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A Sharp Criterion for the Existence of the Density in the Product Formula on Symmetric Spaces of Type $\mathbf{A}_n$

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Abstract. In this paper, we find sharp conditions on $X, Y \in \mathfrak{a}$ for the existence of the density of the measure $\delta^\natural_{e^X} \star \delta^\natural_{e^Y}$ intervening in the product formula for the spherical functions on the symmetric spaces of noncompact type $\mathbf{X} = \mathbf{SL}(n, \mathbf{F})/\mathbf{SU}(n, \mathbf{F})$ where $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. Our results also apply to the symmetric space $\mathbf{E}_6/\mathbf{F}_4$.
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1. Introduction

Let G be a semisimple noncompact connected Lie group with finite centre, K a maximal compact subgroup of G and $\mathbf{X} = G/K$ the corresponding Riemannian symmetric space of noncompact type.

We have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and we choose a maximal Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. In what follows, Σ corresponds to the root system of the pair $(\mathfrak{g}, \mathfrak{a})$ and Σ^+ to a choice of positive roots. This implies that we have chosen a set of simple positive roots $\alpha_1, \ldots, \alpha_r$ where $r = \dim \mathfrak{a}$ is the rank of the symmetric space. We have the root space decomposition $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha$. Recall that $\mathfrak{k}$, the Lie algebra of K, can be described as

$$\mathfrak{k} = \operatorname{span}\{X_\alpha + \theta X_\alpha\colon\ X_\alpha \in \mathfrak{g}_\alpha,\ \alpha \in \Sigma^+ \cup \{0\}\}$$

where θ is the Cartan involution. Let $\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ and denote the groups corresponding to the Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ by A and N respectively.

Let $W = M'/M$ be the Weyl group ($M' \subset K$ is the normalizer of $\mathfrak{a}$ in K i.e. $k \in M'$ if $\operatorname{Ad}(k)\,\mathfrak{a} \subset \mathfrak{a}$ while $M \subset K$ is its centralizer i.e. $k \in M$ if $\operatorname{Ad}(k)\,H = H$ for all $H \in \mathfrak{a}$).

When appropriate we will not distinguish between $w \in W$ and $w \in M' \subset K$. On the other hand, to denote the action of w on $X \in \mathfrak{a}$, we will write $w \cdot X$. We then have $e^{w\cdot X} = \operatorname{Ad}(w)\, e^X$ ([6, Chapter VII, Proposition 2.2]).

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Let $\mathfrak{a}^+ = \{H \in \mathfrak{a}\colon\ \alpha(H) > 0 \,\forall\, \alpha \in \Sigma^+\}$ and $A^+ = \exp(\mathfrak{a}^+)$. For any $g \in G$, $g = k_1\, e^{a(g)}\, k_2$ (Cartan decomposition) where $a(g) \in \overline{\mathfrak{a}^+}$ is uniquely determined by g. Note that $a(k_1\, g\, k_2) = a(g)$ for all $k_i \in K$ and all $g \in G$. We also have $g = k\, e^{\mathcal{H}(g)}\, n$ (Iwasawa decomposition).

If λ is a complex-valued linear form on $\mathfrak{a}$, the corresponding spherical function is

$$\phi_\lambda(e^H) = \int_K e^{(i\,\lambda-\rho)(\mathcal{H}(e^H\, k))}\, dk \tag{1}$$

where $\rho = (1/2)\sum_{\alpha\in\Sigma^+} m_\alpha\, \alpha$ (m_α denotes the multiplicity of the root α). A spherical function, like any K-biinvariant function, can also be considered as a K-invariant function on the Riemannian symmetric space of noncompact type $\mathbf{X} = G/K$. Naturally, such a function is completely determined by its values on A (or on A^+). The books [6, 7] constitute a standard reference on these topics.

In [7, (32), page 480], Helgason shows that, given $X, Y \in \mathfrak{a}$, a Weyl-invariant measure $\mu_{X,Y}$ exists on the Lie algebra $\mathfrak{a}$ such that

$$\phi_\lambda(e^X)\, \phi_\lambda(e^Y) = \int_{\mathfrak{a}} \phi_\lambda(e^H)\, d\mu_{X,Y}(H).$$

It is known [7] that

$$\phi_\lambda(e^X)\, \phi_\lambda(e^Y) = \int_K \phi_\lambda(e^X\, k\, e^Y)\, dk.$$

Consequently, the measure $\mu_{X,Y}$ satisfies

$$\int_K f(a(e^X\, k\, e^Y))\, dk = \int_{\mathfrak{a}} f(H)\, d\mu_{X,Y}(H) \tag{2}$$

for all continuous functions f on $\mathfrak{a}$ which are invariant under the action of W.

We define the kernel $k(H, X, Y)$ in the product formula via the equation

$$\phi_\lambda(e^X)\, \phi_\lambda(e^Y) = \int_{\mathfrak{a}^+} \phi_\lambda(e^H)\, k(H, X, Y)\, \delta(H)\, dH \tag{3}$$

where δ is the density of the invariant measure on $\mathfrak{a}$ in polar coordinates, i.e.,

$$\int_G h(g)\, dg = \int_{\mathfrak{a}^+} h(e^H)\, \delta(H)\, dH$$

for any K-biinvariant function h integrable on G. The existence of this kernel, i.e. the absolute continuity of the measure $\mu_{X,Y}$ with respect to the Lebesgue measure, has been shown previously ([1] in rank one case, [2] in the complex case and [3] in the general case) provided that X, $Y \in \mathfrak{a}^+$.

Let m_K denote the Haar measure of the group K and let $\star$ be the convolution on the group G. For $X \in \mathfrak{a}$ we define $\delta^\natural_{e^X} = m_K \star \delta_{e^X} \star m_K$. Then $m_{X,Y} = \delta^\natural_{e^X} \star \delta^\natural_{e^Y}$ is a K-biinvariant measure on G such that its transport measure on $\overline{\mathfrak{a}^+}$ by the map $g \mapsto a(g)$ is $\mu_{X,Y}\big|_{\overline{\mathfrak{a}^+}}$, multiplied by $|W|$ for the sake of normalization.

Remark 1.1.

1. The density $k(H,X,Y)$ exists if and only if $\mathcal{S}_{X,Y} = a(e^X\,K\,e^Y)$, the support of the measure $\mu_{X,Y}\big|_{\overline{\mathfrak{a}^+}}$, has nonempty interior (refer to [2, 3]). Since the problem is clearly symmetric in X and Y, we note that $a(e^X\,K\,e^Y) = a(e^Y\,K\,e^X)$.

2. The existence of the density of the measure $\mu_{X,Y}$ on $\mathfrak{a}$ is equivalent to the existence of the density of $m_{X,Y}$ on G, with respect to the invariant measure dg. The density of the measure $m_{X,Y}$ exists if and only if its support Ke^XKe^YK has nonempty interior.

