

Restrictions of Certain Degenerate Principal Series of the Universal Covering of the Symplectic Group

Hongyu He*

Communicated by B. Ørsted

Abstract. Let $\widetilde{Sp}(n, \mathbb{R})$ be the universal covering of the symplectic group. In this paper, we study the restrictions of the degenerate unitary principal series $I(\epsilon, t)$ of $\widetilde{Sp}(n, \mathbb{R})$ onto $\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(n-p, \mathbb{R})$. We prove that if $n \geq 2p$, $I(\epsilon, t)|_{\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(n-p, \mathbb{R})}$ is unitarily equivalent to an L^2 -space of sections of a homogeneous line bundle $L^2(\widetilde{Sp}(n-p, \mathbb{R}) \times_{\widetilde{GL}(n-2p, \mathbb{R})} \mathbb{C}_{\epsilon, t+\rho})$ (see Theorem 1.1). We further study the restriction of complementary series $C(\epsilon, t)$ onto $\widetilde{U}(n-p)\widetilde{Sp}(p, \mathbb{R})$. We prove that this restriction is unitarily equivalent to $I(\epsilon, t)|_{\widetilde{U}(n-p)\widetilde{Sp}(p, \mathbb{R})}$ for $t \in i\mathbb{R}$. Our results suggest that the direct integral decomposition of $C(\epsilon, t)|_{\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(n-p, \mathbb{R})}$ will produce certain complementary series for $\widetilde{Sp}(n-p, \mathbb{R})$ ([He09]).

Mathematics Subject Classification 2000: 22E45, 43A85.

Key Words and Phrases: Complementary series, degenerate principal series, symplectic groups, universal covering, branching formula.

1. Introduction

Let $\widetilde{Sp}(n, \mathbb{R})$ be the universal covering of $Sp(n, \mathbb{R})$. $\widetilde{Sp}(n, \mathbb{R})$ is a central extension of $Sp(n, \mathbb{R})$:

$$1 \rightarrow C \rightarrow \widetilde{Sp}(n, \mathbb{R}) \rightarrow Sp(n, \mathbb{R}) \rightarrow 1,$$

where $C \cong \mathbb{Z}$. The unitary dual of C is parametrized by a torus \mathbb{T} . For each $\kappa \in \mathbb{T}$, denote the corresponding unitary character of C by χ^κ . We say that a representation π of $\widetilde{Sp}(n, \mathbb{R})$ is of class κ if $\pi|_C = \chi^\kappa$. Since C commutes with $\widetilde{Sp}(n, \mathbb{R})$, for any irreducible representation π of $\widetilde{Sp}(n, \mathbb{R})$, $\pi|_C = \chi^\kappa$ for some κ .

Denote the projection $\widetilde{Sp}(n, \mathbb{R}) \rightarrow Sp(n, \mathbb{R})$ by p . For any subgroup H of $Sp(n, \mathbb{R})$, denote the full inverse image $p^{-1}(H)$ by \widetilde{H} . We adopt the notation from [Sa]. Let P be the Siegel parabolic subgroup of $Sp(n, \mathbb{R})$. One dimensional characters of \widetilde{P} can be parametrized by (ϵ, t) where $\epsilon \in \mathbb{T}$ and $t \in \mathbb{C}$. Let $I(\epsilon, t)$ be the representation of $\widetilde{Sp}(n, \mathbb{R})$ induced from the one dimensional character $\mathbb{C}_{\epsilon, t}$

*This research is supported in part by the NSF grant DMS 0700809.

parametrized by (ϵ, t) of \tilde{P} . If $t \in i\mathbb{R}$ and $t \neq 0$, $I(\epsilon, t)$ is unitary and irreducible. $I(\epsilon, t)$ is called *unitary degenerate principal series*. If t is real, then $I(\epsilon, t)$ has a nontrivial invariant Hermitian form. Sahi gives a classification of all irreducible unitarizable $I(\epsilon, t)$. If $I(\epsilon, 0)$ is irreducible, there are *complementary series* $C(\epsilon, t)$ for t in a suitable interval ([Sa]). Some of these complementary series are obtained by Kudla-Rallis [KR], Østed-Zhang [OZ], Branson-Østed-Olafsson [BOO], Lee [Lee]. Strictly speaking $C(\epsilon, t)$ should be called *degenerate complementary series* because there are complementary series associated with the principal series, which should be called complementary series ([Kos], [ABPTV]). Throughout this paper, complementary series will mean $C(\epsilon, t)$.

Let $(Sp(p, \mathbb{R}), Sp(n-p, \mathbb{R}))$ be a pair of symplectic groups diagonally embedded in $Sp(n, \mathbb{R})$ (see Definition 5.1). Let $U(n)$ be a maximal compact subgroup such that $Sp(n-p, \mathbb{R}) \cap U(n)$ and $Sp(p, \mathbb{R}) \cap U(n)$ are maximal compact subgroups of $Sp(n-p, \mathbb{R})$ and $Sp(p, \mathbb{R})$ respectively. Denote $Sp(n-p, \mathbb{R}) \cap U(n)$ by $U(n-p)$ and $Sp(p, \mathbb{R}) \cap U(n)$ by $U(p)$. The main results of this paper can be stated as follows.

Theorem 1.1. *Suppose $p \leq n-p$ and $t \in i\mathbb{R}$. Let $P_{p, n-2p}$ be a maximal parabolic subgroup of $Sp(n-p, \mathbb{R})$ with Langlands decomposition $Sp(p, \mathbb{R})GL(n-2p)N_{p, n-2p}$. Let $\mathcal{M}_{\epsilon, t}$ be the homogeneous line bundle*

$$\begin{aligned} \widetilde{Sp}(n-p, \mathbb{R}) \times_{\widetilde{GL}(n-2p)N_{p, n-2p}} \mathbb{C}_{\epsilon, t+\rho} &\rightarrow Sp(n-p, \mathbb{R})/GL(n-2p)N_{p, n-2p} \\ &(\cong Sp(p, \mathbb{R})U(n-p)/U(p)O(n-2p)), \end{aligned} \quad (1)$$

where $\rho = \frac{n+1}{2}$. Let $dg_1 d[k_2]$ be an $Sp(p, \mathbb{R})U(n-p)$ -invariant measure. Then

$$I(\epsilon, t)|_{\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(n-p, \mathbb{R})} \cong L^2(\mathcal{M}_{\epsilon, t}, dg_1 d[k_2]),$$

on which $\widetilde{Sp}(n-p, \mathbb{R})$ acts from the left and $\widetilde{Sp}(p, \mathbb{R})$ acts from the right.

Theorem 1.2. *Let $C(\epsilon, t)$ be a complementary series representation. Suppose that $p \leq n-p$. Then*

$$C(\epsilon, t)|_{\tilde{U}(n-p)\tilde{Sp}(p, \mathbb{R})} \cong I(\epsilon, 0)|_{\tilde{U}(n-p)\tilde{Sp}(p, \mathbb{R})} \cong I(\epsilon, i\lambda)|_{\tilde{U}(n-p)\tilde{Sp}(p, \mathbb{R})} \quad (\lambda \in \mathbb{R}).$$

$p = \lfloor \frac{n}{2} \rfloor$ is the best possible value for such a statement. In particular, for $\widetilde{Sp}(2m+1, \mathbb{R})$

$$I(\epsilon, 0)|_{\widetilde{Sp}(m+1, \mathbb{R})} \not\cong C(\epsilon, t)|_{\widetilde{Sp}(m+1, \mathbb{R})}.$$

To see this, let $L^2(\widetilde{Sp}(n, \mathbb{R}))_\kappa$ be the set of functions with

$$f(zg) = \chi^\kappa(z)f(g) \quad (z \in C, g \in \widetilde{Sp}(n, \mathbb{R}));$$

$$\|f\|^2 = \int_{Sp(n, \mathbb{R})} |f(g)|^2 d[g] < \infty \quad (g \in \widetilde{Sp}(n, \mathbb{R}), [g] \in Sp(n, \mathbb{R})).$$

We say that a representation of class κ is tempered if it is weakly contained in $L^2(\widetilde{Sp}(n, \mathbb{R}))_\kappa$. By studying the leading exponents of $I(\epsilon, 0)$ and $C(\epsilon, t)$, it can be shown that $I(\epsilon, 0)|_{\widetilde{Sp}(m+1, \mathbb{R})}$ is tempered and $C(\epsilon, t)|_{\widetilde{Sp}(m+1, \mathbb{R})}$ is not tempered. Therefore

$$I(\epsilon, 0)|_{\widetilde{Sp}(m+1, \mathbb{R})} \not\cong C(\epsilon, t)|_{\widetilde{Sp}(m+1, \mathbb{R})}.$$

The author would like to thank Professors G. Olafsson and J. Lawson for very helpful discussions.

