$ with χ_0 of order two, we compare with $i_{M_4}^G(\lambda)$, where $\lambda=\sigma_1$ corresponds to the irreducible subquotient of $\nu^{-1}\circ det_{GL(3)}\rtimes\nu\chi_0$ given in Lemma 5.1 (6). Again, by Lemma 5.1 (6) and Remark 5.2, we have

$$r_A^{M_4}\lambda = 2\nu^{-1}\otimes\nu^{-1}\otimes 1\otimes\nu\chi_0+\nu^{-2}\otimes\nu\otimes\nu^{-1}\otimes\nu\chi_0+\nu^2\otimes\nu^{-1}\otimes\nu^{-1}\otimes\nu\chi_0.$$

We take $\pi_1 = L(i_A^{M_2}(1\otimes\nu^{-1}\otimes\chi_0\otimes 1))$ as the common irreducible subquotient.

Reducibility in the case $\chi_0=1$ is similar but a bit easier as one can show reducibility by comparing degenerate principal series from different parabolic subgroups. In this case, the degenerate principal series $i_{G,M_1}(\Omega_1)$ for $s=-\frac{5}{2}$, $i_{G,M_3}(\Omega_3)$ for $s=-\frac{3}{2}$, and $i_{G,M_4}(\Omega_4)$ with $s=-2$–using Ω_i to indicate the character is attached to M_i–have $\pi_1 = L(i_{M_{\alpha_3},A}(\nu^{-1}\otimes 1\otimes\nu^{-1}\otimes\nu^{-1}))$ as a common irreducible subquotient, from which the reducibility of all three may be deduced. Similarly, $i_{G,M_1}(\Omega_1)$ for $s=-\frac{1}{2}$, $i_{G,M_2}(\Omega_2)$ for $s=-\frac{3}{2}$, and $i_{G,M_4}(\Omega_4)$ with $s=-1$ have $\pi_1 = L(i_{M_{\alpha_1,\alpha_3},A}(1\otimes\nu^{-1}\otimes 1\otimes\nu^{-1}))$ as a common irreducible subquotient, from which the reducibility of all three of these may be deduced. The degenerate principal series $i_{G,M_2}(\Omega_2)$ with $s=-\frac{1}{2}$ and $i_{G,M_3}(\Omega_3)$ with $s=-\frac{1}{2}$ have $\pi_1 = L(i_{M_2,A}(1\otimes\nu^{-1}\otimes 1\otimes 1))$ as a common irreducible subquotient, which can be used to show the reducibility of both. Finally, $\pi_1 = L(i_{M_{\alpha_2},A}(\nu^{-1}\otimes 1\otimes\nu^{-1}\otimes\nu^{-1}))$ (resp., $\pi_1 = L(i_{M_{\alpha_2}}(1\otimes\nu^{-1}\otimes\nu^{-\frac{1}{2}}\otimes\nu^{-\frac{1}{2}}))$) appears in both $i_{G,M_1}(\Omega_1)$ for $s=-\frac{7}{2}$ and $i_{G,M_2}(\Omega_2)$ for $s=-\frac{5}{2}$ (resp., $i_{G,M_3}(\Omega_3)$ for $s=-1$ and $i_{G,M_4}(\Omega_4)$ for $s=-\frac{1}{2}$). This directly implies the reducibility of $i_{G,M_2}(\Omega_2)$ for $s=-\frac{5}{2}$ and $i_{G,M_3}(\Omega_3)$ for $s=-1$ as their Jacquet modules contain more terms than those of $i_{G,M_1}(\Omega_1)$ for $s=-\frac{7}{2}$ and $i_{G,M_4}(\Omega_4)$ for $s=-\frac{1}{2}$ (which are are irreducible–see below).

We now turn to showing irreducibility in remaining cases. The basic idea is to start with an irreducible subquotient π_1 of $i_M^G(\Omega)$ and show that $r_A^G(\pi_1) = r_A^G(i_M^G(\Omega))$. This may be effected as follows: suppose we know $r_A^G(\pi_1)\geq\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4$. If, e.g., $\chi_1\neq\nu^{\pm1}$, then $i_A^{M_{\alpha_1}}(\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4)$ is irreducible and is the only irreducible representation of M_{α_1} containing $\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4$ in its Jacquet module. Thus,

$$\begin{gathered} r_{M_{\alpha_1}}^G(\pi_1)\geq i_A^{M_{\alpha_1}}(\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4) \\ \Downarrow \\ r_A^G(\pi_1)\geq\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4+\chi_1^{-1}\otimes\chi_1\chi_2\otimes\chi_3\otimes\chi_4. \end{gathered}$$

In particular, if $\chi_1\neq\nu^{\pm1}$, we have

$$\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4\leq r_A^G(\pi_1)\Rightarrow\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4+\chi_1^{-1}\otimes\chi_1\chi_2\otimes\chi_3\otimes\chi_4\leq r_A^G(\pi_1)$$

as well (suitably interpreted for multiplicities). Similar considerations apply if χ_2,χ_3,χ_4 are different than $\nu^{\pm1}$.

While the above observation is often enough to prove irreducibility (e.g., in the regular case), for the cases at hand, a bit more is needed. We claim the following:

$$r_A^G(\pi_1) \geq 1 \otimes \nu^{-1} \otimes \chi_3 \otimes \chi_4 \Rightarrow r_A^G(\pi_1) \geq 2 \cdot 1 \otimes \nu^{-1} \otimes \chi_3 \otimes \chi_4 + \nu^{-1} \otimes \nu \otimes \nu^{-1}\chi_3 \otimes \chi_4$$

$$r_A^G(\pi_1) \geq 1 \otimes \nu \otimes \chi_3 \otimes \chi_4 \Rightarrow r_A^G(\pi_1) \geq 2 \cdot 1 \otimes \nu \otimes \chi_3 \otimes \chi_4 + \nu \otimes \nu^{-1} \otimes \nu\chi_3 \otimes \chi_4$$

$$r_A^G(\pi_1) \geq \chi_1 \otimes \chi_2 \otimes \nu^{-1} \otimes 1 \Rightarrow r_A^G(\pi_1) \geq 2 \cdot \chi_1 \otimes \chi_2 \otimes \nu^{-1} \otimes 1 + \chi_1 \otimes \nu^{-2}\chi_2 \otimes \nu \otimes \nu^{-1}$$

$$r_A^G(\pi_1) \geq \chi_1 \otimes \chi_2 \otimes \nu \otimes 1 \Rightarrow r_A^G(\pi_1) \geq 2 \cdot \chi_1 \otimes \chi_2 \otimes \nu \otimes 1 + \chi_1 \otimes \nu^{2}\chi_2 \otimes \nu^{-1} \otimes \nu.$$