In [3], we prove that $\mu_{X,Y}$ is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{a}$ provided X, $Y \in \mathfrak{a}^+$. We were able to relax these conditions somewhat; for example, we show that $\mu_{X,Y}$ is absolutely continuous provided one of X or Y is in $\mathfrak{a}^+$ as long as the other is nonzero.

Except in the rank one case (see [1]) and in the complex case (see [2]), little was known about the properties of the density of $\mu_{X,Y}$. The articles [4, 5] provide more information about the density $k(H,X,Y)$.

The objective of this paper is to give sharp conditions on X and Y for the existence of the density $k(H,X,Y)$ in the case of the symmetric spaces $\mathbf{SL}(n,\mathbf{F})/\mathbf{SU}(n,\mathbf{F})$ where $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ (real, complex or quaternion numbers); in other words, in the case of the root system A_{n-1}.

In this setup, the Weyl group is the symmetric group S_n. The vector space $\mathfrak{a}$ is the space of diagonal real matrices with trace 0 and is common for the three cases $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$. If $X = \operatorname{diag}[X_1,\ldots,X_n]$ and $\sigma \in S_n$ then $\sigma\cdot X = \operatorname{diag}[X_{\sigma(1)},\ldots,X_{\sigma(n)}]$.

From the examples of [3], it emerges that the further away (in a heuristic sense) we are from X and Y to belong to $\mathfrak{a}^+$, the least likely it is that the density k will exist. This observation is consistent with the Definition 1.3 below.

We will show that in the case of the symmetric spaces of noncompact type $\mathbf{SL}(n,\mathbf{F})/\mathbf{SU}(n,\mathbf{F})$, the definitive criterion for the existence of the density k is given by the following definition of **eligible** X and Y:

Definition 1.2. We say that $p = [p_1,p_2,\ldots,p_r]$ is a partition of n if $p_1 \ge p_2 \ge \cdots \ge p_r > 0$ and $\sum p_i = n$. We also write the partition $1^n = [\overbrace{1,\ldots,1}^{n}]$.

For any $X \in \mathfrak{a}$, there exists $\sigma \in S_n$ such that

$$\sigma\cdot X = \operatorname{diag}[\,\overbrace{x_1,\ldots,x_1}^{p_1},\overbrace{x_2,\ldots,x_2}^{p_2},\ldots,\overbrace{x_r,\ldots,x_r}^{p_r}\,]$$

and $p_1 \ge p_2 \ge \cdots \ge p_r$ (we suppose that the x_i's are real and distinct; naturally, $p_i \ge 1$ for all i). The partition $p = [p_1,p_2,\ldots,p_r]$ will be said to be associated to X. We then say that p is the configuration of X and that X is a realization of the configuration p.

Definition 1.3. Let $G = \mathbf{SL}(n, \mathbf{F})$ and let X, $Y \in \mathfrak{a}$. Let p, respectively q, be the partitions associated to X and Y. We say that X and Y are **eligible** if

$$p_1 + q_1 \le n \tag{4}$$

and, if $n > 2$,

$$p_2 + q_2 \le n - 1. \tag{5}$$

Remark 1.4. Conditions (4) and (5) are equivalent to (4) together with the condition:

for $n > 2$ even, X and Y are not both associated to the partition $[n/2, n/2]$. This means that if X and Y are both in the Weyl orbit of elements such as $a \begin{bmatrix} I_{n/2} & 0 \\ 0 & -I_{n/2} \end{bmatrix}$, $n > 2$, then they are not eligible.

Remark 1.5. X and Y are clearly eligible if one of X or Y belongs to $\mathfrak{a}^+$ and the other is nonzero. Note that when $n = 2$ and a, $b \neq 0$ then

$$X = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \text{ and } Y = \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}$$

are eligible.

The main result of this paper is the following:

Theorem 1.6. *Let $G = \mathbf{SL}(n, \mathbf{F})$ where $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ and let X, $Y \in \mathfrak{a}$. Then the measure $\mu_{X,Y}$ is absolutely continuous if and only if X and Y are eligible.*

Remark 1.7. Our result also applies to the symmetric space $\mathbf{E}_6/\mathbf{F}_4$ which can be realized as $\mathbf{SL}(3, \mathbf{O})/\mathbf{SU}(3, \mathbf{O})$ where $\mathbf{O}$ corresponds to the octonions.

The theorem is proven in Proposition 2.2 (necessity of the eligibility condition) and in Theorem 3.5 (sufficiency of the eligibility condition).

One can hope that the configurations and the eligibility property can be rephrased in terms of more general notions common for all Riemannian symmetric spaces, e.g. facets or parabolic root sub-systems. This would be useful in view of a possible generalization of Theorem 1.6 to other symmetric spaces.

2. A necessary condition for the existence of the density

In this section, we will show (Proposition 2.2) that if X and Y are not eligible then $\mu_{X,Y}$ does not have a density with respect to the Lebesgue measure. The heuristic behind the proof is simple: when we decompose properly the factor K which appears in $a(e^X\, K\, e^Y)$, a good portion may commute with either e^X or e^Y if X or Y have blocks of repeated diagonal values (noting that $a(k\, g) = a(g) = a(g\, k)$ for every $k \in K$). Indeed, what is left may not be enough to ensure that $a(e^X\, K\, e^Y)$ has dimension equal to the rank $n-1$.

We will therefore decompose K in two ways; one that is justified by Lemma 2.1 and one that corresponds to the Cartan decomposition of $K = \mathbf{SU}(n, \mathbf{F})$ (see equation (9)).

Lemma 2.1. *Suppose that* $1 < q < n$. *We have* $\mathbf{SU}(n, \mathbf{F}) = K_1\, K_2\, K_3$ *where*

$$K_1 = \left\{ \begin{bmatrix} k_1' & 0 \\ 0 & I_{q-1} \end{bmatrix} : k_1' \in \mathbf{SU}(n+1-q, \mathbf{F}) \right\}, \tag{6}$$

$$K_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & k_2' \end{bmatrix} : k_2' \in \mathbf{SU}(n-1, \mathbf{F}) \right\}, \tag{7}$$

$$K_3 = \left\{ \begin{bmatrix} k_3' & 0 \\ 0 & I_{n-q} \end{bmatrix} : k_3' \in \mathbf{SU}(q, \mathbf{F}) \right\}. \tag{8}$$

Proof. Let $k \in \mathbf{SU}(n, \mathbf{F})$. For $i \geq q+1$, let $\mathbf{v}_i = k\, \mathbf{e}_i$ where $\mathbf{e}_i$ is the i-th vector of the standard basis. There exists $k_1 = \begin{bmatrix} k_1' & 0 \\ 0 & I_{q-1} \end{bmatrix} \in K_1$ such that $k_1\, [\mathbf{v}_{q+1}, \ldots, \mathbf{v}_n] = \begin{bmatrix} 0_{1,n-q} \\ C^{(1)}_{n-1,n-q} \end{bmatrix}$ where the notation $M_{i,j}$ means that the matrix M is of size $i \times j$. Indeed, it suffices that the first row of k_1' be a unitary vector in $\mathbf{F}^{n+1-q}$ perpendicular to the $n-q$ vectors $\mathbf{v}_i'$, $i \geq q+1$, where $\mathbf{v}_i'$ contains the first $n+1-q$ entries of $\mathbf{v}_i$.

