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# Scalar Invariants on Special Spaces of Equiaffine Connections

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**Abstract.** The only basic scalar invariant in the general equiaffine geometry is the determinant of the Ricci tensor. For special equiaffine geometries, more scalar invariants can emerge. In this paper, we first investigate invariants of torsion-less connections with constant Christoffel symbols in $\mathbb{R}^2$. For this aim, we calculate invariants of the corresponding representation of the group $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathbb{R}^6$ of Christoffel symbols. As a result, we find three bi-quadratic polynomials forming a Hilbert basis of this representation. An interesting phenomenon (rational involutive maps of higher degree) appears during the calculation. We also study representation of $\mathrm{SL}(2,\mathbb{R})$ on the 9-dimensional space of special equiaffine connections in $\mathbb{R}^3$ and corresponding invariants.
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## 1. Introduction

Let $(M,\nabla)$ be a torsion-free affine manifold. The affine connection $\nabla$ is said to be *equiaffine* if there exists a nonvanishing $n$-form $\omega$ which is parallel with respect to $\nabla$. It is well-known that $\omega$ is determined up to a constant multiplication. Further, a simply connected manifold $(M,\nabla)$ is equiaffine if and only if the Ricci tensor $\mathrm{Ric}^\nabla$ is symmetric ([7]).

If we fix a parallel $n$-form $\omega$, then all bases $\{E_1,\dots E_n\}$ of the tangent space $T_pM$ at $p\in M$ which satisfy $\omega(E_1,\dots,E_n)=1$ are related by transofrmations from the group $\mathrm{SL}(n,\mathbb{R})$. Any polynomial created from the Christoffel symbols of the equiaffine connection $\nabla$ will be equiaffine invariant if it is invariant with respect to the group $\mathrm{SL}(n,\mathbb{R})$. In a general situation, the only well-known polynomial with this property is the determinant of the Ricci tensor.

In the previous works [3] and [4], the present author, O. Kowalski and Z. Vlášek studied homogeneous geodesics for special homogeneous equiaffine connections in dimensions 2 and 3 and they found new equiaffine invariants. Hence we are motivated to look for the full classification of all invariants in these situations, which are, in fact, invariants of the corresponding representations of the

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group $\mathrm{SL}(2,\mathbb{R})$ on the spaces of Christoffel symbols. These results go beyond the framework of equiaffine differential geometry, to the representation theory, but the representation theory is not our main objective. During our study, also remarkable involutive maps related to representations of the group $\mathrm{SL}(2,\mathbb{R})$ appear, we describe them in Section 3 and at the end of Section 6.

The following classification of homogeneous connections on 2-dimensional manifolds was obtained T. Arias-Marco and O. Kowalski in [2]. It is the refinement of the classification by B. Opozda in [9].

**Theorem 1.1.** *Let $\nabla$ be a locally homogeneous affine connection with arbitrary torsion on a 2-dimensional manifold $M$. Then, in a neighborhood $\mathcal{U}$ of each point $m \in M$, either $\nabla$ is locally a Levi-Civita connection of the unit sphere or, there is a system $(u,v)$ of local coordinates and constants $A,B,C,D,E,F,G,H$ such that $\nabla$ is expressed in $\mathcal{U}$ by one of the following formulas:*

$$\begin{array}{lll} typeA: & \nabla_{\partial_u}\partial_u = A\,\partial_u + B\,\partial_v, & \nabla_{\partial_u}\partial_v = C\,\partial_u + D\,\partial_v, \\ & \nabla_{\partial_v}\partial_u = E\,\partial_u + F\,\partial_v, & \nabla_{\partial_v}\partial_v = G\,\partial_u + H\,\partial_v, \\ & & \\ typeB: & \nabla_{\partial_u}\partial_u = \frac{A}{u}\,\partial_u + \frac{B}{u}\,\partial_v, & \nabla_{\partial_u}\partial_v = \frac{C}{u}\,\partial_u + \frac{D}{u}\,\partial_v, \\ & \nabla_{\partial_v}\partial_u = \frac{E}{u}\,\partial_u + \frac{F}{u}\,\partial_v, & \nabla_{\partial_v}\partial_v = \frac{G}{u}\,\partial_u + \frac{H}{u}\,\partial_v. \end{array}$$

Let $\mathcal{H}$ denote the 6-dimensional space of torsion-free affine connections of type A defined on the 2-dimensional Euclidean plane $\mathbb{R}^2[u,v]$. The Christoffel symbols $\Gamma^i_{jk}$ of $\nabla$ are calculated using the basis of coordinate vector fields $\{\partial_u,\partial_v\}$. We change the standard notation of Christoffel symbols to a simplified one:

$$\begin{aligned} \Gamma^1_{11} &= A_1, & \Gamma^2_{11} &= A_2, \\ \Gamma^1_{22} &= B_1, & \Gamma^2_{22} &= B_2, \\ \Gamma^1_{12} = \Gamma^1_{21} &= E_1, & \Gamma^2_{12} = \Gamma^2_{21} &= E_2. \end{aligned} \tag{1}$$

We obtain for the corresponding Ricci matrix with respect to the standard basis $\{\partial_u,\partial_v\}$ the expression

$$\mathrm{Ric} = \begin{pmatrix} -A_1E_2 - A_2B_2 + A_2E_1 + E_2^2 & A_2B_1 - E_1E_2 \\ A_2B_1 - E_1E_2 & -A_1B_1 + B_1E_2 - B_2E_1 + E_1^2 \end{pmatrix}. \tag{2}$$

