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The Component Group of the Automorphism Group of a Simple Lie Algebra and the Splitting of the Corresponding Short Exact Sequence

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Abstract. Let $\mathfrak{g}$ be a simple Lie algebra of finite dimension over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\operatorname{Aut}(\mathfrak{g})$ the finite-dimensional Lie group of its automorphisms. We will calculate the component group $\pi_0(\operatorname{Aut}(\mathfrak{g})) = \operatorname{Aut}(\mathfrak{g})/\operatorname{Aut}(\mathfrak{g})_0$, the number of its conjugacy classes and will show that the corresponding short exact sequence

$$\mathbf{1} \to \operatorname{Aut}(\mathfrak{g})_0 \to \operatorname{Aut}(\mathfrak{g}) \to \pi_0(\operatorname{Aut}(\mathfrak{g})) \to \mathbf{1}$$

is split or, equivalently, there is an isomorphism $\operatorname{Aut}(\mathfrak{g}) \cong \operatorname{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\operatorname{Aut}(\mathfrak{g}))$. Indeed, since $\operatorname{Aut}(\mathfrak{g})_0$ is open in $\operatorname{Aut}(\mathfrak{g})$, the quotient group $\pi_0(\operatorname{Aut}(\mathfrak{g}))$ is discrete. Hence a section $\pi_0(\operatorname{Aut}(\mathfrak{g})) \to \operatorname{Aut}(\mathfrak{g})$ is automatically continuous giving rise to an isomorphism of Lie groups $\operatorname{Aut}(\mathfrak{g}) \cong \operatorname{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\operatorname{Aut}(\mathfrak{g}))$.
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Introduction

The component group $\pi_0(\operatorname{Aut}(\mathfrak{g}))$ and the number of its conjugacy classes occur in the classification of Lie algebras of smooth sections (cf. [11] and [5] for a definition and basic properties), a natural generalization of smooth loop algebras: If $\mathfrak{g}$ is complex simple or real central simple, then the Lie algebras $\Gamma(\mathbb{L})$ of sections of the bundle $\pi : \mathbb{L} \to \mathbb{S}^1$ of Lie algebras with fiber $\mathfrak{g}$ are classified by the homotopy class $\{\mathbb{S}^1; B\operatorname{Aut}(\mathfrak{g})\}_{\simeq}$ which is bijectively mapped to $\#\operatorname{Conj}(\pi_0(\operatorname{Aut}(\mathfrak{g})))$, the number of conjugacy classes of the component group.

The splitting of the sequence

$$\mathbf{1} \to \operatorname{Aut}(\mathfrak{g})_0 \to \operatorname{Aut}(\mathfrak{g}) \to \pi_0(\operatorname{Aut}(\mathfrak{g})) \to \mathbf{1} \qquad (1)$$

is a classical fact for complex simple $\mathfrak{g}$, and $\pi_0(\operatorname{Aut}(\mathfrak{g}))$ is realized in $\operatorname{Aut}(\mathfrak{g})$ as the symmetry group of the corresponding Dynkin diagram. By the close connection of real simple compact Lie algebras to their complexifications, it is no surprise, and it has been shown (see f.i. [8]), that the sequence (1) is split for real simple compact

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$\mathfrak{g}$. However, this result in the two other cases, i.e. $\mathfrak{g}$ being real non-central[1] simple and $\mathfrak{g}$ being real central simple non-compact has not been shown by now.[2]

In this paper, firstly the well-known results for complex simple and real simple compact $\mathfrak{g}$ are stated. Then we present some important tools (Cartan and Polar Decompositon) and lemmas from Heintze's and Groß' structural discussion of involutions of compact Lie algebras in [4]. Following Loos' book on symmetric spaces [12], making use of the classification of symmetric spaces by Helgason (cf. [6]) and using the correspondence of non-compact Lie algebras with Cartan involution $(\mathfrak{g}, \tau)$ to compact Lie algebras with non-trivial involution $(\mathfrak{u}, \sigma)$, we prove the important Proposition 2.14. As one corollary, we obtain all component groups $\pi_0(\mathrm{Aut}(\mathfrak{g}))$ for real central simple non-compact $\mathfrak{g}$ and have also shown the splitting of the short exact sequence (1) in some cases. The proof of the splitting for the remaining non-compact Lie algebras is then done case-by-case: Corollary 2.17 for the remaining non-compact real forms of the $\mathfrak{so}(n, \mathbb{C})$'s, Theorem 2.18 for hermitian Lie algebras with an involution ω from Kobayashi's paper [10], Theorem 2.19 for split Lie algebras and Theorem 2.20 for $\mathfrak{g} \cong \mathfrak{sp}(n, n, \mathbb{H})$. Finally, we treat real non-central simple Lie algebras $\mathfrak{g}$, using results from Djoković's paper [2], where he calculated $\pi_0(\mathrm{Aut}(\mathfrak{g}))$, and prove the splitting of the sequence (1) by applying the splitting of (1) for a real split form $\mathfrak{s}$ of $\mathfrak{g}^{\mathbb{C}}$, i.e. the Lie algebra $\mathfrak{g}$ regarded as a complex Lie algebra.

The exceptional complex simple Lie algebras are denoted by $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$, respectively. The real forms of an exceptional complex simple Lie algebra $\mathfrak{x}_r$ are denoted by $\mathfrak{x}_{r(n)}$, where $n \in \mathbb{Z}$ is the characteristic difference $\dim(\mathfrak{p}) - \dim(\mathfrak{k})$ for a Cartan decoposition $\mathfrak{x}_{r(n)} = \mathfrak{g} = \mathfrak{k} \oplus_{\kappa}^{\tau} \mathfrak{p}$. In particular, we have $n < 0$ for a compact real form $\mathfrak{x}_{r(n)}$.

1. The complex simple case

For complex simple Lie algebras $\mathfrak{g}$, the splitting of the short exact sequence (1) is a classical result (cf. f.i. Theorem 10.6.10 of [14]):

Theorem 1.1. *If $\mathfrak{g}$ is a complex simple Lie algebra, then $\pi_0(\mathrm{Aut}(\mathfrak{g}))$ is isomorphic to the symmetry group of $\mathfrak{g}$'s Dynkin diagram and there is an isomorphism $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$. The following table provides all possibilities[3] for $\pi_0(\mathrm{Aut}(\mathfrak{g}))$.*

2. The real simple case

In the real case, things are more complicated. From now on $\mathfrak{g}$ denotes a real simple Lie algebra.

Remark 2.1. It is a well-known fact (cf. Proposition X.1.5 of [6]) that each real simple Lie algebra fulfills exactly one of the two following conditions:

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[1] An $\mathbb{R}$-Lie algebra $\mathfrak{g}$ is called *central*, if $\mathrm{Cent}(\mathfrak{g}) := \{ f \in \mathrm{End}(\mathfrak{g}) \mid [f, \mathrm{ad}_x] = 0 \text{ for all } x \in \mathfrak{g}\} = \mathbb{R} \cdot \mathbf{1}$ and *non-central* otherwise.

[2] In §4 of [13], Murakami calculated the component group $\pi_0(\mathrm{Aut}(\mathfrak{g}))$ for the real forms $\mathfrak{g}$ of the $\mathfrak{su}(n+1, \mathbb{C})$'s and named typical outer automorphisms. However, he did not show that these representatives form a subgroup isomorphic to $\pi_0(\mathrm{Aut}(\mathfrak{g}))$ in $\mathrm{Aut}(\mathfrak{g})$.

[3] The Lie algebras listed in this and the next tables exhaust all possibilities up to isomorphism.

complex simple $\mathfrak{g}$	$\pi_0(\mathrm{Aut}(\mathfrak{g}))$	$\#\,\mathrm{Conj}(\pi_0(\mathrm{Aut}(\mathfrak{g})))$
$\mathfrak{sl}(n+1,\mathbb{C})$ for $n \geq 2$	$\mathcal{C}_2$	2
$\mathfrak{so}(8,\mathbb{C})$	$\mathcal{S}_3$	3
$\mathfrak{so}(2n,\mathbb{C})$ for $n \geq 5$	$\mathcal{C}_2$	2
$\mathfrak{e}_6$	$\mathcal{C}_2$	2
all others	$\mathbf{1}$	1

Table 1: Component group $\pi_0\left(\mathrm{Aut}(\mathfrak{g})\right)$ for complex simple $\mathfrak{g}$

A. $\mathfrak{g}$ admits a complex structure J and the complex Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}(J)$ is simple.[4] The complexification $\mathfrak{g}_{\mathbb{C}}$ is the direct sum of two simple isomorphic ideals, hence $\mathfrak{g}_{\mathbb{C}}$ is not a complex simple Lie algebra.

B. $\mathfrak{g}_{\mathbb{C}}$ is a complex simple Lie algebra. This is equivalent to the condition that $\mathfrak{g}$ is central simple, i.e. if $\mathrm{Cent}(\mathfrak{g}) = \mathbb{R}\cdot\mathbf{1}$.

2.1. The real central simple case. Now let $\mathfrak{g}$ be real central simple. We need the following lemma to determine $\pi_0(\mathrm{Aut}(\mathfrak{g}))$ in two distinct subcases: the compact and the non-compact case.

Lemma 2.2. *Let $\mathfrak{g}$ be a real semisimple Lie algebra of finite dimension with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus_\kappa^\tau \mathfrak{p}$ and $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ the corresponding compact real Lie algebra with involution $\sigma = \mathrm{id}_{\mathfrak{k}} \oplus -\mathrm{id}_{i\mathfrak{p}}$. Then we have the following:* [5]

1. *The map $B_\tau : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$, $(x,y) \mapsto -\kappa(x,\tau y)$ is a euclidean scalar product. Note that $B_\tau = -\kappa$ for compact $\mathfrak{g}$.*

2. *The B_τ-transposition operator $\mathrm{End}(\mathfrak{g}) \to \mathrm{End}(\mathfrak{g})$, $f \mapsto f^T$ defined by $B_\tau\left(f^T x, y\right) := B_\tau\left(x, fy\right)$ leaves $\mathrm{Aut}(\mathfrak{g})$ invariant. More precisely, $f^T = \tau f^{-1}\tau$ for all $f \in \mathrm{Aut}(\mathfrak{g})$. The subgroup $\mathrm{Aut}(\mathfrak{g})^\tau = \{f \in \mathrm{Aut}(\mathfrak{g}) \mid \tau f \tau = f\}$ is equal to $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{O}(\mathfrak{g}, B_\tau) = \left\{f \in \mathrm{Aut}(\mathfrak{g}) \mid f^{-1} = f^T\right\}$ and smoothly isomorphic to $\mathrm{Aut}(\mathfrak{u})^\sigma$.*

3. *The space $\mathrm{ad}(\mathfrak{p}) = \{\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g} \mid x \in \mathfrak{p}\}$ is included in $\mathrm{Sym}(\mathfrak{g}, B_\tau) = \left\{f \in \mathrm{End}(\mathfrak{g}) \mid f = f^T\right\}$ and the space $\mathrm{ad}(\mathfrak{k}) = \{\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g} \mid x \in \mathfrak{k}\}$ is included in $\mathbf{L}(\mathrm{O}(\mathfrak{g}, B_\tau)) = \left\{f \in \mathrm{End}(\mathfrak{g}) \mid f = -f^T\right\}$.*

Proof. 1. + 3. These are the statements of Lemma 12.1.3 of [7].

2. Let $x, y \in \mathfrak{g}$ and $f \in \mathrm{Aut}(\mathfrak{g})$. Then:

$$\begin{aligned} B_\tau(f^T x, y) &= B_\tau(x, fy) = -\kappa(x, \tau f \tau\tau(y)) = -\kappa\left(\tau f^{-1}\tau(x), \tau y\right) \\ &= B_\tau\left(\tau f^{-1}\tau(x), y\right). \end{aligned}$$

[4]If $\mathfrak{g}$ is real simple with a complex structure J, then the only other complex structure is $-J$ and the corresponding simple complex Lie algebras $\mathfrak{g}^{\mathbb{C}}(J)$ and $\mathfrak{g}^{\mathbb{C}}(-J)$ are isomorphic.

[5]The statements are also true in the rather trivial case of $\mathfrak{g}$ being compact. Then $\mathfrak{g}$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{g} \oplus_\kappa^{\mathbf{1}} \mathbf{0}$ and $(\mathfrak{u}, \sigma) = (\mathfrak{g}, \mathbf{1})$.

Furthermore, we have: $f \in \mathrm{Aut}(\mathfrak{g})^\tau \iff \tau f^{-1}\tau = f^{-1} \iff f^T = f^{-1}$.
Now let $f \in \mathrm{Aut}(\mathfrak{g})^\tau$, $k \in \mathfrak{k}$ and $p \in \mathfrak{p}$. Then f preserves the Cartan decomposition:

$$\begin{aligned} f(k) &= \tau f \tau(k) = \tau(f(k)) \implies f(k) \in \mathfrak{g}^\tau = \mathfrak{k}, \\ f(p) &= \tau f \tau(p) = -\tau(f(p)) \implies f(p) \in \mathfrak{g}^{-\tau} = \mathfrak{p}. \end{aligned}$$

So we may define $\eta : \mathrm{Aut}(\mathfrak{g})^\tau \to \mathrm{Aut}(\mathfrak{u})$ by $\eta(f)(k+ip) := f(k) + if(p)$. Its image is in $\mathrm{Aut}(\mathfrak{u})^\sigma$:

$$\begin{aligned} \sigma\eta(f)\sigma(k+ip) &= \sigma\eta(f)(k-ip) = \sigma(f(k) - if(p)) = f(k) + if(p) \\ &= \eta(f)(k+ip). \end{aligned}$$

The map $\eta : \mathrm{Aut}(\mathfrak{g})^\tau \to \mathrm{Aut}(\mathfrak{u})^\sigma$ is a smooth group morphism and has an inverse defined by $\eta^{-1}(g)(k+ip) = g(k) + ig(p)$, thus is an isomorphism of Lie groups.

■

Remark 2.3. By a classical result (cf. Proposition 1.122 of [9]), for any algebraic subgroup $H' \leq \mathrm{GL}(N,\mathbb{R})$ with $(H')^T = H'$, the map $(H' \cap \mathrm{O}(N,\mathbb{R})) \times (\mathbf{L}H' \cap \mathrm{Sym}(N,\mathbb{R})) \to H'$, $(k',x') \mapsto k'e^{x'}$ is a diffeomorphism. We can transfer this to the case of a euclidean scalar product B on a finite-dimensional vector space V and an algebraic subgroup $H \leq \mathrm{GL}(V)$ stable under B-transpositon and obtain a diffeomorphism $(H \cap \mathrm{O}(V,B)) \times (\mathbf{L}H \cap \mathrm{Sym}(V,B)) \to H$, $(k,x) \mapsto ke^x$.

If $\mathfrak{g}$ is a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus_\kappa^\tau \mathfrak{p}$, we have a euclidean scalar product B_τ on $\mathfrak{g}$ such that $\mathrm{Aut}(\mathfrak{g}) \leq \mathrm{GL}(\mathfrak{g})$ is stable under B_τ-transposition (cf. Lemma 2.2), thus we obtain, for algebraic and B_τ-transposition stable subgroup $H \leq \mathrm{Aut}(\mathfrak{g})$, the diffeomorphism

$$\begin{aligned} \Phi : (H \cap \mathrm{O}(\mathfrak{g}, B_\tau)) \times (\mathbf{L}(H) \cap \mathrm{Sym}(\mathfrak{g}, B_\tau)) &\longrightarrow H, \\ (k,x) &\longmapsto k \exp_{\mathrm{Aut}(\mathfrak{g})}(x). \end{aligned}$$

Hence we obtain $H_0 \cap \mathrm{O}(\mathfrak{g}, B_\tau) = (H \cap \mathrm{O}(\mathfrak{g}, B_\tau))_0$ and, as an application of the Second Isomorphism Theorem, a group isomorphism

$$\begin{aligned} \pi_0\left(H \cap \mathrm{O}(\mathfrak{g}, B_\tau)\right) &\longrightarrow H_0 \cdot \left(H \cap \mathrm{O}(\mathfrak{g}, B_\tau)\right)/H_0 = H/H_0 = \pi_0(H), \\ h \cdot \left(H \cap \mathrm{O}(\mathfrak{g}, B_\tau)\right)_0 &\longmapsto h \cdot H_0. \end{aligned}$$

The complexification of a real simple compact Lie algebra is always a complex simple Lie algebra, so any real simple compact Lie algebra is central simple. We want to show that $\pi_0(\mathrm{Aut}(\mathfrak{g}))$ is then isomorphic to $\pi_0(\mathrm{Aut}(\mathfrak{g}_\mathbb{C}))$.