Using the isomorphisms $M_{\alpha_1,\alpha_2} \cong GL(3,F) \times F^\times$ and $M_{\alpha_3,\alpha_4} \cong F^\times \times GL(3,F)$ (which may be realized, e.g., by viewing M_{α_1,α_2} inside M_4 and using Remark 5.2), the claims follow immediately from the fact that in $GL(3,F)$, one has $\nu^{-\frac{1}{2}} \circ det_{GL(2)} \times \nu^{-1}$ (resp., $\nu^{\frac{1}{2}} \circ det_{GL(2)} St_{GL(2)} \times \nu^{-1}$) irreducible, and it is the only irreducible representation containing $\nu^{-1} \otimes \nu^{-1} \otimes 1$ (resp., $1 \otimes \nu^{-1} \otimes \nu^{-1}$) in its Jacquet module (see [21]). The same argument using the irreducibility of $\nu^{-\frac{1}{2}} \circ det_{GL(2)} \times 1$ (resp., $\nu^{\frac{1}{2}} \circ St_{GL(2)} \times 1$) gives

$$r_A^G(\pi_1) \geq \nu^{-1} \otimes 1 \otimes \chi_3 \otimes \chi_4 \Rightarrow r_A^G(\pi_1) \geq 2 \cdot \nu^{-1} \otimes 1 \otimes \chi_3 \otimes \chi_4 + \nu \otimes \nu^{-1} \otimes \chi_3 \otimes \chi_4$$

$$r_A^G(\pi_1) \geq \nu \otimes 1 \otimes \chi_3 \otimes \chi_4 \Rightarrow r_A^G(\pi_1) \geq 2 \cdot \nu \otimes 1 \otimes \chi_3 \otimes \chi_4 + \nu^{-1} \otimes \nu \otimes \chi_3 \otimes \chi_4$$

$$r_A^G(\pi_1) \geq \chi_1 \otimes \chi_2 \otimes 1 \otimes \nu^{-1} \Rightarrow r_A^G(\pi_1) \geq 2 \cdot \chi_1 \otimes \chi_2 \otimes 1 \otimes \nu^{-1} + \chi_1 \otimes \chi_2 \otimes \nu^{-1} \otimes \nu$$

$$r_A^G(\pi_1) \geq \chi_1 \otimes \chi_2 \otimes 1 \otimes \nu \Rightarrow r_A^G(\pi_1) \geq 2 \cdot \chi_1 \otimes \chi_2 \otimes 1 \otimes \nu + \chi_1 \otimes \chi_2 \otimes \nu \otimes \nu^{-1}.$$

Similarly, using $M_{\alpha_2,\alpha_3} \cong GSp(4,F)$ and the irreducibility of $\nu^{-1} \rtimes triv_1(\chi)$ (see [16])–which is then the only irreducible representation of $GSp(4,F)$ containing $\nu^{-1} \otimes \nu^{-1} \otimes \chi$ in its Jacquet module–we see that

$$\begin{gathered} r_A^G(\pi_1) \geq \chi_1 \otimes 1 \otimes \nu^{-1} \otimes \chi_4 \\ \Downarrow \\ r_A^G(\pi_1) \geq 2\chi_1 \otimes 1 \otimes \nu^{-1} \otimes \chi_4 + \chi_1 \otimes \nu^{-2} \otimes \nu \otimes \nu^{-1}\chi_4 + \nu^{-2}\chi_1 \otimes \nu^2 \otimes \nu^{-1} \otimes \nu^{-1}\chi_4. \end{gathered}$$

We need one additional such fact for the case $M = M_4$, $s = 0$ with χ_0 of order two. In $GSp(4,F)$, let τ_1 denote the unique irreducible subquotient common to both $1 \rtimes triv_1(\chi_0)$ and $\nu^{-\frac{1}{2}} \circ det_{GL(2)} \rtimes \chi_0$; it is the unique irreducible representation containing $\nu^{-1} \otimes 1 \otimes \chi_0$ in its Jacquet module (see [16], noting that Sally-Tadić have $\tau_1 = L(\nu, 1 \rtimes \chi_0)$ using the quotient version of the Langlands classification). Note that the Jacquet module of τ_2 consists of $1 \otimes \nu^{-1} \otimes \chi_0 + 2\nu^{-1} \otimes 1 \otimes \chi_0$. Under $M_{\alpha_2,\alpha_3} \cong F^\times \times GSp(4,F)$, this implies that

$$\begin{gathered} r_A^G(\pi_1) \geq \nu^{-1} \otimes \nu^{-1} \otimes 1 \otimes \chi_0 \\ \Downarrow \\ r_A^G(\pi_1) \geq 2\nu^{-1} \otimes \nu^{-1} \otimes 1 \otimes \chi_0 + \nu^{-2} \otimes \nu \otimes \nu^{-1} \otimes \chi_0. \end{gathered}$$

Using the observations above, one can show irreducibility for the remaining cases. ∎

6. Subrepresentations and Quotients

In this section, we determine the number of irreducible subrepresentations and quotients for the reducible degenerate principal series from the previous sections.

Theorem 6.1. *With notation as in Theorem 3.1, we consider the following cases of reducibility of* $i_M^G(\Omega)$*:*

1. $M = M_1$

   We consider (a) $s = -\frac{11}{2}$ *and* $\chi_0 = 1$*, (b)* $s = -\frac{5}{2}$ *and* $\chi_0 = 1$*, (c)* $s = -\frac{1}{2}$ *and* $\chi_0 = 1$*, and (d)* $s = -\frac{1}{2}$ *and* χ_0 *of order two.*

2. $M = M_2$

   We consider (a) $s = -\frac{7}{2}$ *and* $\chi_0 = 1$*, (b)* $s = -\frac{5}{2}$ *and* $\chi_0 = 1$*, (c)* $s = -\frac{3}{2}$ *and* $\chi_0 = 1$*, (d)* $s = -\frac{1}{2}$ *and* $\chi_0 = 1$*, (e)* $s = -\frac{3}{2}$ *and* χ_0 *of order two, (f)* $s = -\frac{1}{2}$ *and* χ_0 *of order two, and (g)* $s = -\frac{1}{2}$ *and* χ_0 *of order three.*

3. $M = M_3$

   We consider (a) $s = -\frac{5}{2}$ *and* $\chi_0 = 1$*, (b)* $s = -\frac{3}{2}$ *and* $\chi_0 = 1$*, (c)* $s = -1$ *and* $\chi_0 = 1$*, (d)* $s = -\frac{1}{2}$ *and* $\chi_0 = 1$*, (e)* $s = -\frac{3}{2}$ *and* χ_0 *of order two, (f)* $s = -1$ *and* χ_0 *of order two, (g)* $s = -\frac{1}{2}$ *and* χ_0 *of order two, (h)* $s = -\frac{1}{2}$ *and* χ_0 *of order three, and (i)* $s = -\frac{1}{2}$ *and* χ_0 *of order four.*

4. $M = M_4$

   We consider (a) $s = -4$ *and* $\chi_0 = 1$*, (b)* $s = -2$ *and* $\chi_0 = 1$*, (c)* $s = -1$ *and* $\chi_0 = 1$*, and (d)* $s = -2$ *and* χ_0 *of order two.*

In cases (1)(a),(c),(d); (2)(a),(b),(c),(e),(g); (3) all cases; and (4)(a),(b),(d), $i_M^G(\Omega)$ *has a unique irreducible quotient and a unique irreducible subrepresentation, and they are inequivalent. In cases (1)(b),(2)(d),(2)(f) and (4)(c),* $i_M^G(\Omega)$ *has a unique irreducible quotient and a subrepresentation of the form* $\pi_1 \oplus \pi_2$*,* $\pi_1 \not\cong \pi_2$*, with no other irreducible subrepresentations. Again, the irreducible quotient is inequivalent to either irreducible subrepresentation.*

The cases of $s > 0$ *are contragredient to those above.*

Proof. Cases (1)(a), (1)(d), (2)(a), (2)(e), (2)(g), (3)(a), (3)(e), (3)(h), (3)(i), (4)(a), and (4)(d) are regular; the existence of unique irreducible subrepresentations and unique irreducible quotients follows directly from Frobenius reciprocity. The fact that the irreducible subrepresentation and irreducible quotient are inequivalent is immediate from regularity.