We can then find $k_2^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & k_2' \end{bmatrix} \in K_2$ such that $k_2^{(1)} k_1\, [\mathbf{v}_{q+1}, \ldots, \mathbf{v}_n] = \begin{bmatrix} 0_{2,n-q} \\ C^{(2)}_{n-2,n-q} \end{bmatrix}$. Indeed, it suffices that the first row of k_2' be a unitary vector perpendicular to all the columns in $C^{(1)}$. In the same fashion, we can find $k_2^{(2)} = \begin{bmatrix} I_2 & 0 \\ 0 & k_2'' \end{bmatrix} \in K_2$ such that $k_2^{(2)}\, k_2^{(1)}\, k_1\, [\mathbf{v}_{q+1}, \ldots, \mathbf{v}_n] = \begin{bmatrix} 0_{3,n-q} \\ C^{(3)}_{n-3,n-q} \end{bmatrix}$ (the first row of k_2'' being perpendicular to the columns of $C^{(2)}$).

Finally, we obtain

$$\underbrace{\overbrace{k_2^{(q-1)} \ldots k_2^{(1)}}^{k_2}}_{with\ k_2^{(i)} \in K_2} k_1\, [\mathbf{v}_{q+1}, \ldots, \mathbf{v}_n] = \begin{bmatrix} 0_{q,n-q} \\ C^{(q)}_{n-q,n-q} \end{bmatrix}$$

that is, $k_2\, k_1\, k\, [\mathbf{e}_{q+1}, \ldots, \mathbf{e}_n] = \begin{bmatrix} 0 \\ C \end{bmatrix}$, where $C = C^{(q)}$.

Hence, $k_2\, k_1\, k = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ since $k_2 k_1 k \in \mathbf{SU}(n, \mathbf{F})$. This means that $k_2\, k_1\, k \in K_2\, K_3$ and the lemma follows. ∎

We now prove a necessary condition for the existence of the density of the measure $\mu_{X,Y}$:

Proposition 2.2. *If X and Y are not eligible (Definition 1.3) then the measure $\mu_{X,Y}$ is not absolutely continuous with respect to the Lebesgue measure on $\mathfrak{a}$.*

Proof. If one of X or Y is zero then $\mathcal{S}_{X,Y} = X + Y$ and the density does not exist. If $n = 2$ then X and Y are eligible if and only if they are nonzero. In this case, the result is clear. We can therefore assume that $n > 2$ and that X and Y are nonzero.

Suppose first that $n > 2$ is even and that $X = a \begin{bmatrix} I_{n/2} & 0 \\ 0 & -I_{n/2} \end{bmatrix}$, $Y = b \begin{bmatrix} I_{n/2} & 0 \\ 0 & -I_{n/2} \end{bmatrix}$. For $k \in \mathbf{SU}(n, \mathbf{F})$, we have the Cartan decomposition of $\mathbf{SU}(p+q, \mathbf{F})/\mathbf{S}(\mathbf{U}(p, \mathbf{F}) \times \mathbf{U}(q, \mathbf{F}))$ with $p = q = n/2$ (refer to [7, Page 518])

$$k = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} e^{\begin{bmatrix} 0 & R \\ -R & 0 \end{bmatrix}} \begin{bmatrix} A_3 & 0 \\ 0 & A_4 \end{bmatrix} \tag{9}$$

where R is a real diagonal matrix of size $(n/2) \times (n/2)$ and $A_i \in \mathbf{SU}(n/2, \mathbf{F})$, $i = 1, \ldots 4$.

Therefore, with X, Y as above, we have

$$\begin{aligned} a(e^X k\, e^Y) &= a\left(e^X \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} e^{\begin{bmatrix} 0 & R \\ -R & 0 \end{bmatrix}} \begin{bmatrix} A_3 & 0 \\ 0 & A_4 \end{bmatrix} e^Y \right) \\ &= a\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} e^X e^{\begin{bmatrix} 0 & R \\ -R & 0 \end{bmatrix}} e^Y \begin{bmatrix} A_3 & 0 \\ 0 & A_4 \end{bmatrix} \right) \\ &= a\left(e^X e^{\begin{bmatrix} 0 & R \\ -R & 0 \end{bmatrix}} e^Y \right). \end{aligned}$$

As R varies along all real diagonal matrices of size $(n/2) \times (n/2)$, we can only obtain a set of dimension at most $n/2 < n - 1$. Hence, the interior of $a\left(e^X\, K\, e^Y\right)$ is empty.

Suppose now that $p_i + q_j > n$ for some i and j. We can assume that $1 \leq p_i \leq q_j$ (from Remark 1.1). Choose w_1, $w_2 \in W$ such that the first p_i diagonal entries of $w_1 \cdot X$ are X_i and the first q_j diagonal entries of $w_2 \cdot Y$ are Y_j. By applying Lemma 2.1 to the matrix $\tilde{k}$ below, we have

$$\begin{aligned} a\left(e^X k e^Y\right) &= a\left(w_1^{-1} e^{w_1 \cdot X} \overbrace{w_1 k w_2^{-1}}^{\tilde{k}} e^{w_2 \cdot Y} w_2 \right) = a\left(e^{w_1 \cdot X} \tilde{k} e^{w_2 \cdot Y} \right) \\ &= a\left(e^{w_1 \cdot X} \begin{bmatrix} A_1 & 0 \\ 0 & I_{q_j - 1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} A_3 & 0 \\ 0 & I_{n - q_j} \end{bmatrix} e^{w_2 \cdot Y} \right) \end{aligned}$$

(with $A_1 \in \mathbf{SU}(n+1-q_j, \mathbf{F})$, $A_2 \in \mathbf{SU}(n-1, \mathbf{F})$ and $A_3 \in \mathbf{SU}(q_j, \mathbf{F})$)

$$= a\left(\begin{bmatrix} A_1 & 0 \\ 0 & I_{q_j-1} \end{bmatrix} e^{w_1\cdot X} \begin{bmatrix} 1 & 0 \\ 0 & A_2 \end{bmatrix} e^{w_2\cdot Y} \begin{bmatrix} A_3 & 0 \\ 0 & I_{n-q_j} \end{bmatrix}\right) \tag{10}$$

$$= a\left(e^{w_1\cdot X} \begin{bmatrix} 1 & 0 \\ 0 & A_2 \end{bmatrix} e^{w_2\cdot Y}\right). \tag{11}$$