Theorem 2.4. *If $\mathfrak{g}$ is a real simple compact Lie algebra, then there exists a group isomorphism $\pi_0(\mathrm{Aut}(\mathfrak{g})) \cong \pi_0(\mathrm{Aut}(\mathfrak{g}_\mathbb{C}))$. Since we know the latter group by Theorem 1.1, we can list all possibilities for $\pi_0(\mathrm{Aut}(\mathfrak{g}))$ in the following table.*

Furthemore, the embedding $\mathrm{Aut}(\mathfrak{g}) \hookrightarrow \mathrm{Aut}(\mathfrak{g}_\mathbb{C}, \mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g}_\mathbb{C})^\sigma$, where $\sigma\colon \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ is the conjugation with respect to $\mathfrak{g}$ (cf. Proposition 2.1 of [2]), leads to an isomorphism $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$ (cf. Theorem 6.61.(vi) of [8]).

real simple compact $\mathfrak{g}$	$\pi_0(\mathrm{Aut}(\mathfrak{g}))$	$\#\,\mathrm{Conj}(\pi_0(\mathrm{Aut}(\mathfrak{g})))$
$\mathfrak{su}(n+1,\mathbb{C})$ for $n \geq 2$	$\mathcal{C}_2$	2
$\mathfrak{so}(8,\mathbb{R})$	$\mathcal{S}_3$	3
$\mathfrak{so}(2n,\mathbb{R})$ for $n \geq 5$	$\mathcal{C}_2$	2
$\mathfrak{e}_{6(-78)}$	$\mathcal{C}_2$	2
all others	$\mathbf{1}$	1

Table 2: Component group $\pi_0\,(\mathrm{Aut}(\mathfrak{g}))$ for real simple compact $\mathfrak{g}$

Proof. A Cartan decomposition of the real semisimple Lie algebra $(\mathfrak{g}_{\mathbb{C}})^{\mathbb{R}}$ is $\mathfrak{g} \oplus i\mathfrak{g}$ with the involution $\tau : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$, $x+iy \mapsto x-iy$. The following calculation for $f \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ and $y \in \mathfrak{g}$ shows the B_τ-stability of $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$:

$$f^T(iy) = \tau f^{-1}\tau(iy) = \tau f^{-1}(-iy) = -\tau i f^{-1}(y) = i\tau f^{-1}(y) = i\tau f^{-1}\tau(y) = if^T(y).$$

We apply Remark 2.3 to the algebraic subgroup $H = \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}) \leq \mathrm{Aut}\left((\mathfrak{g}_{\mathbb{C}})^{\mathbb{R}}\right)$, obtaining the isomorphism $\pi_0(\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})) \cong \pi_0\,(\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}) \cap \mathrm{O}(\mathfrak{g}_{\mathbb{C}}, B_\tau))$. By

$$\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}) \cap \mathrm{O}(\mathfrak{g}_{\mathbb{C}}, B_\tau) = \{\, f \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}) |\, \tau f \tau = f\},$$

there is a smooth isomorphism of this group and $\mathrm{Aut}(\mathfrak{g})$ by $f \mapsto f_{|\mathfrak{g}}$, so $\pi_0(\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})) \cong \pi_0(\mathrm{Aut}(\mathfrak{g}))$. ■

We need the following statements for the calculation of $\pi_0\,(\mathrm{Aut}(\mathfrak{g}))$ for real central simple non-compact $\mathfrak{g}$. The first four lemmas are due to Groß and Heintze (cf. Appendix A of [4]), the fourth is Theorem IX.5.6 of [6]. Lemma 2.11[6] is proven analogously to Proposition 1.1.9 of [7].

Lemma 2.5. *Let $\mathfrak{u}$ be a real simple compact Lie algebra and $\sigma \in \mathrm{Aut}(\mathfrak{u})$ a non-trivial involution. Then the group morphism $\omega : \pi_0\,(\mathrm{Aut}(\mathfrak{u})^\sigma) \to \pi_0\,(\mathrm{Aut}(\mathfrak{u}^\sigma))$ induced by restriction is injective for inner $\sigma \in \mathrm{Aut}(\mathfrak{u})$ and $\ker(\omega) = \{[\mathrm{id}_{\mathfrak{u}}], [\sigma]\}$ for outer $\sigma \in \mathrm{Aut}(\mathfrak{u})$.*

Lemma 2.6. *The so-called* triality automorphism *$\theta \in \mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))$ is outer, commutes with $\mathrm{Ad}\begin{pmatrix} -\mathbf{1}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_4 \end{pmatrix}$ and is of order 3.*

Remark 2.7. We want to give an explicit description of (one realization of) the triality automorphism:

For the conjugation $\tau : \mathfrak{so}(8,\mathbb{C}) = \mathfrak{so}(8,\mathbb{R}) \oplus i\mathfrak{so}(8,\mathbb{R}) \to \mathfrak{so}(8,\mathbb{C})$, $x+iy \mapsto x-iy$, the map $\mathrm{Aut}(\mathfrak{so}(8,\mathbb{C}))^\tau \to \mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))$, $f \mapsto f_{|\mathfrak{so}(8,\mathbb{R})}$ is an isomorphism of Lie groups. We define an automorphism by means of the standard Chevalley generators[7], i.e. let $\mathfrak{h} := \mathbb{C}h_1 \oplus \mathbb{C}h_2 \oplus \mathbb{C}h_3 \oplus \mathbb{C}h_4$ for $h_1 := E_{12} - E_{21}$, $h_2 := E_{34} - E_{43}$, $h_3 := E_{56} - E_{65}$, $h_4 := E_{78} - E_{87}$ be a Cartan subalgebra in $\mathfrak{so}(8,\mathbb{C})$, where the $E_{ij} - E_{ij}$'s denote the elementary skew-symmetric 8×8-matrices. With the dual elements $h_j^* : \mathfrak{h} \to \mathbb{C}$, $h_i \mapsto \delta_{ij}$ there is a root basis

$$\Pi := \Pi(\mathfrak{so}(8,\mathbb{C}),\mathfrak{h}) := \{\alpha_1 := h_1^* - h_2^*, \alpha_2 := h_2^* - h_3^*, \alpha_3 := h_3^* - h_4^*, \alpha_4 := h_3^* + h_4^*\}$$

[6]Note that this lemma is proven for both, real and complex matrices. It is a generalization of Schur's Lemma applied to the canonical action of $\mathfrak{so}(n,\mathbb{C})$.

[7]By f.i. Theorem 3.2.1 of [3], an automorphism of a simple complex Lie algebra is uniquely determined by its values on Chevalley generators.

with corresponding root spaces $\mathbb{C}x_i = \mathfrak{so}(8,\mathbb{C})^{\alpha_i}$, $\mathbb{C}y_i = \mathfrak{so}(8,\mathbb{C})^{-\alpha_i}$ for the matrices $x_1 := \begin{pmatrix} \mathbf{0} & A & \mathbf{0} & \mathbf{0} \\ -A^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$, $x_2 := \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A & \mathbf{0} \\ \mathbf{0} & -A^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$, $x_3 := \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A \\ \mathbf{0} & \mathbf{0} & -A^T & \mathbf{0} \end{pmatrix}$, $x_4 := \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & B \\ \mathbf{0} & \mathbf{0} & -B^T & \mathbf{0} \end{pmatrix}$, where $A := \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$, $B := \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $y_i := \tau(x_i)$ for $i = 1,2,3,4$. The triality automorphism corresponds to a cyclic permutation of the simple roots $\alpha_1, \alpha_4, \alpha_3$, which leads to the following definition of $\theta' \in \mathrm{Aut}(\mathfrak{so}(8,\mathbb{C}))$: Let $T := \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$ be the map matrix of $\theta'_{|\mathfrak{h}}$ with respect to the basis (h_1, h_2, h_3, h_4) and $\theta(x_1) := x_3$, $\theta(x_2) := x_2$, $\theta(x_3) := x_4$, $\theta(x_4) := x_1$, $\theta(y_1) := y_3$, $\theta(y_2) := y_2$, $\theta(y_3) := y_4$, $\theta(y_4) := y_1$. The entries of T and of the h_i's are real and for $z \in \mathbb{C}$, $i = 1,2,3,4$ we have:

$$\tau\theta'\tau(zx_i) = \tau(\overline{z}\theta'(\tau(x_i))) = z\tau(\theta'(y_i)) = z\tau(x_i) = zy_i = \theta'(zx_i).$$

Analogously, we have $\tau\theta'\tau(zy_i) = \theta'(zy_i)$ for $z \in \mathbb{C}$, $i = 1,2,3,4$. So $\theta' \in \mathrm{Aut}(\mathfrak{so}(8,\mathbb{C}))^\tau$. Set $\theta := \theta'_{|\mathfrak{so}(8,\mathbb{R})}$.

Lemma 2.8. *The four conjugacy classes of involutions of the real simple compact Lie algebra $\mathfrak{e}_{6(-78)}$ are represented by commuting elements. More precisely, if we fix a maximal torus $\mathfrak{t} \leq \mathfrak{e}_{6(-78)}$ and ρ_1 is the corresponding Dynkin diagram involution, then there is an $X \in \mathfrak{t}$ such that $\rho_2 := \rho_1 e^X$ is a representative for the second conjugacy class of outer involutions. The two conjugacy class of inner involutions are represented by $\rho_3 := e^Y$ and $\rho_4 := e^Z$ for some elements $Y, Z \in \mathfrak{t}^{\rho_1}$.*

Lemma 2.9. *The non-trivial element of $\pi_0\left(\mathrm{Aut}(\mathfrak{e}_{7(-133)})^{\sigma_{\mathfrak{e}_{7(7)}}}\right)$ is represented by $e^{\mathrm{ad}(X)}$ for any non-zero element $X \in \mathfrak{e}_{7(-133)}^{-\sigma_{\mathfrak{e}_{7(7)}}}$ such that* $\mathrm{ad}(X)^3 = -\pi^2\,\mathrm{ad}(X)$.

Lemma 2.10. *Let $\mathfrak{u}$ be a real simple compact Lie algebra and $\sigma \in \mathrm{Aut}(\mathfrak{u})$ a non-trivial involution. Then σ is inner if and only if the ranks of $\mathfrak{u}^\sigma$ and $\mathfrak{u}$ conincide. Here, the rank of a real compact Lie algebra is the dimension of any maximal abelian subalgebra or, equivalently, of any Cartan subalgebra.*

Lemma 2.11. *For $n \geq 3$ a matrix $X \in \mathrm{M}(n,\mathbb{K})$ commutes with all elementary skew-symmetric matrices $E_{rs} - E_{sr}$, $1 \leq r < s \leq n$, if and only if $X \in \mathbb{K}\mathbf{1}_n$.*

Proof. Let $\mathcal{ESS}(m) := \{E_{rs} - E_{sr} \,|\, 1 \leq r < s \leq m\}$ be the set of elementary skew-symmetric $m \times m$-matrices for $m \in \mathbb{N}$. For $\ell = 1,\ldots,n$ there is an embedding

$$j_\ell : \mathcal{ESS}(n-1) \longrightarrow \mathcal{ESS}(n)$$

$$y=(y_{i,j}) \longmapsto \begin{pmatrix} y_{1,1} & \cdots & y_{1,\ell-1} & 0 & y_{1,\ell} & \cdots & y_{1,n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{\ell-1,1} & \cdots & y_{\ell-1,\ell-1} & \vdots & y_{\ell-1,\ell} & \cdots & y_{\ell-1,n-1} \\ 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\ y_{\ell,1} & \cdots & y_{\ell,\ell-1} & \vdots & y_{\ell,\ell} & \cdots & y_{\ell,n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{n-1,1} & \cdots & y_{n-1,\ell-1} & 0 & y_{n-1,\ell} & \cdots & y_{n-1,n-1} \end{pmatrix}.$$

Note that $\bigcap_{y\in\mathcal{ESS}(n-1)} \{v \in \mathbb{K}^n | \, j_\ell(y)\cdot v = 0\} = \mathbb{K}e_\ell$ for $(e_1,\ldots,e_n)$, the canonical basis of $\mathbb{K}^n$. So the condition $xX = Xx$ for all $x \in \mathcal{ESS}(n)$ leads to $j_\ell(y)Xe_\ell = Xj_\ell(y)e_\ell = 0$ for all $y \in \mathcal{ESS}(n-1)$, yielding $X(\mathbb{K}e_\ell) \subseteq \mathbb{K}e_\ell$. Thus X is diagonal. Furthermore, commuting with $\mathcal{ESS}(n)$, the matrix X does not have distinct eigenvalues. Thus $X \in \mathbb{K}\mathbf{1}_n$. The other implication is trivial. ■

Corollary 2.12. *If $q \geq 2$ and $X \in \mathrm{O}(2q,\mathbb{R})$ with $\det(X) = -1$, then $\mathrm{Ad}(X) \in \mathrm{Aut}(\mathfrak{so}(2q,\mathbb{R}))$ is outer. If $q \geq 5$, then, since $\pi_0(\mathrm{Aut}(\mathfrak{so}(2q,\mathbb{R}))) \cong \mathcal{C}_2$ by Theorem 2.4, the map $\mathrm{Ad} : \mathrm{O}(2q,\mathbb{R}) \to \mathrm{Aut}(\mathfrak{so}(2q,\mathbb{R}))$ is surjective.*

Proof. The matrix $I := I_{1,2q-1} := \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{2q-1}\end{pmatrix}$ is in $\mathrm{O}(2q,\mathbb{R})\setminus \mathrm{SO}(2q,\mathbb{R})$. Furthermore, $\mathrm{Ad}(I) \neq \mathrm{Ad}(Q)$ as maps on $\mathfrak{so}(2q,\mathbb{R})$ for all $Q \in \mathrm{SO}(2q,\mathbb{R})$, since else, by Lemma 2.11 in the case $\mathbb{K} = \mathbb{R}$, there would exist $r \in \mathbb{R}^\times$ such that $Q^{-1}I = r\mathbf{1}_{2q}$ and $\det(I) = r^{2q}\det(Q) = r^{2q} \neq -1 = \det(I)$. Hence, the image of the map $\mathrm{Ad} : \mathrm{O}(2q,\mathbb{R}) \to \mathrm{Aut}(\mathfrak{so}(2q,\mathbb{R}))$ is strictly larger than $\mathrm{Ad}(\mathrm{SO}(2q,\mathbb{R})) = \mathrm{Aut}(\mathfrak{so}(2q,\mathbb{R}))_0$. We know that $[\mathrm{Aut}(\mathfrak{so}(2q,\mathbb{R})) : \mathrm{Aut}(\mathfrak{so}(2q,\mathbb{R}))_0] = 2$ by Theorem 2.4, hence the surjectivity of Ad follows. ■

Let $\mathfrak{g}$ be real simple non-compact with Cartan involution τ and $\mathfrak{k} = \mathfrak{g}^\tau$, $\mathfrak{p} = \mathfrak{g}^{-\tau}$, $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$, $\sigma = \mathrm{id}_\mathfrak{k} \oplus -\mathrm{id}_{i\mathfrak{p}}$. We will calculate $\pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma)$ and see that it is isomorphic to $\pi_0(\mathrm{Aut}(\mathfrak{g}))$ by using the classification of simply connected symmetric spaces.

Remark 2.13. $\mathrm{Aut}(\mathfrak{u})$ preserves the negative definite Cartan-Killing form $\kappa_\mathfrak{u}$, so it is compact and so are $U_{\mathrm{ad}} := \mathrm{Inn}(\mathfrak{u}) = \mathrm{Aut}(\mathfrak{u})_0$ and $U := \widetilde{U_{\mathrm{ad}}}$ because of Weyl's Theorem (cf. Theorem 11.1.17 of [7]) and the (semi)simplicity of $\mathrm{Der}(\mathfrak{u}) = \mathrm{ad}(\mathfrak{u}) \cong \mathfrak{u}$. We identify $\mathfrak{u}$ with $\mathrm{ad}(\mathfrak{u})$ and in this sense and since the exponential map of any connected Lie group with compact Lie algebra is surjective (cf. Proposition II.6.10 of [6]), the map $\mathrm{Ad}_U : U \to U_{\mathrm{ad}}$, $\mathrm{Ad}_U(\exp_U(y))(x) := e^{\mathrm{ad}(y)}(x)$ is the universal cover of U_{ad}.

Let $\overline{\sigma} : U \to U$ be the unique Lie group morphism such that $\mathbf{L}\overline{\sigma} = \sigma$. The compact subgroup $K := U^{\overline{\sigma}} = \{g \in U | \, \overline{\sigma}(g) = g\}$ is connected with Lie algebra $\mathbf{L}K = \mathfrak{u}^\sigma = \mathfrak{k}$ and the homogeneous space $M := U/K$ is a simply connected compact Riemannian symmetric space (cf. Theorem VII.8.2 of [6]) with compact group of displacements $U_{\mathrm{dis}} := U/\Gamma$ for $\Gamma := \mathrm{Z}(U) \cap K \trianglelefteq \mathrm{Z}(U)$, the maximal normal subgroup contained in K. So $\mathrm{Z}(U_{\mathrm{dis}}) \cong \mathrm{Z}(U)/\Gamma$. Furthermore, U_{dis} is, by Proposition IV.1.7 of [12], isomorphic to the connected component of the isometry group of M and there is also an isomorphism of symmetric spaces $M \cong U_{\mathrm{dis}}/K'$, where $(U_{\mathrm{dis}}^{\varsigma})_0 \subseteq K' \subseteq U_{\mathrm{dis}}^{\varsigma}$ for an involution $\varsigma \in \mathrm{Aut}(U_{\mathrm{dis}})$ such that $\mathbf{L}\varsigma = \mathbf{L}\overline{\sigma}$. The homotopy sequence $\mathbf{1} = \pi_1(M) \to \pi_0(K_2) \to \pi_0(U_{\mathrm{dis}}) = \mathbf{1}$ is exact, so $K' = (U_{\mathrm{dis}}^{\varsigma})_0$ and $\mathbf{L}K = \mathbf{L}K'$.

Let $\rho_2 : U \to U_{\mathrm{dis}}$ be the universal covering morphism of U_{dis}. Since $\Gamma = \ker(\rho_2) \subseteq \ker(\mathrm{Ad}_U) = \mathrm{Z}(U)$, the map $\mathrm{Ad}_{U_{\mathrm{dis}}} : U_{\mathrm{dis}} \to U_{\mathrm{ad}}$ is the unique morphism such that the diagram

$$\begin{tikzcd} U \arrow[rr, "\mathrm{Ad}_U"] \arrow[dr, "\rho_2"'] & & U_{\mathrm{ad}} \\ & U_{\mathrm{dis}} \arrow[ur, "\mathrm{Ad}_{U_{\mathrm{dis}}}"'] & \end{tikzcd}$$

commutes, $\ker(\mathrm{Ad}_{U_{\mathrm{dis}}}) = \mathrm{Z}(U_{\mathrm{dis}}) \cong \mathrm{Z}(U)/\Gamma = \mathrm{Z}(U)/\left(\mathrm{Z}(U) \cap K\right)$ and $\mathrm{Ad}_{U_{\mathrm{dis}}}$ is a covering morphism, implying $\mathrm{Ad}_{U_{\mathrm{dis}}} \circ \exp_{U_{\mathrm{dis}}} = \exp_{U_{\mathrm{ad}}}$.

Tables 3 and 4 provide all possibilities for $\mathfrak{g}$, $\mathfrak{k}$ and $(\mathfrak{u}, \sigma)$ except for isomorphy of the Lie algebras and except for conjugation by automorphisms of the involutions. Furthermore, we make use of Loos' classification symbol X_r^*, where X_r is the root system of the corresponding complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_{\mathbb{C}}$ and we list the groups U, U_{dis} with corresponding centers (cf. Chapter VII of [12] and Table 4 of [3]) and $\mathrm{Z}(M)$, the center of M, an abelian discrete group acting on M (cf. Exercise X.C.4 of [6]: The group $\pi_1(\mathrm{Inn}(\mathfrak{u})/\mathrm{Inn}(\mathfrak{u})_\theta)$ mentioned there is equal to $\mathrm{Z}(M)$ by Proposition III.2.4 and the corollary of Theorem VI.3.6 of [12]).