We organize the remaining cases based on the nature of the result and style of argument. We start with (1)(c). To show that there is a unique irreducible subrepresentation, it suffices (by Frobenius reciprocity) to show that Ω appears with multiplicity one in $r_{M_1}^G(i_{M_1}^G(\Omega))$. For this, it is enough to show that $\nu^4 \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1}$ appears with multiplicity one in $r_A^G(i_{M_1}^G(\Omega))$. That this holds may easily be seen by using the tables in the appendix and Theorem 2.1. To show these have unique irreducible quotients, it suffices to show their contragredients have unique irreducible subrepresentations. This is essentially the same argument, replacing s by $-s$, i.e., using $\nu^5 \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1}$. The multiplicity one of $\nu^5 \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1}$ in the Jacquet module shows that the irreducible quotient is inequivalent to the irreducible subrepresentation (or any other irreducible subquotient).

For (2)(b), we observe that since $\nu^{-1} \otimes \nu^5 \otimes \nu^{-1} \otimes \nu^{-1}$ appears with multiplicity one in $r_A^G(i_{M_2}^G(\Omega))$, there is a unique irreducible quotient which appears with multiplicity one. On the other hand, $\nu^{-1} \otimes 1 \otimes \nu^{-1} \otimes \nu^{-1}$ appears with multiplicity two (one copy associated to $w = 1$, the other to $w = s_2$). However, since $i_A^{M_{\alpha_2}}(\nu^{-1} \otimes 1 \otimes \nu^{-1} \otimes \nu^{-1}) \leq r_{M_{\alpha_2}}^G(i_{M_2}^G(\Omega))$ is irreducible, both copies are associated to the same irreducible subquotient of $i_{M_2}^G(\Omega)$. Thus we have a unique irreducible subrepresentation as well.

For (3)(b), the fact that there is a unique irreducible quotient, and that it appears with multiplicity one, follows immediately from the fact that $\nu^{-1} \otimes \nu^{-1} \otimes \nu^3 \otimes \nu^{-1}$ appears with multiplicity one in $r_A^G \circ i_{M_3}^G(\Omega)$. To see there is a unique irreducible subrepresentation, observe that

$$i_{M_3}^G(\Omega) \hookrightarrow i_A^G(\nu^{-1} \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1}) \cong i_{M_{\alpha_3}}^G \circ i_A^{M_{\alpha_3}}(\nu^{-1} \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1}).$$

Noting that $i_A^{M_{\alpha_3}}(\nu^{-1} \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1})$ is irreducible, we see that
$$i_A^{M_{\alpha_3}}(\nu^{-1} \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1})$$
constitutes Langlands data. It then follows from the Langlands classification that $i_{M_{\alpha_3}}^G \circ i_A^{M_{\alpha_3}}(\nu^{-1} \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1})$ has a unique irreducible subrepresentation, as needed.

For (4)(b), the fact that there is a unique irreducible quotient, and that it appears with multiplicity one, follows immediately from the fact that $\nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^5$ appears with multiplicity one in $r_A^G \circ i_{M_4}^G(\Omega)$. The fact that there is a unique irreducible subrepresentation follows from the fact that $\nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu$ appears with multiplicity one in $r_A^G(i_{M_4}^G(\Omega))$.

For (2)(c), the uniqueness of the irreducible quotient, as well as the fact that it appears with multiplicity one, follows from the observation that $\nu^{-1} \otimes \nu^4 \otimes \nu^{-1} \otimes \nu^{-1}$ appears with multiplicity one in $r_A^G \circ i_{M_2}^G(\Omega)$. Now, let $\Omega' = r_{M_{\alpha_1}}^{M_2}(\Omega)$, so that $r_A^{M_{\alpha_1}}(\Omega') = \nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1}$. Then,

$$\begin{gathered} \Omega \hookrightarrow i_{M_{\alpha_1}}^{M_2}(\Omega') \\ \Downarrow \\ i_{M_2}^G(\Omega) \hookrightarrow i_{M_{\alpha_1}}^G(\Omega') \cong i_{M_{\alpha_1,\alpha_2}}^G \circ i_{M_{\alpha_1}}^{M_{\alpha_1,\alpha_2}}(\Omega'). \end{gathered}$$

As a representation of $M_{\alpha_1,\alpha_2} \cong GL(3,F) \times F^\times$ (viewed as a standard Levi of $M_4 \cong GSp(6,F)$, e.g.), we have $i_{M_{\alpha_1}}^{M_{\alpha_1,\alpha_2}}(\Omega') \cong (\nu^{-\frac{1}{2}} \circ det_{GL(2)} \times \nu^{-1}) \otimes \nu^{-1}$–irreducible by the results of Zelevinsky. Now,

$$r_A^{M_{\alpha_1,\alpha_2}} \circ i_{M_{\alpha_1}}^{M_{\alpha_1,\alpha_2}}(\Omega') = \nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1} + 2 \cdot 1 \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1}.$$

Therefore, by central character considerations and Frobenius reciprocity,

$$i_{M_{\alpha_1}}^{M_{\alpha_1,\alpha_2}}(\Omega') \hookrightarrow i_A^G(1 \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1}).$$

Therefore,

$$\begin{aligned} i_{M_2}^G(\Omega) &\hookrightarrow i_{M_{\alpha_1,\alpha_2}}^G \circ i_{M_{\alpha_1}}^{M_{\alpha_1,\alpha_2}}(\Omega') \\ &\hookrightarrow i_{M_{\alpha_1,\alpha_2}}^G \circ i_A^{M_{\alpha_1,\alpha_2}}(1 \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1}) \\ &\cong i_A^G(1 \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1}) \\ &\cong i_{M_{\alpha_1,\alpha_3}}^G \circ i_A^{M_{\alpha_1,\alpha_3}}(1 \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1}). \end{aligned}$$

The representation $i_A^{M_{\alpha_1,\alpha_3}}(1 \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1})$ is an irreducible representation of M_{α_1,α_3} satisfying the requirements of Langlands data. In particular, $i_{M_{\alpha_1,\alpha_3}}^G \circ i_A^{M_{\alpha_1,\alpha_3}}(1 \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1})$ has a unique irreducible subrepresentation (Langlands subrepresentation) π_1. It then follows that this is the unique irreducible subrepresentation of $i_{M_2}^G(\Omega)$.

Cases (3)(c) and (3)(f) may be addressed simultaneously. Letting χ_0' be either trivial or of order two, we see that $\nu^{-1} \otimes \nu^{-1} \otimes \nu^{\frac{5}{2}}\chi_0' \otimes \nu^{-1}$ appears with multiplicity one in $r_A^G \circ i_{M_3}^G(\Omega)$, from which the uniqueness and multiplicity one of the irreducible quotient follows. To address the subrepresentation claim, we argue as in 2(c). First, observe that by irreducibility, $i_A^{M_{\alpha_3}}(\nu^{-1} \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi_0' \otimes \nu^{-1}) \cong i_A^{M_{\alpha_3}}(\nu^{-1} \otimes 1 \otimes \nu^{-\frac{1}{2}}\chi_0' \otimes \nu^{-\frac{1}{2}}\chi_0')$. Thus,

$$i_M^G\Omega \hookrightarrow i_A^G(\nu^{-1} \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi_0' \otimes \nu^{-1}) \cong i_{M_{\alpha_3}}^G \circ i_A^{M_{\alpha_3}}(\nu^{-1} \otimes 1 \otimes \nu^{-\frac{1}{2}}\chi_0' \otimes \nu^{-\frac{1}{2}}\chi_0').$$

Since $i_A^{M_{\alpha_3}}(\nu^{-1}\otimes 1\otimes \nu^{-\frac{1}{2}}\chi_0'\otimes \nu^{-\frac{1}{2}}\chi_0')$ satisfies the requirements for Langlands data, it follows from the Langlands classification that $i_{M_{\alpha_3}}^G \circ i_A^{M_{\alpha_3}}(\nu^{-1}\otimes 1\otimes \nu^{-\frac{1}{2}}\chi_0'\otimes \nu^{-\frac{1}{2}}\chi_0'$ has a unique irreducible subrepresentation, hence so does $i_M^G(\Omega)$.

