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The Lie Superalgebra of a Supermanifold

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Abstract. We prove a 'superversion' of Shanks and Pursell's classical result stating that any isomorphism of the Lie algebras of compactly supported vector fields is implemented by a diffeomorphism of underlying manifolds. We thus provide a Lie algebraic characterization of supermanifolds and describe explicitly isomorphisms of the Lie superalgebras of supervector fields on supermanifolds.
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1. Introduction

Algebraic characterization of space can be traced back to Gel'fand and Kolmogoroff, who proved in 1939 that two compact topological spaces are homeomorphic if and only if the algebras of continuous functions growing on them are isomorphic. A similar result for second countable smooth manifolds and the algebras of smooth functions is also regarded as classical (for the general case see [4, 8]).

In 1954, Pursell and Shanks [9] substituted the Lie algebra of compactly supported vector fields of a manifold for the commutative associative algebra of smooth functions that any isomorphism of the Lie algebras of compactly supported vector fields is implemented by a diffeomorphism of underlying manifolds. This classical upshot triggered a multitude of papers on similar issues by many different authors, which we extensively depicted in our previous works. In 2004, two of us proved Pursell-Shanks type results for the Lie algebra of differential operators of a manifold, and for the Poisson-Lie algebra of smooth functions on the cotangent bundle that are polynomial along the fibers. Our results indicate once more a "no-go" theorem for the Dirac quantization problem, as they imply that the preceding Lie algebras are not isomorphic – since they have nonisomorphic automorphism groups. Let us mention, hoping that the remark might instigate further progress, that the last observation is tightly related to the Kanel-Kontsevich conjecture that

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maintains that the automorphism groups of the Weyl algebra – modelled on the algebra of differential operators *with polynomial coefficients* – and of the standard *Poisson algebra of polynomials* are isomorphic!

Another landmark in the field of *algebraization of space* is the Gel'fand-Naimark theorem, 1943, which states that any C^*-algebra is isometrically $*$-isomorphic to a C^*-algebra of bounded operators on a Hilbert space. This result is usually viewed as the starting point of noncommutative geometry: the basic idea of this branch is to treat certain noncommutative algebraic structures that arise in Physics as if they were related to some "noncommutative spaces", although there are no such spaces in the usual sense of the word. It is well-known that *algebraically defined noncommutative space* is an important tool in quantization of gravity, i.e. in the attempt to unify the contradictory concepts of gravity (which makes no sense at 0-distance) and quantum theory (which precisely concerns the "0-distance"). Indeed, one of the ways out of this conflicting situations consists in replacing at small distance the usual commutative space by noncommutative space. Another possible remedy, which allows dealing with the mentioned singularities, is the replacement of points by extended geometric objects or strings, viewed as the fundamental constituents of reality. The effort to incorporate fermions, the building blocks of matter, in the spectrum of string theory led to supersymmetry and superspace (resp. $\mathbb{Z}$-graded space) – a particular type of noncommutative space: a supermanifold (resp. $\mathbb{Z}$-graded manifold) is a sheaf of supercommutative (resp. $\mathbb{Z}$-graded commutative) associative algebras that is locally isomorphic with a free Grassmann algebra with coefficients in the functions of Euclidean space.

In the present note, we combine the two aforementioned aspects of algebraization of space – algebraic characterization of usual space and algebraically defined noncommutative space. More precisely, we prove that two Lie superalgebras of supervector fields are isomorphic if and only if the underlying smooth supermanifolds are diffeomorphic as sheafs of supercommutative algebras and describe explicitly automorphism groups of the super Lie algebras of super vector fields. This can be viewed as a 'superversion' of the Shanks and Pursell's classical result and solves an open problem in the geometry of supermanifolds.

The paper is organized as follows. In section 2, we recall that every (smooth) supermanifold $\mathcal{M}$ is (noncanonically) diffeomorphic to the total space of some vector bundle V with reversed parity in the fibres [2], which we denote by ΠV and which is actually the prototype of a $\mathbb{Z}$-graded manifold. Further, we show that the super Lie algebra of vector fields of $\mathcal{M}$ admits a canonical Lie-algebraic filtration, such that the corresponding quotient is isomorphic to the $\mathbb{Z}$-graded Lie algebra of vector fields of ΠV, whatever diffeomorphism is chosen. In section 3, we prove the aforementioned superversion of Shanks and Pursell's classical result in the case of $\mathbb{Z}$-graded manifolds ΠV, and finally, in section 4, we use the results of section 2 to deduce the supercase from the preceding $\mathbb{Z}$-graded case.