In (10), we used the fact that $n+1-q_j \le p_i$ and therefore that $\begin{bmatrix} A_1 & 0 \\ 0 & I_{q_j-1} \end{bmatrix}$ commutes with $e^{w_1\cdot X}$. Similarly, $\begin{bmatrix} A_3 & 0 \\ 0 & I_{n-q_j} \end{bmatrix}$ commutes with $e^{w_2\cdot Y}$. The formula (11) implies that all the elements $H \in a\left(e^X K e^Y\right)$ have a diagonal entry equal to $X_i + Y_j$ and therefore that $a\left(e^X K e^Y\right)$ has empty interior. ■

3. Sufficiency of the eligibility condition

We will first discuss how proving the case $\mathbf{F} = \mathbf{R}$ will suffice. Let $\mathcal{S}^{\mathbf{F}}_{X,Y} = a(e^X\, \mathbf{SU}(n, \mathbf{F})\, e^Y)$. The inclusions

$$\mathbf{SO}(n) = \mathbf{SU}(n, \mathbf{R}) \subset \mathbf{SU}(n) = \mathbf{SU}(n, \mathbf{C}) \subset \mathbf{SU}(n, \mathbf{H})$$

imply that

$$\mathcal{S}^{\mathbf{R}}_{X,Y} \subset \mathcal{S}^{\mathbf{C}}_{X,Y} \subset \mathcal{S}^{\mathbf{H}}_{X,Y} \subset \overline{\mathfrak{a}^+}.$$

By Remark 1.1, if we show that for eligible $X, Y \in \mathfrak{a}$, the set $\mathcal{S}^{\mathbf{R}}_{X,Y}$ has a nonempty interior, it follows that the measure $\mu_{X,Y}$ is absolutely continuous in all three cases $m = 1, 2, 4$.

Remark 3.1. In [5, Remark 2.15], we argued that the three sets $\mathcal{S}^{\mathbf{F}}_{X,Y}$ are actually equal. An important ingredient in that result was the convexity of the set $a(e^X\, K\, e^Y)$, a consequence of [9, Theorem 1.3]. The equality of the three sets $\mathcal{S}^{\mathbf{F}}_{X,Y}$ also follows from [9, Theorem 1.2].

From now on, we consider the symmetric spaces $\mathbf{SL}(n, \mathbf{R})/\mathbf{SO}(n)$. Let $E_{i,j}$ be the matrix of size $n \times n$ which is zero everywhere except for the entry (i, j) where it is 1. Define

$$U_X = \mathfrak{so}(n) + e^X\, \mathfrak{so}(n)\, e^{-X}$$
$$V_X = \operatorname{span}\{E_{i,j} \colon X_i \ne X_j\}, \text{ if } X = \operatorname{diag}[X_1, \ldots, X_n].$$

Lemma 3.2. *The following inclusion holds:*

$$V_X \subset U_X.$$

Proof. Let $i \ne j$. Then $E_{i,j} - E_{j,i} \in \mathfrak{so}(n)$ and $e^X(E_{i,j} - E_{j,i})e^{-X} = e^{X_i - X_j} E_{i,j} - e^{X_j - X_i} E_{j,i}$. It follows that when $X_i \ne X_j$, we have $E_{i,j} \in U_X$. ■

We will also need the following property:

Lemma 3.3. *Let $w \in W$. Then*

$$Ad(w)V_X = V_{w\cdot X}.$$

Proof. The proof is clear using the fact that $V_X = \bigoplus_{\alpha\in\Sigma,\alpha(X)\neq 0} \mathfrak{g}_\alpha$. ∎

Proposition 3.4. *If there exists $k \in \mathbf{SO}(n)$ such that*

$$W_{X,Y}(k) := \mathfrak{so}(n) + V_X + Ad(k)\,V_Y = \mathfrak{sl}(n,\mathbf{R}) \tag{12}$$

then the measure $\mu_{X,Y}$ is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{a}$.

Proof. Consider the analytic map $T\colon K\times K\times K \to \mathbf{SL}(n,\mathbf{R})$ defined by

$$T(k_1,k_2,k_3) = k_1\,e^X\,k_2\,e^Y\,k_3.$$

Observe that $T(K\times K\times K)$ is equal to the support of the measure $m_{X,Y} = \delta^\natural_{e^X}\star\delta^\natural_{e^Y}$. We want to show that the derivative of T is surjective for some choice of $\mathbf{k}=(k_1,k_2,k_3)$. The existence of $\mathbf{k}$ ensures that $T(K\times K\times K)$ contains a nonempty open set (refer to [7, p. 479]), and it follows that $m_{X,Y}$ and $\mu_{X,Y}$ are absolutely continuous (see part 2. of Remark 1.1).

Let $A,B,C\in\mathfrak{so}(n)$. The derivative of T at $\mathbf{k}$ in the direction of (A,B,C) equals

$$\begin{aligned} dT_{\mathbf{k}}(A,B,C) &= \frac{d}{dt}\Big|_{t=0} e^{tA}k_1\,e^X\,e^{tB}k_2\,e^Y\,e^{tC}k_3 \\ &= A\,k_1\,e^X\,k_2\,e^Y\,k_3 + k_1\,e^X\,B\,k_2\,e^Y\,k_3 + k_1\,e^X\,k_2\,e^Y\,C\,k_3. \end{aligned} \tag{13}$$

We now transform the space of all matrices of the form (13) without modifying its dimension:

$$\begin{aligned} &\dim\{A\,k_1\,e^X\,k_2\,e^Y\,k_3 + k_1\,e^X\,B\,k_2\,e^Y\,k_3 + k_1\,e^X\,k_2\,e^Y\,C\,k_3\colon A,B,C\in\mathfrak{so}(n)\} \\ &= \dim\{k_1^{-1}\,A\,k_1\,e^X\,k_2\,e^Y + e^X\,B\,k_2\,e^Y + e^X\,k_2\,e^Y\,C\colon A,B,C\in\mathfrak{so}(n)\} \\ &= \dim\{A\,e^X\,k_2\,e^Y + e^X\,B\,k_2\,e^Y + e^X\,k_2\,e^Y\,C\colon A,B,C\in\mathfrak{so}(n)\} \\ &= \dim\{e^{-X}\,A\,e^X + B + k_2\,e^Y\,C\,e^{-Y}k_2^{-1}\colon A,B,C\in\mathfrak{so}(n)\}. \end{aligned}$$

The vector space in the last line equals

$$e^{-X}\,\mathfrak{so}(n)\,e^X + \mathfrak{so}(n) + k_2\,e^Y\,\mathfrak{so}(n)\,e^{-Y}k_2^{-1} = U_{-X} + \mathrm{Ad}(k_2)U_Y.$$

Note that $U_X = U_{-X}$, so our problem is equivalent to showing that

$$U_X + \mathrm{Ad}(k)U_Y = \mathfrak{sl}(n,\mathbf{R}) \text{ for some } k\in K.$$

By Lemma 3.2, the result follows from (12). ∎

The following theorem is the crucial result of this section:

Theorem 3.5. *Let $G=\mathbf{SL}(n,\mathbf{R})$ and let X, $Y\in\mathfrak{a}$. If X and Y are eligible then there exists a matrix $k\in\mathbf{SO}(n)$ such that $W_{X,Y}(k)=\mathfrak{so}(n)+V_X+Ad(k)\,V_Y=\mathfrak{sl}(n,\mathbf{R})$.*

Before we proceed with the proof of Theorem 3.5, we point out a few useful reductions to the problem.

