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A Lie-Trotter Formula for Riemannian Manifolds and Applications

Jimmie Lawson and Yongdo Lim

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Abstract. In this paper we derive a Lie-Trotter formula for general Riemannian manifolds based on the local existence of a midpoint operation and given in terms of that operation. As a corollary one obtains that continuous maps between Riemannian manifolds that preserve the local midpoint structure are smooth. In particular, this is true for (local) isometries.
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1. Introduction

The Lie-Trotter product formula

$$e^{A+B} = \lim_{n\to\infty} \left(e^{A/n}e^{B/n}\right)^n$$

first formulated by Sophus Lie for square matrices A and B is of great utility and generalizes to various settings such as semigroups of (unbounded) operators (see e.g. Trotter [7]) and Lie theory. For an elementary derivation and basic applications of the formula in the setting of Lie theory, we refer the reader to [4]. In this paper we derive a variant for general Riemannian manifolds based not on multiplication, but on a midpoint operation that always exists locally on Riemannian manifolds. A midpoint version of the Lie-Trotter formua for the special case of positive operators on a Hilbert space together with applications of the formula in that context may be found in [1].

Midpoint formulas are of interest since many geometric concepts can be captured from the midpoint concept, e.g., (geodesically) convex sets, convex functions, nonpositive curvature, etc. In his metric approach to geometry, Busemann typically included an axiom of "between points" [2], of which midpoints are a special case. In [6] midpoints play a prominent role in the metric treatment of symmetric spaces.

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One formulation of the famous fifth problem of Hilbert asked whether a locally euclidean topological group had a differentiable multiplication, i.e., was a Lie group. A related problem is finding general conditions for a continuous function to be smooth. In the last section we apply our Lie-Trotter formula to show this is true for continuous functions between Riemannian manifolds preserving the midpoint operation, in particular for isometries of Riemannian manifolds.

2. Background

Let (M, g) be a connected Riemannian manifold, a smooth connected manifold M with a smooth section g of the positive-definite quadratic forms on the tangent bundle giving each tangent space the structure of a Hilbert space. We recall some basic standard material about Riemannian manifolds (see, for example, Chapters VII-IX of [5] or Chapter II of [3]). Any piecewise smooth curve α on M can be assigned an arc length $L(\alpha)$ by integrating the length $\|\alpha'(t)\| = \langle\alpha'(t), \alpha'(t)\rangle_{\alpha(t)}^{1/2}$ of the tangent vector over the domain of α. The g-distance or simply distance δ is defined by setting $\delta(x, y)$ equal to the infimum over the lengths of all piecewise smooth curves from x to y. The distance function δ is indeed a metric, and the corresponding metric topology agrees with the original topology on M ([5, Proposition VII-6.1]).

Any Riemannian manifold M has a unique connection associated with the metric g, called the Levi-Civita connection or the canonical connection. This connection is typically used to define the geodesics of the manifold, although Lang alternatively (but equivalently) defines them from the canonical spray [5, Section VII-7]. Given $x \in M$ and $v \in T_xM$, the tangent space at x, there exists a unique maximal geodesic β_v defined on an open interval containing 0 and satisfying $x = \beta_v(0)$, $v = {\beta_v}'(0)$. Then $\exp(v)$ is defined if and only if $\beta_v(1)$ is, and in this case $\exp(v) = \beta_v(1)$. The domain of the exponential map is an open subset Ω such that $\{0_x : x \in M\} \subseteq \Omega \subseteq TM$, where 0_x is the 0-vector in the tangent space T_xM at x. Geodesics then have the alternative description $t \mapsto \exp(tv)$, $tv \in \Omega$, the unique maximal geodesic that runs through $x = \exp(0_x) = \pi(v)$ at $t = 0$ with velocity $v \in T_xM$, where $\pi : TM \to M$ is the tangent bundle map. For $v \in T_xM$, the geodesic $t \mapsto \exp(tv)$ has constant speed $\|v\|$, and hence for $v \in \Omega$, this geodesic restricted to $[0, 1]$ has length $\|v\|$. The exponential map $\exp : \Omega \to M$ is a smooth map and the map $\exp_x : T_xM \cap \Omega \to M$ is a diffeomorphism on sufficiently small neighborhoods of 0_x.