We also use the following symbols: cj is the complex conjugation, qj is the quarternionic conjugation and we have the matrices $I_{p,q} := \begin{pmatrix} -\mathbf{1}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q \end{pmatrix}$ and $J_n := \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix}$. If $\mathfrak{g}$ is exceptional, then we write $\sigma_{\mathfrak{g}}$ for the unique involution of $\mathfrak{u}$ such that $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$, $x + iy \mapsto \sigma_{\mathfrak{g}}(x) + i\sigma_{\mathfrak{g}}(y)$ fixes $\mathfrak{g}$ pointwise, i.e. it is the complex conjugation with respect to the real form $\mathfrak{g}$. The unique simply connected real simple compact Lie groups with exceptional compact Lie algebras $\mathfrak{e}_{6(-78)}$, $\mathfrak{e}_{7(-133)}$, $\mathfrak{e}_{8(-248)}$, $\mathfrak{f}_{4(-52)}$, $\mathfrak{g}_{2(-14)}$ are denoted by $E_{6(-78)}$, $E_{7(-133)}$, $E_{8(-248)}$, $F_{4(-52)}$, $G_{2(-14)}$, respectively.

X_r^*	$\mathfrak{g}$	$\mathfrak{k}$	$(\mathfrak{u},\sigma)$
$A_n^{\mathbb{R}}$, $n \geq 1$	$\mathfrak{sl}(n+1,\mathbb{R})$	$\mathfrak{so}(n+1,\mathbb{R})$	$(\mathfrak{su}(n+1,\mathbb{C}),\mathrm{cj})$
$A_n^{\mathbb{C},q}$, $q \leq \frac{n+1}{2} > 1$	$\mathfrak{su}(q,n+1-q,\mathbb{C})$	$\mathfrak{su}(q,\mathbb{C}) \oplus \mathfrak{u}(n+1-q,\mathbb{C})$	$(\mathfrak{su}(n+1,\mathbb{C}),\mathrm{Ad}(I_{n+1-q,q}))$
$A_{2n+1}^{\mathbb{H}}$, $n \geq 1$	$\mathfrak{sl}(n+1,\mathbb{H}) \cong \mathfrak{su}^*(2n+2,\mathbb{C})$	$\mathfrak{sp}(n+1,\mathbb{H})$	$(\mathfrak{su}(2n+2,\mathbb{C}),\mathrm{cj}\circ\mathrm{Ad}(J_{n+1}))$
$B_n^{\mathbb{R},q}$, $q \leq n \geq 2$	$\mathfrak{so}(q,2n+1-q,\mathbb{R})$	$\mathfrak{so}(q,\mathbb{R}) \oplus \mathfrak{so}(2n+1-q,\mathbb{R})$	$(\mathfrak{so}(2n+1,\mathbb{R}),\mathrm{Ad}(I_{2n+1-q,q}))$
$C_n^{\mathbb{R}}$, $n \geq 3$	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{u}(n,\mathbb{C})$	$(\mathfrak{sp}(n,\mathbb{H}),\mathrm{qj})$
$C_n^{\mathbb{H},q}$, $q \leq \frac{n}{2} > 1$	$\mathfrak{sp}(q,n-q,\mathbb{H})$	$\mathfrak{sp}(q,\mathbb{H}) \oplus \mathfrak{sp}(n-q,\mathbb{H})$	$(\mathfrak{sp}(n,\mathbb{H}),\mathrm{Ad}(I_{n-q,q}))$
$D_n^{\mathbb{R},q}$, $q \leq n \geq 4$	$\mathfrak{so}(q,2n-q,\mathbb{R})$	$\mathfrak{so}(q,\mathbb{R}) \oplus \mathfrak{so}(2n-q,\mathbb{R})$	$(\mathfrak{so}(2n,\mathbb{R}),\mathrm{Ad}(I_{2n-q,q}))$
$D_n^{\mathbb{H}}$, $n \geq 5$	$\mathfrak{so}^*(2n,\mathbb{R})$	$\mathfrak{u}(n,\mathbb{C})$	$(\mathfrak{so}(2n,\mathbb{R}),\mathrm{Ad}(J_n))$
$E_{6(6)}$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(4,\mathbb{H})$	$(\mathfrak{e}_{6(-78)},\sigma_{\mathfrak{e}_{6(6)}})$
$E_{6(2)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{su}(6,\mathbb{C})$	$(\mathfrak{e}_{6(-78)},\sigma_{\mathfrak{e}_{6(2)}})$
$E_{6(-14)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10,\mathbb{C}) \oplus \mathbb{R}$	$(\mathfrak{e}_{6(-78)},\sigma_{\mathfrak{e}_{6(-14)}})$
$E_{6(-26)}$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-52)}$	$(\mathfrak{e}_{6(-78)},\sigma_{\mathfrak{e}_{6(-26)}})$
$E_{7(7)}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(8,\mathbb{C})$	$(\mathfrak{e}_{7(-133)},\sigma_{\mathfrak{e}_{7(7)}})$
$E_{7(-5)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{so}(12,\mathbb{R})$	$(\mathfrak{e}_{7(-133)},\sigma_{\mathfrak{e}_{7(-5)}})$
$E_{7(-25)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-78)} \oplus \mathbb{R}$	$(\mathfrak{e}_{7(-133)},\sigma_{\mathfrak{e}_{7(-25)}})$
$E_{8(8)}$	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16,\mathbb{R})$	$(\mathfrak{e}_{8(-248)},\sigma_{\mathfrak{e}_{8(8)}})$
$E_{8(-24)}$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{e}_{7(-133)}$	$(\mathfrak{e}_{8(-248)},\sigma_{\mathfrak{e}_{8(-24)}})$
$F_{4(4)}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{sp}(3,\mathbb{H})$	$(\mathfrak{f}_{4(-52)},\sigma_{\mathfrak{f}_{4(4)}})$
$F_{4(-20)}$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9,\mathbb{R})$	$(\mathfrak{f}_{4(-52)},\sigma_{\mathfrak{f}_{4(-20)}})$
$G_{2(2)}$	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{su}(2,\mathbb{C})$	$(\mathfrak{g}_{2(-14)},\sigma_{\mathfrak{g}_{2(2)}})$

Table 3: Classification of real simple non-compact Lie algebras by symmetric spaces

X_r^*	U	$\mathrm{Z}(U)$	U_{dis}	$\mathrm{Z}(U_{\mathrm{dis}})$	$\mathrm{Z}(M)$
$A_n^{\mathbb{R}}$, odd $n \geq 1$	$\mathrm{SU}(n+1,\mathbb{C})$	$\mathcal{C}_{n+1}$	$\mathrm{SU}(n+1,\mathbb{C})/\mathcal{C}_2\mathbf{1}$	$\mathcal{C}_{\frac{n+1}{2}}$	$\mathcal{C}_{n+1}$
$A_n^{\mathbb{R}}$, even $n \geq 2$	$\mathrm{SU}(n+1,\mathbb{C})$	$\mathcal{C}_{n+1}$	$\mathrm{SU}(n+1,\mathbb{C})$	$\mathcal{C}_{n+1}$	$\mathcal{C}_{n+1}$
$A_n^{\mathbb{C},q}$, $q < \frac{n+1}{2} > 1$	$\mathrm{SU}(n+1,\mathbb{C})$	$\mathcal{C}_{n+1}$	$\mathrm{SU}(n+1,\mathbb{C})/\mathcal{C}_{n+1}\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$A_{2q-1}^{\mathbb{C},q}$, $q > 1$	$\mathrm{SU}(2q,\mathbb{C})$	$\mathcal{C}_{2q}$	$\mathrm{SU}(2q,\mathbb{C})/\mathcal{C}_{2q}\mathbf{1}$	$\mathbf{1}$	$\mathcal{C}_2$
$A_{2n+1}^{\mathbb{H}}$, $n \geq 1$	$\mathrm{SU}(2n+2,\mathbb{C})$	$\mathcal{C}_{2n+2}$	$\mathrm{SU}(2n+2,\mathbb{C})/\mathcal{C}_2\mathbf{1}$	$\mathcal{C}_{n+1}$	$\mathcal{C}_{n+1}$
$B_n^{\mathbb{R},q}$, $q \leq n \geq 2$	$\mathrm{Spin}(2n+1,\mathbb{R})$	$\mathcal{C}_2$	$\mathrm{SO}(2n+1,\mathbb{R})$	$\mathbf{1}$	$\mathcal{C}_2$
$C_n^{\mathbb{R}}$, $n \geq 3$	$\mathrm{Sp}(n,\mathbb{H})$	$\mathcal{C}_2$	$\mathrm{Sp}(n,\mathbb{H})/\mathcal{C}_2\mathbf{1}$	$\mathbf{1}$	$\mathcal{C}_2$
$C_n^{\mathbb{H},q}$, $q < \frac{n}{2} > 1$	$\mathrm{Sp}(n,\mathbb{H})$	$\mathcal{C}_2$	$\mathrm{Sp}(n,\mathbb{H})/\mathcal{C}_2\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$C_{2q}^{\mathbb{H},q}$, $q \geq 2$	$\mathrm{Sp}(2q,\mathbb{H})$	$\mathcal{C}_2$	$\mathrm{Sp}(2q,\mathbb{H})/\mathcal{C}_2\mathbf{1}$	$\mathbf{1}$	$\mathcal{C}_2$
$D_n^{\mathbb{R},q}$, even $n \geq 4$, odd $q < n$	$\mathrm{Spin}(2n,\mathbb{R})$	$\mathcal{C}_2 \times \mathcal{C}_2$	$\mathrm{SO}(2n,\mathbb{R})$	$\mathcal{C}_2$	$\mathcal{C}_2$
$D_n^{\mathbb{R},q}$, even $n \geq 4$, even $q < n$	$\mathrm{Spin}(2n,\mathbb{R})$	$\mathcal{C}_2 \times \mathcal{C}_2$	$\mathrm{SO}(2n,\mathbb{R})/\mathcal{C}_2\mathbf{1}$	$\mathbf{1}$	$\mathcal{C}_2$
$D_q^{\mathbb{R},q}$, even $q \geq 4$	$\mathrm{Spin}(2q,\mathbb{R})$	$\mathcal{C}_2 \times \mathcal{C}_2$	$\mathrm{SO}(2q,\mathbb{R})/\mathcal{C}_2\mathbf{1}$	$\mathbf{1}$	$\mathcal{C}_2 \times \mathcal{C}_2$
$D_n^{\mathbb{R},q}$, odd $n \geq 5$, odd $q < n$	$\mathrm{Spin}(2n,\mathbb{R})$	$\mathcal{C}_4$	$\mathrm{SO}(2n,\mathbb{R})$	$\mathcal{C}_2$	$\mathcal{C}_2$
$D_q^{\mathbb{R},q}$, odd $q \geq 5$	$\mathrm{Spin}(2q,\mathbb{R})$	$\mathcal{C}_4$	$\mathrm{SO}(2q,\mathbb{R})$	$\mathcal{C}_2$	$\mathcal{C}_4$
$D_n^{\mathbb{R},q}$, odd $n \geq 5$, even $q < n$	$\mathrm{Spin}(2n,\mathbb{R})$	$\mathcal{C}_4$	$\mathrm{SO}(2n,\mathbb{R})/\mathcal{C}_2\mathbf{1}$	$\mathbf{1}$	$\mathcal{C}_2$
$D_n^{\mathbb{H}}$, odd $n \geq 5$	$\mathrm{Spin}(2n,\mathbb{R})$	$\mathcal{C}_4$	$\mathrm{SO}(2n,\mathbb{R})/\mathcal{C}_2\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$D_n^{\mathbb{H}}$, even $n \geq 6$	$\mathrm{Spin}(2n,\mathbb{R})$	$\mathcal{C}_2 \times \mathcal{C}_2$	$\mathrm{SO}(2n,\mathbb{R})/\mathcal{C}_2\mathbf{1}$	$\mathbf{1}$	$\mathcal{C}_2$
$E_{6(6)}$	$E_{6(-78)}$	$\mathcal{C}_3$	$E_{6(-78)}$	$\mathcal{C}_3$	$\mathcal{C}_3$
$E_{6(2)}$	$E_{6(-78)}$	$\mathcal{C}_3$	$E_{6(-78)}/\mathcal{C}_3$	$\mathbf{1}$	$\mathbf{1}$
$E_{6(-14)}$	$E_{6(-78)}$	$\mathcal{C}_3$	$E_{6(-78)}/\mathcal{C}_3$	$\mathbf{1}$	$\mathbf{1}$
$E_{6(-26)}$	$E_{6(-78)}$	$\mathcal{C}_3$	$E_{6(-78)}$	$\mathcal{C}_3$	$\mathcal{C}_3$
$E_{7(7)}$	$E_{7(-133)}$	$\mathcal{C}_2$	$E_{7(-133)}/\mathcal{C}_2$	$\mathbf{1}$	$\mathcal{C}_2$
$E_{7(-5)}$	$E_{7(-133)}$	$\mathcal{C}_2$	$E_{7(-133)}/\mathcal{C}_2$	$\mathbf{1}$	$\mathbf{1}$
$E_{7(-25)}$	$E_{7(-133)}$	$\mathcal{C}_2$	$E_{7(-133)}/\mathcal{C}_2$	$\mathbf{1}$	$\mathcal{C}_2$
$E_{8(8)}$	$E_{8(-248)}$	$\mathbf{1}$	$E_{8(-248)}$	$\mathbf{1}$	$\mathbf{1}$
$E_{8(-24)}$	$E_{8(-248)}$	$\mathbf{1}$	$E_{8(-248)}$	$\mathbf{1}$	$\mathbf{1}$
$F_{4(4)}$	$F_{4(-52)}$	$\mathbf{1}$	$F_{4(-52)}$	$\mathbf{1}$	$\mathbf{1}$
$F_{4(-20)}$	$F_{4(-52)}$	$\mathbf{1}$	$F_{4(-52)}$	$\mathbf{1}$	$\mathbf{1}$
$G_{2(2)}$	$G_{2(-14)}$	$\mathbf{1}$	$G_{2(-14)}$	$\mathbf{1}$	$\mathbf{1}$

Table 4: Classification of real simple non-compact Lie algebras by symmetric spaces (continued)

We prove our key proposition by slightly modifying Loos' proof of Theorem VII.4.4 of [12].

Proposition 2.14. *Let $\mathfrak{u}$ be a real simple compact Lie algebra, $\sigma \in \mathrm{Aut}(\mathfrak{u})$ a non-trivial involution, i.e. $\sigma \neq \mathrm{id}_{\mathfrak{u}}$ and $\sigma^2 = \mathrm{id}_{\mathfrak{u}}$, set $\mathfrak{k} := \mathfrak{u}^{\sigma}$ and $\mathfrak{p} := \mathfrak{u}^{-\sigma}$ and let $\mathfrak{g} = \mathfrak{k} + i\mathfrak{p}$ be the corresponding real simple non-compact Lie algebra with Cartan involution $\tau = \mathrm{id}_{\mathfrak{k}} \oplus - \mathrm{id}_{i\mathfrak{p}}$. Then:*

1. *The group $\pi_0((\mathrm{Aut}(\mathfrak{u})_0)^{\sigma})$ is elementary abelian, i.e. isomorphic to $\mathcal{C}_2^k$ for some $k \in \mathbb{N}_0$.*
2. *The identity components of $\mathrm{Aut}(\mathfrak{u})^{\sigma}$ and $(\mathrm{Aut}(\mathfrak{u})_0)^{\sigma}$ conincide.*
3. *The inclusion $\mathrm{Aut}(\mathfrak{u})^{\sigma} \cong \mathrm{Aut}(\mathfrak{g})^{\tau} \hookrightarrow \mathrm{Aut}(\mathfrak{g})$ induces an isomorphism of groups $\pi_0(\mathrm{Aut}(\mathfrak{u})^{\sigma}) \to \pi_0(\mathrm{Aut}(\mathfrak{g}))$ and $\mathrm{Aut}(\mathfrak{u})^{\sigma} \cong (\mathrm{Aut}(\mathfrak{u})^{\sigma})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{u})^{\sigma})$ if and only if $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$.*
4. *There exists a finite group F such that $\mathrm{Aut}(\mathfrak{u})^{\sigma} \cong (\mathrm{Aut}(\mathfrak{u})_0)^{\sigma} \rtimes F$ and $\pi_0(\mathrm{Aut}(\mathfrak{u})^{\sigma}) \cong \mathcal{C}_2^k \rtimes F$.*
5. *The following table provides all possibilities for 2^k, F and $\pi_0(\mathrm{Aut}(\mathfrak{u})^{\sigma})$.*

$(\mathfrak{u}, \sigma)$	2^k	F	$\pi_0(\mathrm{Aut}(\mathfrak{u})^{\sigma})$
$(\mathfrak{su}(2,\mathbb{C}), \mathrm{cj})$	2	$\mathbf{1}$	$\mathcal{C}_2$
$(\mathfrak{su}(n+1,\mathbb{C}), \mathrm{cj})$ *for even* $n \geq 2$	1	$\mathcal{C}_2$	$\mathcal{C}_2$
$(\mathfrak{su}(n+1,\mathbb{C}), \mathrm{cj})$ *for odd* $n \geq 3$	2	$\mathcal{C}_2$	$\mathcal{C}_2 \times \mathcal{C}_2$
$(\mathfrak{su}(n+1,\mathbb{C}), \mathrm{Ad}(I_{n+1-q,q}))$ *for* $q < \frac{n+1}{2} > 1$	1	$\mathcal{C}_2$	$\mathcal{C}_2$
$(\mathfrak{su}(2n+2,\mathbb{C}), \mathrm{Ad}(I_{n+1,n+1}))$ *for* $n \geq 1$	2	$\mathcal{C}_2$	$\mathcal{C}_2 \times \mathcal{C}_2$
$(\mathfrak{su}(2n+2,\mathbb{C}), \mathrm{cj} \circ \mathrm{Ad}(J_{n+1}))$ *for* $n \geq 1$	1	$\mathcal{C}_2$	$\mathcal{C}_2$
$(\mathfrak{so}(2n+1,\mathbb{R}), \mathrm{Ad}(I_{2n+1-q,q}))$ *for* $q \leq n \geq 2$	2	$\mathbf{1}$	$\mathcal{C}_2$
$(\mathfrak{sp}(n,\mathbb{H}), \mathrm{qj})$ *for* $n \geq 3$	2	$\mathbf{1}$	$\mathcal{C}_2$
$(\mathfrak{sp}(2n,\mathbb{H}), \mathrm{Ad}(I_{n,n}))$ *for* $n \geq 1$	2	$\mathbf{1}$	$\mathcal{C}_2$
$(\mathfrak{so}(2n,\mathbb{R}), \mathrm{Ad}(I_{2n-q,q}))$ *for odd* $q < n \geq 4$	1	$\mathcal{C}_2$	$\mathcal{C}_2$
$(\mathfrak{so}(2n,\mathbb{R}), \mathrm{Ad}(I_{2n-q,q}))$ *for even* $q < n \geq 4$	2	$\mathcal{C}_2$	$\mathcal{C}_2 \times \mathcal{C}_2$
$(\mathfrak{so}(2n,\mathbb{R}), \mathrm{Ad}(I_{n,n}))$ *for odd* $n \geq 5$	2	$\mathcal{C}_2$	$\mathcal{C}_2 \times \mathcal{C}_2$
$(\mathfrak{so}(2n,\mathbb{R}), \mathrm{Ad}(I_{n,n}))$ *for even* $n \geq 6$	4	$\mathcal{C}_2$	$\mathcal{D}_4$
$(\mathfrak{so}(8,\mathbb{R}), \mathrm{Ad}(I_{4,4}))$	4	$\mathcal{S}_3$	$\mathcal{S}_4$
$(\mathfrak{so}(4n,\mathbb{R}), \mathrm{Ad}(J_{2n}))$ *for* $n \geq 3$	2	$\mathbf{1}$	$\mathcal{C}_2$
$(\mathfrak{so}(4n+2,\mathbb{R}), \mathrm{Ad}(J_{2n+1}))$ *for* $n \geq 2$	1	$\mathcal{C}_2$	$\mathcal{C}_2$
$\left(\mathfrak{e}_{6(-78)}, \sigma_{\mathfrak{e}_{6(j)}}\right)$ *for* $j = 6, 2, -14, -26$	1	$\mathcal{C}_2$	$\mathcal{C}_2$
$\left(\mathfrak{e}_{7(-133)}, \sigma_{\mathfrak{e}_{7(j)}}\right)$ *for* $j = 7, -25$	2	$\mathbf{1}$	$\mathcal{C}_2$
all others	1	$\mathbf{1}$	$\mathbf{1}$

Proof. We use the notation of Remark 2.13.