For (3)(d), the fact that there is a unique irreducible quotient, and that it appears with multiplicity one, follows immediately from the fact that $\nu^{-1} \otimes \nu^{-1} \otimes \nu^2 \otimes \nu^{-1}$ appears with multiplicity one in $r_A^G \circ i_{M_3}^G(\Omega)$. For the uniqueness of the irreducible subrepresentation, observe that

$$i_{M_3}^G(\Omega) \hookrightarrow i_{M_{\alpha_2,\alpha_4}}^G(\Omega') \cong i_{M_1}^G \circ i_{M_{\alpha_2,\alpha_4}}^{M_1}(\Omega'),$$

where $\Omega' = r_{M_{\alpha_2,\alpha_4}}^{M_3}(\Omega)$ (so that Ω' is a character of M_{α_2,α_4} satisfying $r_A^{M_{\alpha_2,\alpha_4}}(\Omega') = \nu^{-1} \otimes \nu^{-1} \otimes \nu \otimes \nu^{-1}$). Under the isomorphism $M_1 \cong GSpin(7,F)$, $i_{M_{\alpha_2,\alpha_4}}^{M_1}(\Omega')$ corresponds to the degenerate principal series $\nu^{-\frac{3}{2}} \circ det_{GL(2)} \rtimes triv_1(\nu^{-\frac{3}{2}})$; irreducible by Lemma 5.1. Since $1 \otimes \nu^{-1} \otimes 1 \otimes 1 \leq r_A^{M_1} \circ i_{M_{\alpha_2,\alpha_4}}^{M_1}(\Omega')$, central character considerations tell us

$$i_{M_{\alpha_2,\alpha_4}}^{M_1}(\Omega') \hookrightarrow i_A^{M_1}(1 \otimes \nu^{-1} \otimes 1 \otimes 1).$$

Therefore,

$$i_{M_3}^G(\Omega) \hookrightarrow i_{M_1}^G \circ i_{M_{\alpha_2,\alpha_4}}^{M_1}(\Omega') \hookrightarrow i_{M_1}^G \circ i_A^{M_1}(1 \otimes \nu^{-1} \otimes 1 \otimes 1) \cong i_A^G(1 \otimes \nu^{-1} \otimes 1 \otimes 1).$$

Now, $i_A^G(1\otimes\nu^{-1}\otimes 1\otimes 1) \cong i_{M_2}^G \circ i_A^{M_2}(1\otimes\nu^{-1}\otimes 1\otimes 1)$, noting that $i_A^{M_2}(1\otimes\nu^{-1}\otimes 1\otimes 1)$ is irreducible. Since $i_A^{M_2}(1 \otimes \nu^{-1} \otimes 1 \otimes 1)$ constitutes Langlands data, we see that

$i_A^G(1 \otimes \nu^{-1} \otimes 1 \otimes 1)$ admits a unique irreducible subrepresentation, from which it follows immediately that $i_{M_3}^G(\Omega)$ does as well.

For (3)(g), the fact that there is a unique irreducible quotient, and that it appears with multiplicity one, follows immediately from the fact that $\nu^{-1} \otimes \nu^{-1} \otimes \nu^2\chi_0 \otimes \nu^{-1}$ appears with multiplicity one in $r_A^G \circ i_{M_3}^G(\Omega)$. For the uniqueness of the irreducible subrepresentation, observe that

$$i_{M_3}^G(\Omega) \hookrightarrow i_{M_{\alpha_1,\alpha_2}}^G(\Omega') \cong i_{M_4}^G \circ i_{M_{\alpha_1,\alpha_2}}^{M_4}(\Omega'),$$

where $\Omega' = r_{M_{\alpha_1,\alpha_2}}^{M_3}(\Omega)$ (so that Ω' is a character of M_{α_1,α_2} satisfying $r_A^{M_{\alpha_1,\alpha_2}}(\Omega') = \nu^{-1} \otimes \nu^{-1} \otimes \nu\chi_0 \otimes \nu^{-1}$). Now, under the isomorphism $M_4 \cong GSp(6,F)$, $\theta = i_{M_{\alpha_1,\alpha_2}}^{M_4}(\Omega')$ corresponds to the degenerate principal series $\chi_0 \circ det_{GL(3)} \rtimes \nu^{-1}$, hence is irreducible by Lemma 5.1. Now, $1 \otimes \nu^{-1} \otimes \chi_0 \otimes \chi_0 \leq r_A^{M_4}(\theta)$, so by central character considerations, we have $\theta \hookrightarrow i_A^{M_4}(1 \otimes \nu^{-1} \otimes \chi_0 \otimes \chi_0)$. Hence,

$$i_{M_3}^G(\Omega) \hookrightarrow i_{M_4}^G(\theta) \hookrightarrow i_A^G(1 \otimes \nu^{-1} \otimes \chi_0 \otimes \chi_0) \cong i_{M_2}^G \circ i_A^{M_2}(1 \otimes \nu^{-1} \otimes \chi_0 \otimes \chi_0).$$

Since $i_A^{M_2}(1 \otimes \nu^{-1} \otimes \chi_0 \otimes \chi_0)$ satisfies the requirements of Langlands data, it follows that $i_{M_2}^G \circ i_A^{M_2}(1 \otimes \nu^{-1} \otimes \chi_0 \otimes \chi_0)$ has a unique irreducible subrepresentation, so $i_{M_3}^G(\Omega)$ must as well.

In (1)(b), the fact that there is a unique irreducible quotient, and that it appears with multiplicity one, follows immediately from the fact that $\nu^7 \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1}$ appears with multiplicity one in $r_A^G \circ i_{M_1}^G(\Omega)$. For irreducible subrepresentations, observe that $\nu^2 \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1}$ appears in the Jacquet module with multiplicity two (with the second copy associated to $s_1s_2s_3s_2s_1$). Therefore, the above argument then shows that there are at most two irreducible subrepresentations. To see that there are two irreducible subrepresentations, consider $i_{M_{\alpha_2,\alpha_3}}^{M_4}(\Omega')$, where $\Omega' = r_{M_{\alpha_2,\alpha_3}}^{M_1}(\Omega)$ (a character of M_{α_2,α_3}). Recall that $M_4 \cong GSp(6,F)$ and $M_{\alpha_2,\alpha_3} \cong F^\times \times GSp(4,F)$. In particular, $i_{M_{\alpha_2,\alpha_3}}^{M_4}(\Omega')$ is the representation $1_{GL(1)} \rtimes triv_2(\nu^{-1})$ of $GSp(6,F)$. By Lemma 5.1, $1_{GL(1)} \rtimes triv_2(\nu^{-1})$ decomposes as a direct sum of two inequivalent irreducible representations; write $i_{M_{\alpha_2,\alpha_3}}^{M_4}(\Omega) \cong \sigma_1 \oplus \sigma_2$. Then,

$$i_{M_1}^G(\Omega) \hookrightarrow i_{M_{\alpha_2,\alpha_3}}^G(\Omega') \cong i_{M_4}^G(\sigma_1 \oplus \sigma_2).$$

We argue that each $i_{M_4}^G(\sigma_i)$ has an irreducible subrepresentation in common with $i_{M_1}^G(\Omega)$, accounting for the two irreducible subrepresentations of $i_{M_1}^G(\Omega)$.