Let $\beta : [a, b] \to M$ be a geodesic with $\beta(a) = x$, $\beta(b) = y$. We may reparametrize to obtain a geodesic on $[0, 1]$ by setting $\gamma(t) = \beta(a + t(b - a))$ for $0 \le t \le 1$. Then as in the preceding paragraph we may rewrite γ as $\gamma(t) = \exp_x(tv)$ where $v = \gamma'(0)$. This gives us a convenient parametrization of a geodesic stretching between two points.

A geodesic $\beta : [a, b] \to M$ is a minimal geodesic if $L(\beta) \le L(\alpha)$ for every piecewise smooth path joining $\beta(a)$ to $\beta(b)$. This is equivalent to $L(\beta) = \delta(\beta(a), \beta(b))$. Since the geodesic β has some constant speed σ, we have

$$\delta(\beta(a), \beta(b)) = L(\beta) = \sigma(b - a).$$

It is straightforward to see that the restriction of β to any closed subinterval of

$[a, b]$ will again be a minimal geodesic.

An open subset U of M is said to be convex if given $y, z \in U$, there exists a unique minimal geodesic from y to z and this geodesic is contained in U, where uniqueness means unique up to reparametrization. The following is a form of Whitehead's theorem (see [5, Theorem VII-5.8]).

Theorem 2.1. *Let (M, g) be a Riemannian manifold, let $p \in M$. Then there exists $\epsilon > 0$ such that the following are satisfied:*

1. *For $0 < r \leq \epsilon$, the open neighborhood $B_\delta(p, r) = \exp_p B_g(0_p, r)$ (where the second ball is taken in T_pM in the norm arising from g_p).*

2. *Each $B_\delta(p, r)$ is convex, $0 < r \leq \epsilon$.*

3. *The map $\exp_p$ restricted to $B_g(0_p, \epsilon)$ is a diffeomorphism.*

4. *For $0 < r \leq \epsilon$, the mapping $(x, y) \mapsto v$, where $v \in T_xM$ and $t \mapsto \exp_x(tv)$ $(0 \leq t \leq 1)$ is the minimal geodesic joining x and y, is a diffeomorphism from $B_\delta(p, r) \times B_\delta(p, r)$ onto an open subset of Ω, the domain of $\exp$.*

5. *For any $v \in B_g(0_p, \epsilon)$, the mapping $tv \mapsto \exp(tv)$ for $0 \leq t \leq 1$ is an isometry.*

6. *For $0 < r \leq \epsilon/2$, $x, y \in B_\delta(p, r)$, there is exactly one midpoint between x and y.*

Proof. Items (1) and (2) follow from [5, Theorem VII-5.8]. Items (3) and (4) follow from [5, Corollary VII-5.3] applied to the canonical spray for a Riemannian metric.

For (5), let $0 \leq s < r \leq 1$ and let $v \in B_g(0_p, \epsilon)$. It follows from the convexity of $B_\delta(p, \epsilon)$ that the geodesic $t \mapsto \exp(tv)$, $0 \leq t \leq 1$ is a minimal geodesic. Set $\alpha(t) = \exp(tv)$ for $s \leq t \leq r$. Thus $\delta(\exp(sv), \exp(rv)) = L(\alpha) = \|v\|(r - s)$. Since also $\|rv - sv\| = |r - s|\|v\|$, we conclude that $tv \mapsto \exp(tv)$ for $0 \leq t \leq 1$ is an isometry.