1. The closed subgroup $U_{\mathrm{ad}}^{\sigma} = \{ f \in U_{\mathrm{ad}} | \, \sigma f \sigma = f \}$ is compact and so is $\pi_0(U_{\mathrm{ad}}^{\sigma})$. In addition, the latter group is discrete, hence finite. We will show that every element in $\pi_0(U_{\mathrm{ad}}^{\sigma})$ is self-inverse: The Lie algebra of U_{ad}^{σ} is $\mathbf{L}U_{\mathrm{ad}}^{\sigma} = \mathbf{L}K = \mathfrak{k}$,

thus the identity component $(U_{\mathrm{ad}}^{\sigma})_0$ is $\exp_{U_{\mathrm{ad}}^{\sigma}}(\mathfrak{k}) = \exp_{U_{\mathrm{ad}}}(\mathfrak{k})$, so by using the surjectivity of

$$\begin{aligned} \Phi : \exp_{U_{\mathrm{ad}}}(\mathfrak{k}) \times \mathfrak{p} &\longrightarrow \exp_{U_{\mathrm{ad}}}(\mathfrak{u}) = U_{\mathrm{ad}} \\ (f, x) &\longmapsto f \cdot \exp_{U_{\mathrm{ad}}}(x), \end{aligned}$$

(cf. Remark 2.3), we have $U_{\mathrm{ad}}^{\sigma} = (U_{\mathrm{ad}}^{\sigma})_0 \cdot \big(U_{\mathrm{ad}}^{\sigma} \cap \exp_{U_{\mathrm{ad}}}(\mathfrak{p})\big)$ and every element of $\pi_0(U_{\mathrm{ad}}^{\sigma})$ takes the form $\exp_{U_{\mathrm{ad}}}(x) \cdot (U_{\mathrm{ad}}^{\sigma})_0 = \big[\exp_{U_{\mathrm{ad}}}(x)\big]$ for some $x \in \mathfrak{p}$. Thus the calculation

$$\begin{aligned} \big[\exp_{U_{\mathrm{ad}}}(x)\big]^{-1} &= \big[\exp_{U_{\mathrm{ad}}}(-x)\big] = \big[\exp_{U_{\mathrm{ad}}}(\sigma(x))\big] = \big[\sigma \exp_{U_{\mathrm{ad}}}(x)\sigma\big] \\ &= \big[\exp_{U_{\mathrm{ad}}}(x)\big] \end{aligned}$$

shows that every element of $\pi_0(U_{\mathrm{ad}}^{\sigma})$ is self-inverse, hence $\pi_0(U_{\mathrm{ad}}^{\sigma}) \cong \mathcal{C}_2^k$ for some $k \in \mathbb{N}_0$.

In order to derive a formula for k, we consider the following commutative diagram of pointed spaces:

$$\begin{tikzcd} U_{\mathrm{dis}}/K' \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & U_{\mathrm{dis}}/\rho^{-1}(U_{\mathrm{ad}}^{\sigma}) \\ & U_{\mathrm{dis}}/\rho^{-1}((U_{\mathrm{ad}}^{\sigma})_0), \arrow[ur, "\psi"'] & \end{tikzcd}$$

where π, φ, ψ are induced by the identity on U_{dis} and $\rho := \mathrm{Ad}_{U_{\mathrm{dis}}}$. As a first step, we calculate:

$$\begin{aligned} \rho^{-1}\left((U_{\mathrm{ad}}^{\sigma})_0\right) &= \big\{ g \in U_{\mathrm{dis}} \,|\, (\exists x \in \mathfrak{k}) : \rho(g) = \exp_{U_{\mathrm{ad}}}(x) \big\} \\ &= \big\{ g \in U_{\mathrm{dis}} \,|\, (\exists x \in \mathfrak{k}) : \rho(g) = \rho \circ \exp_{U_{\mathrm{dis}}}(x) \big\} \\ &= \big\{ g \in U_{\mathrm{dis}} \,|\, (\exists h \in K') : gh^{-1} \in \mathrm{Z}(U_{\mathrm{dis}}) \big\} = \mathrm{Z}(U_{\mathrm{dis}})K'. \end{aligned}$$

So we have $\ker(\varphi) = \mathrm{Z}(U_{\mathrm{dis}})K'/K' \cong \mathrm{Z}(U_{\mathrm{dis}})/(\mathrm{Z}(U_{\mathrm{dis}}) \cap K') \cong \mathrm{Z}(U_{\mathrm{dis}})$, where the last isomorphism follows from the fact that $\mathrm{Z}(U_{\mathrm{dis}}) \cap K'$ acts trivially on $M = U_{\mathrm{dis}}/K'$.

There exists a bijective group morphism $\rho^{-1}(U_{\mathrm{ad}}^{\sigma})/\rho^{-1}\left((U_{\mathrm{ad}}^{\sigma})_0\right) \to \pi_0(U_{\mathrm{ad}}^{\sigma})$ by $[g] \mapsto [\rho(g)]$: The map is obviously well-defined and its kernel is:

$$\{[g] \,|\, \rho(g) \in (U_{\mathrm{ad}}^{\sigma})_0\} = \mathbf{1} \in \rho^{-1}(U_{\mathrm{ad}}^{\sigma})/\rho^{-1}\left((U_{\mathrm{ad}}^{\sigma})_0\right).$$

Its surjectivity follows from the surjectivity of $\rho_{|\rho^{-1}(U_{\mathrm{ad}}^{\sigma})}^{|U_{\mathrm{ad}}^{\sigma}}$. This shows $\ker(\psi) \cong \pi_0(U_{\mathrm{ad}}^{\sigma})$.

In order to calculate $\ker(\pi)$ we use the bijective map $U_{\mathrm{dis}}/\rho^{-1}(U_{\mathrm{ad}}^{\sigma}) \to U_{\mathrm{ad}}/U_{\mathrm{ad}}^{\sigma}$, $[g] \mapsto [\rho(g)]$, leading to $\widetilde{\pi} : U_{\mathrm{dis}}/K' \to U_{\mathrm{ad}}/U_{\mathrm{ad}}^{\sigma}$, $[g] \mapsto [\rho(g)]$ and, by Proposition III.2.4 and the corollary of Theorem VI.3.6 of [12], the $[\mathbf{1}]$-fiber of this map is $\mathrm{Z}(M)$.

We thus obtain the following formula for the order of the group $\pi_0(U_{\mathrm{ad}}^{\sigma})$:

$$2^k = \frac{\#\,\mathrm{Z}(M)}{\#\,\mathrm{Z}(U_{\mathrm{dis}})},$$

leading to the following preliminary result:

X_r^*	2^k
$A_n^{\mathbb{R}}$, odd $n \geq 1$	2
$A_{2q-1}^{\mathbb{C},q}$, $q > 1$	2
$B_n^{\mathbb{R},q}$, $q \leq n \geq 2$	2
$C_n^{\mathbb{R}}$, $n \geq 3$	2
$C_{2q}^{\mathbb{H},q}$, $q \geq 2$	2
$D_n^{\mathbb{R},q}$, $n \geq 4$, even $q < n$	2
$D_q^{\mathbb{R},q}$, even $q \geq 4$	4
$D_q^{\mathbb{R},q}$, odd $q \geq 5$	2
$D_n^{\mathbb{H}}$, even $n \geq 6$	2
$E_{7(7)}$	2
$E_{7(-25)}$	2
all others	1

2. Any element in U_{ad}^σ is of the form $\exp_{U_{\mathrm{ad}}}(x)$ for some $x \in \mathfrak{k}$. If $f \in \mathrm{Aut}(\mathfrak{u})^\sigma$ is arbitrary, then $f \exp_{U_{\mathrm{ad}}}(x) f^{-1} = \exp_{U_{\mathrm{ad}}}(f(x))$ is again in U_{ad}, so in U_{ad}^σ, hence $U_{\mathrm{ad}}^\sigma \trianglelefteq \mathrm{Aut}(\mathfrak{u})^\sigma$. The Lie algebra of $\mathrm{Aut}(\mathfrak{u})^\sigma$ is $\mathbf{L}\,\mathrm{Aut}(\mathfrak{u})^\sigma = \mathbf{L}U_{\mathrm{ad}}^\sigma = \mathfrak{k}$ and $\mathrm{Aut}(\mathfrak{u})^\sigma$, U_{ad}^σ are closed subgroups of $\mathrm{Aut}(\mathfrak{u})$, so $(\mathrm{Aut}(\mathfrak{u})^\sigma)_0 = (U_{\mathrm{ad}}^\sigma)_0$. The short exact sequence

$$\mathbf{1} \longrightarrow U_{\mathrm{ad}}^\sigma \xrightarrow{\text{incl.}} \mathrm{Aut}(\mathfrak{u})^\sigma \xrightarrow{\text{quot.}} F \longrightarrow \mathbf{1}$$

induces the short exact sequence

$$\mathbf{1} \longrightarrow \pi_0\left(U_{\mathrm{ad}}^\sigma\right) \xrightarrow{\text{incl.}} \pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right) \xrightarrow{\text{quot.}} F \longrightarrow \mathbf{1}$$

and if the former sequence is split, then the latter is split, too.

3. We apply Remark 2.3 to $H = \mathrm{Aut}(\mathfrak{g})$. So we obtain $\mathrm{Aut}(\mathfrak{g})_0 \cap \mathrm{Aut}(\mathfrak{g})^\tau = (\mathrm{Aut}(\mathfrak{g})^\tau)_0$ and $\mathrm{Aut}(\mathfrak{g})_0 \cdot \mathrm{Aut}(\mathfrak{g})^\tau = \mathrm{Aut}(\mathfrak{g})$ and $\pi_0\left(\mathrm{Aut}(\mathfrak{g})^\tau\right) \to \pi_0\left(\mathrm{Aut}(\mathfrak{g})\right)$, $[\omega] \mapsto [\omega]$ is a group isomorphism.

   Furthermore, the isomorphism $\eta : \mathrm{Aut}(\mathfrak{g})^\tau \to \mathrm{Aut}(\mathfrak{u})^\sigma$ from Lemma 2.2.2 induces the isomorphism

   $$\pi_0(\eta) : \pi_0\left(\mathrm{Aut}(\mathfrak{g})^\tau\right) \longrightarrow \pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right), [f] \longmapsto [\eta(f)],$$

   yielding $\pi_0(\mathrm{Aut}(\mathfrak{g})) \cong \pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)$.

   If there are representatives of the elements in $\pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)$ forming a subgroup of $\mathrm{Aut}(\mathfrak{u})^\sigma$ isomorphic to $\pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)$, we may apply η^{-1} to them and they turn into representatives in $\pi_0\left(\mathrm{Aut}(\mathfrak{g})^\tau\right)$ and thus in $\pi_0\left(\mathrm{Aut}(\mathfrak{g})\right)$, showing that $\pi_0\left(\mathrm{Aut}(\mathfrak{g})\right)$ can be injectively embedded into $\mathrm{Aut}(\mathfrak{g})$ with its image intersecting $\mathrm{Aut}(\mathfrak{g})_0$ trivially, i.e. $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$.

   This argument works, conversely, by applying η to the elements in $\pi_0\left(\mathrm{Aut}(\mathfrak{g})^\tau\right)$ forming a subgroup of $\mathrm{Aut}(\mathfrak{g})^\tau$ isomorphic to $\pi_0\left(\mathrm{Aut}(\mathfrak{g})^\tau\right)$.

4. + 5. We will prove $\mathrm{Aut}(\mathfrak{u})^\sigma \cong U_{\mathrm{ad}}^\sigma \rtimes F$, where F is some finite group, by a case-by-case discussion:

Case 1: If $\mathrm{Aut}(\mathfrak{u})$ is connected, i.e.

$$X_r^* \in \left\{ A_1^{\mathbb{R}};\ B_n^{\mathbb{R},q}, q \le n \ge 2;\ C_n^{\mathbb{R}}, n \ge 3;\ C_n^{\mathbb{H},q}, q \le \frac{n}{2} > 1;\right.$$
$$\left. E_{7(7)};\ E_{7(-5)};\ E_{7(-25)};\ E_{8(8)};\ E_{8(-24)};\ F_{4(4)};\ F_{4(-20)};\ G_{2(2)} \right\}$$

by Theorem 2.4, then $\mathrm{Aut}(\mathfrak{u})^\sigma \cong U_{\mathrm{ad}}^\sigma \rtimes F$ follows for trivial F. In these cases, $\mathrm{Z}(U_{\mathrm{dis}})$ is trivial, so $\pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma)$ has order $\#\mathrm{Z}(M) \in \{1,2\}$, thus $\pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma) \in \{\mathbf{1}, \mathcal{C}_2\}$ (cf. Table 4).

Case 2: Let the automorphism $\sigma \in \mathrm{Aut}(\mathfrak{u})$ be outer and $\mathfrak{u} \ne \mathfrak{so}(8,\mathbb{R})$. The former condition is fulfilled if and only if the ranks of $\mathfrak{u}$ and $\mathfrak{k}$ do not conincide by Lemma 2.10, so we obtain:

$$X_r^* \in \left\{ A_n^{\mathbb{R}}, n > 1;\ A_{2n+1}^{\mathbb{H}}, n \ge 1;\ D_n^{\mathbb{R},q}, \text{odd } q \le n > 4;\ E_{6(6)};\ E_{6(-26)} \right\}.$$

In all these cases, we have $k = 0$, except for the case $X_r^* = A_n^{\mathbb{R}}$, odd $n > 1$, and $D_q^{\mathbb{R},q}$, odd $q > 4$, where $k = 1$. Then $\sigma \notin \mathrm{Aut}(\mathfrak{u})_0 = U_{\mathrm{ad}} \supseteq U_{\mathrm{ad}}^\sigma \supseteq (U_{\mathrm{ad}}^\sigma)_0 = (\mathrm{Aut}(\mathfrak{u})^\sigma)_0$, so $[\sigma] \in \pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma) \setminus \pi_0(U_{\mathrm{ad}}^\sigma)$ has order two. We define $F := \{[\mathrm{id}_\mathfrak{u}], [\sigma]\} \le \pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma)$ and a map $p : \mathrm{Aut}(\mathfrak{u})^\sigma \to F$ by $p(\theta) := [\mathrm{id}_\mathfrak{u}]$ if and only if $\theta \in U_{\mathrm{ad}}^\sigma$ and $p(\theta) := [\sigma]$ otherwise. If $\theta, \theta' \in \mathrm{Aut}(\mathfrak{u})^\sigma \setminus U_{\mathrm{ad}}^\sigma$, then the equivalence classes of θ^{-1}, θ' in $\pi_0(\mathrm{Aut}(\mathfrak{u}))$ are different from $[\mathrm{id}_\mathfrak{u}]$, so, by Theorem 2.4, the equivalence classes conincide and $\theta\theta' \in \mathrm{Aut}(\mathfrak{u})_0 \cap \mathrm{Aut}(\mathfrak{u})^\sigma = U_{\mathrm{ad}}^\sigma$. So $p(\theta\theta') = [\mathrm{id}_\mathfrak{u}] = [\sigma]^2 = p(\theta)p(\theta')$ and hence p is a morphism of groups. This morphism has a natural section $s : F \to \mathrm{Aut}(\mathfrak{u})^\sigma$, thus $\mathrm{Aut}(\mathfrak{u})^\sigma \cong U_{\mathrm{ad}}^\sigma \rtimes_\alpha F$ with $\alpha([\sigma])(f) := \sigma f \sigma = f$ for all $f \in U_{\mathrm{ad}}^\sigma$, so $\mathrm{Aut}(\mathfrak{u})^\sigma \cong U_{\mathrm{ad}}^\sigma \times \mathcal{C}_2$.