We can see directly that each connection from $\mathcal{H}$ is equiaffine. Namely, it can be verified that the corresponding parallel 2-form is given by the formula

$$\omega = \exp\Big((A_1+E_2)u + (E_1+B_2)v + const\Big)\cdot du \wedge dv.$$

Let us define the natural representation $\rho$ of the group $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathcal{H}$. For each matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \tag{3}$$

and each connection $\nabla \in \mathcal{H}$ with the Christoffel symbols $A_1, A_2, \ldots, E_2$, the corresponding connection $\bar{\nabla} = \rho(A)(\nabla)$ will be defined as follows: Choose the new Cartesian coordinates $\bar{u}, \bar{v}$ in $\mathbb{R}^2$ by the formula

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = A \cdot \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then the Christoffel symbols $\bar{A}_1, \bar{A}_2, \ldots, \bar{E}_2$ of $\bar{\nabla}$ with respect to the standard basis $\{\partial_u, \partial_v\}$ are calculated as the Christoffel symbols of $\nabla$ with respect to the new vector basis $\{\partial_{\bar{u}}, \partial_{\bar{v}}\}$ The representation $\rho$ is herewith uniquely determined. There is an obvious transformation rule

$$\begin{aligned} \partial_u &= a\partial_{\bar{u}} + c\partial_{\bar{v}}, \\ \partial_v &= b\partial_{\bar{u}} + d\partial_{\bar{v}}. \end{aligned}$$

We use the relations

$$\begin{aligned} \nabla_{\partial_u}\partial_u &= A_1\partial_u + A_2\partial_v = (aA_1 + bA_2)\partial_{\bar{u}} + (cA_1 + dA_2)\partial_{\bar{v}}, \\ \nabla_{\partial_u}\partial_v &= E_1\partial_u + E_2\partial_v = (aE_1 + bE_2)\partial_{\bar{u}} + (cE_1 + dE_2)\partial_{\bar{v}}, \\ \nabla_{\partial_v}\partial_v &= B_1\partial_u + B_2\partial_v = (aB_1 + bB_2)\partial_{\bar{u}} + (cB_1 + dB_2)\partial_{\bar{v}} \end{aligned}$$

and the relations

$$\begin{aligned} \nabla_{\partial_u}\partial_u &= a^2\nabla_{\partial_{\bar{u}}}\partial_{\bar{u}} + 2ac\nabla_{\partial_{\bar{u}}}\partial_{\bar{v}} + c^2\nabla_{\partial_{\bar{v}}}\partial_{\bar{v}}, \\ \nabla_{\partial_u}\partial_v &= ab\nabla_{\partial_{\bar{u}}}\partial_{\bar{u}} + (ad + bc)\nabla_{\partial_{\bar{u}}}\partial_{\bar{v}} + cd\nabla_{\partial_{\bar{v}}}\partial_{\bar{v}}, \\ \nabla_{\partial_v}\partial_u &= b^2\nabla_{\partial_{\bar{u}}}\partial_{\bar{u}} + 2bd\nabla_{\partial_{\bar{u}}}\partial_{\bar{v}} + d^2\nabla_{\partial_{\bar{v}}}\partial_{\bar{v}}. \end{aligned}$$

Using these two systems of equations we express the covariant derivatives $\nabla_{\partial_{\bar{u}}}\partial_{\bar{u}}, \nabla_{\partial_{\bar{u}}}\partial_{\bar{v}}, \nabla_{\partial_{\bar{v}}}\partial_{\bar{v}}$ with respect to the vector fields $\partial_{\bar{u}}, \partial_{\bar{v}}$ and hence we obtain the Christoffel symbols $\bar{A}_i, \bar{B}_i, \bar{E}_i$ expressed through $A_i, B_i, E_i$ and $a, b, c, d$ in the form

$$\begin{aligned} \bar{A}_1 &= ad^2A_1 + bd^2A_2 + ac^2B_1 + bc^2B_2 - 2\,acdE_1 - 2\,bcdE_2, \\ \bar{A}_2 &= cd^2A_1 + d^3A_2 + c^3B_1 + c^2dB_2 - 2\,c^2dE_1 - 2\,cd^2E_2, \\ \bar{B}_1 &= ab^2A_1 + b^3A_2 + a^3B_1 + a^2bB_2 - 2\,a^2bE_1 - 2\,ab^2E_2, \\ \bar{B}_2 &= b^2cA_1 + b^2dA_2 + a^2cB_1 + a^2dB_2 - 2\,abcE_1 - 2\,abdE_2, \\ \bar{E}_1 &= -abdA_1 - b^2dA_2 - a^2cB_1 - abcB_2 + a(bc + ad)E_1 + b(bc + ad)E_2, \\ \bar{E}_2 &= -bcdA_1 - bd^2A_2 - ac^2B_1 - acdB_2 + c(bc + ad)E_1 + d(bc + ad)E_2. \end{aligned} \tag{4}$$

In fact, the right-hand sides in a general form are all divided by the factor $(ad-bc)^2$, but we can use here the identity $ad - bc = 1$ for each matrix $A$ as in formula (3).

The first scalar invariant of this representation is the determinant of the Ricci matrix in formula (2). It is the polynomial

$$\begin{aligned} I_1 &= I_1(A_i, B_i, E_i) = \\ &= (-A_1E_2 - A_2B_2 + A_2E_1 + E_2^2)(-A_1B_1 + B_1E_2 - B_2E_1 + E_1^2) \\ &\quad -(A_2B_1 - E_1E_2)^2. \end{aligned} \tag{5}$$

We denote $\bar{I}_1 = I_1(\bar{A}_i, \bar{B}_i, \bar{E}_i)$ and by the straightforward check using (4) we can verify that $\bar{I}_1 = I_1$.