2. A Lemma on Friedrichs Extension

Let S be a semibounded densely defined symmetric operator on a Hilbert space H . S is said to be *positive* if $(Su, u) > 0$ for every nonzero $u \in \mathcal{D}(S)$. Suppose that S is positive. For $u, v \in \mathcal{D}(S)$, define

$$(u, v)_S = (u, Sv),$$

$$\|u\|_S^2 = (u, Su).$$

Let H_S be the completion of $\mathcal{D}(S)$ under the norm $\| \cdot \|_S$. Clearly $H_{S+I} \subseteq H$ and $H_{S+I} \subseteq H_S$.

The operator $S+I$ has a unique self-adjoint extension $(S+I)_0$ in H , the Friedrichs extension. $(S+I)_0$ has the following properties

- $\mathcal{D}(S) \subseteq \mathcal{D}((S+I)_0) \subseteq H_{S+I} \subseteq H$;
- $(u, v)_{S+I} = (u, (S+I)_0 v)$ for all $u \in H_{S+I}$ and $v \in \mathcal{D}((S+I)_0)$

(see Theorem in Page 335 [RS]). Now consider $(S+I)_0 - I$. It is a self-adjoint extension of S . It is nonnegative. By the spectral decomposition and functional calculus, $(S+I)_0 - I$ has a unique *square root* T (See Pg. 127. 128. [RS]).

Lemma 2.1. *Let S be a positive densely defined symmetric operator. Then the square root of $(S+I)_0 - I$ extends to an isometry from H_S into H .*

Proof. Clearly, the spectrum of T is contained in the nonnegative part of the real line. By spectral decomposition $\mathcal{D}((S+I)_0 - I) = \mathcal{D}((S+I)_0) \subseteq \mathcal{D}(T)$ and $TT = (S+I)_0 - I$. In addition for any $u, v \in \mathcal{D}(S) \subseteq \mathcal{D}((S+I)_0)$,

$$(Tu, Tv) = (u, TTv) = (u, (S+I)_0 v - v) = (u, Sv) = (u, v)_S.$$

So T is an isometry from $\mathcal{D}(S)$ into H . Since $\mathcal{D}(S)$ is dense in H_S , T extends to an isometry from H_S into H . ■

3. Degenerate Principal Series of $\widetilde{Sp}(n, \mathbb{R})$

Fix the Lie algebra:

$$\mathfrak{sp}(n, \mathbb{R}) = \left\{ \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} \mid Y^t = Y, Z^t = Z \right\}$$

and the Siegel parabolic algebra:

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ 0 & -X^t \end{pmatrix} \mid Y^t = Y \right\}.$$

Fix the Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ with

$$\mathfrak{l} = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} \mid X \in \mathfrak{gl}(n, \mathbb{R}) \right\}, \quad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \mid Y^t = Y \right\}.$$

Fix a Cartan subalgebra

$$\mathfrak{a} = \{ \text{diag}(H_1, H_2, \dots, H_n, -H_1, -H_2, \dots, -H_n) \mid H_i \in \mathbb{R} \}.$$

Let $Sp(n, \mathbb{R})$ be the symplectic group and P be the Siegel parabolic subgroup. Set $U(n) = Sp(n, \mathbb{R}) \cap O(2n)$ where $O(2n)$ is the standard orthogonal group. Let LN be the Levi decomposition of P and A be the analytic group generated by the Lie algebra \mathfrak{a} . Clearly, $L \cong GL(n, \mathbb{R})$ and $L \cap U(n) \cong O(n)$. On the covering group, we have $\tilde{L} \cap \tilde{U}(n) = \tilde{O}(n)$. Recall that

$$\tilde{U}(n) = \{(x, g) \mid g \in U(n), \exp 2\pi i x = \det g, x \in \mathbb{R}\}.$$

Therefore

$$\tilde{O}(n) = \{(x, g) \mid g \in O(n), \exp 2\pi i x = \det g, x \in \mathbb{R}\}.$$

Notice that for $g \in O(n)$, $\det g = \pm 1$ and $x \in \frac{1}{2}\mathbb{Z}$. We have the following exact sequence

$$1 \rightarrow SO(n) \rightarrow \tilde{O}(n) \rightarrow \frac{1}{2}\mathbb{Z} \rightarrow 1.$$

Consequently, we have

$$1 \rightarrow GL_0(n, \mathbb{R}) \rightarrow \tilde{L} \rightarrow \frac{1}{2}\mathbb{Z} \rightarrow 1.$$

In fact,

$$\tilde{L} = \{(x, g) \mid g \in L, \exp 2\pi i x = \frac{\det g}{|\det g|}, x \in \mathbb{R}\}.$$

The one dimensional unitary characters of $\frac{1}{2}\mathbb{Z}$ are parametrized by the one dimensional torus T . Identify T with $[0, 1)$. Let μ^ϵ be the character of $\frac{1}{2}\mathbb{Z}$ corresponding to $\epsilon \in [0, 1)$. Now each character μ^ϵ yields a character of \tilde{L} , which in turn, yields a character of \tilde{P} . For simplicity, we retain μ^ϵ to denote the character on \tilde{L} and \tilde{P} . Let ν be the det-character on \tilde{L}_0 , i.e.,

$$\nu(x, g) = |\det g| \quad (x, g) \in \tilde{L}. \quad (2)$$

Let

$$I(\epsilon, t) = \text{Ind}_{\tilde{P}}^{\tilde{Sp}(n, \mathbb{R})} \mu^\epsilon \otimes \nu^t$$

be the normalized induced representation with $\epsilon \in [0, 1)$ and $t \in \mathbb{C}$. This is Sahi's notation in the case of the universal covering of the symplectic group ([Sa]). $I(\epsilon, t)$ is a degenerate principal series representation. Clearly, $I(\epsilon, t)$ is unitary

when $t \in i\mathbb{R}$.

When t is real and $I(\epsilon, t)$ is unitarizable, the unitary representation, often denoted by $C(\epsilon, t)$, is called a *complementary series representation*. Various complementary series of $Sp(n, \mathbb{R})$ and its metaplectic covering was determined explicitly or implicitly by Kudla-Rallis, Ørsted-Zhang, Brason-Olafsson-Ørsted and others. See [KR], [BOO], [OZ] and the references therein. The complete classification of the complementary series of the universal covering is due to Sahi.

Theorem 3.1 (Thm A, [Sa]). *Suppose that t is real. For n even, $I(\epsilon, t)$ is irreducible and unitarizable if and only if $0 < |t| < |\frac{1}{2} - |2\epsilon - 1||$. For n odd and $n > 1$, $I(\epsilon, t)$ is irreducible and unitarizable if and only if $0 < |t| < \frac{1}{2} - |\frac{1}{2} - |2\epsilon - 1||$.*

One can easily check that the complementary series exist if $\epsilon \neq 0, \frac{1}{2}$ for n odd and $n > 1$; if $\epsilon \neq \frac{1}{4}, \frac{3}{4}$ for n even. It is interesting to note that complementary series always exist unless $I(\epsilon, t)$ descends into a representation of the metaplectic group. For the metaplectic group $Mp(2n+1, \mathbb{R})$, there are two complementary series $I(\frac{1}{4}, t)$ ($0 < t < \frac{1}{2}$) and $I(\frac{3}{4}, t)$ ($0 < t < \frac{1}{2}$). For the metaplectic group $Mp(2n, \mathbb{R})$, there are two complementary series $I(0, t)$ ($0 < t < \frac{1}{2}$) and $I(\frac{1}{2}, t)$ ($0 < t < \frac{1}{2}$). These four complementary series are the “longest”.