Case 3: $X_r^* \in \left\{ D_4^{\mathbb{R},1}, D_4^{\mathbb{R},3} \right\}$, i.e. $(\mathfrak{u},\sigma) = \begin{cases} (\mathfrak{so}(8,\mathbb{R}), \mathrm{Ad}(I_{7,1})), \text{ or} \\ (\mathfrak{so}(8,\mathbb{R}), \mathrm{Ad}(I_{5,3})). \end{cases}$

The automorphism group of the compact subalgebra $\mathfrak{k} = \mathfrak{u}^\sigma$ is

$$\mathrm{Aut}(\mathfrak{so}(1,\mathbb{R})) \times \mathrm{Aut}(\mathfrak{so}(7,\mathbb{R})) \cong \mathrm{Aut}(\mathfrak{so}(7,\mathbb{R}))$$

or

$$\mathrm{Aut}(\mathfrak{so}(3,\mathbb{R})) \times \mathrm{Aut}(\mathfrak{so}(5,\mathbb{R})),$$

but in both cases $\mathrm{Aut}(\mathfrak{k})$ is connected, because the Dynkin diagrams A_1, B_2 and B_3 have no non-trivial symmetries. Since $\sigma \in \mathrm{Aut}(\mathfrak{u})$ is outer, Lemma 2.5 implies that $\pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma) = \{[\mathrm{id}_\mathfrak{u}], [\sigma]\}$ and, since U_{ad}^σ is connected in both cases, we have a short exact sequence as follows:

$$\mathbf{1} \longrightarrow U_{\mathrm{ad}}^\sigma \xrightarrow{\text{incl.}} \mathrm{Aut}(\mathfrak{u})^\sigma \xrightarrow{p} \{[\mathrm{id}_\mathfrak{u}], [\sigma]\} \longrightarrow \mathbf{1},$$

where $p(\theta) := [\mathrm{id}_\mathfrak{u}]$ if and only if $\theta \in U_{\mathrm{ad}}^\sigma = (U_{\mathrm{ad}}^\sigma)_0 = (\mathrm{Aut}(\mathfrak{u})^\sigma)_0$ and $p(\theta) := [\sigma]$ otherwise. The sequence possesses a natural section $s : \{[\mathrm{id}_\mathfrak{u}], [\sigma]\} \to \mathrm{Aut}(\mathfrak{u})^\sigma$, thus $\mathrm{Aut}(\mathfrak{u})^\sigma \cong U_{\mathrm{ad}}^\sigma \rtimes_\alpha \{[\mathrm{id}_\mathfrak{u}], [\sigma]\}$ with $\alpha([\sigma])(f) := \sigma f \sigma = f$ for all $f \in U_{\mathrm{ad}}^\sigma$, so $\mathrm{Aut}(\mathfrak{u})^\sigma \cong U_{\mathrm{ad}}^\sigma \times \mathcal{C}_2$.

Case 4: It remains to consider the case where $\mathrm{Aut}(\mathfrak{u})$ is disconnected, but σ is inner. We have the following subcases:

i. $X_r^* = A_n^{\mathbb{C},q}$, i.e. $(\mathfrak{u},\sigma) = (\mathfrak{su}(n+1,\mathbb{C}), \mathrm{Ad}(I_{n+1-q,q}))$, for an integer $q \leq \frac{n+1}{2} > 1$:
Let $\mathrm{cj} : \mathfrak{su}(n+1,\mathbb{C}) \to \mathfrak{su}(n+1,\mathbb{C})$ be the complex conjugation. It commutes with σ because $I_{n+1-q,q}$ is real. The rank of $\mathfrak{u}^{\mathrm{cj}} = \mathfrak{so}(n+1,\mathbb{R})$ is $\frac{n}{2}$ for even n and $\frac{n+1}{2}$ for odd n, so never equal to n, the rank of $\mathfrak{u}$. Lemma 2.10 implies that $[\mathrm{cj}] \in \pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma) \setminus \pi_0(U_{\mathrm{ad}}^\sigma)$ has order two. We define $F := \{[\mathrm{id}_\mathfrak{u}], [\mathrm{cj}]\} \leq \pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma)$ and a map $p : \mathrm{Aut}(\mathfrak{u})^\sigma \to F$ by $p(\theta) := [\mathrm{id}_\mathfrak{u}]$ if and only if $\theta \in U_{\mathrm{ad}}^\sigma$ and $p(\theta) := [\mathrm{cj}]$ otherwise. If $\theta, \theta' \in \mathrm{Aut}(\mathfrak{u})^\sigma \setminus U_{\mathrm{ad}}^\sigma$, then the equivalence classes of θ^{-1}, θ' in $\pi_0(\mathrm{Aut}(\mathfrak{u}))$ are different from $[\mathrm{id}_\mathfrak{u}]$, so, by Theorem 2.4, the equivalence classes conincide and $\theta\theta' \in \mathrm{Aut}(\mathfrak{u})_0 \cap \mathrm{Aut}(\mathfrak{u})^\sigma = U_{\mathrm{ad}}^\sigma$. So $p(\theta\theta') = [\mathrm{id}_\mathfrak{u}] = [\mathrm{cj}]^2 = p(\theta)p(\theta')$ and hence p is a morphism of groups. This morphism has a natural section $s : F \to \mathrm{Aut}(\mathfrak{u})^\sigma$, thus $\mathrm{Aut}(\mathfrak{u})^\sigma \cong U_{\mathrm{ad}}^\sigma \rtimes F$.

ii. $X_r^* = D_q^{\mathbb{R},q}$, i.e. $(\mathfrak{u},\sigma) = (\mathfrak{so}(2q,\mathbb{R}), \mathrm{Ad}(I_{q,q}))$, for an even integer $q \geq 6$:
By Corollary 2.12 we have:

$$\begin{aligned}\mathrm{Aut}(\mathfrak{u})^\sigma &= \{\mathrm{Ad}(X) |\, X \in \mathrm{O}(2q,\mathbb{R}) \text{ and } I_{q,q} X I_{q,q} = \pm X\} \\ &= \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \text{ or } X = \begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix} \right. \\ &\qquad \left. \text{with } A, B \in \mathrm{O}(q,\mathbb{R}) \right\}.\end{aligned}$$

Since $\mathrm{Inn}(\mathfrak{u}) = U_{\mathrm{ad}} = \{\mathrm{Ad}(X) | X \in \mathrm{SO}(2q,\mathbb{R})\}$, it follows that

$$\begin{aligned}U_{\mathrm{ad}}^\sigma = &\left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \text{ or } X = \begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix} \right. \\ &\qquad \left. \text{with } A, B \in \mathrm{O}(q,\mathbb{R}), \det(A) = \det(B) \right\}.\end{aligned}$$

By $\mathfrak{u}^\sigma = \mathfrak{so}(q,\mathbb{R}) \oplus \mathfrak{so}(q,\mathbb{R})$ and the surjectivity of the exponential maps $\mathfrak{u}^\sigma \to (U_{\mathrm{ad}}^\sigma)_0$ and $\mathfrak{so}(q,\mathbb{R}) \to \mathrm{SO}(q,\mathbb{R})$ we have:

$$\begin{aligned}(U_{\mathrm{ad}}^\sigma)_0 = (\mathrm{Aut}(\mathfrak{u})^\sigma)_0 = &\left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \right. \\ &\qquad \left. \text{with } A, B \in \mathrm{SO}(q,\mathbb{R}) \right\}.\end{aligned}$$

The short exact sequence

$$\begin{aligned}&\mathbf{1} \to (U_{\mathrm{ad}}^\sigma)_0 \to U_{\mathrm{ad}}^\sigma \\ &\quad \to \pi_0(U_{\mathrm{ad}}^\sigma) = \{[\mathrm{Ad}(X_1)], [\mathrm{Ad}(X_2)], [\mathrm{Ad}(X_3)], [\mathrm{Ad}(X_4)]\} \cong \mathcal{C}_2^2 \to \mathbf{1}\end{aligned}$$

with the representatives $X_1 = \begin{pmatrix} \mathbf{1}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q \end{pmatrix}$, $X_2 = \begin{pmatrix} I_{1,q-1} & \mathbf{0} \\ \mathbf{0} & I_{1,q-1} \end{pmatrix}$ and $X_3 = \begin{pmatrix} \mathbf{0} & \mathbf{1}_q \\ -\mathbf{1}_q & \mathbf{0} \end{pmatrix}$ and $X_4 = \begin{pmatrix} \mathbf{0} & I_{1,q-1} \\ -I_{1,q-1} & \mathbf{0} \end{pmatrix}$ has the section $\pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma) \to \mathrm{Aut}(\mathfrak{u})^\sigma$, $[\mathrm{Ad}(X_i)] \mapsto \mathrm{Ad}(X_i)$. The eight components of $\mathrm{Aut}(\mathfrak{u})^\sigma$ are represented by $\mathrm{Ad}(X_i)$, $i = 1, \ldots, 8$ with $X_1, \ldots, X_4$ as above and $X_5 = \begin{pmatrix} I_{1,q-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q \end{pmatrix}$, $X_6 = \begin{pmatrix} \mathbf{1}_q & \mathbf{0} \\ \mathbf{0} & -I_{1,q-1} \end{pmatrix}$,

$X_7 = \begin{pmatrix} \mathbf{0} & I_{1,q-1} \\ \mathbf{1}_q & \mathbf{0} \end{pmatrix}$, $X_8 = \begin{pmatrix} \mathbf{0} & \mathbf{1}_q \\ -I_{1,q-1} & \mathbf{0} \end{pmatrix}$. The group $\pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)$ of eight elements has exactly two elements of order four, $[\mathrm{Ad}(X_7)]$ and $[\mathrm{Ad}(X_8)]$, and, by $[\mathrm{Ad}(X_3X_5)] = [\mathrm{Ad}(X_8)] \neq [\mathrm{Ad}(X_7)] = [\mathrm{Ad}(X_5X_3)]$, it is not abelian, hence is isomorphic to $\mathcal{D}_4$. We obtain the split short exact seqence

$$\begin{aligned}\mathbf{1} \to U_{\mathrm{ad}}^\sigma \to \mathrm{Aut}(\mathfrak{u})^\sigma \to \mathrm{Aut}(\mathfrak{u})^\sigma/U_{\mathrm{ad}}^\sigma &= \{[\mathrm{Ad}(X_1)]\,, [\mathrm{Ad}(X_5)]\} \\ &\cong \mathcal{C}_2 \to \mathbf{1}.\end{aligned}$$

iii. $X_r^* = D_q^{\mathbb{R},n}$, i.e. $(\mathfrak{u}, \sigma) = (\mathfrak{so}(2q, \mathbb{R}), \mathrm{Ad}(I_{2n-q,q}))$, for an even integer $q < n \geq 5$:
In analogy with the previous subcase we have:

$$\begin{aligned}\mathrm{Aut}(\mathfrak{u})^\sigma &= \{\,\mathrm{Ad}(X) |\, X \in \mathrm{O}(2n, \mathbb{R}) \text{ and } I_{2n-q,q} X I_{2n-q,q} = \pm X\} \\ &= \left\{\mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}\right. \\ &\qquad \left. \text{with } A \in \mathrm{O}(2n-q, \mathbb{R}), B \in \mathrm{O}(q, \mathbb{R})\right\},\end{aligned}$$

where the second equality follows from the fact that a matrix of the form $X = \begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix}$ for $A \in M_{2n-q,q}(\mathbb{R}), B \in M_{q,2n-q}(\mathbb{R})$ is singular if $q < n$. By $U_{\mathrm{ad}} = \{\mathrm{Ad}(X) | X \in \mathrm{SO}(2n, \mathbb{R})\}$, it follows that

$$\begin{aligned}U_{\mathrm{ad}}^\sigma = &\left\{\mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}\right. \\ &\quad \left. \text{with } A \in \mathrm{O}(2q-n, \mathbb{R}), B \in \mathrm{O}(q, \mathbb{R}), \det(A) = \det(B)\right\}.\end{aligned}$$

By $\mathfrak{u}^\sigma = \mathfrak{so}(2n-q, \mathbb{R}) \oplus \mathfrak{so}(q, \mathbb{R})$ and the surjectivity of the exponential maps $\mathfrak{u}^\sigma \to (U_{\mathrm{ad}}^\sigma)_0$, $\mathfrak{so}(2n-q, \mathbb{R}) \to \mathrm{SO}(2n-q, \mathbb{R})$ and $\mathfrak{so}(q, \mathbb{R}) \to \mathrm{SO}(q, \mathbb{R})$ we have:

$$\begin{aligned}(U_{\mathrm{ad}}^\sigma)_0 = (\mathrm{Aut}(\mathfrak{u})^\sigma)_0 = &\left\{\mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}\right. \\ &\quad \left. \text{with } A \in \mathrm{SO}(2n-q, \mathbb{R}), B \in \mathrm{SO}(q, \mathbb{R})\right\}.\end{aligned}$$

The short exact sequence

$$\mathbf{1} \to (U_{\mathrm{ad}}^\sigma)_0 \to U_{\mathrm{ad}}^\sigma \to \pi_0\left(U_{\mathrm{ad}}^\sigma\right) = \{[\mathrm{Ad}(X_1)]\,, [\mathrm{Ad}(X_2)]\} \cong \mathcal{C}_2 \to \mathbf{1}$$

with representatives $X_1 = \begin{pmatrix} \mathbf{1}_{2n-q} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q \end{pmatrix}$, $X_2 = \begin{pmatrix} I_{1,2n-q-1} & \mathbf{0} \\ \mathbf{0} & I_{1,q-1} \end{pmatrix}$ has the section $\pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right) \to \mathrm{Aut}(\mathfrak{u})^\sigma$, $[\mathrm{Ad}(X_i)] \mapsto \mathrm{Ad}(X_i)$. The four components of $\mathrm{Aut}(\mathfrak{u})^\sigma$ are represented by $\mathrm{Ad}(X_i)$, $i = 1, \ldots, 4$ with X_1, X_2 as above and $X_3 = \begin{pmatrix} I_{1,2n-q-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q \end{pmatrix}$ and $X_4 = \begin{pmatrix} \mathbf{1}_{2n-q} & \mathbf{0} \\ \mathbf{0} & I_{1,q-1} \end{pmatrix}$. The group $\pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)$ of four elements has no element of order four, hence is isomorphic to $\mathcal{C}_2 \times \mathcal{C}_2$. We obtain the split short exact seqence

$$\mathbf{1} \to U_{\mathrm{ad}}^{\sigma} \to \mathrm{Aut}(\mathfrak{u})^{\sigma} \to \mathrm{Aut}(\mathfrak{u})^{\sigma}/U_{\mathrm{ad}}^{\sigma} = \{[\mathrm{Ad}(X_1)], [\mathrm{Ad}(X_3)]\} \\ \cong \mathcal{C}_2 \to \mathbf{1}.$$

iv. $X_r^* = D_n^{\mathbb{H}}$, i.e. $(\mathfrak{u}, \sigma) = (\mathfrak{so}(2n, \mathbb{R}), \mathrm{Ad}(J_n))$, for $n \geq 5$:
In analogy with the two previous subcases we have:

$$\begin{aligned}\mathrm{Aut}(\mathfrak{u})^{\sigma} &= \left\{ \mathrm{Ad}(X) \middle| X \in \mathrm{O}(2n, \mathbb{R}) \text{ and } J_n X J_n^{-1} = \pm X \right\} \\ &= \Big\{ \mathrm{Ad}(X) \,\Big|X \in \mathrm{O}(2n, \mathbb{R})^{J_n} = \mathrm{U}(n, \mathbb{C}) \\ &\qquad \text{or } X \in \mathrm{O}(2n, \mathbb{R})^{-J_n} = I_{n,n}\,\mathrm{U}(n, \mathbb{C}) \Big\},\end{aligned}$$

where we consider $\mathrm{U}(n, \mathbb{C})$ as the subgroup of $\mathrm{O}(2n, \mathbb{R})$ fixed by the conjugation with J_n. The relation $\det_{\mathbb{R}}(X) = |\det_{\mathbb{C}}(X)|^2$ tells us that the matrices in $\mathrm{U}(n, \mathbb{C})$ have real determinant 1, whereas the matrices in $I_{n,n}\,\mathrm{U}(n, \mathbb{C})$ have real determinant $(-1)^n$, so we have the following decomposition into closed open subsets:

$$\mathrm{Aut}(\mathfrak{u})^{\sigma} = \{\mathrm{Ad}(X) \,|X \in \mathrm{U}(n, \mathbb{C})\} \cup \{\mathrm{Ad}(X) \,|X \in I_{n,n}\,\mathrm{U}(n, \mathbb{C})\}$$

Since $U_{\mathrm{ad}} = \{\mathrm{Ad}(X) | X \in \mathrm{SO}(2n, \mathbb{R})\}$, it follows that

$$\begin{aligned}U_{\mathrm{ad}}^{\sigma} = &\underbrace{\{\mathrm{Ad}(X) \,|X \in \mathrm{U}(n, \mathbb{C})\}}_{=(U_{\mathrm{ad}}^{\sigma})_0 = (\mathrm{Aut}(\mathfrak{u})^{\sigma})_0} \\ &\cup \left\{ \mathrm{Ad}(X) \middle| X \in I_{n,n}\,\mathrm{U}(n, \mathbb{C}), \det_{\mathbb{R}}(X) = 1 \right\}.\end{aligned}$$

If n is odd, then $k = 0$ and $U_{\mathrm{ad}}^{\sigma} = (U_{\mathrm{ad}}^{\sigma})_0$. We obtain the split short exact seqence

$$\mathbf{1} \to U_{\mathrm{ad}}^{\sigma} \to \mathrm{Aut}(\mathfrak{u})^{\sigma} \to \mathrm{Aut}(\mathfrak{u})^{\sigma}/U_{\mathrm{ad}}^{\sigma} = \{[\mathrm{id}_{\mathfrak{u}}], [\mathrm{Ad}(I_{n,n})]\} \\ \cong \mathcal{C}_2 \to \mathbf{1}.$$

If n is even, then $k = 1$ and $\mathrm{Aut}(\mathfrak{u})^{\sigma} = U_{\mathrm{ad}}^{\sigma}$.