To find another invariant is not so obvious. In [3], the authors investigated, among others, homogeneous geodesics of homogeneous affine connections defined on $\mathbb{R}^2[u,v]$. Here a *homogeneous geodesic* is a (possibly reparametrized) geodesic which is an orbit of a 1-parameter group of affine transformations, i.e., $\gamma(t) = \exp(tW)(p)$, where $W \neq 0$ is an element of the Lie algebra of the of the group of all affine diffeomorphisms and $p \in M$ is a fixed point. Each connection $\nabla \in \mathcal{H}$ is invariant with respect to all translations of the plane and hence it admits a 2-dimensional space of affine Killing vector fields of the form $X = x\partial_u + y\partial_v$, where $x, y$ are arbitrary parameters. In this situation, a Killing vector field $X = x\partial_u + y\partial_v$ is a geodesic vector field (i.e., every integral curve of $X$ is homogeneous geodesic) if and only if the equation $\nabla_X X = kX$ is satisfied, where $k$ is a constant. This equation leads to the system of equations

$$\begin{aligned} A_1x^2 + B_1y^2 + 2E_1xy &= kx, \\ A_2x^2 + B_2y^2 + 2E_2xy &= ky. \end{aligned} \tag{6}$$

If we put $k = 0$, these equations characterize the existence of a geodesic Killing field whose integral curves are homogeneous geodesics which do not require a reparametrization. We can calculate the resultant of these equations in the variable $x$ and the resultant in the variable $y$. These resultants are $y^4 I_2$ and $x^4 I_2$, respectively, where

$$\begin{aligned} I_2 &= I_2(A_i, B_i, E_i) = \\ &= 4(A_1E_2 - A_2E_1)(B_1E_2 - B_2E_1) + (A_1B_2 - A_2B_1)^2. \end{aligned} \tag{7}$$

The polynomial $I_2$ is an invariant of the group $\mathrm{SL}(2,\mathbb{R})$ again. Indeed, denote by $\bar{I}_2$ the polynomial $I_2(\bar{A}_i, \bar{B}_i, \bar{E}_i)$ and by the computer check we easily verify $\bar{I}_2 = I_2$.

We see that both polynomials $I_1$ and $I_2$ are invariants with respect to the representation $\rho$ of the group $\mathrm{SL}(2,\mathbb{R})$. We are going to investigate general invariants of this representation in Section 5.

## 2. Infinitesimal transformations for the representation $\rho$

Let us identify the space $\mathcal{H}$ of connections with the space $\mathbb{R}^6[A_1, A_2, B_1, B_2, E_1, E_2]$ under the condition that the Christoffel symbols are calculated with respect to the *original* Cartesian coordinate system $(u, v)$. Let the 1-parameter group $g_t$ of transformations be acting on $\mathbb{R}^6$ by

$$g_t \cdot (A_1, A_2, \ldots, E_2) = (A_1(t), A_2(t), \ldots, E_2(t)). \tag{8}$$

The corresponding Killing vector field is

$$X = A_1'(0)\frac{\partial}{\partial A_1} + A_2'(0)\frac{\partial}{\partial A_2} + \ldots + E_2'(0)\frac{\partial}{\partial E_2}. \tag{9}$$

Consider the following generating 1-dimensional subgroups of the group $\mathrm{SL}(2,\mathbb{R})$:

$$g_1(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, g_2(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, g_3(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \tag{10}$$

For these 1-dimensional subgroups $g_i(t), i = 1,2,3$, we obtain, by a routine calculations, the following infinitesimal transformations (left-invariant vector fields) on $\mathcal{H} = \mathbb{R}^6[A_1, A_2, B_1, B_2, E_1, E_2]$ corresponding to the representation $\rho$ of the group $\mathrm{SL}(2,\mathbb{R})$

$$\begin{aligned}
X_1 &= -A_1\frac{\partial}{\partial A_1} - 3A_2\frac{\partial}{\partial A_2} + 3B_1\frac{\partial}{\partial B_1} + B_2\frac{\partial}{\partial B_2} + E_1\frac{\partial}{\partial E_1} - E_2\frac{\partial}{\partial E_2},\\
X_2 &= -2E_1\frac{\partial}{\partial A_1} + (A_1 - 2E_2)\frac{\partial}{\partial A_2}\\
&\quad + B_1\frac{\partial}{\partial B_2} - B_1\frac{\partial}{\partial E_1} + (-B_2 + E_1)\frac{\partial}{\partial E_2},\\
X_3 &= A_2\frac{\partial}{\partial A_1} + (B_2 - 2E_1)\frac{\partial}{\partial B_1}\\
&\quad - 2E_2\frac{\partial}{\partial B_2} + (-A_1 + E_2)\frac{\partial}{\partial E_1} - A_2\frac{\partial}{\partial E_2}.
\end{aligned} \tag{11}$$

By the straightforward check, it can be verified that the polynomials $I_1$ and $I_2$ satisfy $X_i(I_1) = X_i(I_2) = 0$ for all the operators $X_1, X_2, X_3$ given by (11). Also by the straightforward calculation, it can be verified that the Lie bracket of these operators satisfy the standard relations for the algebra $\mathfrak{sl}(2,\mathbb{R})$

$$\begin{aligned}
[X_1, X_2] &= 2X_2,\\
[X_1, X_3] &= -2X_3,\\
[X_2, X_3] &= X_1.
\end{aligned} \tag{12}$$

Using Maple program for each of the operators $X_1, X_2, X_3$, we find always immediately a basis of invariants consisting of 5 polynomials in the Christoffel symbols. Some of these invariants are obvious and most of them can be calculated by hand. For the operator $X_1$, the invariants are

$$w_1 = A_1B_2, \quad w_2 = A_2B_1, \quad w_3 = A_1E_1, \quad w_4 = B_2E_2, \quad w_5 = A_1^3B_1, \tag{13}$$

for the operator $X_2$, we have

$$\begin{aligned}
&u_1 = B_1, \quad u_2 = B_2 + E_1, \quad u_3 = A_1B_1 - E_1^2, \quad u_4 = B_1E_2 - B_2E_1,\\
&u_5 = B_1(A_2B_1 - A_1B_2) + 2E_1(B_2E_1 - B_1E_2)
\end{aligned} \tag{14}$$

and for the operator $X_3$ we have

$$\begin{aligned}
&v_1 = A_2, \quad v_2 = A_1 + E_2, \quad v_3 = A_2B_2 - E_2^2, \quad v_4 = A_2E_1 - A_1E_2,\\
&v_5 = A_2(A_2B_1 - A_1B_2) + 2E_2(A_1E_2 - A_2E_1).
\end{aligned} \tag{15}$$

Seemingly, it would be more advantageous to calculate the invariants of the operators $X_2 + X_3$ and $X_2 - X_3$. Yet, in such cases, Maple program gives no answer. This is an indication that a systematic calculation of the invariants of the representation $\rho$ of $\mathrm{SL}(2,\mathbb{R})$ might be a hard problem. This was really the case, as we shall see later.