For $n = 1$, the situation is quite different. The difference was pointed out in [KR]. For example, there are Bargmann’s complementary series representation for $I(0, t)$ ($t \in (0, \frac{1}{2})$). The classification of the complementary series of $\widetilde{Sp}(1, \mathbb{R})$ can be found in [Bar], [Puk], [Howe].

Since our restriction theorem only makes sense for $n \geq 2$, we will **assume** $n \geq 2$ **from now on**. The parameters for the complementary series of $\widetilde{Sp}(n, \mathbb{R})$ are illustrated in Fig. 1.

4. The generalized compact model and The Intertwining Operator

Recall that

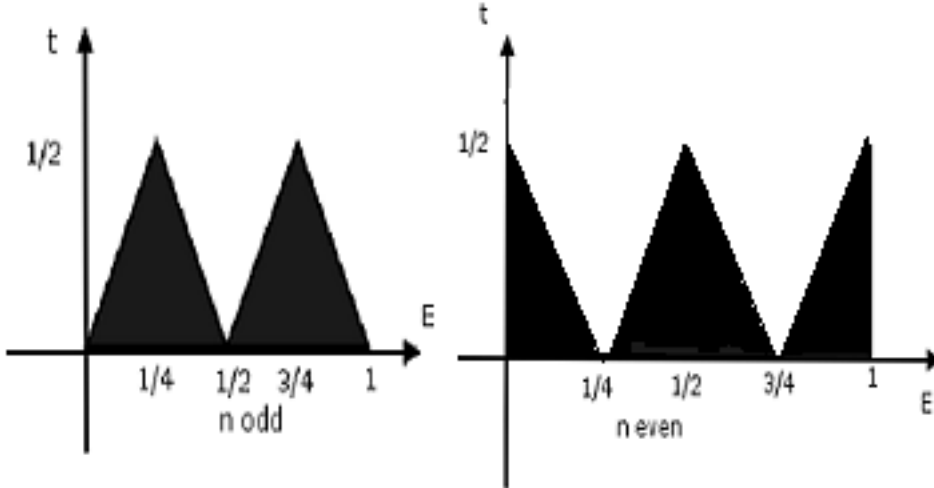
$$I^\infty(\epsilon, t) = \{f \in C^\infty(\widetilde{Sp}(n, \mathbb{R})) \mid f(gln) = (\mu^\epsilon \otimes \nu^{t+\rho})(l^{-1})f(g), \\ (g \in \widetilde{Sp}(n, \mathbb{R}), l \in \tilde{L}, n \in N)\} \quad (3)$$

where $\rho = \frac{n+1}{2}$. Let $X = \widetilde{Sp}(n, \mathbb{R})/\tilde{P}$. Then $I^\infty(\epsilon, t)$ consists of smooth sections of the homogeneous line bundle $\mathcal{L}_{\epsilon, t}$

$$\widetilde{Sp}(n, \mathbb{R}) \times_{\tilde{P}} \mathbb{C}_{\mu^\epsilon \otimes \nu^{t+\rho}} \rightarrow X.$$

Since $X \cong \tilde{U}(n)/\tilde{O}(n)$, $\tilde{U}(n)$ acts transitively on X . The function $f \in I^\infty(\epsilon, t)$ is uniquely determined by $f|_{\tilde{U}(n)}$ and vice versa. Moreover, the homogeneous vector bundle $\mathcal{L}_{\epsilon, t}$ can be identified with $\mathcal{K}_{\epsilon, t}$

$$\tilde{U}(n) \times_{\tilde{O}(n)} \mathbb{C}_{\mu^\epsilon \otimes \nu^{t+\rho}}|_{\tilde{O}(n)} \rightarrow X$$

Figure 1: Complementary Parameters (E, t)

naturally. Notice that the homogeneous line bundle $\mathcal{K}_{\epsilon, t}$ does not depend on the parameter t . We denote this line bundle by \mathcal{K}_{ϵ} . The representation $I^{\infty}(\epsilon, t)$ can then be modeled on smooth sections of \mathcal{K}_{ϵ} . This model will be called *the generalized compact model*.

Let $d[k]$ be the normalized $\tilde{U}(n)$ -invariant measure on X . The generalized compact model equips the smooth sections of $\mathcal{K}_{\epsilon, t}$ with a natural pre-Hilbert structure

$$(f_1, f_2)_X = \int_{[k] \in X} f_1(k) \overline{f_2(k)} d[k],$$

where $k \in \tilde{U}(n)$ and $[k] \in X$. It is easy to verify that $f_1(k) \overline{f_2(k)}$ is a function of $[k]$ and it does not depend on any particular choice of k . Notice that our situation is different from the compact model since $\tilde{U}(n)$ is not compact. We denote the completion of I^{∞} with respect to $(\cdot, \cdot)_X$ by $I_X(\epsilon, t)$.

Secondly, the action of $\tilde{U}(n)$ on \mathcal{K}_{ϵ} induces an orthogonal decomposition of $I_X(\epsilon, t)$:

$$I_X(\epsilon, t) = \hat{\oplus}_{\alpha \in 2\mathbb{Z}^n} V(\alpha + \epsilon(2, 2, \dots, 2)),$$

where $V(\alpha + \epsilon(2, 2, \dots, 2))$ is an irreducible finite dimensional representation of $\tilde{U}(n)$ with highest weight $\alpha + \epsilon(2, 2, \dots, 2)$ and α satisfies

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n.$$

This is essentially a consequence of Helgason's theorem. Let

$$V(\epsilon, t) = \oplus_{\alpha \in 2\mathbb{Z}^n} V(\alpha + \epsilon(2, 2, \dots, 2)).$$

$V(\epsilon, t)$ possesses an action of the Lie algebra $\mathfrak{sp}(n, \mathbb{R})$. It is the Harish-Chandra module of $I(\epsilon, t)$. Clearly, $V(\epsilon, t) \subset I^\infty(\epsilon, t) \subset I_X(\epsilon, t)$.

For each t , there is an $\widetilde{Sp}(n, \mathbb{R})$ -invariant sesquilinear pairing of $I_X(\epsilon, t)$ and $I_X(\epsilon, -\bar{t})$, namely,

$$(f_1, f_2) = \int_X f_1(k) \overline{f_2(\bar{k})} d[k],$$

where $f_1 \in I_X(\epsilon, t)$ and $f_2 \in I_X(\epsilon, -\bar{t})$. If $t \in i\mathbb{R}$, we obtain a $\widetilde{Sp}(n, \mathbb{R})$ -invariant Hermitian form which is exactly $(,)_X$. Since $(,)_X$ is positive definite, $I_X(\epsilon, t)$ is a unitary representation of $\widetilde{Sp}(n, \mathbb{R})$.

For each real t , the form $(,)$ gives an $\mathfrak{sp}(n, \mathbb{R})$ -invariant sesquilinear pairing of $V(\epsilon, t)$ and $V(\epsilon, -t)$. In addition, there is an intertwining operator

$$A(\epsilon, t) : V(\epsilon, t) \rightarrow V(\epsilon, -t)$$

which preserves the action of $\mathfrak{sp}(n, \mathbb{R})$ (see for example [BOO]). Define a Hermitian structure $(,)_{\epsilon, t}$ on $V(\epsilon, t)$ by

$$(u, v)_{\epsilon, t} = (A(\epsilon, t)u, v), \quad (u, v \in V(\epsilon, t)).$$

Clearly, $(,)_{\epsilon, t}$ is $\mathfrak{sp}(n, \mathbb{R})$ -invariant. So $A(\epsilon, t)$ induces an invariant Hermitian form on $V(\epsilon, t)$.