2. The Lie algebra of supervector fields

Let $\mathcal{M}$ be a smooth supermanifold of dimension (s, r) over the body M. Here we understand the supermanifold as a ringed space: the standard manifold M

of dimension s is equipped with a sheaf $\mathcal{O}_M$ of superalgebras which is locally isomorphic to $C^\infty(\mathbb{R}^s)\otimes\Lambda^\bullet(\xi^1,\dots,\xi^r)$. Sections of this sheaf form a superalgebra $\mathcal{A}=\mathcal{A}_0\oplus\mathcal{A}_1$ of *smooth functions* on the supermanifold $\mathcal{M}$.

An important result of smooth supergeometry [2] (see also [7]) asserts that there exists a vector bundle V of rank r over M, such that $\mathcal{M}$ is diffeomorphic as a supermanifold to ΠV, that is, to the total space of V with the reversed parity of fibres. This implies that the algebra of smooth functions on $\mathcal{M}$ is isomorphic (as a commutative superalgebra) to the algebra of functions on ΠV, which is canonically identified with $\Gamma(\Lambda^\bullet V^*)$. This isomorphism is not canonical but it gives us an identification

$$\mathcal{A}=\mathcal{A}_0\oplus\mathcal{A}_1\simeq\Gamma(\Lambda^\bullet V^*)\,, \tag{2.1}$$

with

$$\mathcal{A}_0=\bigoplus_{i\geq 0}\mathcal{A}^{2i}\,,\ \ \mathcal{A}_1=\bigoplus_{i\geq 0}\mathcal{A}^{2i+1}\,,\ \ \text{where } \mathcal{A}^k\simeq\Gamma(\Lambda^k V^*)\,. \tag{2.2}$$

The choice of an isomorphism (2.1) provides therefore an additional $\mathbb{Z}-$grading in $\mathcal{A}$ which is compatible with the given super-structure. Such a grading uniquely determines the *Euler vector field*, that is, an operator ϵ satisfying the following property (the definition of ϵ implements the Leibnitz rule):

$$\mathcal{A}^k=\{a\in\mathcal{A}\ \mid\ \epsilon(a)=ka\}\,. \tag{2.3}$$

Denote with $\mathfrak{g}$ the super Lie algebra of vector fields on $\mathcal{M}$, the even and odd parts of which are $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively,

$$\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1\ .$$

To the end of this section an isomorphism 2.1 is chosen. Moreover, we will assume that the rank r of the vector bundle V is at least 1, otherwise we are in the standard purely even situation.

Proposition 2.1. *The adjoint action of the Euler vector field supplies $\mathfrak{g}$ with a $\mathbb{Z}$-grading compatible with the Lie super structure such that*

$$\mathfrak{g}_0=\bigoplus_{i\geq 0}\mathfrak{g}^{2i}\,,\ \ \mathfrak{g}_1=\bigoplus_{i\geq -1}\mathfrak{g}^{2i+1}\,,\ \ \text{where } \mathfrak{g}^k:\ =\{X\in\mathfrak{g}\ \mid\ [\epsilon,X]=kX\}\,. \tag{2.4}$$

Any super vector field X admits a unique homogeneous decomposition $X=\sum_{m\geq -1}X_m$ with respect to the Euler vector field, $[\epsilon,X_m]=mX_m$. In local coordinates this decomposition is given by the polynomial degree of ξ^a:

$$X_m:=\sum_{a_1<\ldots<a_m}\sum_i f^i_{a_1\ldots a_m}(x)\xi^{a_1}\ldots\xi^{a_m}\frac{\partial}{\partial x^i}+ \tag{2.5}$$
$$+\sum_{b_1<\ldots<b_{m+1}}\sum_c g^c_{b_1\ldots b_{m+1}}(x)\xi^{b_1}\ldots\xi^{b_{m+1}}\frac{\partial}{\partial\xi^c}\,.$$

Here $f^i_{b_1\ldots b_r}(x)$ and $g^b_{a_1\ldots a_r}(x)$ are smooth functions of x.