## 3. A remarkable involutive map

We put further

$$u_6 = E_1, \qquad v_6 = E_2 \tag{16}$$

and consider $u_i$, or $v_i$, respectively, as the new coordinates. We can express the Christoffel symbols $A_1, \dots, E_2$ in $u_i$ or in $v_i$. We get the inverse transformation to the transformation given by the formulas (14) and (16), or to the transformation given by (15) and (16), respectively, in the forms

$$\begin{aligned}
A_1 &= (u_3 + u_6^2)/u_1, \\
A_2 &= (u_2 u_3 + u_2 u_6^2 - u_3 u_6 + 2u_4 u_6 + u_5 - u_6^3)/u_1^2, \\
B_1 &= u_1, \\
B_2 &= u_2 - u_6, \\
E_1 &= u_6, \\
E_2 &= (u_2 u_6 + u_4 - u_6^2)/u_1
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
A_1 &= v_2 - v_6, \\
A_2 &= v_1, \\
B_1 &= (v_2 v_3 + v_2 v_6^2 - v_3 v_6 + 2v_4 v_6 + v_5 - v_6^3)/v_1^2, \\
B_2 &= (v_3 + v_6^2)/v_1, \\
E_1 &= (v_2 v_6 + v_4 - v_6^2)/v_1, \\
E_2 &= v_6.
\end{aligned} \tag{18}$$

Using these formulas under the conditions $u_1 = B_1 \neq 0, v_1 = A_2 \neq 0$, we can express $u_i$ in $v_i$ or vice versa. We obtain the formulas

$$\begin{aligned}
u_1 &= \frac{v_2 v_3 + v_2 v_6^2 - v_3 v_6 + 2v_4 v_6 + v_5 - v_6^3}{v_1^2}, \\
u_2 &= \frac{v_2 v_6 + v_3 + v_4}{v_1}, \\
u_3 &= \frac{v_2^2 v_3 - 2v_2 v_3 v_6 + v_2 v_5 + v_3 v_6^2 - v_4^2 - v_5 v_6}{v_1^2}, \\
u_4 &= \frac{-v_3 v_4 + v_4 v_6^2 + v_5 v_6}{v_1^2}, \\
u_5 &= [4v_2 v_3 v_4 v_6 + v_2 v_3 v_5 - v_2 v_5 v_6^2 + 2v_3 v_4^2 - 4v_3 v_4 v_6^2 - v_3 v_5 v_6 \\
&\quad +2v_4^2 v_6^2 + 2v_4 v_5 v_6 + v_5^2 + v_5 v_6^3]/v_1^3, \\
u_6 &= \frac{v_2 v_6 + v_4 - v_6^2}{v_1}
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
v_1 &= \frac{u_2 u_3 + u_2 u_6^2 - u_3 u_6 + 2u_4 u_6 + u_5 - u_6^3}{u_1^2}, \\
v_2 &= \frac{u_2 u_6 + u_3 + u_4}{u_1}, \\
v_3 &= \frac{u_2^2 u_3 - 2u_2 u_3 u_6 + u_2 u_5 + u_3 u_6^2 - u_4^2 - u_5 u_6}{u_1^2},
\end{aligned}$$

$$\begin{aligned}
v_4 &= \frac{-u_3u_4 + u_4u_6^2 + u_5u_6}{u_1^2},\\
v_5 &= [4u_2u_3u_4u_6 + u_2u_3u_5 - u_2u_5u_6^2 + 2u_3u_4^2 - 4u_3u_4u_6^2 - u_3u_5u_6\\
&\quad +2u_4^2u_6^2 + 2u_4u_5u_6 + u_5^2 + u_5u_6^3]/u_1^3,\\
v_6 &= \frac{u_2u_6 + u_4 - u_6^2}{u_1}.
\end{aligned} \tag{20}$$

We see that these transformations are involutive and they map the set $\mathcal{H}^+ = \{(A_1, A_2, B_1, B_2, E_1, E_2) \in \mathbb{R}^6, A_2 \neq 0, B_1 \neq 0\}$ onto itself.

Involutive transformations play an important role in integrable dynamical systems. See e.g. [1], [5], [6], [10] and the references inside. Unfortunately, all the works known to the present author investigate in general, or apply to dynamics, involutive automorphisms of the type $\mathbb{R}^2 \to \mathbb{R}^2$ or involutive transformations of the real projective plane. Probably, no systematic studies in higher dimensions are known.