To this end, recall that $r_A^G(i_{M_1}^G(\Omega))$ contains $\nu^2 \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1}$ with multiplicity two. Now, observe that $r_A^{M_4}(\sigma_1) = \nu^2 \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1} + \nu^{-2} \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1}$ and $r_A^{M_4}(\sigma_2) = \nu^2 \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1} + \nu^{-2} \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1} + 2\nu^{-1} \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1}$. One can then directly check (using Theorem 2.1) that $r_A^G(i_{M_4}^G(\sigma_i))$, $i = 1,2$ each contains $\nu^2 \otimes \nu^{-1} \otimes \nu^{-1} \otimes \nu^{-1}$ with multiplicity one. Let π_i denote the irreducible subrepresentation of $i_{M_4}^G(\sigma_i)$ (noting that it is the unique irreducible subrepresentation of $i_{M_4}^G(\sigma_i)$ by Frobenius reciprocity). Without loss of generality, we may let π_1 be the one having $r_A^G(\pi_1) \geq 2\nu^{-1} \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1}$ (and the fact that this is true for only one of the π_i shows that they are inequivalent).

Note that by Jacquet module considerations, π_i appears with multiplicity one in $i_{M_4}^G(\sigma_i)$ (and not in $i_{M_4}^G(\sigma_{3-i})$). It then follows that π_1 and π_2 must also appear as subrepresentations of $i_{M_1}^G(\Omega)$–consider the subspace $(V_{\pi_1} \oplus V_{\pi_2}) \cap V_{i_{M_1}^G(\Omega)}$ in $V_{i^G_{M_{\alpha_2,\alpha_3}}(\Omega')}$. The result follows.

For (2)(d), we observe that since $\nu^{-1} \otimes \nu^3 \otimes \nu^{-1} \otimes \nu^{-1}$ appears with multiplicity one in $r_A^G(i_{M_2}^G(\Omega))$, there is a unique irreducible quotient and it appears with multiplicity one. To show there are two irreducible subrepresentations, first let π_1 be the representation with Langlands data $i_A^{M_2}(1 \otimes \nu^{-1} \otimes 1 \otimes 1)$. Since $r_A^G(\pi_1)$ contains all 12 copies of $1 \otimes \nu^{-1} \otimes 1 \otimes 1$, we see that $r_{M_1}^G(\pi_1)$ contains both copies of θ_1, where under the isomorphism $M_1 \cong GSpin(7,F)$, $\theta_1 \cong \nu^{-\frac{3}{2}} \circ det_{GL(2)} \rtimes triv_1(\nu^{-\frac{3}{2}})$ (irreducible by Lemma 5.1). Therefore, $r_A^G(\pi_1)$ must contain (at least) 8 of the 10 copies of $\nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes 1$ in $r_A^G \circ i_{M_2}^G(\Omega)$. In the notation of Lemma 5.1 (2), $r_{M_4}^G(\pi_1) \geq 3\sigma_1 + 2\sigma_2$ (noting that the $3\sigma_1$ and $4\sigma_2$ in $r_{M_4}^G \circ i_{M_2}^G(\Omega)$ account for all 10 copies of $\nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes 1$). In particular, this forces $r_A^G(\pi_1) \geq r_A^{M_4}(\sigma_2) \geq \nu^{-1} \otimes \nu^2 \otimes \ \nu^{-1} \otimes \nu^{-1}$.

By central character considerations,

$$\begin{gathered} \pi_1 \hookrightarrow i_A^G(\nu^{-1} \otimes \nu^2 \otimes \nu^{-1} \otimes \nu^{-1}) \\ \Downarrow \text{ (Lemma 5.5 [12])} \\ \pi_1 \hookrightarrow i_{M_2}^G(\lambda) \end{gathered}$$

for some irreducible $\lambda \leq i_A^{M_2}(\nu^{-1} \otimes \nu^2 \otimes \nu^{-1} \otimes \nu^{-1})$. Since $r_A^G \circ i_{M_2}^G(\Omega) \geq 12 \cdot 1 \otimes \nu^{-1} \otimes 1 \otimes 1$, and $r_A^G \circ i_A^G(\nu^{-1} \otimes \nu^2 \otimes \nu^{-1} \otimes \nu^{-1})$ contains only 12 copies of $1 \otimes \ \nu^{-1} \otimes 1 \otimes 1$, we see that $\lambda = \Omega$, as needed.

Next, a comparison with the case $s = -\frac{1}{2}$ from M_3 shows that $r_A^G(\pi_1)$ contains only 8 copies of $\nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes 1$, while $r_A^G \circ i_{M_2}^G(\Omega)$ has a total of ten copies. Since $i_A^{M_{\alpha_4}}(\nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes 1)$ is irreducible, the two remaining copies of $\nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes 1$ are associated to the same irreducible subquotient of $i_{M_2}^G(\Omega)$; let π_2 denote this subquotient. In the notation of Lemma 5.1 (2), we have $r_{M_4}^G(\pi_2)$ contains two copies of σ_2. Now, let λ be the irreducible representation of M_{α_2} having $r_A^{M_2}(\lambda) = \nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes 1$. We have (e.g., from the Langlands classification for $M_4 \cong GSp(6,F)$ and induction in stages)

$$\begin{gathered} \sigma_2 \hookrightarrow i_{M_{\alpha_2}}^{M_4}(\lambda) \\ \Downarrow \\ i_{M_4}^G(\sigma_2) \hookrightarrow i_{M_{\alpha_2}}^G(\lambda) \cong i_{M_{\alpha_2,\alpha_4}}^G \circ i_{M_{\alpha_2}}^{M_{\alpha_2,\alpha_4}}(\lambda). \end{gathered}$$

We note that $i_{M_{\alpha_2}}^{M_{\alpha_2,\alpha_4}}(\lambda)$ is irreducible. Further, we claim that $i_{M_{\alpha_2}}^{M_{\alpha_2,\alpha_4}}(\lambda)$ is the Langlands data for π_2. In particular, by central character considerations, we have

$$\begin{gathered} \pi_2 \hookrightarrow i_A^G(\nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes 1) \cong i_{M_{\alpha_2}}^G \circ i_A^{M_{\alpha_2}}(\nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes 1) \\ \Downarrow \text{ (Lemma 5.5 [12])} \\ \pi_2 \hookrightarrow i_{M_{\alpha_2}}^G(\lambda) \text{ or } \pi_2 \hookrightarrow i_{M_{\alpha_2}}^G(\tau), \end{gathered}$$

where $r_A^{M_2}(\tau) = 1 \otimes \nu^{-1} \otimes 1 \otimes 1$. However, $i_{M_{\alpha_2}}^G(\tau) \hookrightarrow i_A^G(1 \otimes \nu^{-1} \otimes 1 \otimes \ \nu^{-1})$ contains π_1 as unique irreducible subrepresentation (by the Langlands classification). Thus,

$\pi_2 \hookrightarrow i^G_{M_{\alpha_2}}(\lambda)$, as claimed. Now, (from the Langlands classification for M_4, e.g.), we have

$$\begin{gathered} \sigma_2 \hookrightarrow i^{M_4}_{M_{\alpha_2}}(\lambda) \\ \Downarrow \text{ (induction in stages)} \\ i^G_{M_4}(\sigma_2) \hookrightarrow i^G_{M_{\alpha_2}}(\lambda) \cong i^G_{M_{\alpha_2,\alpha_4}}(i^{M_{\alpha_2,\alpha_4}}_{M_{\alpha_2}}(\lambda)). \end{gathered}$$

Since $i^G_{M_{\alpha_2,\alpha_4}}(i^{M_{\alpha_2,\alpha_4}}_{M_{\alpha_2}}(\lambda))$ is the standard module containing π_2 as unique irreducible subrepresentation, we see that π_2 is the unique irreducible subrepresentation of $i^G_{M_{\alpha_2}}(\lambda)$, hence also the unique irreducible subrepresentation of $i^G_{M_4}(\sigma_2)$.