Now $A(\epsilon, t)$ can also be realized as an unbounded operator on $I_X(\epsilon, t)$ as follows. For each $f \in V(\epsilon, t)$, define $A_X(\epsilon, t)f$ to be the unique section of $\mathcal{L}_{\epsilon, t}$ such that

$$(A_X(\epsilon, t)f)|_{\tilde{U}(n)} = (A(\epsilon, t)f)|_{\tilde{U}(n)}.$$

Notice that $A_X(\epsilon, t)f \in I(\epsilon, t)$ and $A(\epsilon, t)f \in I(\epsilon, -t)$. They differ by a multiplier.

Now $A_X(\epsilon, t)$ is an unbounded operator on the Hilbert space $I_X(\epsilon, t)$. The following fact is well-known in many different forms. I state it in a way that is convenient for later use.

Lemma 4.1. *Let $t \in \mathbb{R}$. $I(\epsilon, t)$ is unitarizable if and only if $A_X(\epsilon, t)$ extends to a self-adjoint operator on $I_X(\epsilon, t)$ with spectrum on the nonnegative part of the real axis.*

The spectrum of $A_X(\epsilon, t)$ was computed in [BOO] and [OZ] explicitly for special cases and in [Sa] implicitly. In particular, $A_X(\epsilon, t)$ restricted onto each $\tilde{U}(n)$ -type is a scalar multiplication and the scalar is bounded by a polynomial on the highest weight. We obtain

Lemma 4.2 ([WV]). *$A_X(\epsilon, t)$ extends to an unbounded operator from $I^\infty(\epsilon, t)$ to $I^\infty(\epsilon, t)$.*

This lemma follows from a standard argument that the norm of each $\tilde{U}(n)$ -component in the Peter-Weyl expansion of any smooth section of \mathcal{K}_ϵ decays rapidly with respect to the highest weight. It is true in general (see [WV]).

5. Actions of $\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R})$

Suppose that $p + q = n$ and $p \leq q$. Fix a standard basis

$$\{e_1, e_2, \dots, e_p; e_1^*, e_2^*, \dots, e_p^*\}$$

for the symplectic form Ω_p on \mathbb{R}^{2p} . Fix a standard basis

$$\{f_1, f_2, \dots, f_q; f_1^*, f_2^*, \dots, f_q^*\}$$

for the symplectic form Ω_q on \mathbb{R}^{2q} .

Definition 5.1. Let $Sp(p, \mathbb{R})$ be the symplectic group preserving Ω_p and $Sp(q, \mathbb{R})$ be the symplectic group preserving Ω_q . Let

$$\Omega = \Omega_p - \Omega_q$$

and $Sp(n, \mathbb{R})$ be the symplectic group preserving Ω . We say that the pair $(Sp(p, \mathbb{R}), Sp(q, \mathbb{R}))$ is *diagonally embedded* in $Sp(n, \mathbb{R})$.

We shall make a remark here. In [Henu], $\Omega = \Omega_p + \Omega_q$. $Sp(p, \mathbb{R})Sp(q, \mathbb{R})$ is embedded differently there. The effect of this difference is an involution τ on the representation level.

Let $P_{p, q-p}$ be the subgroup of $Sp(q, \mathbb{R})$ that preserves the linear span of $\{f_{p+1}, \dots, f_q\}$. Choose the Levi factor $GL(q-p)Sp(p, \mathbb{R})$ to be the subgroup of $P_{p, q-p}$ that preserves the span of $\{f_{p+1}^*, \dots, f_q^*\}$. In particular the $Sp(p, \mathbb{R})$ factor can be identified with the symplectic group of

$$\text{span}\{f_1, \dots, f_p; f_1^*, \dots, f_p^*\},$$

which will be identified with the standard $Sp(p, \mathbb{R})$. More precisely, for $x \in Sp(p, \mathbb{R})$, by identify e_i with f_i and e_i^* with f_i^* and extending x trivially on $f_{p+1}, \dots, f_q; f_{p+1}^*, \dots, f_q^*$, we obtain the identification

$$x \in Sp(p, \mathbb{R}) \rightarrow \dot{x} \in Sp(q, \mathbb{R}). \quad (4)$$

Now fix a Lagrangian Grassmanian

$$x_0 = \text{span}\{e_1 + f_1, \dots, e_p + f_p, e_1^* + f_1^*, \dots, e_p^* + f_p^*, f_{p+1}, \dots, f_q\}.$$

Then the stabilizer $Sp(q, \mathbb{R})_{x_0} = GL(q-p)N_{p, q-p}$ where $N_{p, q-p}$ is the nilradical of $P_{p, q-p}$. Put

$$\Delta(Sp(p, \mathbb{R})) = \{(u, \dot{u}) \mid u \in Sp(p, \mathbb{R})\} \subseteq Sp(p, \mathbb{R})Sp(q, \mathbb{R})$$

and

$$H = \Delta(Sp(p, \mathbb{R}))GL(q-p)N_{p, q-p}.$$

Lemma 5.2 ([Henu]). *Let $p \leq q$ and $p + q = n$. Let X_0 be the $Sp(p, \mathbb{R}) \times Sp(q, \mathbb{R})$ -orbit generated by x_0 . Then X_0 is open and dense in X and $[Sp(p, \mathbb{R})Sp(q, \mathbb{R})]_{x_0} = H$.*

Notice here that X_0 depends on (p, q) . Let $P = Sp(n, \mathbb{R})_{x_0}$. The smooth representation $I^\infty(\epsilon, t)$ consists of smooth sections of $\mathcal{L}_{\epsilon, t}$:

$$\widetilde{Sp}(n, \mathbb{R}) \times_{\bar{P}} \mathbb{C}_{\mu^\epsilon \otimes \nu^{t+\rho}} \rightarrow X.$$

Consider the subgroup $\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R})$ in $\widetilde{Sp}(n, \mathbb{R})$. Notice that $\widetilde{Sp}(p, \mathbb{R}) \cap \widetilde{Sp}(q, \mathbb{R}) \cong \mathbb{Z}$. So $\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R})$ is not a direct product, but rather the product of the two groups as sets.

Definition 5.3. For any $f \in I_X(\epsilon, t)$, define

$$f_{X_0} = f|_{\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R})}.$$

Let $I_{c, X_0}^\infty(\epsilon, t)$ be the set of smooth sections of $\mathcal{L}_{\epsilon, t}$ that are compactly supported in X_0 .

Clearly f_{X_0} is a smooth section of

$$\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R}) \times_{\bar{H}} \mathbb{C}_{\mu^\epsilon \otimes \nu^{t+\rho}} \rightarrow X_0.$$

Notice that $\Delta(Sp(p, \mathbb{R}))$ sits inside of $SL(n, \mathbb{R}) \subseteq GL(n, \mathbb{R}) \subseteq P$. The universal covering of $Sp(n, \mathbb{R})$ splits over $SL(n, \mathbb{R}) \subseteq P$. Similarly the universal covering of $Sp(q, \mathbb{R})$ also splits over $N_{p, q-p}$. So we have

$$\tilde{H} \cong \Delta(Sp(p, \mathbb{R}))\widetilde{GL}(q-p)N_{p, q-p},$$

where $\widetilde{GL}(q-p)N_{p, q-p} \subseteq \widetilde{Sp}(q, \mathbb{R})$. In particular, $\mu^\epsilon \otimes \nu^{t+\rho}|_{\Delta(Sp(p, \mathbb{R}))N_{p, q-p}}$ is trivial and $\mu^\epsilon \otimes \nu^{t+\rho}|_{\widetilde{GL}(q-p)}$ is essentially the restriction from $\widetilde{GL}(p+q)$ to $\widetilde{GL}(q-p)$. If $p = q$, then $GL(0)$ will be the identity element. So $\widetilde{GL}(0)$ is just C . We have