## 4. Structure of orbits

Let us fix a matrix

$$W = \begin{pmatrix} X & Y \\ Z & -X \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}) \tag{21}$$

and consider the action of $\exp(tW)$ on $\mathcal{H}$. We are going to investigate the orbits of this action and thus we start with the calculation of its isotropy subgroups at points of $\mathcal{H}$. The Taylor polynomial of the first order of the group $\exp(tW)$ is the 1-parameter system of matrices

$$g(t) = \begin{pmatrix} 1 + tX & tY \\ tZ & 1 - tX \end{pmatrix} \subset GL(2, \mathbb{R}). \tag{22}$$

We will act by the matrices $g(t)$ and use the equations (4), whose right-hand sides must be divided by $(ad - bc)^2$ for $g(t) \subset GL(2, \mathbb{R})$, to write down explicitly the conditions

$$\begin{aligned}
\bar{A}_1 = A_1, \quad \bar{B}_1 = B_1, \quad \bar{E}_1 = E_1,\\
\bar{A}_2 = A_2, \quad \bar{B}_2 = B_2, \quad \bar{E}_2 = E_2.
\end{aligned} \tag{23}$$

Each of these equalities splits into four equalities corresponding to the coefficients at $t, t^2, t^3$ and $t^4$. We will consider only the coefficients at $t$, which give the conditions

$$\begin{aligned}
-A_1X + A_2Y - 2E_1Z &= 0,\\
-3A_2X + (A_1 - 2E_2)Z &= 0,\\
3B_1X + (B_2 - 2E_1)Y &= 0,\\
B_2X - 2E_2Y + B_1Z &= 0,\\
E_1X + (E_2 - A_1)Y - B_1Z &= 0,\\
-E_2X - A_2Y + (E_1 - B_2)Z &= 0.
\end{aligned} \tag{24}$$

The same relations remain valid if we start with any Taylor polynomial of higher degree because all contributions of higher degree terms belong to the class $O(t^2)$. Now we will consider the system of equations (24) for fixed coefficients $X, Y, Z$ and look for the Christoffel symbols $A_1, A_2, \ldots, E_2$ which satisfy it. The matrix of this homogeneous system of equations is

$$M = \begin{pmatrix} -X & Y & 0 & 0 & -2Z & 0 \\ Z & -3X & 0 & 0 & 0 & -2Z \\ 0 & 0 & 3X & Y & -2Y & 0 \\ 0 & 0 & Z & X & 0 & -2Y \\ -Y & 0 & -Z & 0 & X & Y \\ 0 & -Y & 0 & -Z & Z & -X \end{pmatrix}. \tag{25}$$

The determinant of this matrix is equal to

$$D = \det(M) = -9\,(X^2 + YZ)^3. \tag{26}$$

**Lemma 4.1.** *The unique connection with* 3*-dimensional isotropy group is the standard Euclidean connection.*

**Proof.** If some connection has 3-dimensional isotropy group, then the corresponing Christoffel symbols $A_1, A_2, \ldots, E_2$ must satisfy the system of equations given by the matrix $M$ in (25) for any $X, Y, Z$. It is clear that, in the generic case $X^2 + YZ \neq 0$, the matrix $M$ is regular and the solution is only the connection with zero Christoffel symbols. ■

**Theorem 4.2.** *The* 1*-dimensional isotropy groups are generated by the matrices* $W \in \mathfrak{sl}(2, \mathbb{R})$ *given by* (21) *for the triplets* $(X, Y, Z) \neq (0, 0, 0)$ *which satisfy* $X^2 + YZ = 0$*. For each such triplet, there is a* 2*-parameter system of connections whose isotropy group is generated by the corresponding matrix* $W$*. The set* $\bar{\mathcal{M}}$ *of connections whose isotropy group depends at least on one parameter is closed in* $\mathcal{H}$ *and the set* $\mathcal{U} = \mathcal{H} \setminus \bar{\mathcal{M}}$ *is a smooth manifold.*

**Proof.** Each triplet $(X, Y, Z)$ such that $X \neq 0$ can be normalized so that we put $X = 1$ and $Y = -1/Z$. Thus we consider the isotropy group

$$g_Z(t) = \begin{pmatrix} 1+t & -t/Z \\ tZ & 1-t \end{pmatrix} \tag{27}$$

for a fixed $Z \neq 0$ and let it act on $\mathcal{H}$. Either calculating by hand or using Maple program to the homogeneous system of equations (24) whose matrix is $M$, with $X = 1$ and $Y = -1/Z$, we obtain the following family of relations for the Christoffel symbols

$$\begin{aligned} A_1 &= -B_1 Z^2 - 2E_1 Z, \\ A_2 &= B_1 Z^3, \end{aligned}$$

$$\begin{aligned} B_2 &= 3B_1Z + 2E_1, \\ E_2 &= -2B_1Z^2 - E_1Z. \end{aligned} \tag{28}$$

It determines a 2-dimensional subspace $\mathcal{M}(Z)$ of $\mathcal{H}$. (It can be checked that this system of connections satisfies also the conditions corresponding to the coefficients at $t^2, t^3$ and $t^4$ in the equalities (23).) We are now looking for the topological closure of the set $\mathcal{M} = \bigcup_{Z\neq 0} \mathcal{M}(Z)$ in the space $\mathcal{H} = \mathbb{R}^6[A_1, A_2, \ldots, E_2]$. The limit case for $Z \to 0$ in (28) gives the 2-dimensional subspace $\mathcal{M}(0)$ given by the conditions

$$A_1 = A_2 = E_2 = 0, \qquad B_2 = 2E_1, \tag{29}$$

which is obviously that with the isotropy group $g_2(t)$. There is still a problem how to manage correctly the limit cases $Z \to \infty$ and $Z \to -\infty$. We shall describe this procedure. Let us substitute $Z = -1/Y$ into the formulas (28) and remove all denominators. We are left with the system

$$\begin{aligned} B_1 &= -A_1Y^2 + 2E_1Y, \\ B_1 &= -A_2Y^3, \\ 3B_1 &= (2E_1 - B_2)Y, \\ 2B_1 &= E_1Y - E_2Y^2. \end{aligned} \tag{30}$$

Now we will express the variables $A_1, B_1, B_2, E_1$ in $A_2, E_2$ and $Y$. We keep the equation $(30_2)$ and substitute it into $(30_4)$. We obtain

$$\begin{aligned} B_1 &= -A_2Y^3, \\ E_1 &= -2A_2Y^2 + E_2Y. \end{aligned} \tag{31}$$