Next, let Ω' denote the character of M_{α_1,α_3} having $r_A^{M_{\alpha_1,\alpha_3}}(\Omega') = \nu^{-1} \otimes \nu^2 \otimes \nu^{-1} \otimes \nu^{-1}$. Note that under the isomorphism $M_4 \cong GSp(6,F)$ (Remark 5.2), we have $i^{M_4}_{M_{\alpha_1,\alpha_3}}(\Omega') \cong \nu^{\frac{1}{2}} \circ det_{GL(2)} \rtimes triv_1(\nu^{-1})$. By Lemma 5.1, this induced representation has σ_1 and σ_2 as its irreducible subquotients. Since $r_A^{M_4}(\sigma_2)$ contains a copy of $\nu^{-1} \otimes \nu^2 \otimes \nu^{-1} \otimes \nu^{-1}$ (using Remark 5.2 to convert the results in Lemma 5.1), but $r_A^{M_4}(\sigma_1)$ does not, we see that σ_2 is the unique irreducible subrepresentation of $i^{M_4}_{M_{\alpha_1,\alpha_3}}(\Omega')$. Therefore,

$$\begin{gathered} \sigma_2 \hookrightarrow i^{M_4}_{M_{\alpha_1,\alpha_3}}(\Omega') \\ \Downarrow \text{ (induction in stages)} \\ \pi_2 \hookrightarrow i^G_{M_4}(\sigma_2) \hookrightarrow i^G_{M_4} \circ i^{M_4}_{M_{\alpha_1,\alpha_3}}(\Omega') \cong i^G_{M_2} \circ i^{M_2}_{M_{\alpha_1,\alpha_3}}(\Omega') \\ \Downarrow \text{ (Lemma 5.5 [12])} \\ \pi_2 \hookrightarrow i^G_{M_2}(\eta) \end{gathered}$$

for some irreducible $\eta \leq i^{M_2}_{M_{\alpha_1,\alpha_3}}(\Omega')$. Now, observe that

$$i^{M_2}_{M_{\alpha_1,\alpha_3}}(\Omega') = \eta_1 + \eta_2,$$

where $\eta_1 = \Omega$, so that $r_A^{M_2}(\eta_1) = \nu^{-1} \otimes \nu^2 \otimes \nu^{-1} \otimes \nu^{-1}$, and η_2 has $r_A^{M_2}(\eta_2) = \nu^{-1} \otimes \nu^2 \otimes \nu^{-2} \otimes \nu + \nu^{-1} \otimes \nu^{-2} \otimes \nu^2 \otimes \nu^{-1}$. Calculating $W^{M_2,A} \cdot (\nu^{-1} \otimes \nu^2 \otimes \nu^{-2} \otimes \nu + \nu^{-1} \otimes \nu^{-2} \otimes \nu^2 \otimes \nu^{-1})$, we see that $r_A^G \circ i^G_{M_2}(\eta_2)$ does not contain $\nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes 1$. (In fact, $i^G_{M_2}(\Omega)$ accounts for 10 copies of $\nu^{-1} \otimes \nu \otimes \nu^{-1} \otimes 1$; the remaining two copies from $i^G_A(\nu^{-1} \otimes \nu^2 \otimes \nu^{-1} \otimes \nu^{-1})$ occur in $i^G_{M_2}\zeta$, where ζ is the irreducible representation of M_2 having $r_A^{M_2}(\zeta) = \nu \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1}$.) Therefore, we have

$$\pi_2 \hookrightarrow i^G_{M_2}(\theta_1) = i^G_{M_2}(\Omega),$$

as needed. We remark that π_1 and π_2 are clearly inequivalent.

We now show that there are no other irreducible subrepresentations of $i^G_{M_2}(\Omega)$. Now, suppose π is an irreducible subrepresentation. Then, in the notation of Lemma 5.1,

$$\begin{gathered} \pi \hookrightarrow i^G_{M_2}(\Omega) \hookrightarrow i^G_{M_{\alpha_1,\alpha_3}}(\Omega') \cong i^G_{M_4} \circ i^{M_4}_{M_{\alpha_1,\alpha_3}}(\Omega') \\ \Downarrow \text{ (Lemma 5.5 [12])} \\ \pi \hookrightarrow i^G_{M_4}(\sigma_1) \text{ or } i^G_{M_4}(\sigma_2). \end{gathered}$$

As noted above, $i^G_{M_4}(\sigma_2)$ has π_2 as unique irreducible subrepresentation. Since $r_A^{M_4}(\sigma_1)$ contains copies of $1 \otimes \nu^{-1} \otimes 1 \otimes 1$, by central character considerations,

$$\begin{gathered} \sigma_1 \hookrightarrow i^{M_4}_A(1 \otimes \nu^{-1} \otimes 1 \otimes 1) \\ \Downarrow \\ i^G_{M_1}(\sigma_1) \hookrightarrow i^G_{M_4} \circ i^{M_4}_A(1 \otimes \nu^{-1} \otimes 1 \otimes 1) \cong i^G_{M_2}(i^{M_2}_A(1 \otimes \nu^{-1} \otimes 1 \otimes 1)), \end{gathered}$$

which (by the Langlands classification) has π_1 as unique irreducible subrepresentation. Therefore, the only possibilities are $\pi \cong \pi_1$ or π_2. Further, we note that by Jacquet module considerations, π_1 and π_2 each appear only once in $i_{M_2}^G(\Omega)$, so cannot appear more than once as subrepresentations.

In (2)(f), the fact that there is a unique irreducible quotient, and that it appears with multiplicity one, follows immediately from the fact that $\nu^{-1} \otimes \nu^3\chi_0 \otimes \nu^{-1} \otimes \nu^{-1}$ appears with multiplicity one in $r_A^G \circ i_{M_2}^G(\Omega)$. For irreducible subrepresentations, observe that

$$i_{M_2}^G(\Omega) \hookrightarrow i_{M_{\alpha_3,\alpha_4}}^G(\theta) \cong i_{M_1}^G(i_{M_{\alpha_3,\alpha_4}}^{M_1}(\theta)),$$