Reductions

1. The result of Theorem 3.5 only depends on the configurations of X and Y, i.e., on the partitions p and q associated to X and Y.

   Indeed, suppose that we have (12) for X and Y fixed. Let $w_1,w_2\in W$. Then, using Lemma 3.3, we get

   $$\mathrm{Ad}(w_1)\mathfrak{so}(n)+V_{w_1\cdot X}+\mathrm{Ad}(w_1kw_2^{-1})V_{w_2\cdot Y}=\mathrm{Ad}(w_1)\mathfrak{sl}(n,\mathbf{R})$$

   which gives $W_{w_1\cdot X,w_2\cdot Y}(w_1kw_2^{-1})=\mathfrak{sl}(n,\mathbf{R})$.

2. Suppose that $W_{X,Y}(k)=\mathfrak{sl}(n,\mathbf{R})$ for some k and that X has the configuration $p=[p_1,\dots,p_r]$. Suppose that $\tilde{X}$ has the configuration

   $$p=[p_1,\dots,p_i',p_i'',\dots,p_r]$$

   where $p_i=p_i'+p_i''$. Then $W_{\tilde{X},Y}(k)=\mathfrak{sl}(n,\mathbf{R})$. This is a consequence of the inclusion $V_X\subset V_{\tilde{X}}$. The same observation holds for Y.

3. Without loss of generality, we may assume that $p_1\le q_1$ (see part 1. of Remark 1.1).

4. We can always assume that the configuration of Y consists of two blocks. Indeed, suppose that X and Y corresponding to partitions p and q are eligible and that $p_1\le q_1$. Then $p_1+q_2+\dots+q_s\le q_1+q_2+\dots+q_s=n$. Therefore, using the reduction 2 above, we can consider instead of p and q, the partitions p and $q'=[q_1,n-q_1]$ if $q_1\ge n-q_1$ or $q'=[n-q_1,q_1]$ if $q_1<n-q_1$, which are also eligible, except in the case

   $$p=[n/2,n/2],\quad q=[n/2,q_2,\dots,q_s],\ s\ge 3.$$

   In that situation, we will exchange the roles of X and Y.

5. In order to simplify the proof, it is convenient to consider symmetrized matrices $E_{i,j}$. Let $E^S_{i,j}=E_{i,j}+E_{j,i}$ if $i\ne j$ and let $E^S_{i,i}=E_{i,i}$. We shall denote by V^S_X the subspace of symmetric matrices in V_X

   $$V^S_X=\mathrm{span}\{E^S_{i,j}\colon X_i\ne X_j\},\ \text{if } X=\mathrm{diag}[X_1,\dots,X_n].$$

   Similarly we define V^S_Y and $\mathfrak{sl}^S(n,\mathbf{R})$ (the space of real symmetric matrices of trace 0). If we show that

   $$W^S_{X,Y}(k):=V^S_X+\mathrm{Ad}(k)\,V^S_Y=\mathfrak{sl}^S(n,\mathbf{R})$$

   then (12) follows by $V^S_X\subset V_X, V^S_Y\subset V_Y,\mathfrak{so}(n)\subset W_{X,Y}(k)$ and $\mathfrak{so}(n)\oplus\mathfrak{sl}^S(n,\mathbf{R})=\mathfrak{k}\oplus\mathfrak{p}=\mathfrak{sl}(n,\mathbf{R})$.

Taking into account all the reductions we will prove by induction the following result:

Theorem 3.6. *Let $G = \mathbf{SL}(n, \mathbf{R})$ and let X, $Y \in \mathfrak{a}$ with respective configurations p and q, where $q = [q_1, q_2]$ and $p_1 \leq q_1$. If X and Y are eligible then there exists a matrix $k \in \mathbf{SO}(n)$ such that*

$$V_X^S + Ad(k)\, V_Y^S = \mathfrak{sl}^S(n, \mathbf{R}). \tag{14}$$

In the proof we will use the following definition and properties of *total* matrices and technical Lemma 3.10:

Definition 3.7. We will say that the $n \times n$ matrix A is **total** if all the submatrices

$$A_I = (a_{ij})_{i,j \in I}, \quad \emptyset \neq I \subset \{1, 2, \dots, n\}$$

are nonsingular.

Proposition 3.8. *The matrices which are total in $K = \mathbf{SO}(n)$ form an open dense subset of K with Haar measure 1.*

Proof. Let $I \subset \{1, 2, \dots, n\}$. We define $f_I(A) = \det A_I$ and $f(A) = \prod_I f_I(A)$. Then $f : \mathbf{SO}(n) \mapsto \mathbf{R}$ is a non-vanishing analytic function ($f(I_n) = 1$). The zeros of f form a closed set of zero Haar measure in K (refer to [3, Proposition 2.2]). ■

Lemma 3.9. *Let $1 \leq r < n$ and $S = \operatorname{span}\{E_{1,r+1}^S, \dots, E_{1,n}^S\}$. Denote $\pi_S : \mathfrak{sl}^S(n, \mathbf{R}) \mapsto S$ the orthogonal projection. If $k = \begin{bmatrix} 1 & 0 \\ 0 & k' \end{bmatrix} \in K$ where $k' \in \mathbf{SO}(n-1)$ is total then*

$$\pi_S(Ad(k)S) = S.$$

Proof. Let $k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_{r-1,r-1} & B_{r-1,n-r} \\ 0 & C_{n-r,r-1} & D_{n-r,n-r} \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 0 & 0 & U_{1,n-r} \\ 0 & 0 & 0 \\ U_{1,n-r}^T & 0 & 0 \end{bmatrix} \in S$.

Then we compute easily that

$$\pi_S(k\mathbf{s}k^t) = \begin{bmatrix} 0 & 0 & U\,D^T \\ 0 & 0 & 0 \\ D\,U^T & 0 & 0 \end{bmatrix}.$$

Since k' is total, we have $\det(D) \neq 0$ and therefore the rank of π_S equals $n-r$. ■

For $i \neq j$, let $e_{i,j}(\theta) \in \mathbf{SO}(n)$ be defined as

$$e_{i,j}(\theta) = \cos\theta\,(E_{i,i} + E_{j,j}) + \sin\theta\,(E_{j,i} - E_{i,j}) + \sum_{k \neq i,j} E_{k,k}.$$

Note that $e_{i,j}(\theta) = e_{j,i}(-\theta)$.