Proof. Suppose we are given a local trivialization of V over M, that is, an open cover by coordinate charts $\{U_\alpha, x^i\}$ together with a local frame of the restriction of V to each U_α, denoted by (e_a), where $a = 1, \ldots, r$. Combining these data, we obtain a local coordinate description of $\mathcal{M}$: $\{U_\alpha, x^i, \xi^a\}$, where ξ^a are dual to e_a thought of as odd coordinates. By construction of the trivialization, the change of coordinates over double overlaps is linear with respect to odd coordinates, so the aforementioned Euler vector field is well defined, and in any coordinate system reads

$$\epsilon = \sum_a \xi^a \frac{\partial}{\partial \xi^a} \,. \tag{2.6}$$

It is now a standard task in local coordinates to write any vector field in the form (2.5). ■

Remark 2.2. Apparently, each vector field of degree -1 can be identified with a section of V, thus $\mathfrak{g}^{-1}$ is naturally isomorphic to $\Gamma(V)$ which acts on $\mathcal{A}$ by contractions, provided we identify $\mathcal{A}$ with $\Gamma(\Lambda^\bullet V^*)$. On the other hand, a super vector field of degree 0, which can be written as

$$X = \sum_i f^i(x) \frac{\partial}{\partial x^i} + \sum_{a,b} g_a^b(x) \xi^a \frac{\partial}{\partial \xi^b} \,, \tag{2.7}$$

defines a general infinitesimal automorphism of the vector bundle V. The vector fields from $\mathfrak{g}^0$ can be therefore identified with the sections of the Lie algebroid of infinitesimal automorphism of V, called sometimes the *Atiyah algebroid* of V. This identification respects the bracket, i.e. is a Lie algebra isomorphism. The corresponding anchor map $\rho : \mathfrak{g}^0 \to \mathcal{X}(M)$ from the Lie algebra $\mathfrak{g}^0$ of the Atiyah algebroid into the Lie algebra $\mathcal{X}(M)$ of vector fields on M in local coordinates reads

$$\rho(X) = \rho\left(\sum_i f^i(x) \frac{\partial}{\partial x^i} + \sum_{a,b} g_a^b(x) \xi^a \frac{\partial}{\partial \xi^b}\right) = \sum_i f^i(x) \frac{\partial}{\partial x^i} \tag{2.8}$$

and is also a Lie algebra homomorphism.

Suppose we are given a maximal ideal of $\mathfrak{g}_0$, denoted by $\mathfrak{g}'$, the elements of which act as ad-nilpotent operators in $\mathfrak{g}$, that is, for each $X \in \mathfrak{g}'$ there exists a non-negative integer m, such that $ad_X^m(Y) = 0$ for all $Y \in \mathfrak{g}$.

Proposition 2.3. *The ideal $\mathfrak{g}'$ is related to the $\mathbb{Z}-$grading by the formula:*

$$\mathfrak{g}' = \bigoplus_{i>0} \mathfrak{g}^{2i} \,. \tag{2.9}$$

Since the ideal $\mathfrak{g}'$ is defined in purely super Lie algebraic terms, the latter space in fact does no depend on the introduced $\mathbb{Z}-$graduation.

Proof. Let us consider the image of $\mathfrak{g}'$ under the projection

$$\pi_0 \colon \mathfrak{g}_0 \to \mathfrak{g}_0 / \oplus_{i>0} \mathfrak{g}^{2i} \simeq \mathfrak{g}^0 \,,$$

which is fixed by the choice of an isomorphism $\mathcal{M} \simeq \Pi V$ (2.1). Apparently, $\pi_0(\mathfrak{g}')$ is a maximal $ad-$nilpotent ideal of $\mathfrak{g}^0$. Let us apply the anchor map ρ to this ideal. For each $X \in \pi_0(\mathfrak{g}')$, the vector field $\rho(X)$ has to be $ad-$nilpotent as well. It is easy to see that such a vector field is necessarily zero, as any vector field on a standard (even) manifold can be written locally as the coordinate vector field ∂_{x_1} in a neighborhood of any point at which it does not vanish.

Hence we conclude that $\pi_0(\mathfrak{g}') \subset \mathrm{Ker}\rho$. But $\mathrm{Ker}\rho$ is the bundle of Lie algebras over M, the fiber of which is isomorphic to $\mathfrak{gl}(V_z)$ at any $z \in M$. Thus $\pi_0(\mathfrak{g}')$ evaluated at z is necessarily a subset of the ideal of scalar operators, the only ideal of $\mathfrak{gl}(V_z)$, which implies that $\pi_0(\mathfrak{g}') \subset A\epsilon$. Here ϵ is the Euler vector field and $A \colon = \mathcal{A}^0$ is the algebra of functions on M.