Lemma 5.4. *The identification (4)*

$$x \in Sp(p, \mathbb{R}) \rightarrow \dot{x}Sp(q, \mathbb{R})$$

lifts naturally to $\widetilde{Sp}(p, \mathbb{R}) \rightarrow \widetilde{Sp}(q, \mathbb{R})$. Let $\phi \in I^\infty(\epsilon, t)$. Then

$$\phi(g_1, g_2) = \phi(1, g_2g_1^{-1}) \quad (g_1 \in \widetilde{Sp}(p, \mathbb{R}), g_2 \in \widetilde{Sp}(q, \mathbb{R})).$$

In addition

$$\phi(1, g_2h) = \mu^\epsilon \otimes \nu^{t+\rho}(h^{-1})\phi(1, g_2) \quad (h \in \widetilde{GL}(q-p)N_{p, q-p}).$$

Now let us consider the action of $\widetilde{Sp}(p, \mathbb{R})$ and $\widetilde{Sp}(q, \mathbb{R})$ on $I(\epsilon, t)$. By Lemma 5.4, we obtain

Lemma 5.5. *Let $\phi \in I^\infty(\epsilon, t)$ and $h_1 \in \widetilde{Sp}(p, \mathbb{R})$ and $g_2 \in \widetilde{Sp}(q, \mathbb{R})$. Then*

$$[I(\epsilon, t)(h_1)\phi](1, g_2) = f(1, g_2 h_1).$$

In particular the restriction map

$$\phi \in I^\infty(\epsilon, t) \rightarrow \phi|_{\widetilde{Sp}(q, \mathbb{R})} \in C^\infty(\widetilde{Sp}(q, \mathbb{R}) \times_{\widetilde{GL}(q-p)N_{p, q-p}} \mathbb{C}^{\mu^\epsilon \otimes \nu^{t+\rho}})$$

intertwines the left regular action of $\widetilde{Sp}(p, \mathbb{R})$ on $I^\infty(\epsilon, t)$ with the right regular action of $\widetilde{Sp}(p, \mathbb{R})$ on $C^\infty(\widetilde{Sp}(q, \mathbb{R}) \times_{\widetilde{GL}(q-p)N_{p, q-p}} \mathbb{C}^{\mu^\epsilon \otimes \nu^{t+\rho}})$.

Obviously, the restriction map also intertwines the left regular actions of $\widetilde{Sp}(q, \mathbb{R})$.

6. Mixed Model

Now fix complex structures on \mathbb{R}^{2p} and \mathbb{R}^{2q} and inner products $(,)_p$, $(,)_q$ such that

$$\Omega_p = \mathfrak{S}(,)_p, \quad \Omega_q = -\mathfrak{S}(,)_q.$$

Let $U(p)$ and $U(q)$ be the unitary groups preserving $(,)_p$ and $(,)_q$ respectively. $U(p)$ and $U(q)$ are maximal compact subgroups of $Sp(p, \mathbb{R})$ and $Sp(q, \mathbb{R})$. Let $U(n)$ be the unitary group preserving $(,)_p + (,)_q$. Then $U(n)$ is a maximal compact subgroup of $Sp(n, \mathbb{R})$. In addition,

$$U(p) = Sp(p, \mathbb{R}) \cap U(n) \quad U(q) = Sp(q, \mathbb{R}) \cap U(n).$$

Identify $U(q) \cap P_{p, q-p}$ with $O(q-p)U(p)$. Recall that $X_0 \cong Sp(q, \mathbb{R})/GL(q-p)N_{p, q-p}$. The group $Sp(p, \mathbb{R})$ acts on X_0 freely from the right. We obtain a principal fibration

$$Sp(p, \mathbb{R}) \rightarrow X_0 \rightarrow Sp(q, \mathbb{R})/P_{p, q-p} \cong U(q)/O(q-p)U(p).$$

This fibration allows us to visualize the action of $\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R})$ on $I^\infty(\epsilon, t)$. Let dg_1 be a Haar measure on $Sp(p, \mathbb{R})$ and $d[k_2]$ be an invariant probability measure on $U(q)/O(q-p)U(p)$. Then $dg_1 d[k_2]$ defines an $U(q)Sp(p, \mathbb{R})$ invariant measure on X_0 .

Definition 6.1. Let $M = Sp(p, \mathbb{R})U(q) \subset Sp(p, \mathbb{R})Sp(q, \mathbb{R}) \subset Sp(n, \mathbb{R})$. Elements in X_0 are parametrized by a pair $(g_1, [k_2])$ for $(g_1, k_2) \in M$. For each $g \in \widetilde{Sp}(n, \mathbb{R})$, write $g = \tilde{u}(g)p(g)$ where $\tilde{u}(g) \in \tilde{U}(n)$ and $p(g) \in P_0$, the identity component of \tilde{P} . For each $(g_1, k_2) \in (\widetilde{Sp}(p, \mathbb{R}), \tilde{U}(q))$, we have

$$g_1 k_2 = \tilde{u}(g_1 k_2)p(g_1 k_2) = k_2 \tilde{u}(g_1)p(g_1).$$

The component \tilde{u} defines a map from \tilde{M} to $\tilde{U}(n)$. In particular, \tilde{u} induces a map from $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$ to $\tilde{U}(n)/\tilde{O}(n)$ which will be denoted by j . The map j parametrizes the open dense subset X_0 in X by

$$([g_1], [k_2]) \in \widetilde{Sp}(p, \mathbb{R})/C \times \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p).$$

Change the variables on X_0 from $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$ to $\tilde{U}(n)/\tilde{O}(n)$. Let $J([g_1], [k_2])$ be the Jacobian:

$$\frac{dj([g_1], [k_2])}{d[g_1]d[k_2]}.$$

J can be regarded as a function on $Sp(p, \mathbb{R})U(q)$ or $Sp(p, \mathbb{R})U(q)/U(p)O(q-p)$, even though it is defined as a function on the covering. Denote the line bundle

$$\widetilde{Sp}(q, \mathbb{R}) \times_{\widetilde{GL}(q-p)N_{p,q-p}} \mathbb{C}_{\mu^\epsilon \otimes \nu^{t+\rho}} \rightarrow X_0.$$

by $\mathcal{M}_{\epsilon,t}$. Denote the line bundle

$$\tilde{M} \times_{\tilde{O}(q-p)\tilde{U}(p)} \mathbb{C}_{\mu^\epsilon} \rightarrow \tilde{M}/\tilde{O}(q-p)\tilde{U}(p) \cong X_0.$$

by \mathcal{M}_ϵ .

Clearly, $I_{c,X_0}^\infty(\epsilon, t) \subset I^\infty(\epsilon, t)$. Consider the restriction of $(,)_X$ onto $I_{c,X_0}^\infty(\epsilon, t)$. We are interested in expressing $(,)_X$ as an integral on $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$. This boils down to a change of variables from $\tilde{U}(n)/\tilde{O}(n)$ to $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$. We have

Lemma 6.2. *Let $\Delta_t(g_1, k_2) = \nu(p(g_1))^{t+\bar{t}+2\rho} J([g_1], [k_2])$ (see Equ.(2)). Then for every $f_1, f_2 \in I^\infty(\epsilon, t)$ we have*

$$(f_1, f_2)_X = \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1 k_2) \overline{f_2(g_1 k_2)} \Delta_t(g_1, k_2) d[g_1] d[k_2]$$

where $g_1 \in \widetilde{Sp}(p, \mathbb{R})$, $k_2 \in \tilde{U}(q)$, $[g_1] \in Sp(p, \mathbb{R})$ and $[k_2] \in \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p)$. Furthermore, $\Delta_t(g_1, k_2)$ is a nonnegative right $\tilde{O}(q-p)\tilde{U}(p)$ -invariant function on \tilde{M} .