For (6), we can always find exactly one midpoint on the unique minimal geodesic connecting them (see the beginning of the next section). Suppose m were another. Then $\delta(x, m) = (1/2)\delta(x, y) \leq (1/2)(\delta(x, p) + \delta(p, y)) < (1/2)(2r) = r$. Thus $\delta(m, p) \leq \delta(m, x) + \delta(x, p) < r + r = 2r \leq \epsilon$. Since $B_\delta(p, \epsilon)$ is convex, there exist minimal geodesics from x to m and from y to m, each of length $(1/2)\delta(x, y)$. Their piecewise combination will have length $\delta(x, y)$. By [5, Theorem VII-6.2] some reparametrization of this curve must be the unique minimal geodesic. But this contradicts the fact that m did not belong to this geodesic.

■

We call the balls $B_\delta(x, r) = \exp_x B_g(0_x, r)$ and $B_g(0_x, r)$ that satisfy (1)-(6) *normal balls.*

3. Midpoints

For a Riemannian manifold (M, g) we may define midpoints in the same way that it is done for general metric spaces: m is a midpoint for x, y if $\delta(x, m) = \delta(m, y) = \delta(x, y)/2$. In general points x, y may have no, one, or multiple midpoints. However, there is a distinguished one in the case that there exists a unique minimal geodesic stretching from x to y. Indeed let β a parametrization of this minimal geodesic with domain $[a, b]$ and speed σ. Then for $c = (a + b)/2$,

$$\delta(a, c) = L(\beta|_{[a,c]}) = \sigma(c - a) = \frac{1}{2}\sigma(b - a) = \frac{1}{2}\delta(a, b),$$

and similarly $\delta(c, b) = (1/2)\delta(a, b)$ and no other point on the geodesic is a midpoint. We denote this uniquely determined midpoint for those pairs (x, y) that lie on a unique minimal geodesic by $x\#y$. Note that $x\#y = y\#x$ and trivially $x\#x = x$.

We restrict our study of the midpoint operation $x\#y$ to normal balls. Since these are convex, we have that $x\#y$ is defined and back in the ball for all points x, y in the ball. By Theorem 2.1(6) $x\#y$ is the unique midpoint between x and y. Our Trotter formula will be derived in terms of this operation.

It is frequently convenient to think of a Riemannian manifold M as pointed by fixing some $\varepsilon \in M$.

Lemma 3.1. *Let $B = B_\delta(\varepsilon, r) = \exp_\varepsilon B_g(0_\varepsilon, r)$ be a normal ball around $\varepsilon \in M$. The operation $\# : B \times B \to B$ is smooth.*

Proof. For $x, y \in B$, let $v(x, y) \in T_x M$ be chosen so that $\exp_x\big(v(x, y)\big) = y$. By Theorem 2.1(4), the map v exists and is smooth from $B \times B$ into TM. We claim $x\#y = exp((1/2)v(x, y))$, which will establish the smoothness of $\#$, since the right hand side is a composition of smooth maps.

Consider the geodesic $\beta(t) = \exp(tv(x, y))$ for $0 \le t \le 1$. By Theorem 2.1(4), β is the unique minimal geodesic joining x, y in B. Hence by our previous discussion $x\#y$ is the image of the midpoint $1/2$ of $[0, 1]$, i.e., $x\#y = \beta(1/2) = \exp((1/2)v(x, y))$. ■

In the following theorem and proof, we work in a pointed Riemannian manifold (M, g, ε) and denote $\exp_\varepsilon$ as simply $\exp$. For further notational convenience, we restrict to normal balls of $0 = 0_\varepsilon$ in $T_\varepsilon M$ and ε in M. Set log to be the inverse of exp on these neighborhoods.

Theorem 3.2 (Lie-Trotter product formula). *Let $u, v \in T_\varepsilon$, where (M, g, ε) is a pointed Riemannian manifold. Then for $u, v \in T_\varepsilon M$,*

$$\begin{aligned} u + v &= \lim_{n\to\infty} n \log\big(\exp(2u/n)\#\exp(2v/n)\big) \\ &= \lim_{t\to 0^+} \frac{1}{t} \log\big(\exp(2tu\#2tv)\big). \end{aligned}$$

Proof. Set $E := T_\varepsilon M$. Define for $2u, 2v \in B_g(0_\varepsilon, r) \subseteq E$, a normal ball, $u * v = \log\big(\exp(2u)\#\exp(2v)\big)$. Then from Lemma 3.1 the binary operation $*$ is differentiable.