Let us assume that $X \in \mathfrak{g}_0$, then $\pi_0(X) = f\epsilon$ for some smooth function $f \in A$. Therefore $X = f\epsilon +$ terms of order ≥ 2. On the other hand, $[f\epsilon, Y] = -fY$ for each $f \in A$ and $Y \in \mathfrak{g}^{-1}$, hence $ad_X^m(Y) = (-f)^m Y +$ terms of order ≥ 1. Thus we conclude that X is $ad-$nilpotent if and only if $f = 0$, therefore $\pi_0(X) = 0$ and $\mathfrak{g}' \subset \mathrm{Ker}\pi_0$. The kernel of π_0 is an ideal of $\mathfrak{g}_0$, consisting of $ad-$nilpotent elements; but $\mathfrak{g}'$ has to be maximal, which immediately implements the identity $\mathfrak{g}' = \mathrm{Ker}\pi_0$. ■

There is an additional conclusion drawn from the above proof which we formulate as a separate proposition that we will use later.

Proposition 2.4. *The maximal Lie ideal in $\mathfrak{g}^0$ of elements acting ad-nilpotently on $\mathfrak{g}^0$ consists of vector fields of the form $f\epsilon$ with $f \in A$ being a smooth function on the body M.*

Let us introduce (inductively) the following subspaces:

$$\mathfrak{g}^{(p+2)} \colon = [\mathfrak{g}', \mathfrak{g}^{(p)}] \,, \quad \text{where } \mathfrak{g}^{(-1)} \colon = \mathfrak{g}_1 \,, \ \mathfrak{g}^{(0)} \colon = \mathfrak{g}_0 \,. \tag{2.10}$$

Proposition 2.5. *Let us assume that $r = \mathrm{rk}V > 2$ or $\dim M > 0$ and $\mathrm{rk}V > 1$ Then*

$$\mathfrak{g}^{(p)} = \bigoplus_{i \geq 0} \mathfrak{g}^{p+2i} \,. \tag{2.11}$$

independently on the choice of the isomorphism (2.1).

Proof. It follows from the equality $[\mathfrak{g}^p, \mathfrak{g}^q] = \mathfrak{g}^{p+q}$, which is true for all p, q (except for $p = q = 0$ in the pure odd case when M is a point). Indeed, let X be a super vector field of degree $p + q$, then given a local cover $\{U_\alpha\}$ of M, we decompose X into a sum of X_α, such that $\mathrm{supp}X_\alpha \subset M$. It is enough to find Y_α^k, Z_α^k of the degree p and q, respectively, with the support in U_α for each α, such that $X_\alpha = \sum_k [Y_\alpha^k, Z_\alpha^k]$. Now we use the local representation (2.5) of X. ■

We still make the assumption that $r = \mathrm{rk}V > 2$ or $\dim M > 0$ and $\mathrm{rk}V > 1$, leaving the low rank cases, for which the next two corollaries fail, to separate considerations.

Corollary 2.6. *The filtration of* $\mathfrak{g}$ *by* $\mathfrak{g}^{(p)}$ *respects the super Lie algebra structure, i.e.* $[\mathfrak{g}^{(p)}, \mathfrak{g}^{(q)}] \subset \mathfrak{g}^{(p+q)}$. *The graded Lie superalgebra* $\mathfrak{g}^{\mathrm{gr}}$*, associated with the given filtration,*

$$\mathfrak{g}^{\mathrm{gr}} = \bigoplus_{p \geq -1} (\mathfrak{g}^{\mathrm{gr}})^p\,, \quad \text{where } (\mathfrak{g}^{\mathrm{gr}})^p = \mathfrak{g}^{(p)}/\mathfrak{g}^{(p+2)}\,,$$

equipped with the bracket naturally induced by the bracket in $\mathfrak{g}$*, is isomorphic to* $\mathfrak{g}$ *as a* $\mathbb{Z}-$*graded superalgebra, independently on the choice of (2.1).*

Since the filtration $\mathfrak{g}^{(p)}$ is canonical, thus preserved by any automorphism of $\mathfrak{g}$, any automorphism ψ of the Lie superalgebra $\mathfrak{g}$ induces an automorphism $\mathrm{p}(\psi)$ of the graded Lie algebra $\mathfrak{g}^{\mathrm{gr}}$ by

$$\mathrm{p}(\psi)([X]) = [\psi(X)]\,, \tag{2.12}$$

where $[X]$ is the coset of $X \in \mathfrak{g}^{(p)}$.