v. $X_r^* = D_4^{\mathbb{R},2}$, i.e. $(\mathfrak{u}, \sigma) = (\mathfrak{so}(8, \mathbb{R}), \mathrm{Ad}(I_{6,2}))$:
The automorphism group of the compact subalgebra $\mathfrak{k} = \mathfrak{u}^{\sigma}$ is $\mathrm{Aut}(\mathfrak{so}(2, \mathbb{R}) \oplus \mathfrak{so}(6, \mathbb{R})) \cong \mathrm{Aut}(\mathbb{R} \oplus \mathfrak{su}(4, \mathbb{C}))$, so $\pi_0(\mathrm{Aut}(\mathfrak{k})) \cong \mathcal{C}_2 \times \mathcal{C}_2$, since the Dynkin diagram A_3 only has one non-trivial symmetry. Since $\sigma \in \mathrm{Aut}(\mathfrak{u})$ is inner, Lemma 2.5 implies that $\pi_0(\mathrm{Aut}(\mathfrak{u})^{\sigma})$ is isomorphic to a subgroup of $\mathcal{C}_2 \times \mathcal{C}_2$ and since $k = 1$, we either have $\pi_0(\mathrm{Aut}(\mathfrak{u})^{\sigma}) \cong \mathcal{C}_2$ or $\pi_0(\mathrm{Aut}(\mathfrak{u})^{\sigma}) \cong \mathcal{C}_2 \times \mathcal{C}_2$.
By $\mathrm{Aut}(\mathfrak{so}(8, \mathbb{C})) \supseteq \mathrm{Ad}(\mathrm{O}(8, \mathbb{C}))$, we have at least:

$$\begin{aligned}\mathrm{Aut}(\mathfrak{u})^{\sigma} \supseteq \Big\{ &\mathrm{Ad}(X) \Big| X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \\ &\text{with } A \in \mathrm{O}(6, \mathbb{R}), B \in \mathrm{O}(2, \mathbb{R}) \Big\},\end{aligned}$$

Since $U_{\mathrm{ad}} = \{\mathrm{Ad}(X) | X \in \mathrm{SO}(8, \mathbb{R})\}$, it follows that

$$\begin{aligned}U_{\mathrm{ad}}^{\sigma} = \Big\{ &\mathrm{Ad}(X) \Big| X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \\ &\text{with } A \in \mathrm{O}(6, \mathbb{R}), B \in \mathrm{O}(2, \mathbb{R}), \det(A) = \det(B) \Big\}.\end{aligned}$$

By $\mathfrak{u}^\sigma = \mathfrak{so}(6,\mathbb{R}) \oplus \mathfrak{so}(2,\mathbb{R})$ and the surjectivity of the exponential maps $\mathfrak{u}^\sigma \to (U_{\mathrm{ad}}^\sigma)_0$, $\mathfrak{so}(6,\mathbb{R}) \to \mathrm{SO}(6,\mathbb{R})$ and $\mathfrak{so}(2,\mathbb{R}) \to \mathrm{SO}(2,\mathbb{R})$ we have:

$$(U_{\mathrm{ad}}^\sigma)_0 = (\mathrm{Aut}(\mathfrak{u})^\sigma)_0 = \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \right. \\ \left. \text{with } A \in \mathrm{SO}(6,\mathbb{R}), B \in \mathrm{SO}(2,\mathbb{R}) \right\}.$$

But we already know that $\pi_0 \left(U_{\mathrm{ad}}^\sigma\right) \cong \mathcal{C}_2$, implying the equality

$$U_{\mathrm{ad}}^\sigma = \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \right. \\ \left. \text{with } A \in \mathrm{O}(6,\mathbb{R}), B \in \mathrm{O}(2,\mathbb{R}), \det(A) = \det(B) \right\}.$$

We consider $X' = \begin{pmatrix} \mathbf{1}_6 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 1 \\ \mathbf{0} & 1 & 0 \end{pmatrix}$. Hence $\mathrm{Ad}(X') \in \mathrm{Aut}(\mathfrak{u}) \backslash U_{\mathrm{ad}}^\sigma$, and we calculate:

$$\begin{aligned} \sigma \,\mathrm{Ad}(X')\sigma(x) &= \mathrm{Ad}(I_{6,2} X' I_{6,2})(x) \\ &= \mathrm{Ad}\left(\begin{pmatrix} -\mathbf{1}_6 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} X' \begin{pmatrix} -\mathbf{1}_6 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \right)(x) \\ &= \mathrm{Ad}(X')(x). \end{aligned}$$

So we see that $[\mathrm{Aut}(\mathfrak{u})^\sigma : U_{\mathrm{ad}}^\sigma] > 1$, so $[\mathrm{Aut}(\mathfrak{u})^\sigma : U_{\mathrm{ad}}^\sigma] = 2$, yielding the equality

$$\mathrm{Aut}(\mathfrak{u})^\sigma = \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \right. \\ \left. \text{with } A \in \mathrm{O}(6,\mathbb{R}), B \in \mathrm{O}(2,\mathbb{R}) \right\}.$$

We obtain the split short exact seqence

$$\mathbf{1} \to U_{\mathrm{ad}}^\sigma \to \mathrm{Aut}(\mathfrak{u})^\sigma \to \mathrm{Aut}(\mathfrak{u})^\sigma / U_{\mathrm{ad}}^\sigma = \{[\mathrm{id}_{\mathfrak{u}}], [\mathrm{Ad}(X')]\} \cong \mathcal{C}_2 \to \mathbf{1}.$$

vi. $X_r^* = D_4^{\mathbb{R},2}$, i.e. $(\mathfrak{u}, \sigma) = (\mathfrak{so}(8,\mathbb{R}), \mathrm{Ad}(I_{4,4}))$:
The automorphism group of the compact subalgebra $\mathfrak{k} = \mathfrak{u}^\sigma$ is

$$\begin{aligned} &\mathrm{Aut}(\mathfrak{so}(4,\mathbb{R}) \oplus \mathfrak{so}(4,\mathbb{R})) \\ &\cong \mathrm{Aut}(\mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{su}(2,\mathbb{C})), \end{aligned}$$

so $\pi_0(\mathrm{Aut}(\mathfrak{k}))$ is isomorphic the symmetry group of the Dynkin diagram with four disconnected nodes, hence isomorphic to $\mathcal{S}_4$. Since $\sigma \in \mathrm{Aut}(\mathfrak{u})$ is inner, Lemma 2.5 implies that $\pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)$ is isomorphic to a subgroup of $\mathcal{S}_4$. Furthermore, we know that $\mathcal{C}_2^2 \cong \pi_0\left(U_{\mathrm{ad}}^\sigma\right) \trianglelefteq \pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)$, thus the Third Isomorphism Theorem yields that the order of $\mathrm{Aut}(\mathfrak{u})^\sigma / U_{\mathrm{ad}}^\sigma \cong \pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right) / \pi_0\left(U_{\mathrm{ad}}^\sigma\right)$ is at most six.

By Lemma 2.6, there is an element $[\theta]$ of order three in the quotient $\mathrm{Aut}(\mathfrak{u})^\sigma/U_{\mathrm{ad}}^\sigma \cong \pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)/\pi_0\left(U_{\mathrm{ad}}^\sigma\right)$ of order three. The automorphism $\mathrm{Ad}(Y)$ with $Y = \begin{pmatrix} \mathbf{1}_4 & \mathbf{0} \\ \mathbf{0} & -I_{3,1} \end{pmatrix}$ commutes with $\sigma = \mathrm{Ad}(I_{4,4})$ and the rank of $\mathfrak{u}^{\mathrm{Ad}(Y)} = \mathfrak{so}(7,\mathbb{R})$ is different from the rank of $\mathfrak{u}$, thus, by Lemma 2.10, the order of $[\mathrm{Ad}(Y)] \in \mathrm{Aut}(\mathfrak{u})^\sigma/U_{\mathrm{ad}}^\sigma$ is two. Hence $\mathrm{Aut}(\mathfrak{u})^\sigma/U_{\mathrm{ad}}^\sigma$ is a group of six elements and $\pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)$ has 24 elements, so $\pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right) \cong \mathcal{S}_4$.
The Second Isomorphism Theorem yields

$$\begin{aligned}\mathrm{Aut}(\mathfrak{u})^\sigma/U_{\mathrm{ad}}^\sigma = \mathrm{Aut}(\mathfrak{u})^\sigma/\left(U_{\mathrm{ad}} \cap \mathrm{Aut}(\mathfrak{u})^\sigma\right) &\cong \left(U_{\mathrm{ad}} \cdot \mathrm{Aut}(\mathfrak{u})^\sigma\right)/U_{\mathrm{ad}} \\ &\leq \pi_0\left(\mathrm{Aut}(\mathfrak{u})\right).\end{aligned}$$

Since $\pi_0\left(\mathrm{Aut}(\mathfrak{u})\right) \cong \mathcal{S}_3$ by Theorem 2.4, we even have $\mathrm{Aut}(\mathfrak{u})^\sigma/U_{\mathrm{ad}}^\sigma \cong \pi_0\left(\mathrm{Aut}(\mathfrak{u})\right) \cong \mathcal{S}_3$. We obtain the short exact sequence

$$\begin{aligned}\mathbf{1} &\to U_{\mathrm{ad}}^\sigma \to \mathrm{Aut}(\mathfrak{u})^\sigma \\ &\to \mathrm{Aut}(\mathfrak{u})^\sigma/U_{\mathrm{ad}}^\sigma \\ &= \left\{[\mathrm{id}_{\mathfrak{u}}], [\theta], \left[\theta^2\right], [\mathrm{Ad}(Y)], [\theta\,\mathrm{Ad}(Y)], \left[\theta^2\,\mathrm{Ad}(Y)\right]\right\} \cong \mathcal{S}_3 \to \mathbf{1},\end{aligned}$$

which is split: We take $\theta' \in \mathrm{Aut}(\mathfrak{so}(8,\mathbb{C}))$ and $\theta \in \mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))$ as described in and with the notation of Remark 2.7. The subgroup $\Gamma \leq \mathrm{Aut}(\mathfrak{u})^\sigma$ generated by $\mathrm{Ad}(Y)$ and θ is mapped onto $\mathcal{S}_3$ by a group morphism. So the First Isomorphism Theorem yields that $\mathcal{S}_3$ is isomorphic to a quotiemt of Γ, thus it suffices to show that the order of Γ is at most six.
We calculate, for the standard Chevalley generators h_i, x_i, y_i, where $i = 1,2,3,4$, of $\mathfrak{so}(8,\mathbb{C})$ and $\mathfrak{x} := \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3 \oplus \mathbb{C}x_4$:

$$T_1 := \left[\theta'_{|\mathfrak{h}}\right]_{(h_1,\ldots,h_4)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix},$$

$$T_2 := \left[\theta'_{|\mathfrak{x}}\right]_{(x_1,\ldots,x_4)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$Y_1 := \left[\mathrm{Ad}(Y)_{|\mathfrak{h}}\right]_{(h_1,\ldots,h_4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$Y_2 := \left[\mathrm{Ad}(Y)_{|\mathfrak{x}}\right]_{(x_1,\ldots,x_4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The identities $T_1Y_1T_1 = Y_1$ and $T_2Y_2T_2 = Y_2$ show[8] that $\theta'\,\mathrm{Ad}(Y)\theta' =$

[8]The automorphisms are uniquely determined by their values on the h_i' s and x_i's and by commuting with the conjugation $\tau : \mathfrak{so}(8,\mathbb{R}) + i\mathfrak{so}(8,\mathbb{R}) = \mathfrak{so}(8,\mathbb{C}) \to \mathfrak{so}(8,\mathbb{C})$.

$\mathrm{Ad}(Y)$ on $\mathfrak{u}_{\mathbb{C}}$, so also $\theta\,\mathrm{Ad}(Y)\theta = \mathrm{Ad}(Y)$ on $\mathfrak{u}$, hence

$$\mathrm{Ad}(Y)\left\{\mathbf{1},\theta,\theta^2\right\}\mathrm{Ad}(Y) = \left\{\mathbf{1},\theta,\theta^2\right\},$$

thus $\{\mathbf{1},\theta,\theta^2\} \trianglelefteq \Gamma$ and $\{\mathbf{1},\mathrm{Ad}(Y)\}\cdot\{\mathbf{1},\theta,\theta^2\}$ is a subgroup of Γ, yielding $\{\mathbf{1},\mathrm{Ad}(Y)\}\cdot\{\mathbf{1},\theta,\theta^2\} = \Gamma$, showing $\#\Gamma \leq 6$ and hence $\Gamma \cong \mathcal{S}_3$.

vii. $X_r^* \in \left\{E_{6(2)}, E_{6(-14)}\right\}$:
We use the the notation of Lemma 2.8. The involution σ is inner, so we have $\sigma = e^Y = \exp_{\mathrm{Aut}(\mathfrak{u})}(\mathrm{ad}_{\mathfrak{u}}(Y))$ or $\sigma = e^Z = \exp_{\mathrm{Aut}(\mathfrak{u})}(\mathrm{ad}_{\mathfrak{u}}(Z))$, respectively. Since $\mathrm{Aut}(\mathfrak{u})^\sigma/U_{\mathrm{ad}}^\sigma$ is isomorphic to a subgroup of $\pi_0\left(\mathrm{Aut}(\mathfrak{u})\right)$, which is isomorphic to $\mathcal{C}_2$ by Theorem 2.4, and the non-trivial Dynkin diagram involution ρ_1 is outer and commutes with e^Y and e^Z, we obtain the split short exact seqence

$$\mathbf{1} \to U_{\mathrm{ad}}^\sigma \to \mathrm{Aut}(\mathfrak{u})^\sigma \to \mathrm{Aut}(\mathfrak{u})^\sigma/U_{\mathrm{ad}}^\sigma = \{[\mathrm{id}_{\mathfrak{u}}],[\rho_1]\} \cong \mathcal{C}_2 \to \mathbf{1}.$$

■

Corollary 2.15. *If $\mathfrak{g}$ is a real central simple non-compact Lie algebra, then the following table provides all possibilities for $\pi_0(\mathrm{Aut}(\mathfrak{g}))$.*

$\mathfrak{g}$	$\pi_0(\mathrm{Aut}(\mathfrak{g}))$	$\#\,\mathrm{Conj}(\pi_0(\mathrm{Aut}(\mathfrak{g})))$
$\mathfrak{sl}(2,\mathbb{R})$	$\mathcal{C}_2$	2
$\mathfrak{sl}(n+1,\mathbb{R})$ *for even* $n\geq 2$	$\mathcal{C}_2$	2
$\mathfrak{sl}(n+1,\mathbb{R})$ *for odd* $n\geq 3$	$\mathcal{C}_2\times\mathcal{C}_2$	4
$\mathfrak{su}(q,n+1-q,\mathbb{C})$ *for* $q<\frac{n+1}{2}>1$	$\mathcal{C}_2$	2
$\mathfrak{su}(n+1,n+1,\mathbb{C})$ *for* $n\geq 1$	$\mathcal{C}_2\times\mathcal{C}_2$	4
$\mathfrak{sl}(n+1,\mathbb{H}) = \mathfrak{su}^*(2n+2,\mathbb{C})$ *for* $n\geq 1$	$\mathcal{C}_2$	2
$\mathfrak{so}(q,2n+1-q,\mathbb{R})$ *for* $q\leq n\geq 2$	$\mathcal{C}_2$	2
$\mathfrak{sp}(2n,\mathbb{R})$ *for* $n\geq 3$	$\mathcal{C}_2$	2
$\mathfrak{sp}(n,n,\mathbb{H})$ *for* $n\geq 1$	$\mathcal{C}_2$	2
$\mathfrak{so}(q,2n-q,\mathbb{R})$ *for odd* $q<n\geq 4$	$\mathcal{C}_2$	2
$\mathfrak{so}(q,2n-q,\mathbb{R})$ *for even* $q<n\geq 4$	$\mathcal{C}_2\times\mathcal{C}_2$	4
$\mathfrak{so}(n,n,\mathbb{R})$ *for odd* $n\geq 5$	$\mathcal{C}_2\times\mathcal{C}_2$	4
$\mathfrak{so}(n,n,\mathbb{R})$ *for even* $n\geq 6$	$\mathcal{D}_4$	5
$\mathfrak{so}(4,4,\mathbb{R})$	$\mathcal{S}_4$	5
$\mathfrak{so}^*(4n,\mathbb{R})$ *for* $n\geq 3$	$\mathcal{C}_2$	2
$\mathfrak{so}^*(4n+2,\mathbb{R})$ *for* $n\geq 2$	$\mathcal{C}_2$	2
$\mathfrak{e}_{6(j)}$ *for* $j=6,2,-14,-26$	$\mathcal{C}_2$	2
$\mathfrak{e}_{7(j)}$ *for* $j=7,-25$	$\mathcal{C}_2$	2
all others	$\mathbf{1}$	1

Corollary 2.16. *If $\mathfrak{g}$ is a real central simple non-compact Lie algebra such that $\pi_0(\mathrm{Aut}(\mathfrak{g})) = \mathbf{1}$ or $\mathfrak{g}$ is isomorphic to one of the following:*

1. $\mathfrak{sl}(n+1,\mathbb{R})$ *for even* $n\geq 2$*,*

2. $\mathfrak{su}(q,n+1-q,\mathbb{C})$ *for* $q<\frac{n+1}{2}>1$*,*

3. $\mathfrak{sl}(n+1,\mathbb{H}) = \mathfrak{su}^*(2n+2,\mathbb{C})$ *for* $n \geq 1$,

4. $\mathfrak{so}(q,2n-q,\mathbb{R})$ *for odd* $q < n \geq 4$,

5. $\mathfrak{so}^*(4n+2,\mathbb{R})$ *for* $n \geq 2$,

6. $\mathfrak{e}_{6(j)}$ *for* $j = 6, 2, -14, -26$,

then there is an isomorphism $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$.

Proof. By Proposition 2.14, we have $(\mathrm{Aut}(\mathfrak{u})_0)^\sigma = ((\mathrm{Aut}(\mathfrak{u})_0)^\sigma)_0 = (\mathrm{Aut}(\mathfrak{u})^\sigma)_0$ in the mentioned cases, yielding $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$. ■

Corollary 2.17. *If* $\mathfrak{g}$ *is a real central simple non-compact Lie algebra isomorphic to one of the following:*

1. $\mathfrak{so}(q,2n+1-q,\mathbb{R})$ *for* $q \leq n \geq 2$,

2. $\mathfrak{so}(q,2n-q,\mathbb{R})$ *for even* $q < n \geq 4$,

3. $\mathfrak{so}(n,n,\mathbb{R})$ *for odd* $n \geq 5$,

4. $\mathfrak{so}(n,n,\mathbb{R})$ *for even* $n \geq 6$,

5. $\mathfrak{so}(4,4,\mathbb{R})$,

6. $\mathfrak{so}^*(4n,\mathbb{R})$ *for* $n \geq 3$,

then $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$.