We compute $u*0$. The map $\beta(t)=\exp(2ut)$, $0\le t\le 1$, is a geodesic in $B_\delta(\varepsilon,r)$ from ε to $\exp(2u)$ and hence must be the minimal geodesic. It carries $1/2$ to $\exp(u)$, and so $\exp(u)=\varepsilon\#\exp(2u)$. Thus $u*0=\log(\exp(u))=u$. Similarly $0*v=0$. Thus the partials of $(u,v)\mapsto u*v$ at $(0,0)$ are given by the projections $\pi_1,\pi_2: E\times E\to E$. Since the derivative is the sum of the partials, we have $d*_{(0,0)}(u,v)=u+v$, which also gives the directional derivative of $*: E\times E\to E$ at $(0,0)$, in the direction (u,v). But this directional derivative computed directly as a limit is given by the right side of the equation in the theorem. ■

We note that a proof analogous to the preceding is a way of deriving the Lie-Trotter formula in the theory of Lie groups.

4. Continuity implies differentiability

In this section we show that continuous maps locally preserving the operation $\#$ are smooth in quite general circumstances. We first need to develop some machinery associated with the study of midpoint-preserving maps.

Lemma 4.1. *Let A be a subset of $[0,1]$ containing $\{0,1\}$ and closed under the operation of taking midpoint. The A contains all dyadic rationals and if closed, is equal to $[0,1]$.*

Proof. By hypothesis A contains all dyadic rationals between 0 and 1 with denominator 2^0 and contains all those with denominator 2^{n+1} if it contains those with denominator 2^n. By induction A contains all dyadic rationals, is thus dense, and hence all of $[0,1]$ if closed. ■

Lemma 4.2. *Let $\gamma:[0,b]\to E$, $b>0$ be a continuous, midpoint preserving map with $\gamma(0)=0$ into a topological vector space E. Then there exists $v\in E$ such $\gamma(t)=tv$ is a linear extension of γ to all of $\mathbb{R}$.*

Proof. Suppose $b=1$. Let $v=\gamma(1)$. We consider the set A of all $t\in[0,1]$ such that $\gamma(t)=tv$. Clearly $0,1\in A$. If $a,b\in A$, then

$$\gamma\big((a+b)/2\big)=(1/2)\big(\gamma(a)+\gamma(b)\big)=(1/2)(av+bv)=(1/2)(a+b)v.$$

Thus A is closed under taking midpoints. By Lemma 4.1, $A=[0,1]$, since A is easily seen to be closed.

In case $b\neq 1$, we first scale by $1/b$ from $[0,1]$ to $[0,b]$ and apply the preceding paragraph to the composition of γ with the scaling. ■

For two Riemannian manifolds M,N, we say that $f:M\to N$ *is locally midpoint preserving* if for each $x\in M$, there exists an open set U containing x such that $f(y\#z)=f(y)\#f(z)$ for all $y,z\in U$.

Theorem 4.3. *Let X,Y be Riemannian manifolds and let $\sigma:X\to Y$ be a continuous and locally midpoint preserving map. Then σ is smooth. Furthermore, if the exponential maps for X,Y are analytic, then σ is analytic.*

Proof. We choose a distinguished point $\varepsilon \in X$, take $\sigma(\varepsilon)$ for the distinguished point of Y, and define $\tilde{\sigma}$ locally near $0_\varepsilon \in T_\varepsilon X$ by $\tilde{\sigma} = \log_{\sigma(\varepsilon)} \circ \sigma \circ \exp_\varepsilon$.