Corollary 2.7. *The formula (2.12) defines a group homomorphism*

$$\mathrm{p}\colon Aut_{\mathbb{Z}_2}(\mathfrak{g}) \to Aut_{\mathbb{Z}}(\mathfrak{g}^{\mathrm{gr}})\,, \tag{2.13}$$

where the former and the latter groups consist of all automorphisms preserving $\mathbb{Z}_2$ *on* $\mathfrak{g}$ *and* $\mathbb{Z}-$*grading on* $\mathfrak{g}^{\mathrm{gr}}$*, respectively.*

3. Lie algebras of $\mathbb{Z}$-graded manifolds associated with vector bundles

Let $\mathcal{M} = \Pi V$ and $\mathcal{N} = \Pi W$ be $\mathbb{Z}$-graded manifolds associated with vector bundles $V \to M$ and $W \to N$ respectively. Let $\mathcal{A} = \oplus_{i \geq 0} \Gamma(\Lambda^i V^*)$ and $\mathcal{B} = \oplus_{i \geq 0} \Gamma(\Lambda^i W^*)$ be the graded algebras of smooth functions on $\mathcal{M}$ and $\mathcal{N}$ (Grassmann algebras of multi-sections of dual bundles), respectively. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding $\mathbb{Z}$-graded Lie algebras of vector fields. Let us also assume that $\dim M$ and $\dim N$ are non-zero or the ranks $\mathrm{rk}V$ and $\mathrm{rk}W$ are both positive and different from 2.

Theorem 3.1. *For any isomorphism* $\psi\colon \mathfrak{g} \to \mathfrak{h}$ *of the* $\mathbb{Z}$*-graded Lie algebras of vector fields on* ΠV *and* ΠW*, respectively, there exists an isomorphism of vector bundles* $\phi\colon V \to W$ *such that* $\psi(X) = (\phi^*)^{-1} \circ X \circ \phi^*$*, where* $\phi^*\colon \mathcal{B} \to \mathcal{A}$ *is the isomorphism of the* $\mathbb{Z}$*-graded algebras of smooth functions (multi-sections of the dual bundles) induced by* ϕ.

Proof. Let us restrict the isomorphism ψ to $\mathfrak{g}^0$. According to proposition 2.4, the subspaces $A\epsilon$ and $B\epsilon$, where A and B are the algebras of functions on M and N, respectively, are the (uniquely determined) maximal ideals of $\mathfrak{g}^0$ and $\mathfrak{h}^0$

acting nilpotently on $\mathfrak{g}^0$ and $\mathfrak{h}^0$ respectively, therefore $B\epsilon = \psi(A\epsilon)$. Of course, here we identify the two Euler vector fields in our bundles as uniquely determined by the $\mathbb{Z}-$graduation. This implies the existence of a bijective map $\widetilde{\psi}\colon B \to A$, such that

$$\psi(f\epsilon) = \widetilde{\psi}^{-1}(f)\epsilon\,, \ \ \forall f \in A\,. \tag{3.1}$$

Taking into account that $[X, f\epsilon] = \rho(X)(f)\epsilon$, where $\rho(X)$ is the anchor of $X \in \mathfrak{g}^0$, we immediately obtain the following property

$$\rho(\psi(X)) = \widetilde{\psi}^{-1}\rho(X)\widetilde{\psi}\,, \tag{3.2}$$

which implies that the conjugation by $\widetilde{\psi}$ induces a Lie algebra isomorphism $\rho(\mathfrak{g}^0) \to \rho(\mathfrak{h}^0)$. On the other hand, these Lie algebras consist of all vector fields on M and N, correspondingly. Using the classical result on the Lie algebras of vector fields [3] (see also [9, 1]), we conclude that the conjugation by $\widetilde{\psi}$ coincides with the conjugation by some diffeomorphism $h : M \to N$. One can also conclude the latter fact from a theorem in [5] applied directly to the Atiyah algebroid $\mathfrak{g}^0$. Any operator, acting on smooth functions and commuting with all vector fields, is necessarily a constant, which implies that $\widetilde{\psi}(h^*)^{-1}$ is the operator of multiplication by a non-zero constant λ, and thus $\widetilde{\psi} = \lambda h^*$. But the Euler vector fields are uniquely defined, so they are associated by the isomorphism which yields $\lambda = 1$.