Proof. We compute

$$\begin{aligned} & \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1 k_2) \overline{f_2(g_1 k_2)} \Delta_t(g_1, k_2) d[g_1] d[k_2] \\ &= \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(\tilde{u}(g_1 k_2)) \overline{f_2(\tilde{u}(g_1 k_2))} \nu(p(g_1))^{-t-\bar{t}-2\rho} \Delta_t(g_1, k_2) d[g_1] d[k_2] \\ &= \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(\tilde{u}(g_1 k_2)) \overline{f_2(\tilde{u}(g_1 k_2))} \nu(p(g_1))^{-t-\bar{t}-2\rho} \Delta_t(g_1, k_2) J^{-1}(g_1, k_2) dj([g_1], [k_2]) \\ &= \int_{X_0} f_1(\tilde{u}) \overline{f_2(\tilde{u})} d[\tilde{u}] = (f_1, f_2)_X. \end{aligned} \tag{5}$$

Since $\nu(p(g_1))$ and $J([g_1], [k_2])$ remain the same when we multiply k_2 on the right by $\tilde{O}(q-p)\tilde{U}(p)$, $\Delta_t(g_1, k_2)$ is a nonnegative right $\tilde{O}(q-p)\tilde{U}(p)$ -invariant function. ■

Combining with Lemma 5.5, we obtain

Corollary 6.3. *As representations of $\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R})$,*

$$I_X(\epsilon, t) \cong L^2(\mathcal{M}_{\epsilon,t}, \Delta_t d[g_1] d[k_2]).$$

For each $f_1, f_2 \in I_{c, X_0}^\infty(\epsilon, t)$, define

$$(f_1, f_2)_{M,t} = \int_{\widetilde{M}/\widetilde{O}(q-p)\widetilde{U}(p)} f_1(g_1 k_2) \overline{f_2(g_1 k_2)} \Delta_t(g_1 k_2) d[g_1] d[k_2],$$

$$(f_1, f_2)_M = \int_{\widetilde{M}/\widetilde{O}(q-p)\widetilde{U}(p)} f_1(g_1, k_2) \overline{f_2(g_1 k_2)} d[g_1] d[k_2].$$

The completion of $I_{c, X_0}^\infty(\epsilon, t)$ under $(\cdot, \cdot)_{M,t}$ is $L^2(\mathcal{M}_{\epsilon,t}, \Delta_t d[g_1] d[k_2])$. We call $L^2(\mathcal{M}_{\epsilon,t}, \Delta_t d[g_1] d[k_2])$, the mixed model. We denote it by $I_M(\epsilon, t)$. On $I_M(\epsilon, t)$, the actions of $\widetilde{Sp}(p, \mathbb{R})$ and $\widetilde{Sp}(q, \mathbb{R})$ are easy to manipulate.

7. Mixed Model for Unitary Principal Series

Lemma 7.1. *If $t \in i\mathbb{R}$, then $\Delta_t(g_1, k_2)$ is a constant and $(\cdot, \cdot)_{M,t}$ is a constant multiple of $(\cdot, \cdot)_M$.*

Proof. Let $t \in i\mathbb{R}$. Let $f_1, f_2 \in I^\infty(\epsilon, t)$ and $h \in \widetilde{Sp}(p, \mathbb{R})$. Recall that X_0 is parametrized by a pair $[g_1] \in \widetilde{Sp}(p, \mathbb{R})/C$ and $[k_2] \in \widetilde{U}(q)/\widetilde{O}(q-p)\widetilde{U}(p)$. By Lemma 6.2, we have

$$\begin{aligned} & (I(\epsilon, t)(h)f_1, I(\epsilon, t)(h)f_2)_X \\ &= \int_{X_0} f_1(h^{-1}g_1 k_2) \overline{f_2(h^{-1}g_1 k_2)} \Delta_t(g_1, k_2) d[g_1] d[k_2] \\ &= \int_{X_0} f_1(g_1 k_2) \overline{f_2(g_1, k_2)} \Delta_t(hg_1, k_2) d[g_1] d[k_2] \end{aligned} \quad (6)$$

Since $I(\epsilon, t)$ is unitary, $(I(\epsilon, t)(h)f_1, I(\epsilon, t)(h)f_2)_X = (f_1, f_2)_X$. We have

$$\int_{X_0} f_1(g_1 k_2) \overline{f_2(g_1 k_2)} \Delta_t(hg_1, k_2) d[g_1] d[k_2] = \int_{X_0} f_1(g_1 k_2) \overline{f_2(g_1 k_2)} \Delta_t(g_1, k_2) d[g_1] d[k_2].$$

It follows that $\Delta_t(hg_1, k_2) = \Delta_t(g_1, k_2)$ for any $h \in \widetilde{Sp}(p, \mathbb{R})$. Similarly, we obtain $\Delta_t(g_1, k k_2) = \Delta_t(g_1, k_2)$ for any $k \in \widetilde{U}(q)$. Hence, $\Delta_t(g_1, k_2)$ is a constant for purely imaginary t . \blacksquare

Combining with Corollary. 6.3, we obtain

Theorem 7.2. *Let $t \in i\mathbb{R}$. The restriction map $f \rightarrow f_{X_0}$ induces an isometry between $I(\epsilon, t)$ and $L^2(\mathcal{M}_{\epsilon,t}, d[g_1] d[k_2])$. In addition, this isometry intertwines the actions of $\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R})$. So as $\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R})$ representations,*

$$I(\epsilon, t) \cong L^2(\mathcal{M}_{\epsilon,t}, d[g_1] d[k_2]);$$

and as $\widetilde{Sp}(p, \mathbb{R})\widetilde{U}(q)$ representations,

$$I(\epsilon, t) \cong L^2(\mathcal{M}_\epsilon, d[g_1] d[k_2]).$$

Notice that $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$ does not depend on the parameter t . The following corollary is automatical.

Corollary 7.3. *Suppose that $p + q = n$ and $p \leq q$. For t real,*

$$I(\epsilon, it)|_{\widetilde{Sp}(p, \mathbb{R})\widetilde{U}(q)} \cong I(\epsilon, 0)|_{\widetilde{Sp}(p, \mathbb{R})\widetilde{U}(q)} \cong L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2]).$$

For t a nonzero real number, $\Delta_t(g, k)$ is not a constant. So $C(\epsilon, t)$ cannot be modeled naturally on $L^2(\mathcal{M}_{\epsilon, t}, d[g_1]d[k_2])$. Nevertheless, we have

Theorem 7.4 (Main Theorem). *Suppose that $p + q = n$ and $p \leq q$. Given a complementary series representation $C(\epsilon, t)$,*

$$C(\epsilon, t)|_{\widetilde{Sp}(p, \mathbb{R})\widetilde{U}(q)} \cong I(\epsilon, 0)|_{\widetilde{Sp}(p, \mathbb{R})\widetilde{U}(q)} \cong L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2]).$$

In other words, there is an isometry between $C(\epsilon, t)$ and $I(\epsilon, 0)$ that intertwines the actions of $\widetilde{U}(q)$ and of $\widetilde{Sp}(p, \mathbb{R})$.

We shall postpone the proof of this theorem to the next section. We will first derive some corollaries from Lemma 7.1 concerning Δ and $\nu(g_1)$.

Corollary 7.5. *$J([g_1], [k_2]) = c\nu(p(g_1))^{-2\rho}$ for a constant c and $\Delta_t(g_1, k_2) = c\nu(p(g_1))^{t+\bar{t}}$. So both Δ_t and $J([g_1], [k_2])$ do not depend on k_2 . Furthermore,*

$$I(\epsilon, t) \cong L^2(\mathcal{M}_{\epsilon, t}, \nu(p(g_1))^{t+\bar{t}}d[g_1]d[k_2]) = I_M(\epsilon, t). \quad (7)$$

$\nu(p(g_1))$ is a function on $\widetilde{Sp}(p, \mathbb{R})/C$. So it can be regarded as a function on $Sp(p, \mathbb{R})$.