Proof. We will see that, in the mentioned cases and with the notation of Remark 2.13, there are representatives of the elements in $\pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma)$ which form a subgroup of $\mathrm{Aut}(\mathfrak{u})^\sigma$ isomorphic to $\pi_0(\mathrm{Aut}(\mathfrak{u})^\sigma)$. Hence, by Proposition 2.14.3, we have $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$.

1. $\mathfrak{g} \cong \mathfrak{so}(q,2n+1-q,\mathbb{R})$ for $q \leq n \geq 2$:
   $\mathrm{Aut}(\mathfrak{so}(2n+1,\mathbb{R}))$ is connected by Theorem 2.4, so $\mathrm{Ad}(\mathrm{SO}(2n+1,\mathbb{R})) = \mathrm{Aut}(\mathfrak{so}(2n+1,\mathbb{R}))$. Thus, in analogy with the proof of Proposition 2.14.4, Case 4, iii, we know that
   $$\pi_0(\mathrm{Aut}(\mathfrak{so}(2n+1,\mathbb{R}))^\sigma) = \{\mathbf{1}, [\mathrm{Ad}(X_2)]\} \cong \mathcal{C}_2$$
   for $X_2 = \begin{pmatrix} I_{1,2n-q} & \mathbf{0} \\ \mathbf{0} & I_{1,q-1} \end{pmatrix}$.

2. $\mathfrak{g} \cong \mathfrak{so}(q,2n-q,\mathbb{R})$ for even $q < n \geq 4$:
   By the proof of Proposition 2.14.4, Case 4, iii and v, we know that
   $$\pi_0(\mathrm{Aut}(\mathfrak{so}(2n,\mathbb{R}))^\sigma) = \{\mathbf{1}, [\mathrm{Ad}(X_2)], [\mathrm{Ad}(X_3)], [\mathrm{Ad}(X_4)]\} \cong \mathcal{C}_2 \times \mathcal{C}_2$$
   for $X_2 = \begin{pmatrix} I_{1,2n-q-1} & \mathbf{0} \\ \mathbf{0} & I_{1,q-1} \end{pmatrix}$, $X_3 = \begin{pmatrix} I_{1,2n-q-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q \end{pmatrix}$, $X_4 = X_2X_3$.

3. $\mathfrak{g} \cong \mathfrak{so}(n,n,\mathbb{R})$ for odd $n \geq 5$:
   In analogy with the proof of Proposition 2.14.4, Case 4, ii, we have:

$$\begin{aligned}
&\mathrm{Aut}(\mathfrak{so}(2n,\mathbb{R}))^\sigma \\
&= \{\mathrm{Ad}(X) | \, X \in \mathrm{O}(2n,\mathbb{R}) \text{ and } I_{n,n} X I_{n,n} = \pm X\} \\
&= \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \text{ or } X = \begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix} \text{ with } A, B \in \mathrm{O}(n,\mathbb{R}) \right\}, \\
&(\mathrm{Aut}(\mathfrak{so}(2n,\mathbb{R}))_0)^\sigma \\
&= \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \text{ with } A, B \in \mathrm{O}(n,\mathbb{R}), \det(A) = \det(B) \right\} \\
&\cup \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix} \text{ with } A, B \in \mathrm{O}(n,\mathbb{R}), \det(A) = -\det(B) \right\}, \\
&((\mathrm{Aut}(\mathfrak{so}(2n,\mathbb{R}))_0)^\sigma)_0 = (\mathrm{Aut}(\mathfrak{so}(2n,\mathbb{R}))^\sigma)_0 \\
&= \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \text{ with } A, B \in \mathrm{SO}(n,\mathbb{R}) \right\}.
\end{aligned}$$

   We know that n is odd, $\mathrm{Ad}(A) = \mathrm{Ad}(-A)$ for each $A \in \mathrm{O}(n,\mathbb{R})$ and $\mathrm{SO}(n,\mathbb{R}) \to \mathrm{O}(n,\mathbb{R}) \setminus \mathrm{SO}(n,\mathbb{R})$, $A \mapsto -A$ is a bijection, so we can also write:

$$\begin{aligned}
&(\mathrm{Aut}(\mathfrak{so}(2n,\mathbb{R}))_0)^\sigma \\
&= \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \text{ or } X = \begin{pmatrix} \mathbf{0} & A \\ -B & \mathbf{0} \end{pmatrix} \text{ with } A, B \in \mathrm{SO}(n,\mathbb{R}) \right\}.
\end{aligned}$$

   Thus we have

$$\pi_0\left(\mathrm{Aut}(\mathfrak{so}(2n,\mathbb{R}))^\sigma\right) = \{\mathbf{1}, [\mathrm{Ad}(X_3)], [\mathrm{Ad}(X_5)], [\mathrm{Ad}(X_8)]\} \cong \mathcal{C}_2 \times \mathcal{C}_2$$

   for $X_3 = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix}$, $X_5 = \begin{pmatrix} I_{1,n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_n \end{pmatrix}$, $X_8 = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -I_{1,n-1} & \mathbf{0} \end{pmatrix}$.

4. $\mathfrak{g} \cong \mathfrak{so}(n,n,\mathbb{R})$ for even $n \geq 6$:
   By the proof of Proposition 2.14.4, Case 4, ii, we know that

$$\pi_0\left(\mathrm{Aut}(\mathfrak{so}(2n,\mathbb{R}))^\sigma\right) = \{\mathbf{1}, [\mathrm{Ad}(X_2)], \ldots, [\mathrm{Ad}(X_8)]\} \cong \mathcal{D}_4$$

   for $X_2 = \begin{pmatrix} I_{1,n-1} & \mathbf{0} \\ \mathbf{0} & I_{1,n-1} \end{pmatrix}$, $X_3 = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix}$, $X_4 = \begin{pmatrix} \mathbf{0} & I_{1,n-1} \\ -I_{1,n-1} & \mathbf{0} \end{pmatrix}$, $X_5 = \begin{pmatrix} I_{1,n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_n \end{pmatrix}$, $X_6 = \begin{pmatrix} \mathbf{1}_n & \mathbf{0} \\ \mathbf{0} & -I_{1,n-1} \end{pmatrix}$, $X_7 = \begin{pmatrix} \mathbf{0} & I_{1,n-1} \\ \mathbf{1}_n & \mathbf{0} \end{pmatrix}$, $X_8 = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -I_{1,n-1} & \mathbf{0} \end{pmatrix}$.

5. $\mathfrak{g} \cong \mathfrak{so}(4,4,\mathbb{R})$:

In analogy with the proof of Proposition 2.14.4, Case 4, ii, we have:

$$
\begin{aligned}
&\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))^\sigma \\
&\supseteq \{\mathrm{Ad}(X) |\, X \in \mathrm{O}(8,\mathbb{R}) \text{ and } I_{4,4} X I_{4,4} = \pm X\} \\
&= \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \text{ or } X = \begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix} \text{ with } A, B \in \mathrm{O}(4,\mathbb{R}) \right\}, \\
&(\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))_0)^\sigma \\
&= \Bigg\{ \mathrm{Ad}(X) \,\Bigg|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \text{ or } X = \begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix} \\
&\qquad \text{with } A, B \in \mathrm{O}(4,\mathbb{R}), \det(A) = \det(B) \Bigg\}, \\
&((\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))_0)^\sigma)_0 = (\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))^\sigma)_0 \\
&= \left\{ \mathrm{Ad}(X) \,\middle|\, X = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \text{ with } A, B \in \mathrm{SO}(4,\mathbb{R}) \right\}, \\
&\pi_0\left((\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))_0)^\sigma\right) \\
&= \{\mathbf{1}, [\mathrm{Ad}(X_2)], [\mathrm{Ad}(X_3)], [\mathrm{Ad}(X_4)]\} \cong \mathcal{C}_2^2
\end{aligned}
$$

for $X_2 = \begin{pmatrix} I_{3,1} & \mathbf{0} \\ \mathbf{0} & I_{3,1} \end{pmatrix}$, $X_3 = \begin{pmatrix} \mathbf{0} & \mathbf{1}_4 \\ -\mathbf{1}_4 & \mathbf{0} \end{pmatrix}$, $X_4 = \begin{pmatrix} \mathbf{0} & I_{3,1} \\ -I_{3,1} & \mathbf{0} \end{pmatrix}$. By the proof of Proposition 2.14.4, Case 4, vi, we know that

$$
\begin{aligned}
&\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))^\sigma / (\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))_0)^\sigma \\
&\cong \left\{\mathbf{1}, [\theta], \left[\theta^2\right], [\mathrm{Ad}(Y)], [\theta\,\mathrm{Ad}(Y)], \left[\theta^2\,\mathrm{Ad}(Y)\right]\right\} \cong \mathcal{S}_3
\end{aligned}
$$

for the matrix $Y = \begin{pmatrix} \mathbf{1}_4 & \mathbf{0} \\ \mathbf{0} & -I_{3,1} \end{pmatrix}$ and the triality automorphism θ as described in and with the notation of Remark 2.7 and that the corresponding short exact sequence is split. Furthermore, $\pi_0\left(\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))^\sigma\right) \cong \mathcal{S}_4$.

We will show that also the short exact sequence

$$
\mathbf{1} \to (\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))^\sigma)_0 \to \mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))^\sigma \to \pi_0\left(\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))^\sigma\right) \cong \mathcal{S}_4 \to \mathbf{1}
$$

is split by showing that the subgroup Γ of $\mathrm{Aut}(\mathfrak{so}(8,\mathbb{R}))^\sigma$ generated by $\{\mathrm{Ad}(X_2), \mathrm{Ad}(X_3), \mathrm{Ad}(Y), \theta\}$ is isomorphic to $\mathcal{S}_4$. We already know that Γ is mapped onto $\mathcal{S}_4$ by a group morphism. So the First Isomorphism Theorem yields that $\mathcal{S}_4$ is isomorphic to a quotiemt of Γ, thus it suffices to show that the order of Γ is at most 24.

First consider the subgroup $S \leq \Gamma$ generated by $\{\mathrm{Ad}(X_2), \mathrm{Ad}(X_3), \mathrm{Ad}(Y)\}$. With $a := \mathrm{Ad}(X_2)$, $b := \mathrm{Ad}(X_3)$, $c := \mathrm{Ad}(Y)$ we see that $ab = ba$ and $ac = ca$ and hence each product of three different elements is equal to abc or acb. We calculate:

$$
\begin{aligned}
cb &= \mathrm{Ad}\left(\begin{pmatrix} \mathbf{1}_4 & \mathbf{0} \\ \mathbf{0} & -I_{3,1} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbf{1}_4 \\ -\mathbf{1}_4 & \mathbf{0} \end{pmatrix}\right) \\
&= \mathrm{Ad}\begin{pmatrix} \mathbf{0} & \mathbf{1}_4 \\ I_{3,1} & \mathbf{0} \end{pmatrix} = \mathrm{Ad}\left(\begin{pmatrix} \mathbf{0} & I_{3,1} \\ \mathbf{1}_4 & \mathbf{0} \end{pmatrix}\begin{pmatrix} I_{3,1} & \mathbf{0} \\ \mathbf{0} & I_{3,1} \end{pmatrix}\right) = bca = abc.
\end{aligned}
$$

Multiplying by a from the left yields: $acb = bc$. So the fact that also b, c are of order two yields: $S = \{\mathbf{1}, a, b, c, ab, ac, bc, cb\}$. In S, the elements $d := ab$

and $e := bc$ are of order two and four, respectively, and fulfill the relation $de = (ab)(bc) = ac = ca = (cb)(ba) = (cabbccab)(ab) = e^3 d$, hence $S \cong \mathcal{D}_4$ is of order eight.

We calculate, for the standard Chevalley generators h_i, x_i, y_i, where $i = 1, 2, 3, 4$, of $\mathfrak{so}(8, \mathbb{C})$ and $x_5 := \frac{1}{4}[y_1, [y_2, y_4]]$, $x_6 := \frac{1}{16}[[x_1, x_2], [x_3, [x_4, x_2]]]$, $x_7 := \frac{1}{4}[y_3, [y_2, y_1]]$, $x_8 := \frac{1}{4}[y_4, [y_2, y_3]]$ and $\mathfrak{x} := \mathbb{C}x_1 \oplus \ldots \oplus \mathbb{C}x_8$:

$$T_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \; D_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; T_2 = \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \end{pmatrix},$$

where $T_1 := \left[\theta'_{|\mathfrak{h}}\right]_{(h_1,\ldots,h_4)}$, $D_1 := \left[d_{|\mathfrak{h}}\right]_{(h_1,\ldots,h_4)}$, $E_1 := \left[e_{|\mathfrak{h}}\right]_{(h_1,\ldots,h_4)}$, $T_2 := \left[\theta'_{|\mathfrak{x}}\right]_{(x_1,\ldots,x_8)}$, $D_2 := \left[\theta'_{|\mathfrak{x}}\right]_{(x_1,\ldots,x_8)}$, $E_2 := \left[\theta'_{|\mathfrak{x}}\right]_{(x_1,\ldots,x_8)}$. The identities $T_\ell D_\ell = D_\ell E_\ell^2 T_\ell$ and $T_\ell E_\ell = D_\ell E_\ell T_\ell^2$ for $\ell = 1, 2$ show that $\theta d = de^2\theta$ and $\theta e = de\theta^2$ hold on $\mathfrak{so}(8, \mathbb{R})$, hence:

$$\Gamma = \left\{d^i e^j \theta^k : i \in \{0,1\}, j \in \{0,1,2,3\}, k \in \{0,1,2\}\right\},$$

showing $\#\Gamma \leq 24$ and thus $\Gamma \cong \mathcal{S}_4$.

6. $\mathfrak{g} \cong \mathfrak{so}^*(4n, \mathbb{R})$ for $n \geq 3$:
By the proof of Proposition 2.14.4, Case 4, iv, we have:

$$\begin{aligned} (\mathrm{Aut}(\mathfrak{so}(4n,\mathbb{R}))^\sigma)_0 &= \{\mathrm{Ad}(X)\,|X \in \mathrm{U}(2n,\mathbb{C})\}, \\ \mathrm{Aut}(\mathfrak{so}(4n,\mathbb{R}))^\sigma &= (\mathrm{Aut}(\mathfrak{so}(4n,\mathbb{R}))_0)^\sigma \\ &= (\mathrm{Aut}(\mathfrak{so}(4n,\mathbb{R}))^\sigma)_0 \\ &\quad \cup \left\{\mathrm{Ad}(X)\,\middle|\, X \in I_{2n,2n}\,\mathrm{U}(2n,\mathbb{C}), \det_{\mathbb{R}}(X) = 1\right\}. \end{aligned}$$

Thus we have:

$$\pi_0\left(\mathrm{Aut}(\mathfrak{so}(4n,\mathbb{R}))^\sigma\right) = \{\mathbf{1}, [\mathrm{Ad}(I_{2n,2n})]\} \cong \mathcal{C}_2.$$

■

The split real forms of complex simple Lie algebras and the hermitian Lie algebras both form important classes of real central simple Lie algebras. We treat them in the next two theorems.

Theorem 2.18. *If $\mathfrak{g}$ is a hermitian Lie algebra, i.e. isomorphic to one of* $\mathfrak{sl}(2,\mathbb{R})$*,* $\mathfrak{su}(q,n+1-q,\mathbb{C})$ *for* $q \leq \frac{n+1}{2} > 1$*,* $\mathfrak{so}^*(2n+6,\mathbb{R})$*,* $\mathfrak{so}(2,n+2,\mathbb{R})$ *or* $\mathfrak{sp}(2n+4,\mathbb{R})$ *for some* $n \in \mathbb{N}$ *or isomorphic to* $\mathfrak{e}_{6(-14)}$ *or* $\mathfrak{e}_{7(-25)}$*, then* $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$.

Proof. Let $\mathfrak{g} = \mathfrak{k} \oplus_\kappa^\tau \mathfrak{p}$ be a Cartan decomposition, G a Lie simply connected Lie group with Lie algebra $\mathfrak{g}$ and $K \leq G$ a connected compact subalgebra with Lie algebra $\mathfrak{k}$. Since $\mathfrak{g}$ is hermitian, $\mathfrak{z}(\mathfrak{k}) = \mathbb{R} \cdot z$ for some $0 \neq z \in \mathfrak{k}$. Kobayashi has shown (cf. Lemma 2.4 of [10]) the existence of an involutive Lie group automorphism of G such that its derivative in $\mathbf{1} \in G$ is a Lie algebra automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ with $\omega(z) = -z$ and $\omega\tau = \tau\omega$.

By Corollary 2.16 and Corollary 2.17, we have to prove the statement of the theorem only if $\mathfrak{g}$ is isomorphic to one of $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{e}_{7(-25)}$ or $\mathfrak{sp}(2n+4,\mathbb{R})$ for some $n \in \mathbb{N}$. By $\pi_0(\mathrm{Aut}(\mathfrak{g})) \cong \mathcal{C}_2$, it suffices to show that $\omega \in \mathrm{Aut}(\mathfrak{g})$ is outer.

Supposing ω inner, there are $n \in \mathbb{N}$, $k_1, \ldots, k_n \in \mathfrak{k}$ and $p \in \mathfrak{p}$ such that $\omega = e^{\mathrm{ad}(p)} e^{\mathrm{ad}_{\mathfrak{g}}(k_n)} \cdots e^{\mathrm{ad}_{\mathfrak{g}}(k_1)}$ (cf. Cartan Decomposition Theorem 12.1.7 of [7]), thus $\omega(z) = e^{\mathrm{ad}(p)}(z)$.