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & Y \\ {\scriptstyle \exp_\varepsilon}\big\uparrow & & \big\uparrow{\scriptstyle \exp_{\sigma(\varepsilon)}} \\ T_\varepsilon X & \xrightarrow{\tilde{\sigma}} & T_{\sigma\varepsilon} Y. \end{array}$$

"Locally" means choosing a normal ball $B = B_\delta(\varepsilon, r) = \exp_\varepsilon(B_g(0_\varepsilon, r))$ such that $\sigma(B)$ is contained in a normal ball around $\sigma(\varepsilon)$, so that log can be defined as the inverse of $\exp_{\sigma(\varepsilon)}$. We note that $\tilde{\sigma}$ is the composition of smooth maps, hence smooth on $B_g(0_\varepsilon, r)$. We proceed in steps.

(1) We show for $u \in B_g(0_\varepsilon, r)$, there exists $w \in T_{\sigma(\varepsilon)}$ such that $\tilde{\sigma}(tu) = tw$for $t \in [0,1]$. The mapping $t \mapsto \exp_\varepsilon(tu)$ resp. $tu \mapsto \exp_\varepsilon(tu)$, $0 \le t \le 1$ is a mimimal geodesic resp. an isometry by Theorem 2.1, and hence is midpoint preserving from $\{tu : 0 \le t \le 1\}$ to $\{\exp_\varepsilon(tu) : 0 \le t \le 1\}$. Let $x = \exp(u)$, let $y = \sigma(x)$, and let $v = \log_{\sigma(\varepsilon)}(y)$. Then also $t \mapsto \exp(tv)$ resp. $tv \mapsto \exp(tv)$, $0 \le t \le 1$ defines a minimal geodesic resp. an isometry from $[0,1]v$ to $\exp([0,1]v)$. It follows that $\exp([0,1]v)$ is closed under the midpoint operation $\#$ and that $\log_{\sigma(\varepsilon)}$ is midpoint-preserving from $\exp([0,1]v)$ to $[0,1]v$. The map $t \to tu \to \exp(tu) \to \sigma(\exp(tu))$ from $[0,1]$ into Y is a composition of midpoint preserving maps, hence midpoint preserving. Since it carries 0_ε to $\sigma(\varepsilon)$ and u to $y = \exp_{\sigma(\varepsilon)}(v)$, one argues directly from Lemma 4.1 that all dyadic rationals are carried into $\exp_{\sigma(\varepsilon)}([0,1]v)$, and hence all of $[0,1]$ is, since $\exp_{\sigma(\varepsilon)}([0,1]v)$ is compact and the composition is continuous. It follows directly that $\sigma(\exp_\varepsilon([0,1]u) \subseteq \exp_{\sigma(\varepsilon)}([0,1]v)$. Combining all this information together, we conclude that $\tilde{\sigma} = \log_{\sigma(\varepsilon)} \circ \sigma \circ \exp_\varepsilon$ restricted to $[0,1]u$ is midpoint preserving. Thus $t \mapsto \tilde{\sigma}(tu)$ is continuous, midpoint preserving, and carries 0_ε to $0_{\sigma(\varepsilon)}$. By Lemma 4.2 there exists $w \in T_{\sigma(\varepsilon)}$ such that $\tilde{\sigma}(tu) = tw$ for all $t \in [0,1]$.

(2) We show that we can uniquely extend $\tilde{\sigma}$ so that $\tilde{\sigma}$ is globally defined, continuous, and positively homogeneous, namely $\tilde{\sigma}(u) = m\tilde{\sigma}(u/m)$, $m >> 0$. Indeed let $u \in T_\varepsilon X$ and choose $0 < m < n$ such that $u/m, u/n \in B_g(0_\varepsilon, r)$. Then $u/n = s(u/m)$ for $s = m/n < 1$. By Step 1, there exists $w \in T_{\sigma(\varepsilon)}$ such that $\tilde{\sigma}(t(u/m)) = tw$ for all $t \in [0,1]$. It follows that

$$n\tilde{\sigma}(u/n) = (m/s)\tilde{\sigma}(su/m) = (m/s)sw = mw = m\tilde{\sigma}(u/m).$$

Thus definition of $\tilde{\sigma}(u) = m\tilde{\sigma}(u/m)$ is independent of the real number $m > 0$, as long as m is sufficiently large so that $\|u/m\| < r$.