Corollary 7.6. *$\nu(p(g_1))^{-\rho} \in L^2(Sp(p, \mathbb{R}))$ and $\nu(p(g_1))^{-1}$ is a bounded positive function.*

Proof. Since X is compact,

$$\int_{Sp(p, \mathbb{R})} \nu(p(g_1))^{-2\rho} dg_1 = C \int_{\widetilde{M}/\widetilde{O}(q-p)\widetilde{U}(p)} J([g_1], [k_2])d[g_1]d[k_2] = C \int_{\widetilde{U}(n)/\widetilde{O}(n)} 1d[k] < \infty.$$

So $\nu(p(g_1))^{-\rho} \in L^2(Sp(p, \mathbb{R}))$. Now we need to compute $\nu(g_1)$. Recall that P is defined to be the stabilizer of

$$x_0 = \text{span}\{e_1 + f_1, \dots, e_p + f_p, e_1^* + f_1^*, \dots, e_p^* + f_p^*, f_{p+1}, \dots, f_q\}.$$

So $j(g_1, 1)$ is the following Lagrangian

$$\text{span}\{g_1 e_1 + f_1, \dots, g_1 e_p + f_p, g_1 e_1^* + f_1^*, \dots, g_1 e_p^* + f_p^*, f_{p+1}, \dots, f_q\}.$$

The action of $\widetilde{U}(n)$ will not change the volume of the n -dimensional cube spanned by the basis above. So $\nu(p(g_1))$, as the determinant character, is equal to the volume of the n -dimensional cube, up to a constant. Hence

$$\nu(p(g_1)) = [2^{-n} \det(g_1 g_1^t + I)]^{\frac{1}{2}}.$$

Clearly, $\nu(p(g_1))^{-1}$ is bounded and positive. ■

This corollary is easy to understand in terms of compactification. Notice that the map j , without the covering,

$$Sp(p, \mathbb{R})U(q)/U(p)O(q-p) \rightarrow U(n)/O(n)$$

is an analytic compactification. Hence the Jacobian $J(g_1, [k_2])$ should be positive and bounded above. Since $J(g_1, [k_2]) = c\nu(p(g_1))^{-2\rho}$, $\nu(p(g_1))^{-1}$ must also be positive and bounded above. The situation here is similar to [He02] (see Appendix) and [He06] (Theorem 2.3). It is not clear that $j(g_1, 1)$ gets mapped onto $U(2p)/O(2p)$ though.

If $f \in I_M(\epsilon, t_1)$ and $h > 0$, by Cor. 7.6 and Equation (7), we have $\|f\|_{M, t_1-h} \leq C\|f\|_{M, t_1}$. So $I_M(\epsilon, t_1) \subset I_M(\epsilon, t_1 - h)$.

Corollary 7.7. *Suppose that $h > 0$. Then $I_M(\epsilon, t_1) \subset I_M(\epsilon, t_1 - h)$.*

8. “Square Root” of the Intertwining Operator

Suppose from now on $t \in \mathbb{R}$. For $f \in I^\infty(\epsilon, t)|_{\tilde{M}}$, define a function on \tilde{M} ,

$$(A_M(\epsilon, t)f)(g_1k_2) = A(\epsilon, t)f(g_1k_2) \quad (g_1 \in \widetilde{Sp}(p, \mathbb{R}), k_2 \in \tilde{U}(q)).$$

So $A_M(\epsilon, t)$ is the “restriction” of $A(\epsilon, t)$ onto \tilde{M} . $A_M(\epsilon, t)$ is not yet an unbounded operator on $I_M(\epsilon, t)$. In fact, for $t > 0$, $A_M(\epsilon, t)$ does not behave well and it is not clear whether $A_M(\epsilon, t)$ can be realized as an unbounded operator on $I_M(\epsilon, t)$. The function $A_M(\epsilon, t)f$ differs from $A_X(\epsilon, t)f$.

Lemma 8.1. *For $t \in \mathbb{R}$ and $f \in I^\infty(\epsilon, t)$,*

$$(A_M(\epsilon, t)f|_{\tilde{M}})(g_1k_2) = (A_X(\epsilon, t)f)(g_1k_2)\nu(p(g_1))^{2t} = (A_X(\epsilon, t)f)(g_1k_2)\Delta_t(g_1, k_2).$$

This Lemma is due to the fact that $A_X(\epsilon, t)f \in I(\epsilon, t)$ but $A(\epsilon, t)f \in I(\epsilon, -t)$.

Let $f \in I^\infty(\epsilon, t)$. In terms of the mixed model, the invariant Hermitian form $(\cdot, \cdot)_{\epsilon, t}$ can be written as follows:

$$(f, f)_{\epsilon, t} = (A_X(\epsilon, t)f, f)_X = \int_{\tilde{M}/\tilde{O}(q-p)U(p)} A_M(\epsilon, t)f|_{\tilde{M}} \overline{f}|_{\tilde{M}} d[g_1]d[k_2].$$

This follows from Lemma 8.1 and Lemma 6.2. We obtain

Lemma 8.2. *For $f_1, f_2 \in I^\infty(\epsilon, t)$, $(f_1, f_2)_{\epsilon, t} = (A_M(\epsilon, t)f_1|_{\tilde{M}}, f_2|_{\tilde{M}})_M$.*

Theorem 8.3. *If $t < 0$ and $C(\epsilon, t)$ is a complementary series representation, then $A_M(\epsilon, t)$ is a positive and densely defined symmetric operator. Its self-adjoint extension $(A_M(\epsilon, t) + I)_0 - I$ has a unique square root which extends to an isometry from $C(\epsilon, t)$ onto*

$$L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2]).$$

Proof. Let $t < 0$. Put

$$\mathcal{H} = L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2]).$$

Let $f \in I^\infty(\epsilon, t)$. Then $A_M(\epsilon, t)(f|_{\tilde{M}})(g_1 k_2) = \nu(p(g_1))^{2t} A_X(\epsilon, t)f(g_1 k_2)$. By Lemma 8.1, Cor. 7.6 and Lemma 6.2, we have

$$\begin{aligned} & \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} A_M(\epsilon, t)(f|_{\tilde{M}})\overline{A_M(\epsilon, t)(f|_{\tilde{M}})}d[g_1]d[k_2] \\ &= \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} \nu(p(g_1))^{2t} |(A_X(\epsilon, t)f)(g_1 k_2)|^2 \nu(p(g_1))^{2t} d[g_1]d[k_2] \\ &= \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} \nu(p(g_1))^{2t} |(A_X(\epsilon, t)f)(g_1 k_2)|^2 \Delta_t(g_1, k_2) d[g_1]d[k_2] \quad (8) \\ &\leq C \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} |A_X(\epsilon, t)f(g_1 k_2)|^2 \Delta_t(g_1, k_2) d[g_1]d[k_2] \\ &= C(A_X(\epsilon, t)f, A_X(\epsilon, t)f)_X < \infty. \end{aligned}$$

Therefore, $A_M(\epsilon, t)(f|_{\tilde{M}}) \in \mathcal{H}$. Let $\mathcal{D} = I^\infty(\epsilon, t)|_{\tilde{M}}$. Clearly, \mathcal{D} is dense in \mathcal{H} . So $A_M(\epsilon, t)$ is a densely defined unbounded operator. It is positive and symmetric by Lemma 8.2.

Definition 8.4. Define $\mathcal{U}(\epsilon, t) = ((A_M(\epsilon, t) + I)_0 - I)^{\frac{1}{2}}$.

Now $(f, g)_{\epsilon, t} = (A_M(\epsilon, t)f|_{\tilde{M}}, g|_{\tilde{M}})_M$ for any $f, g \in I^\infty(\epsilon, t)$. So $C(\epsilon, t) = \mathcal{H}_{A_M(\epsilon, t)}$. By Lemma 2.1, $\mathcal{U}(\epsilon, t)$, mapping from $C(\epsilon, t)$ into \mathcal{H} , is an isometry.