where θ is the character of $M_{\alpha_3\alpha_4}$ having $r_A^{M_{\alpha_3,\alpha_4}}(\theta) = \nu^{-1} \otimes \nu^2\chi_0 \otimes \nu^{-1} \otimes \nu^{-1}$. Under $M_1 \cong GSpin(7,F)$, $i_{M_{\alpha_3,\alpha_4}}^{M_1}(\theta)$ corresponds to the degenerate principal series $\nu^{-1} \circ det_{GL(3)} \rtimes \nu^{-2}\chi_0$ (see Note 5.2). As this representation is irreducible (see Lemma 5.1), we have $\nu^{-1} \circ det_{GL(3)} \rtimes \nu^{-2}\chi_0 \cong \nu^{-1}\chi_0 \circ det_{GL(3)} \rtimes \nu^{-2}\chi_0$ (noting that $\nu^{-1}\chi_0 \circ det_{GL(3)} \otimes \nu^{-2}\chi_0 = w_0'(\nu^{-1} \circ det_{GL(3)} \otimes \nu^{-2}\chi_0)$, where w_0' is the long double-coset representative). This translates to $i_{M_{\alpha_3,\alpha_4}}^{M_1}(\theta) \cong i_{M_{\alpha_3,\alpha_4}}^{M_1}(\theta')$, where θ' is the character of M_{α_3,α_4} having $r^{M_{\alpha_3,\alpha_4}}(\theta') = \nu^{-1}\chi_0 \otimes \nu^2\chi_0 \otimes \nu^{-1} \otimes \nu^{-1}$. Therefore,

$$i_{M_2}^G(\Omega) \hookrightarrow i_{M_1}^G(i_{M_{\alpha_3,\alpha_4}}^{M_1}(\theta)) \cong i_{M_1}^G(i_{M_{\alpha_3,\alpha_4}}^{M_1}(\theta')) \cong i_{M_2}^G(i^{M_2}M_{\alpha_3,\alpha_4}(\theta')).$$

Since $i_{M_{\alpha_3,\alpha_4}}^{M_2}(\theta')$ is irreducible, we have $i_{M_{\alpha_3,\alpha_4}}^{M_2}(\theta') \cong i_{M_{\alpha_3,\alpha_4}}^{M_2}(\theta'')$, where $\theta'' = s_1\theta'$ (so that $r_A^{M_{\alpha_3,\alpha_4}}(\theta'') = \nu\chi_0 \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1}$). Thus,

$$i_{M_2}^G(\Omega) \hookrightarrow i_{M_2}^G(i_{M_{\alpha_3,\alpha_4}}^{M_2}(\theta')) \cong i_{M_2}^G(i_{M_{\alpha_3,\alpha_4}}^{M_2}(\theta'')) \cong i_{M_1}^G(i_{M_{\alpha_3,\alpha_4}}^{M_1}(\theta'')).$$

We note that under the isomorphism $M_1 \cong GSpin(7,F)$ (see Note 5.2), we have $i_{M_{\alpha_3,\alpha_4}}^{M_1}(\theta')$ corresponds to $\nu^{-1}\chi_0 \circ det_{GL(3)} \rtimes \nu^{-1}$, hence has two irreducible subrepresentations, which we also denote by σ_1 and σ_2 (see Lemma 5.1). Using Note 5.2 to translate back to the M_1 setting, we have $r_A^{M_1}(\sigma_1) = \nu^2\chi_0 \otimes \nu^{-1} \otimes 1 \otimes \nu^{-1} + \nu\chi_0 \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1} + \nu^2\chi_0 \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1}$ and $r_A^G(\sigma_2) = \nu\chi_0 \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1}$. Since $r_A^G \circ i_A^G(\nu\chi_0 \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1})$ decomposes with multiplicity two–in particular, contains $\nu\chi_0 \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1}$ with multiplicity two–we see that $i_{M_2}^G(\Omega)$ must have nontrivial intersection with both $i_{M_1}^G(\sigma_1)$ and $i_{M_1}^G(\sigma_2)$. More precisely, if we let π_1 (resp., π_2) denote the irreducible subquotient of $i_{M_1}^G(\sigma_1)$ (resp., $i_{M_1}^G(\sigma_2)$) containing $\nu\chi_0 \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1}$ in its Jacquet module, we see that π_1 and π_2 appear as subquotients of $i_{M_2}^G(\Omega)$. Note that $\pi_1 = L(i_A^{M_2}(\chi_0 \otimes \nu^{-1}\chi_0 \otimes \chi_0 \otimes \chi_0))$ appears with multiplicity one in $i_A^G(\nu\chi_0 \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1})$, so $\pi_2 \ncong \pi_1$ and also appears with multiplicity one in $i_A^G(\nu\chi_0 \otimes \nu \otimes \nu^{-1} \otimes \nu^{-1})$. Since π_1 (resp., π_2) is the unique irreducible subrepresentation of $i_{M_1}^G(\sigma_1)$ (resp., $i_{M_1}^G(\sigma_2)$)–an easy consequence of Frobenius reciprocity–we have

$$\pi_i \hookrightarrow i_{M_1}^G(\sigma_i) \hookrightarrow i_{M_1}^G \circ i_{M_{\alpha_3,\alpha_4}}^{M_1}(\theta'').$$

Since $i_{M_2}^G(\Omega) \hookrightarrow i_{M_1}^G \circ i_{M_{\alpha_3,\alpha_4}}^{M_1}(\theta'')$, it then follows that π_1 and π_2 appear as subrepresentations of $i_{M_2}^G(\Omega)$ (consider the subspace $V_{i_{M_2}^G(\Omega)} \cap (V_{\pi_1} + V_{\pi_2})$ inside $V_{i_{M_{\alpha_3,\alpha_4}}^G(\theta'')}$).

In (4)(c), the fact that there is a unique irreducible quotient, and that it appears with multiplicity one, follows immediately from the fact that $\nu^{-1}\otimes\nu^{-1}\otimes\nu^{-1}\otimes\nu^4$ appears with multiplicity one in $r_A^G\circ i_{M_4}^G(\Omega)$. For irreducible subrepresentations, observe that $\nu^{-1}\otimes\nu^{-1}\otimes\nu^{-1}\otimes\nu^2$ appears in the Jacquet module with multiplicity two (with the second copy associated to $s_4s_3s_2s_3s_1s_2s_3s_4$). Therefore, the above argument shows that there are at most two irreducible subrepresentations. We now check that there are two irreducible subrepresentations. To this end, let Ω' and Ω'' be the characters of M_{α_1,α_2} defined by $r_A^{M_{\alpha_1,\alpha_2}}(\Omega')=\nu^{-1}\otimes\nu^{-1}\otimes\nu^{-1}\otimes\nu^2$ and $r_A^{M_{\alpha_1,\alpha_2}}(\Omega'')=\nu^{-1}\otimes\nu^{-1}\otimes\nu\otimes\nu^{-2}$. Under the isomorphism $M_4\cong GSp(6,F)$, $i_{M_{\alpha_1,\alpha_2}}^{M_4}(\Omega'')\cong 1_{GL(3)}\rtimes\nu^2$. By Lemma 5.1, write $i_{M_{\alpha_1,\alpha_2}}^{M_4}(\Omega')\cong\sigma_1\oplus\sigma_2$. Since $i_{M_{\alpha_1,\alpha_2}}^{M_3}(\Omega')\cong i_{M_{\alpha_1,\alpha_2}}^{M_3}(\Omega'')$ (by irreducibility), we have

$$\begin{aligned} i_{M_4}^G(\Omega) &\hookrightarrow i_{M_4}^G(i_{M_{\alpha_1,\alpha_2}}^{M_4}(\Omega'))\\ &\cong i_{M_3}^G(i_{M_{\alpha_1,\alpha_2}}^{M_3}(\Omega'))\\ &\cong i_{M_3}^G(i_{M_{\alpha_1,\alpha_2}}^{M_3}(\Omega''))\\ &\cong i_{M_4}^G(i_{M_{\alpha_1,\alpha_2}}^{M_4}(\Omega''))\\ &\cong i_{M_4}^G(\sigma_1\oplus\sigma_2). \end{aligned}$$

We argue that each $i_{M_4}^G(\sigma_i)$ has an irreducible subrepresentation in common with $i_{M_4}^G(\Omega)$, accounting for the two irreducible subrepresentations of $i_{M_4}^G(\Omega)$.