Further, we substitute both equations (31) into $(30_1)$ and $(30_3)$. After the simplification, we obtain

$$\begin{aligned} A_1 &= -3A_2Y + 2E_2, \\ B_2 &= -A_2Y^2 + 2E_2Y. \end{aligned} \tag{32}$$

Summarizing, formulas (31) and (32) describe each set $\mathcal{M}(Z)$ through the variable $Y = -1/Z$. This system of equations now enables to study the limits $Z \to \infty$ and $Z \to -\infty$ in a smart form. Indeed, putting $Y = 0$ in (31) and (32), we get the special 2-dimensional subspace $\mathcal{M}(\infty)$ given by the conditions

$$B_1 = B_2 = E_1 = 0, \qquad A_1 = 2E_2, \tag{33}$$

which is obviously that with the isotropy group $g_3(t)$. We deduce that the closure of $\mathcal{M}$ consists of all connections which admit at least 1-dimensional isotropy group. Hence, $\mathcal{H} \setminus \bar{\mathcal{M}}$ is an open set in $\mathcal{H}$, thus a smooth manifold, which is composed of 3-dimensional orbits with respect to the representation $\rho$. ■

**Remark 4.3.** The standard Euclidean connection also belongs to the 3-parameter system of connections described by the formulas (28).

## 5. Complete set of invariants

According to [8], a representation of a Lie group is called *semi-regular* if all orbits have the same dimension.

**Theorem 5.1** ([8]). *Let a Lie group $G$ act semi-regularly on the $n$-dimensional manifold $M$ with $s$-dimensional orbits. At each point $x \in M$, there exist $m - s$ functionally independent local invariants $I_1, \dots, I_{n-s}$, defined on a neighbourhood of $x$.*

According to Theorem 5.1 and Theorem 4.2, our representation $\rho$ of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{U}$ admits (locally) 3 independent invariants. The present author have made much effort to determine the *third* local invariant $I_3$, in addition to $I_1$ given by the formula (5) and $I_2$ given by the formula (7), by a direct mathematical way, using partial invariants (14) and (15). This lead to a difficult problem of "separation of variables": find all pairs of functions $f, g$ of 5 variables such that $f(u_1, \dots, u_5) = g(v_1, \dots, v_5)$. Unfortunately, solving this particular problem by this method was not successfull. Finally, let us use two reasonable assumptions and the computer force.

We notice that the known invariants $I_1$ and $I_2$ can be written in terms of particular invariants $u_i$ in the form

$$\begin{aligned} I_1 &= -\frac{u_2^2 u_3 u_4 + u_2 u_3 u_5 + u_2 u_4 u_5 - u_3^2 u_4 + 2u_3 u_4^2 - u_4^3 + u_5^2}{u_1^2}, \\ I_2 &= \frac{4u_3 u_4^2 + u_5^2}{u_1^2} \end{aligned} \tag{34}$$

and we will assume that the invariant $I_3$ can be also written in the form

$$I_3 = \frac{P(u_2, \dots, u_5)}{u_1^2}. \tag{35}$$

When we explore carefully the formulas (34), we see that both numerators written in variables $u_2, \dots, u_5$ are, after substitution from formulas (14), homogeneous polynomials in $A_1, \dots, E_2$ of degree 6. Both numerators are divisible by $u_1^2$ and after this division we obtain a polynomial of degree 4 in variables $A_1, \dots, E_2$ (see formulas (5) and (7)). We will look for the numerator $P(u_2, \dots, u_5)$ in the invariant $I_3$ in the same form. There are 14 monomials in $u_2, \dots, u_5$ which have degree 6 in $A_1, \dots, E_2$. These are

$$\begin{gathered} u_2^6, u_2^4 u_3, u_2^4 u_4, u_2^3 u_5, u_2^2 u_3^2, u_2^2 u_3 u_4, u_2^2 u_4^4, \\ u_2 u_3 u_5, u_2 u_4 u_5, u_3^3, u_3^2 u_4, u_3 u_4^2, u_4^3, u_5^2. \end{gathered} \tag{36}$$

The polynomial $P$ from (35) must be a linear combination with constant coefficients of these monomials. Let us further assume that the coefficients by these monomials are not too high. When we admit only the coefficients $0, 1, -1, 2$, we need to check $4^{14}$ possible polynomials. With the contemporary personal computer and Maple, this would take approximately one month. Fortunately, we had good luck earlier and we found the invariant

$$
\begin{aligned}
I_3 &= [u_2^4 u_3 + u_2^3 u_5 + 2u_2^2 u_3^2 + 2u_2^2 u_3 u_4 - u_2^2 u_4^2 + u_2 u_3 u_5 \\
&\quad + u_2 u_4 u_5 + u_3^3 + 2u_3^2 u_4 + u_3 u_4^2]/u_1^2 = \\
&= (A_1^2 + A_1 E_2 + A_2 B_2 + A_2 E_1)(A_1 B_1 + B_1 E_2 + B_2^2 + B_2 E_1) \\
&\quad - (A_1 E_1 + B_2 E_2 + 2E_1 E_2)^2.
\end{aligned} \tag{37}
$$

It can be checked directly that $I_3$ satisfies differential equations $X_i(I_3) = 0$, where $X_i$ are given by the equations (11), and also the condition $\bar{I}_3 = I_3$. The question about the geometrical meaning of this invariant remains open.