Suppose that $\mathcal{U}(\epsilon, t)$ is not onto. Let $f \in \mathcal{H}$ such that for any $u \in \mathcal{D}(\mathcal{U}(\epsilon, t))$,

$$(f, \mathcal{U}(\epsilon, t)u)_M = 0.$$

Notice that

$$I^\infty(\epsilon, t)|_{\tilde{M}} \subset \mathcal{D}((A_M(\epsilon, t) + I)_0 - I) \subset \mathcal{D}(\mathcal{U}(\epsilon, t)),$$

and

$$\mathcal{U}(\epsilon, t)\mathcal{U}(\epsilon, t) = (A_M(\epsilon, t) + I)_0 - I.$$

In particular,

$$\mathcal{U}(\epsilon, t)I^\infty(\epsilon, t)|_{\tilde{M}} \subset \mathcal{D}(\mathcal{U}(\epsilon, t)).$$

It follows that

$$\begin{aligned} & (f, A_M(\epsilon, t)I^\infty(\epsilon, t)|_{\tilde{M}})_M \\ &= (f, ((A_M(\epsilon, t) + I)_0 - I)I^\infty(\epsilon, t)|_{\tilde{M}})_M \\ &= (f, \mathcal{U}(\epsilon, t)\mathcal{U}(\epsilon, t)I^\infty(\epsilon, t)|_{\tilde{M}})_M \quad (9) \\ &= 0. \end{aligned}$$

Let $f_{\epsilon, t}$ be a function such that $f_{\epsilon, t}|_{\tilde{M}} = f$ and

$$f_{\epsilon, t}(gln) = (\mu^\epsilon \otimes \nu^{t+\rho})(l^{-1})f_{\epsilon, t}(g) \quad (l \in \tilde{L}, n \in N).$$

By Lemma 8.2, $\forall u \in V(\epsilon, t)$,

$$0 = (f, A_M(\epsilon, t)(u|_{\tilde{M}}))_M = (f_{\epsilon, t}, A_X(\epsilon, t)u)_X = (f_{\epsilon, t}, u)_{\epsilon, t}.$$

This equality is to be interpreted as an equality of integrals according to the definitions of $(\cdot, \cdot)_M$ and $(\cdot, \cdot)_X$. Since $A_X(\epsilon, t)$ acts on $\tilde{U}(n)$ -types in $V(\epsilon, t)$ as scalars, $A_X(\epsilon, t)V(\epsilon, t) = V(\epsilon, t)$. We now have

$$(f_{\epsilon, t}, V(\epsilon, t))_X = 0.$$

In particular, $f_{\epsilon, t}|_{\tilde{U}(n)} \in L^1(X)$. By Peter-Weyl Theorem, $f_{\epsilon, t} = 0$. We see that $\mathcal{U}(\epsilon, t)$ is an isometry from $C(\epsilon, t)$ onto $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$. ■

The Hilbert space $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$ is the mixed model for $I(\epsilon, 0)$ restricted to \tilde{M} . We now obtain an isometry from $C(\epsilon, t)$ onto $I(\epsilon, 0)$. Within the mixed model, the action of $I(\epsilon, t)(g_1 k_2)$ is simply the left regular action and it is independent of t . We obtain

Lemma 8.5. *Suppose $t < 0$. Let $g \in \tilde{U}(q)$. Let $L(g)$ be the left regular action on $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$. As an operator on $I^\infty(\epsilon, t)|_{\tilde{M}}$, $L(g)$ commutes with $A_M(\epsilon, t)$. Furthermore, $L(g)$ commutes with $(A_M(\epsilon, t) + I)_0 - I$. Similar statement holds for $g \in \tilde{Sp}(p, \mathbb{R})$.*

Proof. Let $g \in \tilde{M}$. Both $A_M(\epsilon, t)$ and $L(g)$ are well-defined operator on $I^\infty(\epsilon, t)|_{\tilde{M}}$. Regarding $A(\epsilon, t)I(\epsilon, t)(g) = I(\epsilon, -t)(g)A(\epsilon, t)$ as operators on the mixed model $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$, we have

$$A_M(\epsilon, t)L(g) = L(g)A_M(\epsilon, t).$$

It follows that

$$L(g)^{-1}(A_M(\epsilon, t) + I)L(g) = (A_M(\epsilon, t) + I).$$

Since $L(g)$ is unitary, $L(g)^{-1}(A_M(\epsilon, t) + I)_0 L(g) = (A_M(\epsilon, t) + I)_0$. In fact, $(A_M(\epsilon, t) + I)_0$ can be defined as the inverse of $(A_M(\epsilon, t) + I)^{-1}$, which exists and is bounded. So $L(g)$ commutes with both $(A_M(\epsilon, t) + I)^{-1}$ and $(A_M(\epsilon, t) + I)_0$. ■

Lemma 8.6. *We have, for $g \in \tilde{M}$, $\mathcal{U}(\epsilon, t)I(\epsilon, t)(g) = I(\epsilon, 0)(g)\mathcal{U}(\epsilon, t)$.*

Proof. Recall from Theorem 7.2 that the action of \tilde{M} on the mixed model is independent of t . It suffices to show that on the mixed model, $\mathcal{U}(\epsilon, t)$ commutes with $L(g)$ for any $g \in \tilde{M}$. By Lemma 8.5,

$$L(g)^{-1}[(A_M(\epsilon, t) + I)_0 - I]L(g) = (A_M(\epsilon, t) + I)_0 - I.$$

Since $L(g)$ is unitary on $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$, both sides are positive self-adjoint operators. Taking square roots, we obtain $L(g)^{-1}\mathcal{U}(\epsilon, t)L(g) = \mathcal{U}(\epsilon, t)$. ■

Theorem 7.2 is proved.

References

- [ABPTV] Adam, J., D. Barbasch, A. Paul, P. Trapa, and D. Vogan *Unitary Shimura correspondences for split real groups*, J. Amer. Math. Soc. **20** (2007), 701–751.
- [Bar] Bargmann, V. *Irreducible unitary representations of the Lorentz group*, Annals of Math. **48** (1947), 568–640.
- [BOO] Branson, T., G. Olafsson, and B. Ørsted, *Spectrum Generating Operators and Intertwining Operators for Representations Induced from a Maximal Parabolic Subgroup*, Journal of Functional Analysis **135** (1996), 163–205.
- [He02] He, H., *Compactification of Classical Groups*, Communications in Analysis and Geometry **10** (2002), 709–740.
- [He06] —, *Functions on Symmetric Spaces and Oscillator Representation*, Journal of Functional Analysis **244** (2007), 536–564.
- [Henu] —, *Unipotent Representations and Quantum Induction*, Preprint, <http://www.arxiv.org/math.GR/0210372>, (2002).
- [He09] —, *Certain Induced Complementary Series of the Universal Covering of the Symplectic Group*, submitted (2009).
- [Howe] Howe, R., and E.-C. Tan, “Non-Abelian Harmonic Analysis: Applications of $SL(2, \mathbb{R})$,” Springer-Verlag, 1992.
- [Kos] Kostant, B., *On the existence and irreducibility of certain series of representations*, Bull. Amer. Math. Soc. **75** (1969), 627–642.
- [KR] Kudla, S., and S. Rallis, *Degenerate Principal Series and Invariant Distribution*, Israel Journal of Mathematics **69** (1990), 25–45.
- [Lee] Lee, S. T., *Degenerate principal series representations of $Sp(2n, R)$* , Compositio Math. **103** (1996), 123–151.
- [OZ] Ørsted, B., and G. Zhang, *Generalized Principal Series Representations and Tube Domain*, Duke Math. Journal **78** (1995), 335–357.
- [RS] Riesz, F., and B. Sz.-Nagy, “Functional Analysis,” New York, Ungar 1955.
- [Puk] Pukánszky, L., *The Plancherel formula for the universal covering group of $SL(R, 2)$* , Math. Ann. **156** (1964), 96–143.

- [Sa] Sahi, S., *Unitary Representations on the Shilov Boundary of a Symmetric Tube Domain*, Representation Theory of Groups and Algebras, Comtemp. Math. **145**, Amer. Math. Soc., Providence, RI, (1993), 275–286.
- [WV] Wallach, N., and D. Vogan, *Intertwining Operators for Real Reductive Groups*, Advances in Mathematics **82** (1990), 203–243.

Hongyu He
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803, USA
livingstone@alum.mit.edu

Received July 6, 2009
and in final form February 8, 2010