Now we restrict the automorphism ψ to $\mathfrak{g}^{-1}$; thus we obtain a non-degenerate linear map $\Gamma(V) \to \Gamma(W)$. If we proved that for each $s \in \Gamma(V)$ and $f \in A$, the following property holds: $\psi(fs) = (h^*)^{-1}(f)\psi(s)$, the restriction of ψ would have been induced by a bundle map $V \to W$ covering h. Indeed, $fs = [s, f\epsilon]$. On the other hand, $\psi(f\epsilon) = \widetilde{\psi}^{-1}(f)\epsilon$, so that

$$\psi(fs) = \psi[s, f\epsilon] = [\psi(s), \psi(f\epsilon)] = [\psi(s), \widetilde{\psi}^{-1}(f)\epsilon] = \widetilde{\psi}^{-1}(f)\psi(s)\,.$$

Since $\widetilde{\psi}$ is an automorphism of the algebra of functions, $\psi(fs) = (h^*)^{-1}(f)\psi(s)$. The bundle map $V \to W$, induced by the restriction of ψ to $\mathfrak{g}^{-1}$, can be uniquely extended to a diffeomorphism ϕ of $\mathcal{M}$ over $\mathbb{Z}$. The uniqueness follows from the isomorphism (2.1) because the algebras of functions on $\mathcal{M}$ and $\mathcal{N}$ are freely generated by $\Gamma(V^*)$ and $\Gamma(W^*)$, respectively. Taking into account that the representation of $\mathfrak{g}^0$ on the space of sections of V is faithful, we immediately obtain the required property of Theorem 3.1 for $\mathfrak{g}^0 \oplus \mathfrak{g}^{-1}$.

The last step is proving that the extension of the restriction of ψ to $\mathfrak{g}^0 \oplus \mathfrak{g}^{-1}$ is unique. Suppose there exists another Lie algebra morphism ψ', satisfying the property

$$\psi(X) = \psi'(X)\,, \quad \forall X \in \mathfrak{g}^0 \oplus \mathfrak{g}^{-1}\,. \tag{3.3}$$

Then for each $Z \in \mathfrak{g}^k$, where $k \geq 0$, and $X_i \in \mathfrak{g}^{-1}$, $i = 1, \ldots, k$, we have

$$[X_1, \ldots, [X_k, Z] \ldots] \in \mathfrak{g}^0\,.$$

We use (3.3) to show that $\psi\left([X_1,\ldots,[X_l,Z]\ldots]\right)=\psi'\left([X_1,\ldots,[X_k,Z]\right)$. On the other hand $\psi(X_i)=\psi'(X_i)$ and ψ is an invertible map, therefore

$$[X_1,\ldots,[X_k,\psi(Z)-\psi'(Z)]\ldots]=0\,.$$

But it is easy to see in local coordinates that $[\mathfrak{g}^{-1},Z]=0$ for $Z\in\mathfrak{g}^k,\ k>0$, implies $Z=0$, so we get inductively $\psi(Z)=\psi'(Z)$. ∎

4. Lie algebras of supermanifolds: General case

Let $\mathcal{M}$ and $\mathcal{N}$ be smooth supermanifolds, $\mathcal{A}$ and $\mathcal{B}$ the algebras of functions on $\mathcal{M}$ and $\mathcal{N}$, respectively, and $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding super Lie algebras of vector fields. Suppose that the supermanifolds are supplied with a compatible $\mathbb{Z}-$grading as in (2.1), which means that there exist vector bundles $V\to M$ and $W\to N$, such that $\mathcal{M}\simeq\Pi V$ and $\mathcal{N}\simeq\Pi W$ in the category of $\mathbb{Z}-$graded manifolds. Let us also assume that $\dim M$ and $\dim N$ are non-zero and the rank of the corresponding bundles is greater than 1 or $\mathrm{rk}V$ and $\mathrm{rk}W$ are both greater than 2.

Theorem 4.1. *For any isomorphism $\psi\colon\mathfrak{g}\to\mathfrak{h}$ of the super Lie algebras of vector fields on $\mathcal{M}$ and $\mathcal{N}$ respectively there exists a diffeomorphism $\phi:\mathcal{M}\to\mathcal{N}$ such that $\psi(X)=(\phi^*)^{-1}\circ X\circ\phi^*$, where ϕ^* is the isomorphism of the corresponding superalgebras of smooth functions functions induced by ϕ.*

Proof. Taking into account that the isomorphism of super vector fields ψ preserves the canonical filtration in $\mathfrak{g}$ and $\mathfrak{h}$, determined by the correspondent maximal ad-nilpotent ideals, we obtain a unique bundle map $V\to W$ as in Theorem 3.1. This bundle map induces an isomorphism χ of super Lie algebras $\mathfrak{g}\to\mathfrak{h}$, which is also an isomorphism of the corresponding $\mathbb{Z}-$graded Lie algebras, associated to the filtration, such that $\psi^{-1}\chi\colon\mathfrak{g}\to\mathfrak{g}$ has a trivial coset.