**Theorem 5.2.** *Polynomials $I_1, I_2, I_3$ form a Hilbert basis of global scalar invariants of the representation $\rho$ of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{U}$.*

**Proof.** It remains to check the functional independence of the invariants $I_1, I_2, I_3$. The independence of $I_1$ and $I_2$ is clear. We put $A_1 = B_1 = E_1 = E_2 = 0$ and we easily see that $I_1 = I_2 = 0$ and $I_3 = A_2 B_2^3$. We see that also $I_3$ is independent of $I_1$ and $I_2$. ∎

## 6. Representation of SL(2, ℝ) on $R^9$

Let us now consider the space of torsion-free affine connections with constant Christoffel symbols on $\mathbb{R}^3[u,v,w]$. We denote

$$
\begin{aligned}
&\Gamma_{11}^i = A_i, \qquad \Gamma_{22}^i = B_i, \qquad \Gamma_{33}^i = C_i, \\
&\Gamma_{12}^i = \Gamma_{21}^i = E_i, \quad \Gamma_{13}^i = \Gamma_{31}^i = F_i, \quad \Gamma_{23}^i = \Gamma_{32}^i = G_i.
\end{aligned} \tag{38}
$$

Geodesic Killing vector fields and homogeneous geodesics of these connections were investigated among others in [4]. The equations which characterize the condition that the Killing vector field $X = x\partial_u + y\partial_v + z\partial_w$ satisfies $\nabla_X X = kX$ and it is a geodesic vector field are now of the form

$$
\begin{aligned}
x^2 A_1 + y^2 B_1 + z^2 C_1 + 2\,xyE_1 + 2\,xzF_1 + 2\,yzG_1 &= kx \\
x^2 A_2 + y^2 B_2 + z^2 C_2 + 2\,xyE_2 + 2\,xzF_2 + 2\,yzG_2 &= ky \\
x^2 A_3 + y^2 B_3 + z^2 C_3 + 2\,xyE_3 + 2\,xzF_3 + 2\,yzG_3 &= kz.
\end{aligned} \tag{39}
$$

We now put $z = 0$ and restrict ourselves on the plane $\mathbb{R}^2[u,v]$. The equations (39) are in the form

$$
\begin{aligned}
x^2 A_1 + y^2 B_1 + 2\,xyE_1 &= kx \\
x^2 A_2 + y^2 B_2 + 2\,xyE_2 &= ky \\
x^2 A_3 + y^2 B_3 + 2\,xyE_3 &= 0,
\end{aligned} \tag{40}
$$

where $k \in \mathbb{R}$. If we assume $y \neq 0$, we may normalize $y = 1$, express the parameter $k$ from the second equation and substitute it into the first one. We obtain

$$
\begin{aligned}
A_2\,x^3 + (2\,E_2 - A_1)\,x^2 + (B_2 - 2\,E_1)\,x - B_1 &= 0, \\
A_3\,x^2 + 2\,E_3\,x + B_3 &= 0.
\end{aligned} \tag{41}
$$

If we assume that $A_2 \neq 0 \neq A_3$, the resultant of the polynomials in (41) is

$$\begin{aligned} I_4 \;=\;& {A_1}^2 A_3\, {B_3}^2 + {A_2}^2 {B_3}^3 + {A_3}^3 {B_1}^2 \\ &+A_1\Big(2\,A_2\,{B_3}^2 E_3 - 2\,{A_3}^2 B_1\,B_3 \\ &\qquad +A_3\left(4\,B_1\,{E_3}^2 + 2\,B_2\,B_3\,E_3 - 4\,{B_3}^2 E_2 - 4\,B_3\,E_1\,E_3\right)\Big) \\ &+A_2\Big(A_3\left(-6\,B_1\,B_3\,E_3 - 2\,B_2\,{B_3}^2 + 4\,{B_3}^2 E_1\right) \\ &\qquad +8\,B_1\,{E_3}^3 + 4\,B_2\,B_3\,{E_3}^2 - 4\,{B_3}^2 E_2\,E_3 - 8\,B_3\,E_1\,{E_3}^2\Big) \\ &+{A_3}^2\Big(2B_1\left(B_2\,E_3 + 2\,B_3\,E_2 - 2\,E_1\,E_3\right) \\ &\qquad +{B_2}^2 B_3 - 4\,B_2\,B_3\,E_1 + 4\,B_3\,{E_1}^2\Big) \\ &+A_3\Big(-8\,B_1\,E_2\,{E_3}^2 - 4\,B_2\,B_3\,E_2\,E_3 + 4\,{B_3}^2 {E_2}^2 + 8\,B_3\,E_1\,E_2\,E_3\Big). \end{aligned} \tag{42}$$

This resultant is an invariant of the following representation $\rho'$ of the group $\mathrm{SL}(2,\mathbb{R})$. We denote by $\mathcal{H}'$ the space $\mathbb{R}^9[A_i, B_i, E_i]$ of constant Christoffel symbols corresponding to the covariant differentiations in the coordinate plane $\mathbb{R}^2[u,v]$ of $\mathbb{R}^3[u,v,w]$. We will call the elements of this family *relative connections* on $\mathbb{R}^2[u,v]$ with respect to $\mathbb{R}^3[u,v,w]$. We define the representation $\rho'$ in the similar way as in the first section. Now we use the matrices of the form

$$A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{43}$$

We have the obvious transformation rule

$$\begin{aligned} \partial_u &= a\partial_{\bar u} + c\partial_{\bar v}, \\ \partial_v &= b\partial_{\bar u} + d\partial_{\bar v}, \\ \partial_w &= \partial_{\bar w} \end{aligned} \tag{44}$$

and in the analogous way, we calculate the new Christoffel symbols in the new basis $\{\partial_{\bar u}, \partial_{\bar v}, \partial_{\bar w}\}$ in the form