Lemma 3.10. *The following formulas hold (different letters are assumed to represent different values):*

$$\begin{aligned}
Ad(e_{i,j}(\theta))\, E^S_{k,l} &= E^S_{k,l},\\
Ad(e_{i,j}(\theta))\, E^S_{j,l} &= -\sin\theta\, E^S_{i,l} + \cos\theta\, E^S_{j,l},\\
Ad(e_{i,j}(\theta))\, E^S_{i,l} &= \cos\theta\, E^S_{i,l} + \sin\theta\, E^S_{j,l},\\
Ad(e_{i,j}(\theta))\, E^S_{i,j} &= -\sin(2\,\theta)\,(E_{i,i} - E_{j,j}) + \cos(2\,\theta)\, E^S_{i,j},\\
Ad(e_{i,j}(\theta))\, E_{i,i} &= \cos^2\theta\, E_{i,i} + \sin^2\,\theta\, E_{j,j} + \frac{1}{2}\,\sin(2\,\theta)\, E^S_{i,j},\\
Ad(e_{i,j}(\theta))\, E_{j,j} &= \sin^2\theta\, E_{i,i} + \cos^2\,\theta\, E_{j,j} - \frac{1}{2}\,\sin(2\,\theta)\, E^S_{i,j}.
\end{aligned}$$

Proof. The formulas of the lemma follow by direct elementary computations. ∎

Proof of Theorem 3.6: We will show a slightly more general version of the formulas (12) and (14). We will not suppose that the traces $\operatorname{tr} X$ and $\operatorname{tr} Y$ are zero. Let $\mathrm{Diag}(n) \equiv \mathbf{R}^n$ denote the space of all diagonal $n \times n$ real matrices. For any $X, Y \in \mathrm{Diag}(n)$, the definition of eligibility, of the spaces V_X, V_Y, the statement of Lemma 3.3 and all the preceding reductions remain valid.

We will prove the following statement:

For all eligible $X, Y \in \mathrm{Diag}(n)$, there exists $k \in \mathbf{SO}(n)$ such that

$$V^S_X + \mathrm{Ad}(k)\, V^S_Y = \mathfrak{sl}^S(n, \mathbf{R}) \tag{15}$$

using induction with respect to $n \geq 2$. In particular, this provides a new proof for the cases $X, Y \in \mathfrak{a}^+$, $X \in \mathfrak{a}^+$ and $Y \neq 0$ or $X \neq 0$ and $Y \in \mathfrak{a}^+$, a result that is known for all Riemannian symmetric spaces from [3].

Case $n = 2$. The configurations of eligible X and Y are $p = q = [1, 1]$. We then have $V_X = V_Y = \mathrm{span}\{E^S_{1,2}\}$. By Lemma 3.10, if we take $k = e_{1,2}(\theta)$ with $\theta \in (0, \pi/2)$, we have

$$W_{X,Y}(k) = \mathrm{span}\{E^S_{1,2}, E_{1,1} - E_{2,2}\} = \mathfrak{sl}^S(2, \mathbf{R}).$$

Choice of predecessors X', Y' for eligible $X, Y \in \mathrm{Diag}(n)$. Suppose that X, Y have configurations $p = [p_1, p_2, \ldots, p_r], q = [q_1, q_2]$ respectively, with $p_1 \leq q_1$. Denote by x_i the p_i equal diagonal components of X, $i = 1, \ldots, r$, and by y_i the q_i equal diagonal components of Y, $i = 1, 2$. We will prove the formula (15) for the following orderings of elements of X and Y:

$$X = \mathrm{diag}[x_1, x_2, \ldots, x_r, \overbrace{x_1, \ldots, x_1}^{p_1-1}], \qquad Y = \mathrm{diag}[\overbrace{y_1, y_1, \ldots, y_1}^{q_1}, \overbrace{y_2, \ldots, y_2}^{n-q_1}]. \tag{16}$$

We define $X', Y' \in \mathrm{Diag}(n-1)$ by suppressing the first elements of X and Y

$$X' = \mathrm{diag}[x_2, \ldots, x_r, \overbrace{x_1, \ldots, x_1}^{p_1-1}], \qquad Y' = \mathrm{diag}[\overbrace{y_1, \ldots, y_1}^{q_1-1}, \overbrace{y_2, \ldots, y_2}^{n-q_1}].$$

We apply this procedure as long as $p_1>1$, $q_1>1$ and $n\ge 2$. It can be described as the *shortening by 1 of the longest blocks of X and Y*. After a finite number of such shortenings, one or both of reduced vectors will have a configuration 1^m, $m\ge 2$. Shortening of 1^m then leads to 1^{m-1}. In the last step of the reductions, we will end up with the configurations $[1,1]$ and $[1,1]$ for X and Y, considered in the first step of the induction.
X' and Y' are eligible. In order to show this, we consider several possible cases. We call p' and q' the configurations of X' and Y' and p_1', q_1' their maximal elements.

- If $p_1>p_2$ and $q_1>q_2$, then $p_1'=p_1-1$ and $q_1'=q_1-1$. We have $p_1'+q_1'=p_1+q_1-2\le n-2$. The last inequality guarantees that $p'=[(n-1)/2,(n-1)/2]=q'$ does not happen.
- If $p_1>p_2$ and $q_1=q_2=n/2$ then $p_1'=p_1-1$ and $q_1'=q_2=n/2$. The sum $p_1'+q_1'=p_1+q_1-1\le n-1$. Note that we then have configuration $q'=[n/2,(n/2)-1]\ne[(n-1)/2,(n-1)/2]$.
- If $p_1=p_2$ and $q_1>q_2$, then $p_1'=p_2=p_1$ and $q_1'=q_1-1$. We get $p_1'+q_1'=p_1+q_1-1\le n-1$. If $p_1>1$, then the configuration p' contains two different bloc lengths p_2-1 and p_2, so it is not equal to $[(n-1)/2,(n-1)/2]$. If $p_1=1$, then $p=1^n$ and $p'=1^{n-1}$ or $n=2$.
- If $p_1=p_2$ and $q_1=q_2=n/2$ then $p_1<n/2$ unless $n=2$. We have $p_1'+q_1'=p_1+q_1<n$ and $q'\ne[(n-1)/2,(n-1)/2]$.

From $\mathfrak{sl}(n-1,\mathbf{R})$ to $\mathfrak{sl}(n,\mathbf{R})$. The induction hypothesis can be applied to X' and Y'. Let $k'\in SO(n-1)$ be such that

$$V_{X'}^S+\mathrm{Ad}(k')V_{Y'}^S=\mathfrak{sl}^S(n-1,\mathbf{R}). \tag{17}$$

It is easy to see that (17) remains true in an open neighbourhood containing k'. Using Proposition 3.8, this allows to assume that k' is total.