Now we use the result of Corollary 2.7. It is sufficient to prove that the kernel of $\mathrm{p}\colon Aut_{\mathbb{Z}_2}(\mathfrak{g})\to Aut_{\mathbb{Z}}(\mathfrak{g}^{\mathrm{gr}})$ consists of automorphisms induced by super diffeomorphisms. Assume that ψ belongs to the kernel of p, that is, for each $X\in\mathfrak{g}^k$,

$$\psi(X)=X+\psi_2(X)+\psi_4(X)+\ldots\,,\quad\text{where }\psi_{2m}(X)\in\mathfrak{g}^{k+2m}\,. \tag{4.1}$$

Let us denote $\psi_2(\epsilon)$ by Y, then for each $X\in\mathfrak{g}^k$ the following identity holds,

$$[\psi(\epsilon),\psi(X)]=[\epsilon+Y+\ldots,X+\psi_2(X)+\ldots]=kX+[\epsilon,\psi_2(X)]+[Y,X]+\ldots \tag{4.2}$$

where "..." are the higher degree terms. On the other hand,

$$\psi[\epsilon,X]=k\psi(X)=kX+k\psi_2(X)+\ldots$$

and $[\epsilon,\psi_2(X)]=(k+2)\psi_2(X)$, therefore $\psi_2(X)=-\frac{1}{2}[Y,X]$. We exponentiate $\frac{1}{2}Y$ to a super automorphism of $\mathcal{A}$ by use of the exponential series, which obviously converges because of the nilpotency of Y. Then the new automorphism $\psi^{(1)}:\,=$

$\psi \circ \mathrm{Ad}(\exp \frac{1}{2}Y)$ is equal to the identity up to the 2d order, i.e. for each X of the degree k,

$$\psi^{(1)}(X) - X \in \mathfrak{g}^{(k+4)} .$$

Let us repeat this procedure by induction (the number of steps will be certainly finite because the Lie algebra $\mathfrak{g}^{\mathrm{gr}}$ is finitely-graded). Finally we obtain a decomposition of ψ into a (finite) product of $\mathrm{Ad}(\exp Y_j)$ for vector fields Y_j of degree $2j$, which implies that ψ is induced by a the pullback of super diffeomorpism of $\mathcal{M}$. ■

5. Lie algebras of supermanifolds: Exceptional low rank cases

Now let us consider the exceptional case, when $\mathrm{rk}V = 1$ or $\dim M = 0$ and $\mathrm{rk}V \leq 2$. Whatever isomorphism of the form (2.1) is chosen, $\mathfrak{g}_0 = \mathfrak{g}^0$, $\mathfrak{g}_1 = \mathfrak{g}^{-1} \oplus \mathfrak{g}^1$ such that $\mathfrak{g}^{\pm 1}$ can be identified with the spaces of smooth sections of certain vector bundles over M. In particular, $\mathfrak{g}^{-1}$ is always isomorphic to $\Gamma(V)$ and

$$\mathfrak{g}^1 \simeq \begin{cases} \Gamma(V^* \otimes TM)\,, & \text{if} \qquad \mathrm{rk}V = 1 \\ \Gamma(\Lambda^2 V^* \otimes V)\,, & \text{if} \quad \dim M = 0\,,\ \mathrm{rk}V = 2 \end{cases} \tag{5.1}$$

Only in this situation the canonical ideal $\mathfrak{g}'$ is zero, thus if $\mathfrak{g}$ is isomorphic to the super Lie algebra $\mathfrak{h}$ of vector fields on another supermanifold $\mathcal{N}$, the manifold $\mathcal{N}$ has to satisfy the same conditions if the ranks are concerned.