$$\begin{aligned} &\bar A_1 = ad^2 A_1 + bd^2 A_2 + ac^2 B_1 + bc^2 B_2 - 2\,acdE_1 - 2\,bcdE_2, \\ &\bar A_2 = cd^2 A_1 + d^3 A_2 + c^3 B_1 + c^2 dB_2 - 2\,c^2 dE_1 - 2\,cd^2 E_2, \\ &\bar A_3 = d^2 A_3 + c^2 B_3 - 2\,cdE_3, \\ &\bar B_1 = ab^2 A_1 + b^3 A_2 + a^3 B_1 + a^2 bB_2 - 2\,a^2 bE_1 - 2\,ab^2 E_2, \\ &\bar B_2 = b^2 cA_1 + b^2 dA_2 + a^2 cB_1 + a^2 dB_2 - 2\,abcE_1 - 2\,abdE_2, \\ &\bar B_3 = b^2 A_3 + a^2 B_3 - 2\,abE_3, \\ &\bar E_1 = -abdA_1 - b^2 dA_2 - a^2 cB_1 - abcB_2 + a(bc+ad)E_1 + b(bc+ad)E_2, \\ &\bar E_2 = -bcdA_1 - bd^2 A_2 - ac^2 B_1 - acdB_2 + c(bc+ad)E_1 + d(bc+ad)E_2, \\ &\bar E_3 = -bdA_3 - acB_3 + (ad+bc)\,E_3. \end{aligned} \tag{45}$$

Again, the right-hand sides in a general form are all divided by the factor $(ad-bc)^2$, but we can use the identity $ad - bc = 1$ for each matrix $A$ as in formula (3). The relations between $\bar C_i, \bar F_i, \bar G_i$ and $C_i, F_i, G_i$ can be also derived, but they are not necessary here. The computer check verifies easily the relation $\bar I_4 = I_4$.

From the formulas (45) it can be easily seen that the representation space $\mathcal{H}' = \mathbb{R}^9[A_i, B_i, E_i]$ of $\rho'$ decomposes as a direct sum $\mathbb{R}^6[A_1, A_2, B_1, B_2, E_1, E_2] \oplus \mathbb{R}^3[A_3, B_3, E_3] = \mathcal{H} \oplus \widetilde{\mathcal{H}}$, where $\rho' \mid_{\mathcal{H}} = \rho$ (see formulas (4)). Let us denote $\tilde{\rho} = \rho' \mid_{\widetilde{\mathcal{H}}}$ and investigate this subrepresentation in the same way as the representation $\rho$ before.

We easily derive the infinitesimal generators

$$\begin{aligned} \tilde{X}_1 &= -2A_3\frac{\partial}{\partial A_3} + 2B_3\frac{\partial}{\partial B_3}, \\ \tilde{X}_2 &= -2E_3\frac{\partial}{\partial A_3} - B_3\frac{\partial}{\partial E_3}, \\ \tilde{X}_3 &= -2E_3\frac{\partial}{\partial B_3} - A_3\frac{\partial}{\partial E_3} \end{aligned} \tag{46}$$

and the invariants of each of them, which are (in the similar notation as for $\rho$)

$$\begin{aligned} \tilde{w}_1 = A_3B_3, &\qquad \tilde{w}_2 = E_3, \\ \tilde{u}_1 = B_3, &\qquad \tilde{u}_2 = A_3B_3 - E_3^2, \\ \tilde{v}_1 = A_3, &\qquad \tilde{v}_2 = A_3B_3 - E_3^2. \end{aligned} \tag{47}$$

We see at once that

$$I_5 = A_3B_3 - E_3^2 \tag{48}$$

is an invariant of the representation $\tilde{\rho}$. The equality $\bar{I}_5 = I_5$ can be checked also easily. If we denote further $u_3 = E_3$ and $v_3 = E_3$, we obtain (in the same way as for $\rho$) another involutive map of the set $\widetilde{\mathcal{H}}^+ = \{(A_3, B_3, E_3) \in \mathbb{R}^3, A_3 \neq 0, B_3 \neq 0\}$ onto itself. This map is given by the formulas

$$v_1 = \frac{u_2 + u_3^2}{u_1}, \quad v_2 = u_2, \quad v_3 = u_3 \tag{49}$$

and

$$u_1 = \frac{v_2 + v_3^2}{v_1}, \quad u_2 = v_2, \quad u_3 = v_3. \tag{50}$$

**Problem 6.1.** Does every representation of the group SL,$\mathbb{R}$) on a space $\mathbb{R}^n$ give rise to a rational involutive mapping of $\mathbb{R}^n \setminus \mathcal{D}$ onto itself, where $\mathcal{D}$ is a subset of measure zero?

Let us now investigate the structure of orbits. We act again by the Taylor polynomial of the first order of the 1-parameter group $\exp(tW)$ on $(A_3, B_3, E_3)$. The conditions $\bar{A}_3 = A_3, \bar{B}_3 = B_3, \bar{E}_3 = E_3$ give now the system of equations

$$\begin{aligned} A_3X + E_3Z &= 0, \\ B_3X - E_3Y &= 0, \\ A_3Y + B_3Z &= 0 \end{aligned} \tag{51}$$

(which is analogous to the system (24) for the representation $\rho$). We see immediately that the solution of this system is $(X, Y, Z) = (E_3, B_3, -A_3)$. We conclude that any 1-parameter subgroup generated by the triple $(X, Y, Z)$ preserves the element $(A_3, B_3, E_3) = (-Z, Y, X)$ of $\widetilde{\mathcal{H}}$ and, conversely, any element $(A_3, B_3, E_3)$ of $\widetilde{\mathcal{H}}$ is preserved by the 1-parameter subgroup generated by $(X, Y, Z) = (E_3, B_3, -A_3)$. Consequently, dimension of each orbit is equal to two and there is just one invariant with respect to the representation $\tilde{\rho}$, namely the invariant $I_5$ given by the formula (48).

Now we return to the representation $\rho'$ on $\mathbb{R}^9$. The general dimension of orbits is equal to $\max\{2, 3\} = 3$ and there exist 6 independent invariants. We know the invariants $I_1, \ldots, I_5$ and it is easy to check their functional independence. Finding the last invariant of this representation and its geometrical meaning remains an open problem.

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