To this end, recall that $r_A^G(i_{M_4}^G(\Omega))$ contains $\nu^{-1}\otimes\nu^{-1}\otimes\nu\otimes\nu^{-2}$ with multiplicity two. Now, observe that $r_A^{M_4}(\sigma_1)=\nu^{-1}\otimes\nu^{-1}\otimes\nu\otimes\nu^{-2}+2\cdot\nu^{-1}\otimes\nu\otimes\nu^{-1}\otimes\nu^{-1}+4\cdot 1\otimes\nu^{-1}\otimes 1\otimes\nu^{-1}$ and $r_A^{M_4}(\sigma_2)=\nu^{-1}\otimes\nu^{-1}\otimes\nu\otimes\nu^{-2}$ (use Note 5.2). One can then directly check (using Theorem 2.1) that $r_A^G(i_{M_4}^G(\sigma_i))$, $i=1,2$ each contain $\nu^{-1}\otimes\nu^{-1}\otimes\nu^{-1}\otimes\nu^2$ with multiplicity one. Let π_i denote the irreducible subrepresentation of $i_{M_4}^G(\sigma_i)$ (noting that it is the unique irreducible subrepresentation of $i_{M_4}^G(\sigma_i)$ by Frobenius reciprocity). Without loss of generality, we may let π_1 be the one having $r_A^G(\pi_1)\geq 4\cdot 1\otimes\nu^{-1}\otimes 1\otimes\nu^{-1}$ (and the fact that this is true for only one of the π_i shows that they are inequivalent). Note that by Jacquet module considerations, π_i appears with multiplicity one in $i_{M_4}^G(\sigma_i)$ (and not in $i_{M_4}^G(\sigma_{3-i})$). It then follows that π_1 and π_2 must also appear as subrepresentations of $i_{M_4}^G(\Omega)$ (consider the subspace $(V_{\pi_1}+V_{\pi_2})\cap V_{i_{M_4}^G(\Omega)}$ in $V_{i_{M_{\alpha_1,\alpha_2}}^G(\Omega'')}$). The proposition follows. ■

Remark 6.2. The proof shows a bit more–one also sees that the unique irreducible quotient appears with multiplicity one in the induced representation.

A. Double-coset representatives

We give the double coset representatives from $W^{M_i,A}$ below. For W^{M_i,M_j}, we first note that $W^{A,M_j}=\{w^{-1}\,|\,w\in W^{M_j,A}\}$. We have $W^{M_i,M_j}=W^{M_i,A}\cap W^{A,M_j}$. To compare elements of $W^{M_i,A}$ and W^{A,M_j}, one may need to use the relations on W: $s_1s_2s_1 = s_2s_1s_2$, $s_2s_3s_2s_3 = s_3s_2s_3s_2$, $s_3s_4s_3 = s_4s_3s_4$, and $s_ks_\ell = s_\ell s_k$ if $|k-\ell| > 1$. For W^{M_i,N_j} as in section 4, consider an element $w \in W^{M_i,A}$. There are two possibilities: either (1) $s_jw \notin W^{M_i,A}$, in which case $w \in W^{M_i,N_j}$, or (2) $s_jw \in W^{M_i,A}$, in which case only the shorter of w, s_jw is in W^{M_i,N_j}. Again, the relations on the generators may be needed to check this.

$$
\begin{aligned}
W^{M_1,A} = \; & \{id, 1, 21, 321, 2321, 4321, 12321, 42321, 142321, 342321, 1342321,\\
& 2342321, 12342321, 23412321, 123412321, 323412321, 1323412321,\\
& 4323412321, 14323412321, 23123412321, 423123412321,\\
& 3423123412321, 23423123412321, 123423123412321\}
\end{aligned}
$$

$$
\begin{aligned}
W^{M_2,A} = \; & \{id, 2, 12, 32, 432, 312, 232, 4312, 4232, 2312, 1232, 42312, 41232,\\
& 12312, 32312, 34232, 412312, 432312, 312312, 342312, 341232,\\
& 234232, 4312312, 4342312, 2312312, 3412312, 2342312, 2341232,\\
& 1234232, 42312312, 43412312, 42342312, 23412312, 12342312,\\
& 12341232, 32341232, 423412312, 412342312, 432341232,\\
& 342312312, 123412312, 323412312, 323432312, 312341232,\\
& 4123412312, 4323412312, 4312341232, 2342312312, 3234312312,\\
& 3123412312, 3123432312, 2312341232, 43234312312, 43123412312,\\
& 2312341232, 12342312312, 32342312312, 31234312312,\\
& 23123412312, 23123432312, 432342312312, 431234312312,\\
& 423123412312, 312342312312, 231234312312, 323123432312,\\
& 342312341232, 4312342312312, 4231234312312, 4323123432312,\\
& 2312342312312, 3231234312312, 3423123412312, 2342312341232,\\
& 42312342312312, 43231234312312, 32312342312312,\\
& 34231234312312, 23423123412312, 12342312341232,\\
& 432312342312312, 434231234312312, 342312342312312,\\
& 234231234312312, 123423123412312, 4342312342312312,\\
& 4234231234312312, 2342312342312312, 1234231234312312,\\
& 42342312342312312, 41234231234312312, 12342312342312312,\\
& 412342312342312312, 323432312342312312,\\
& 3123432312342312312, 23123432312342312312\}
\end{aligned}
$$

$$\begin{aligned}
W^{M_3,A} = \ & \{id, 3, 23, 43, 123, 323, 423, 1323, 1423, 4323, 3423, 14323, 13423,\\
& 23123, 34323, 23423, 134323, 123423, 323123, 423123, 234123,\\
& 234323, 1234123, 1234323, 4323123, 3423123, 2343123, 3234123,\\
& 3234323, 12343123, 13234123, 13234323, 34323123, 23423123,\\
& 32343123, 43234123, 123423123, 132343123, 143234123,\\
& 234323123, 323423123, 432343123, 231234123, 231234323,\\
& 1234323123, 1323423123, 1432343123, 3234323123, 4323423123,\\
& 2312343123, 4231234123, 3231234323, 13234323123, 14323423123,\\
& 12312343123, 43234323123, 32312343123, 42312343123,\\
& 34231234123, 43231234323, 143234323123, 132312343123,\\
& 142312343123, 231234323123, 432312343123, 342312343123,\\
& 234231234123, 1432312343123, 1342312343123, 1234231234123,\\
& 3231234323123, 4231234323123, 3432312343123, 2342312343123,\\
& 13432312343123, 12342312343123, 43231234323123,\\
& 34231234323123, 23423123423123, 23432312343123,\\
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& 234231234323123, 234323123423123, 1234231234323123,\\
& 1234323123423123, 2343231234323123, 3234323123423123,\\
& 12343231234323123, 13234323123423123, 32343231234323123,\\
& 132343231234323123, 231234323123423123,\\
& 2312343231234323123, 32312343231234323123\}
\end{aligned}$$

$$\begin{aligned}
W^{M_4,A} = \ & \{id, 4, 34, 234, 1234, 3234, 13234, 43234, 143234, 231234, 3231234,\\
& 4231234, 43231234, 34231234, 343231234, 234231234, 1234231234,\\
& 2343231234, 12343231234, 32343231234, 132343231234,\\
& 2312343231234, 32312343231234, 432312343231234\}
\end{aligned}$$

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