Lemma 5.1. *The vector fields $\pm\epsilon$ are the only elements of the form $f\epsilon$, where f is a smooth function on M, the restriction of the adjoint action of which to $\mathfrak{g}_1$ has only eigenvalues ± 1.*

Proof. Indeed, for each $Y \in \mathfrak{g}^{\pm 1}$ one has $[f\epsilon, Y] = \pm Y$. It is obviously true for $\mathfrak{g}^{-1}$, whatever the base manifold M is taken, and for $\mathfrak{g}^1$ in the case of $\dim M = 0$. Let us consider the remaining case of $\mathfrak{g}^1$ when $\dim M > 0$ and $\mathrm{rk}V = 1$. Suppose we are given a local coordinate chart with coordinates $\{x^i, \xi\}$ where the only ξ is odd. Then the restriction of $f\epsilon$ and Y to the local chart is $f(x)\xi\partial_\xi$ and ξZ, respectively, where Z is a local vector field on M. Now the simple computation gives

$$[f\epsilon, Y] = [f\epsilon, \xi Z] = f\xi Z + \xi^2 [f\epsilon, Z] = fY\,,$$

which finishes the proof of lemma. ■

Proposition 5.2. *Let $\psi\colon \mathfrak{g} \to \mathfrak{h}$ be a $\mathbb{Z}_2-$graded isomorphism of the Lie algebras of vector fields on ΠV and ΠW, respectively. Then ψ is the composition of two isomorphisms, $\psi = \phi_* \circ \psi_0$, where ϕ_* is induced by an isomorphism of vector bundles $\phi\colon V \to W$ (as in Theorem 3.1) and*

- *ψ_0 is uniquely determined by a bundle isomorphism $V \to V^* \otimes TM$, if* $\mathrm{rk}V = 1$*;*

- *ψ_0 is uniquely determined by a bundle isomorphism $V \to \Lambda^2 V^* \otimes V$, if $\dim M = 0$ and $\mathrm{rk} V = 2$;*

- *$\psi_0 = id$ in the other cases.*

Proof. Apparently, $\psi(\mathfrak{g}^0) = \mathfrak{g}^0$. Following the similar ideas as in Theorem 3.1, we immediately prove that $\psi(f\epsilon) = \lambda(h^*)^{-1}(f)\epsilon$ for each smooth function f on M where λ is some non-zero constant and $h\colon M \to N$ is a diffeomorphism. Using Lemma 5.1, we get $\lambda = \pm 1$.

If $\lambda = 1$ then we are in the situation of Theorem 3.1, which means that ψ is implemented by a bundle isomorphism $V \to W$.

Suppose $\lambda = -1$, then ψ exchanges $\mathfrak{g}^{\pm 1}$ and $\mathfrak{h}^{\mp 1}$. Applying the same argument as in Theorem 3.1, we conclude that the restriction of ψ to $\mathfrak{g}^{-1}$ is induced by a bundle map covering the diffeomorphism h. In particular, if $\mathrm{rk} V = 1$ then the restriction of ψ to $\mathfrak{g}^{-1}$ is induced by a bundle isomorphism $V \to W^* \otimes TN$, which implements the dimension property $\dim N = \dim M = 1$ because of the rank argument. If $\mathrm{rk} V = 2$ and $\dim M = 0$, we obtain (in the same way) a non-degenerate linear map $V \to \Lambda^2 W^* \otimes W$.

In both acceptable cases, when $\dim M$ equals to 1 or 0, $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ are isomorphic as vector bundles on M. Combining any bundle isomorphism $\mathfrak{g}^{-1} \to \mathfrak{g}^1$, which covers the identity diffeomorphism $M \to M$, with ψ, we get a bundle isomorphism $\phi : V \to W$ which covers h. In particular, this implies that $\mathcal{M}$ and $\mathcal{N}$ are diffeomorphic as smooth supermanifolds. Now we can decompose ψ as $\phi_* \circ \psi_0$ where ϕ_* is the isomorphism of vector fields induced by the diffeomorphism ϕ and ψ_0 is an automorphism of $\mathfrak{g}$ which replaces ϵ with $-\epsilon$, thus $\mathfrak{g}^1$ with $\mathfrak{g}^{-1}$.

As we have seen above, ψ_0 inspires a bundle map $V \to V^* \otimes TM$ if $\dim M = 1$ and $V \to \Lambda^2 V^* \otimes V$ if $\dim M = 0$. Since the representation of $\mathfrak{g}_0$ in sections of V is faithful and any bundle isomorphism $V \to V$, commuting with the adjoint action of all sections of $\mathfrak{g}_0$, is the identity, we conclude that ψ_0 is uniquely fixed by its restriction to $\mathfrak{g}^{-1}$ or by a non-degenerate section of either $(V^*)^{\otimes 2} \otimes TM$ or $\Lambda^2 V^* \otimes V^* \otimes V$ (depending on the dimension of M). ∎

A general corollary independent on the rank of the bundles in question is now the following.

Corollary 5.3. *Two supermanifolds are diffeomorphic if and only if their super Lie algebras of vector fields are isomorphic.*

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