Recall that $X=\mathrm{diag}[x_1,X']$ and $Y=\mathrm{diag}[y_1,Y']$. We “shift” and embed the spaces $V_{X'}^S$ and $V_{Y'}^S$ in $\mathfrak{sl}^S(n,\mathbf{R})$ in the following way:

$$\tilde V_{X'}^S=\mathrm{span}\{E_{i+1,j+1}^S\in\mathfrak{sl}(n,\mathbf{R}):X_i'\ne X_j'\},\ \tilde V_{Y'}^S=\mathrm{span}\{E_{i+1,j+1}^S\in\mathfrak{sl}(n,\mathbf{R}):Y_i'\ne Y_j'\},$$

and we define $k_0\in\mathbf{SO}(n)$ by

$$k_0=\begin{bmatrix}1 & 0_{1,n-1}\\ 0_{n-1,1} & k'\end{bmatrix}.$$

Then (17) implies that

$$\tilde V_{X'}^S+\mathrm{Ad}(k_0)\tilde V_{Y'}^S=\begin{bmatrix}0 & 0_{1,n-1}\\ 0_{n-1,1} & \mathfrak{sl}^S(n-1,\mathbf{R})\end{bmatrix}. \tag{18}$$

New elements in V_X and V_Y. In order to prove (15), we must now use the elements of V_X^S and V_Y^S which do not come from $\tilde V_{X'}^S$ or $\tilde V_{Y'}^S$. We define

$$N_X=\{E_{1,2}^S,\dots,E_{1,n-p_1+1}^S\},\qquad N_Y=\{E_{1,q_1+1}^S,\dots,E_{1,n}^S\}.$$

We have $V_X^S = \mathrm{span}(N_X) \oplus \tilde V_{X'}^S$ and $V_Y^S = \mathrm{span}(N_Y) \oplus \tilde V_{Y'}^S$. Let $P = N_X \cap N_Y$. Observe that $n - p_1 + 1 \geq q_1 + 1$, since $n \geq p_1 + q_1$, thus $E_{1,q_1+1} \in P$. Note that $(N_X \setminus P) \cup N_Y = \{E_{1,2}^S, \ldots, E_{1,n}^S\}$. Define

$$V_X' := \mathrm{span}(N_X \setminus P) \oplus \tilde V_{X'}^S.$$

We shall prove (15) in three steps.
Step 1. We have

$$V_X' + \mathrm{Ad}(k_0)V_Y^S = \begin{bmatrix} 0 & \mathbf{R}^{n-1} \\ \mathbf{R}^{n-1} & \mathfrak{sl}^S(n-1,\mathbf{R}) \end{bmatrix} \cap \mathfrak{sl}^S(n,\mathbf{R}). \tag{19}$$

Proof of Step 1. By matrix multiplication we check that

$$\mathrm{Ad}(k_0)N_Y \subset \begin{bmatrix} 0 & \mathbf{R}^{n-1} \\ \mathbf{R}^{n-1} & 0_{n-1,n-1} \end{bmatrix}. \tag{20}$$

We use Lemma 3.9 with $r = q_1$ and $S = \mathrm{span}(N_Y)$. Given that $\{E_{1,2}^S, \ldots, E_{1,q_1}^S\} \subset V_X'$, it follows that

$$S = \pi_S(\mathrm{Ad}(k_0)S) \subset \mathrm{span}\{E_{1,2}^S, \ldots, E_{1,q_1}^S\} + \mathrm{Ad}(k_0)S \subset V_X' + \mathrm{Ad}(k_0)V_Y^S.$$

and the equation (18) gives (19).
Step 2. For $\theta > 0$ sufficiently small, we have

$$\mathrm{Ad}(e_{1,q_1+1}(\theta))V_X' + \mathrm{Ad}(k_0)V_Y^S = \begin{bmatrix} 0 & \mathbf{R}^{n-1} \\ \mathbf{R}^{n-1} & \mathfrak{sl}^S(n-1,\mathbf{R}) \end{bmatrix} \cap \mathfrak{sl}^S(n,\mathbf{R}). \tag{21}$$

Proof of Step 2. The formula (21) holds for $\theta = 0$ by (19). By Lemma 3.10 and by (20), for any θ we have the inclusion

$$\mathrm{Ad}(e_{1,q_1+1}(\theta))V_X' + \mathrm{Ad}(k_0)V_Y^S \subset \begin{bmatrix} 0 & \mathbf{R}^{n-1} \\ \mathbf{R}^{n-1} & \mathfrak{sl}^S(n-1,\mathbf{R}) \end{bmatrix} \cap \mathfrak{sl}^S(n,\mathbf{R}). \tag{22}$$

Let $d(\theta) := \dim(\mathrm{Ad}(e_{1,q_1+1}(\theta))V_X' + \mathrm{Ad}(k_0)V_Y^S)$.
We have $d(0) = n-1+\dim(\mathfrak{sl}^S(n-1,\mathbf{R}))$. The equality $d(\theta) = d(0)$ is equivalent to non-nullity of an appropriate determinant continuous in θ. Thus $d(\theta) = n - 1 + \dim(\mathfrak{sl}^S(n-1,\mathbf{R})$ holds for θ in a neighborhood of 0. Together with (22), this implies that formula (21) holds for θ small enough.
Step 3. Fix $\theta \in (0, \pi/2)$ for which formula (21) holds and denote $k_1 = e_{1,q_1+1}(\theta)$. Then

$$\mathrm{Ad}(k_1)V_X^S + \mathrm{Ad}(k_0)V_Y^S = \mathfrak{sl}(n,\mathbf{R}). \tag{23}$$

Proof of Step 3. Recall that $E_{1,q_1+1} \in P \subset N_X \subset V_X^S$. By Lemma 3.10,

$$\mathrm{Ad}(k_1)E_{1,q_1+1}^S = -\sin(2\,\theta)\,(E_{1,1} - E_{q_1+1,q_1+1}) + \cos(2\,\theta)\,E_{1,q_1+1}^S,$$

so the following inclusion deduced from (21) is strict:

$$\begin{bmatrix} 0 & \mathbf{R}^{n-1} \\ \mathbf{R}^{n-1} & \mathfrak{sl}^S(n-1,\mathbf{R}) \end{bmatrix} \cap \mathfrak{sl}^S(n,\mathbf{R}) \subset \mathrm{Ad}(k_1)V_X^S + \mathrm{Ad}(k_0)V_Y^S.$$

Consequently,

$$\dim\left(\begin{bmatrix} 0 & \mathbf{R}^{n-1} \\ \mathbf{R}^{n-1} & \mathfrak{sl}^S(n-1,\mathbf{R}) \end{bmatrix} \cap \mathfrak{sl}^S(n,\mathbf{R})\right) < \dim(\mathrm{Ad}(k_1)V_X^S + \mathrm{Ad}(k_0)V_Y^S).$$

This implies that $\dim(\mathrm{Ad}(k_1)V_X^S + \mathrm{Ad}(k_0)V_Y^S) = \dim(\mathfrak{sl}^S(n,\mathbf{R}))$ and the formula (23) follows.