Let $B_\tau : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the euclidean scalar product from Lemma 2.2. Then $(\mathrm{ad}(p))^{2\ell+1}(z) \in \mathfrak{p} = \mathfrak{k}^{\perp_{B_\tau}}$ for all $\ell \in \mathbb{N}_0$, so $B_\tau(z,\omega(z)) = B_\tau(z,\omega'(z))$ for the map

$$\omega' : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad x \longmapsto \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} \left(\mathrm{ad}_{\mathfrak{g}}(p)\right)^{2\ell}(x) = \cosh\left(\mathrm{ad}(p)\right)(x).$$

The B_τ-symmetry of $\mathrm{ad}(p)$ yields:

$$B_\tau\left(x, (\mathrm{ad}(p))^{2\ell}(x)\right) = B_\tau\left((\mathrm{ad}(p))^{\ell}(x), (\mathrm{ad}(p))^{\ell}(x)\right) \geq 0$$

for all $x \in \mathfrak{g}$ and $\ell \in \mathbb{N}_0$. So we know that $B_\tau(z,\omega(z)) = B_\tau(z,\omega'(z)) \geq B_\tau(z,z)$ is positive, contradicting the fact that $\omega(z) = -z$. So ω is outer. ■

Theorem 2.19. *If $\mathfrak{g}$ is a split real forms of its complexification, i.e. isomorphic to one of* $\mathfrak{sl}(n+1,\mathbb{R})$*,* $\mathfrak{so}(n+1,n+2,\mathbb{R})$*,* $\mathfrak{sp}(2n+4,\mathbb{R})$*,* $\mathfrak{so}(n+3,n+3,\mathbb{R})$ *for some* $n \in \mathbb{N}$ *or isomorphic to* $\mathfrak{e}_{6(6)}$*,* $\mathfrak{e}_{7(7)}$*,* $\mathfrak{e}_{8(8)}$*,* $\mathfrak{f}_{4(4)}$ *or* $\mathfrak{g}_{2(2)}$*, then* $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$.

Proof. By Corollary 2.16, Corollary 2.17 and Theorem 2.18, we have to prove the statement of the theorem only if $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(n+1,\mathbb{R})$ for some odd $n \geq 3$ or $\mathfrak{e}_{7(7)}$. We will show this by finding, with the notation of Remark 2.13, representatives of the elements in $\pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)$ forming a subgroup of $\mathrm{Aut}(\mathfrak{u})^\sigma$ isomorphic to $\pi_0\left(\mathrm{Aut}(\mathfrak{u})^\sigma\right)$.

1. $\mathfrak{g} \cong \mathfrak{sl}(n+1,\mathbb{R})$ for odd $n \geq 3$:
By the proof of Proposition 2.14.1 and 2.14.4, Case 2, we know that the group $(\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))_0)^{\mathrm{cj}}$ has two connected components and there is a short exact sequence

$$\mathbf{1} \longrightarrow (\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))_0)^{\mathrm{cj}} \longrightarrow \mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))^{\mathrm{cj}} \longrightarrow \{\mathbf{1},[\mathrm{cj}]\} \longrightarrow \mathbf{1}$$

which is split because cj is of order two in $\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))^{\mathrm{cj}}$. Since $\mathrm{SU}(n+1,\mathbb{C})$ is compact and connected, we know that $\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))_0 = \{\mathrm{Ad}(X) |\, X \in \mathrm{SU}(n+1,\mathbb{C})\}$, so we can write:

$$\begin{aligned}(\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))_0)^{\mathrm{cj}} &= \{\mathrm{Ad}(X) |\, X \in \mathrm{SU}(n+1,\mathbb{C})\}^{\mathrm{cj}} \\ &= \{\mathrm{Ad}(X) |\, X \in \mathrm{U}(n+1,\mathbb{C})\}^{\mathrm{cj}} \\ &= \big\{\mathrm{Ad}(X) |\, X \in \mathrm{U}(n+1,\mathbb{C}) \text{ and } X^{-1}\overline{X}x = xX^{-1}\overline{X} \\ &\quad \text{for all } x \in \mathfrak{su}(n+1,\mathbb{C}) \supseteq \mathfrak{so}(n+1,\mathbb{R})\big\}.\end{aligned}$$

By Lemma 2.11 in the case $\mathbb{K} = \mathbb{C}$, we may write:

$$\begin{aligned}(\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))_0)^{\mathrm{cj}} &= \big\{\mathrm{Ad}(X) |\, X \in \mathrm{U}(n+1,\mathbb{C}) \text{ and } X^{-1}\overline{X} \in \mathbb{C}\mathbf{1}_{n+1}\big\} \\ &= \{\mathrm{Ad}(X) |\, X \in \mathrm{U}(n+1,\mathbb{C}) \\ &\quad \text{and } \overline{X} = \lambda X \text{ for some } \lambda \in \mathbb{C}\big\}.\end{aligned}$$

For $X \in \mathrm{U}(n+1,\mathbb{C})$ the condition $\overline{X} = -X$ implies $X = iR$ and hence $\mathrm{Ad}(X) = \mathrm{Ad}(R)$ for some real-valued matrix R which is then orthogonal. The condition $\overline{X} = \lambda X$ for some $\lambda \in \mathbb{C}\backslash\{-1\}$ yields that $\mathrm{Ad}(X) = \mathrm{Ad}(|\mu|\, Y)$ for $\mu := \frac{2}{1+\lambda}$ and the real-valued matrix $Y := \frac{X+\overline{X}}{2} = \mu^{-1}X$. But also $|\mu|\, Y$ is orthogonal:

$$(|\mu|\, Y)^T (|\mu|\, Y) = |\mu|^2\, Y^T Y = \overline{(\mu Y)^T}\, (\mu Y) = \overline{(X)^T} X = \mathbf{1}_{n+1}.$$

So we have:

$$(\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))_0)^{\mathrm{cj}} = \{\mathrm{Ad}(X) |\, X \in \mathrm{O}(n+1,\mathbb{R})\}.$$

By Corollary 2.12, the involution $\mathrm{Ad}(I_{1,n}) \in (\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))_0)^{\mathrm{cj}}$ is not in the connected component of $(\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))_0)^{\mathrm{cj}}$. Since cj and $\mathrm{Ad}(I_{1,n})$ commute, they form an elementary abelian subgroup of order four in the group $\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))^{\mathrm{cj}}$ and we have a split short exact sequence

$$\begin{aligned}\mathbf{1} &\longrightarrow \left(\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))^{\mathrm{cj}}\right)_0 = \left((\mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))_0)^{\mathrm{cj}}\right)_0 \\ &\longrightarrow \mathrm{Aut}(\mathfrak{su}(n+1,\mathbb{C}))^{\mathrm{cj}} \longrightarrow \{\mathbf{1},[\mathrm{cj}],[\mathrm{Ad}(I_{1,n})],[\mathrm{cj}\,\mathrm{Ad}(I_{1,n})]\} \longrightarrow \mathbf{1}.\end{aligned}$$

2. $\mathfrak{g} \cong \mathfrak{e}_{7(7)}$:
By the proof of Proposition 2.14.1, and 2.14.4, Case 2, we know that $\left(\mathrm{Aut}(\mathfrak{e}_{7(-133)})_0\right)^{\sigma_{\mathfrak{e}_{7(7)}}} = \mathrm{Aut}(\mathfrak{e}_{7(-133)})^{\sigma_{\mathfrak{e}_{7(7)}}}$ has two connected components. By Lemma 2.9, the non-trivial element of $\pi_0\left(\mathrm{Aut}(\mathfrak{e}_{7(-133)})^{\sigma_{\mathfrak{e}_{7(7)}}}\right)$ is represented by $e^{\mathrm{ad}(X)}$ for any non-zero element $X \in \mathfrak{e}_{7(-133)}^{-\sigma_{\mathfrak{e}_{7(7)}}}$ such that $\mathrm{ad}(X)^3 =$

$-\pi^2\,\mathrm{ad}(X)$. So the minimal polynomial of $\mathrm{ad}(X)$ is $p(\lambda)=\lambda(\lambda^2+\pi^2)$, yielding the existence of a basis of $\mathfrak{g}$ with respect to which $\mathrm{ad}(X)$ is represented by a block diagonal matrix $\mathrm{diag}\left(0,\ldots,0,\begin{pmatrix}0&\pi\\-\pi&0\end{pmatrix},\ldots,\begin{pmatrix}0&\pi\\-\pi&0\end{pmatrix}\right)$, hence $\mathrm{diag}\left(1,\ldots,1,\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\ldots,\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right)$ represents $e^{\mathrm{ad}(X)}$, which is of order two in $\mathrm{Aut}(\mathfrak{e}_{7(-133)})^{\sigma_{\mathfrak{e}_{7(7)}}}$.

$\blacksquare$

There is only one isomorphism class of real central simple Lie algebras $\mathfrak{g}$ left, for which we have not yet proved the isomorphism $\mathrm{Aut}(\mathfrak{g})\cong\mathrm{Aut}(\mathfrak{g})_0\rtimes\pi_0(\mathrm{Aut}(\mathfrak{g}))$.

Theorem 2.20. *If $\mathfrak{g}$ is a real central simple Lie algebra which is isomorphic to $\mathfrak{sp}(n,n,\mathbb{H})$ for some $n\in\mathbb{N}$, then* $\mathrm{Aut}(\mathfrak{g})\cong\mathrm{Aut}(\mathfrak{g})_0\rtimes\pi_0(\mathrm{Aut}(\mathfrak{g}))$.

Proof. We know that $(\mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{H}))_0)^{\mathrm{Ad}(I_{n,n})}=\mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{H}))^{\mathrm{Ad}(I_{n,n})}$ has two connected components by the proof of Proposition 2.14.1, and 2.14.4, Case 2. Since $\mathrm{Sp}(2n,\mathbb{H})$ is compact and connected, we know that $\mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{H}))_0=\{\mathrm{Ad}(X)|\,X\in\mathrm{Sp}(2n,\mathbb{H})\}$, so we can write:

$$(\mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{H}))_0)^{\mathrm{Ad}(I_{n,n})}=\{\mathrm{Ad}(X)|\,X\in\mathrm{Sp}(2n,\mathbb{H})\}^{\mathrm{Ad}(I_{n,n})}.$$

Note that $X\in\mathrm{Sp}(2n,\mathbb{H})$ means that X preserves the standard hermitian form $\langle(x_i)_i,(y_i)_i\rangle:=\sum_i\mathrm{qj}(x_i)y_i$ on $\mathbb{H}^{2n}$. Then also $I_{n,n}\in\mathrm{Sp}(2n,\mathbb{H})$, thus also $XI_{n,n}X^{-1}I_{n,n}\in\mathrm{Sp}(2n,\mathbb{H})$ and we can write:

$$\begin{aligned}&(\mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{H}))_0)^{\mathrm{Ad}(I_{n,n})}=\\&\left\{\mathrm{Ad}(X)|\,X\in\mathrm{Sp}(2n,\mathbb{H})\text{ and }\mathrm{Ad}(XI_{n,n}X^{-1}I_{n,n})\in\overbrace{\ker\left(\mathrm{Ad}_{\mathrm{Sp}(2n,\mathbb{H})}\right)}^{=\mathrm{Z}(\mathrm{Sp}(2n,\mathbb{H}))=\{\pm\mathbf{1}_{2n}\}}\right\}\\&=\left\{\mathrm{Ad}(X)|\,X\in\mathrm{Sp}(2n,\mathbb{H})\text{ and }\mathrm{Ad}\left(XI_{n,n}X^{-1}I_{n,n}\right)=\pm\mathbf{1}_{2n}\right\}.\end{aligned}$$

So, for $J_n:=\begin{pmatrix}\mathbf{0}&\mathbf{1}_n\\-\mathbf{1}_n&\mathbf{0}\end{pmatrix}$, we have $\mathrm{Ad}(J_n)\in(\mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{H}))_0)^{\mathrm{Ad}(I_{n,n})}$, which is of order two. It remains to show that $\mathrm{Ad}(J_n)\notin\left((\mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{H}))_0)^{\mathrm{Ad}(I_{n,n})}\right)_0=\left(\mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{H}))^{\mathrm{Ad}(I_{n,n})}\right)_0$. By Lemma 2.5, the map

$$\begin{aligned}\pi_0\left(\mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{H}))^{\mathrm{Ad}(I_{n,n})}\right)&\to\pi_0\left(\mathrm{Aut}(\mathfrak{sp}(n,\mathbb{H})\oplus\mathfrak{sp}(n,\mathbb{H}))\right)\\&\cong\pi_0\left(\mathrm{Aut}(\mathfrak{sp}(n,\mathbb{H}))\times\mathrm{Aut}(\mathfrak{sp}(n,\mathbb{H}))\right)\cong\mathcal{C}_2\end{aligned}$$

induced by restriction is injective. But $\mathrm{Ad}(J_n)_{|\mathfrak{sp}(n,\mathbb{H})\oplus\mathfrak{sp}(n,\mathbb{H})}$ represents the only non-trivial class in $\pi_0\left(\mathrm{Aut}(\mathfrak{sp}(n,\mathbb{H})\oplus\mathfrak{sp}(n,\mathbb{H}))\right)$ because

$$\mathrm{Ad}(J_n)_{|\mathfrak{sp}(n,\mathbb{H})\oplus\mathfrak{sp}(n,\mathbb{H})}(x,y)=(y,x)$$

for all $x,y\in\mathfrak{sp}(n,\mathbb{H})$. So $\mathrm{Ad}(J_n)\in\mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{H}))^{\mathrm{Ad}(I_{n,n})}$ is outer. $\blacksquare$

2.2. The real non-central simple case. We now turn to Case A of Lemma 2.1, i.e. the real non-central simple case. Here, Theorem 2.21 due to Djoković (cf. Proposition 4.1 and 7.1 of [2]), helps us to connect the group $\pi_0(\mathrm{Aut}(\mathfrak{g}))$ to $\pi_0(\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}}))$.

Theorem 2.21. *If $\mathfrak{g}$ is a real simple Lie algebra admitting a complex structure with conjugation σ, then there are two isomorphisms of Lie groups as follows:*

$$\phi = \phi_\sigma : \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}}) \rtimes \mathcal{C}_2 \longrightarrow \mathrm{Aut}(\mathfrak{g}),\ \big(f, (-1)^\ell\big) \longmapsto \sigma^\ell \circ f,$$
$$\eta = \eta_\sigma : \pi_0(\mathrm{Aut}(\mathfrak{g})) \longrightarrow \pi_0(\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})) \times \mathcal{C}_2.$$

Corollary 2.22. *If $\mathfrak{g}$ is a real simple Lie algebra admitting a complex structure, then $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$ and we have $\#\,\mathrm{Conj}(\pi_0(\mathrm{Aut}(\mathfrak{g}))) = \#\,\mathrm{Conj}(\pi_0(\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}}))) \cdot 2$.*

Proof. The subgroups $\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})_0 = \big\langle e^x : x \in \mathbf{L}\big(\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})\big)\big\rangle_{\text{group}}$ and $\mathrm{Aut}(\mathfrak{g})_0 = \langle e^x : x \in \mathbf{L}\,(\mathrm{Aut}(\mathfrak{g}))\rangle_{\text{group}}$ of $\mathrm{Aut}(\mathfrak{g})$ coincide, since $\mathbf{L}\big(\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})\big) = \mathrm{Der}(\mathfrak{g}) = \mathbf{L}\,(\mathrm{Aut}(\mathfrak{g}))$ by the surjectivity of $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$.

We fix a split real form $\mathfrak{s}$ of the complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and a corresponding conjugation σ, i.e. $\sigma : \mathfrak{g}^{\mathbb{C}} = \mathfrak{s}_{\mathbb{C}} = \mathfrak{s} + i\mathfrak{s} \to \mathfrak{g}^{\mathbb{C}}$, $r + is \mapsto r - is$. Let ϕ and η be given by Theorem 2.21 and $\omega : \pi_0\,(\mathrm{Aut}(\mathfrak{s})) = \pi_0(\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})) \to \mathrm{Aut}(\mathfrak{s})$ be a section of the short exact sequence

$$\mathbf{1} \to \mathrm{Aut}(\mathfrak{s})_0 \to \mathrm{Aut}(\mathfrak{s}) \to \pi_0\,(\mathrm{Aut}(\mathfrak{s})) \to \mathbf{1}$$

given by Theorem 2.19. Define a morphism $\mathrm{ex}_{\mathfrak{s}} : \mathrm{Aut}(\mathfrak{s}) \to \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})^\sigma \subseteq \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ by $\mathrm{ex}_{\mathfrak{s}}(g)(r + is) := g(r) + ig(s)$ for $g \in \mathrm{Aut}(\mathfrak{s})$ and $r, s \in \mathfrak{s}$. We will now show that for the short exact sequence

$$\mathbf{1} \to \mathrm{Aut}(\mathfrak{g})_0 \to \mathrm{Aut}(\mathfrak{g}) \to \pi_0\,(\mathrm{Aut}(\mathfrak{g})) \to \mathbf{1}$$

the map $\gamma := \phi \circ (\mathrm{ex}_{\mathfrak{s}} \times \mathrm{id}_{\mathcal{C}2}) \circ (\omega \times \mathrm{id}_{\mathcal{C}2}) \circ \eta$ is a section:

Let $f \in \mathrm{Aut}(\mathfrak{g})$. Then $((\omega \times \mathrm{id}_{\mathcal{C}2}) \circ \eta)\,([f]) \in \mathrm{Aut}(\mathfrak{s}) \times \mathcal{C}_2$ is a homomorphic image of $[f] \in \pi_0(\mathrm{Aut}(\mathfrak{g}))$ and can be written as $(g, (-1)^\ell)$ for some $g \in \mathrm{Aut}(\mathfrak{s})$ and $\ell \in \{0, 1\}$. Let, for $f' \in \mathrm{Aut}(\mathfrak{g})$, be $(g', (-1)^{\ell'})$ be a corresponding pair. Then $\gamma([f]) = (\phi \circ (\mathrm{ex}_{\mathfrak{s}} \times \mathrm{id}_{\mathcal{C}2}))\,(g, (-1)^\ell) = \phi(\mathrm{ex}_{\mathfrak{s}}(g), (-1)^\ell)$ and $\gamma([f']) = \phi(\mathrm{ex}_{\mathfrak{s}}(g'), (-1)^{\ell'})$ and so:

$$\begin{aligned}\gamma([f]) \circ \gamma([f']) &= \phi(\mathrm{ex}_{\mathfrak{s}}(g), (-1)^\ell) \circ \phi(\mathrm{ex}_{\mathfrak{s}}(g'), (-1)^{\ell'}) = \sigma^\ell \circ \mathrm{ex}_{\mathfrak{s}}(g) \circ \sigma^{\ell'} \circ \mathrm{ex}_{\mathfrak{s}}(g') \\ &= \sigma^{\ell+\ell'} \circ \mathrm{ex}_{\mathfrak{s}}(gg') = \gamma([f]\,[f']).\end{aligned}$$

The second statement of the corollary immediately follows from Theorem 2.21. ∎

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