For $u \in T_\varepsilon M$, fix $m > 0$ such that $\|u/m\| < r$. Then there is an open neighborhood Q of u such that $\|v/m\| < r$ for $v \in V$. Then on Q the extension of $\tilde{\sigma}$ is given by $v \mapsto m\tilde{\sigma}(u/m)$, which is continuous.

For $u \in T_\varepsilon M$ and $s > 0$, pick $m > s$ such that $\|su/m\| < r$. Then

$$\tilde{\sigma}(su) = m\tilde{\sigma}(su/m) = m(s/m)\tilde{\sigma}(u) = s\tilde{\sigma}(u).$$

Thus s is positively homogeneous.

(3) We use the Trotter formula to show that $\tilde{\sigma}$ is additive. We temporarily abbreviate the exponential functions $\exp_\varepsilon$ and $\exp_{\sigma(\varepsilon)}$ by $\exp$, distinguishing them by context, and the corresponding log functions by $\log$. We first use the positive homogeneity and the local equality $\tilde{\sigma} = \log \circ \sigma \circ \exp$ in various equivalent forms to calculate for $u, v \in T_\varepsilon X$ and n large:

$$\begin{aligned}
\tilde{\sigma}\Big(n \log\big(\exp(2u/n) \# \exp(2v/n)\big)\Big) &= n(\tilde{\sigma} \circ \log)\big(\exp(2u/n) \# \exp(2v/n)\big) \\
&= n(\log \circ \sigma)\big(\exp(2u/n) \# \exp(2v/n)\big) \\
&= n \log\big(\sigma(\exp(2u/n)) \# \sigma(\exp(2v/n))\big) \\
&= n \log\big(\exp(\tilde{\sigma}(2u/n)) \# \exp(\tilde{\sigma}(2v/n))\big) \\
&= n \log\big(\exp((2/n)\tilde{\sigma}(u)) \# \exp((2/n)\tilde{\sigma}(v))\big).
\end{aligned}$$

We thus have by the Trotter formula for $u, v \in T_\varepsilon X$:

$$\begin{aligned}
\tilde{\sigma}(u+v) &= \tilde{\sigma}\Big(\lim_{n\to\infty} n \log\big(\exp(2u/n) \# \exp(2v/n)\big)\Big) \\
&= \lim_{n\to\infty} \tilde{\sigma}\Big(n \log\big(\exp(2u/n) \# \exp(2v/n)\big)\Big) \\
&= \lim_{n\to\infty} n \log\big(\exp((2/n)\tilde{\sigma}(u)) \# \exp((2/n)\tilde{\sigma}(v))\big) \\
&= \tilde{\sigma}(u) + \tilde{\sigma}(v).
\end{aligned}$$

(4) Since $\tilde{\sigma}$ is continuous, additive, and homogeneous for positive scalars, it follows easily that it is a continuous linear mapping, hence smooth. Near ε, $\sigma = \exp \circ \tilde{\sigma} \circ \log$ and is thus smooth. Since ε was an arbitrary choice for the distinguished point, the map σ is smooth in a neighborhood of every point, hence smooth. If each exponential map $\exp_x$ is analytic, then the local inverse log is also analytic since the derivative at 0_x is invertible (indeed, it is the identity). Thus the composition $\sigma = \exp \circ \tilde{\sigma} \circ \log$ is locally analytic, hence analytic. ∎

Since local isometries are locally midpoint preserving (since locally the midpoints are unique by Theorem 2.1(6)), we have immediately:

Corollary 4.4. *A local isometry, in particular an isometry, between Riemannian manifolds is smooth.*

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Jimmie Lawson
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803, USA
lawson@math.lsu.edu

Yongdo Lim
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
ylim@knu.ac.kr

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