Proof of (15). From the formula (23) we get

$$V_X^S + \mathrm{Ad}(k_1^{-1}k_0)V_Y^S = \mathfrak{sl}^S(n,\mathbf{R}),$$

thus (15) is true for $k_1^{-1}k_0$. This ends the proof of Theorem 3.6. Together with the necessity condition proved in Section 2, the proof of the main Theorem 1.6 is completed. ∎

We illustrate the proof with an example.

Example 3.11. Let $n=5$. Consider X and Y with configurations $[2,2,1]$ and $[3,1,1]$ respectively. The vectors X and Y are eligible. By reduction 2, we can suppose that the configurations of X and Y are $[2,2,1]$ and $[3,2]$ respectively. We order X and Y in the following way:

$$X = \mathrm{diag}[x_1,x_2,x_2,x_3,x_1], \qquad Y = \mathrm{diag}[y_1,y_1,y_1,y_2,y_2].$$

The predecessors of X and Y in the induction procedure are

$$X' = \mathrm{diag}[x_2,x_2,x_3,x_1], \qquad Y' = \mathrm{diag}[y_1,y_1,y_2,y_2]$$

and their configurations are $[2,1,1]$ and $[2,2]$ respectively.

Using induction, we know that there exists $k' \in \mathbf{SO}(4)$ which can be chosen to be total such that

$$V_{X'}^S + \mathrm{Ad}(k')\,V_{Y'}^S = \mathfrak{sl}^S(4,\mathbf{R}).$$

Define $k_0 = \begin{bmatrix} 1 & 0 \\ 0 & k' \end{bmatrix}$. We shift the spaces $V_{X'}^S$ and $V_{Y'}^S$

$$\tilde V_{X'}^S = \mathrm{span}\{E_{2,4}^S, E_{2,5}^S, E_{3,4}^S, E_{3,5}^S, E_{4,5}^S\}, \quad \tilde V_{Y'}^S = \mathrm{span}\{E_{2,4}^S, E_{2,5}^S, E_{3,4}^S, E_{3,5}^S\}$$

and we deduce from the induction hypothesis the analogue of the formula (18), namely

$$\tilde V_{X'}^S + \mathrm{Ad}(k_0)\tilde V_{Y'}^S = \begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{sl}^S(4,\mathbf{R}) \end{bmatrix}.$$

The "new" elements in V_X and V_Y are

$$N_X = \{E_{1,2}, E_{1,3}, E_{1,4}\}, \quad N_Y = \{E_{1,4}, E_{1,5}\}$$

and their intersection $P = N_X \cap N_Y = \{E_{14}\}$. The space V_X' is defined by

$$V_X' := \mathrm{span}\{E_{1,2}, E_{1,3}\} \oplus \tilde V_{X'}^S.$$

In Step 1 of the proof of Theorem 3.6 we show that

$$V'_X + \mathrm{Ad}(k_0)\, V_Y^S = \left\{ \left[\begin{array}{c|cccc} 0 & a & b & c & d \\ \hline a & & & & \\ b & & & & \\ c & & \mathfrak{sl}^S(4,\mathbf{R}) & & \\ d & & & & \end{array}\right] : \ a,b,c,d \in \mathbf{R} \right\}.$$

In Step 2, we justify the fact that acting with $e_{1,4}(\theta)$ on V'_X does not affect the last equality if θ is small enough

$$\mathrm{Ad}(e_{1,4}(\theta))V'_X + \mathrm{Ad}(k_0)\, V_Y^S = \left\{ \left[\begin{array}{c|cccc} 0 & a & b & c & d \\ \hline a & & & & \\ b & & & & \\ c & & \mathfrak{sl}^S(4,\mathbf{R}) & & \\ d & & & & \end{array}\right] : \ a,b,c,d \in \mathbf{R} \right\}.$$

Finally, in the Step 3, we adjoin the element $E_{1,4}$ to V'_X. Observing that

$$\mathrm{Ad}(e_{1,4}(\theta))E_{1,4}^S = -\sin(2\,\theta)\,(E_{1,1} - E_{4,4}) + \cos(2\,\theta)\, E_{1,4}^S,$$

we justify the fact that a diagonal element missing in $\mathrm{Ad}(e_{1,4}(\theta))V'_X + \mathrm{Ad}(k_0)\, V_Y^S$ to make it equal to $\mathfrak{sl}^S(5,\mathbf{R})$ is now generated in $\mathrm{Ad}(e_{1,4}(\theta))V_X^S + \mathrm{Ad}(k_0)\, V_Y^S$. Finally we get

$$V_X^S + \mathrm{Ad}(e_{1,4}(-\theta)k_0)\, V_Y^S = \mathfrak{sl}^S(5,\mathbf{R})$$

for the vectors $X = \mathrm{diag}[x_1, x_2, x_2, x_3, x_1]$ and $Y = \mathrm{diag}[y_1, y_1, y_1, y_2, y_2]$. If we want X to be ordered $\tilde{X} = [x_1, x_1, x_2, x_2, x_3]$, we use $w_1 \in W$ exchanging the first and the fifth coordinates when acting on X. We can write w_1 as the following matrix of $\mathbf{SO}(5)$:

$$w_1 = e_{1,2}(\pi/2)\, e_{2,3}(\pi/2)\, e_{3,4}(\pi/2)\, e_{4,5}(\pi/2).$$

Finally

$$V_{\tilde{X}}^S + \mathrm{Ad}(w_1^{-1}\, e_{1,4}(-\theta)k_0)\, V_Y^S = \mathfrak{sl}^S(5,\mathbf{R}).$$

If we go deeper in this induction procedure, in order to get the case X' and Y', we order their components as follows

$$\mathrm{diag}\,[x_2, x_3, x_1, x_2] \quad \text{and} \quad \mathrm{diag}\,[y_1, y_1, y_2, y_2]$$

and we boil down to

$$X'' = \mathrm{diag}[x_3, x_1, x_2] \quad \text{and} \quad Y'' = \mathrm{diag}[y_1, y_2, y_2]$$

with configurations $[1,1,1]$ and $[2,1]$. One more reduction of $''$ and $''$ with their components ordered as $\mathrm{diag}[x_1, x_2, x_3]$ and $\mathrm{diag}[y_2, y_2, y_1]$ leads to the initial case with configurations $p = q = [1,1]$.

If we go forward in the induction procedure, the case of configurations $[2,2,1], [3,2]$ that we have proved, implies Theorem 3.6 for configurations $[3,2,1]$ and $[3,3]$, etc.

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