Journal of Lie Theory
Volume **20** (2010) 347–374

Locally Precompact Groups: (Local) Realcompactness and Connectedness

W. W. Comfort[1] and Gábor Lukács[2]

Communicated by K. H. Hofmann

Abstract. A theorem of A. Weil asserts that a topological group embeds as a (dense) subgroup of a locally compact group if and only if it contains a non-empty precompact open set; such groups are called *locally precompact*. Within the class of locally precompact groups, the authors classify those groups with the following topological properties:
(a) Dieudonné completeness;
(b) local realcompactness;
(c) realcompactness;
(d) hereditary realcompactness;
(e) connectedness;
(f) local connectedness;
(g) zero-dimensionality.

They also prove that an abelian locally precompact group occurs as the quasi-component of a topological group if and only if it is *precompactly generated*, that is, it is generated algebraically by a precompact subset.
Mathematics Subject Classification 2000: 22A05, 54H11; 22B05, 22C05.
Key Words and Phrases: Precompact group, precompactly generated group, locally precompact group, Weil completion, Dieudonné complete group, locally Dieudonné complete group, realcompact group, locally realcompact group, connected group, locally connected group, ω-balanced group, quasi-component.

0. Introduction

A subset X of a topological group G is *precompact* if for every neighborhood U of the identity in G, there is a finite $S\subseteq X$ such that $X\subseteq (SU)\cap(US)$. It is easily seen that every subgroup (indeed, every subset) of a compact group is precompact. The local version of that statement, and its converse, are the content of a theorem of A. Weil: A topological group G embeds as a (dense) subgroup of a locally compact group $\widetilde{G}$ if and only if G is *locally precompact* in the sense that some non-empty open subset of G is precompact (cf. [63]). For such a group G, the Weil completion

[1] The first author gratefully acknowledges generous hospitality from the University of Manitoba in January 2008.
[2] The second author gratefully acknowledges the generous financial support received from NSERC and the University of Manitoba, which enabled him to do this research.

ISSN 0949–5932 / $2.50

$\widetilde{G}$ is unique in the obvious sense, and it coincides with the two-sided completion introduced by Raĭkov in 1946 (cf. [48]). For background on these completions, see [49], [63], [48], [31, (4.11)-(4.15)], and [42, Section 1.3].

As the bibliographies in the monographs [31], [32], [34], and [42] attest, there is a huge literature devoted to the study and characterization of the locally compact groups that enjoy additional special topological properties. We work here in a parallel vein, but now in the class of locally precompact groups. Most of our results, when restricted back to the locally compact case, will be unsurprising, and in some cases familiar, to the reader.

The paper is organized as follows. After introductory material in §1, we characterize in §2 those locally precompact groups that are locally realcompact (Theorem 2.22); they are the Dieudonné complete groups, or equivalently, the groups that are G_δ-closed in their completions. In §3, we find internal (intrinsic) characterizations of those locally precompact groups that are hereditarily realcompact (Theorems 3.5), while in §4 we address the relations among connectedness properties of locally precompact groups (emphasizing the locally pseudocompact case) and their completions; here, the principal result is that a locally pseudocompact group is locally connected if and only if its completion is locally connected (Theorem 4.15). §5 is devoted to proving that within the class of locally precompact abelian groups, the groups A that are topologically isomorphic to a group of the form $(\widetilde{G})_0 \cap G$ with G locally pseudocompact are exactly the precompactly generated groups (Theorem 5.6); thus, in particular, every connected precompact abelian group A is topologically isomorphic to the connected component of a pseudocompact group. That theorem was established in [7, 7.6] when A in addition is torsion-free, and was developed further in [12, 3.6].

1. Definitions, notations, and preliminaries

All topological spaces here are assumed to be Tychonoff. Except when specifically noted, no algebraic assumptions are imposed on the groups; in particular, our groups are not necessarily abelian. A "neighborhood" of a point means an *open* set containing the point. The collection of neighborhoods of the identity in a topological group G is denoted by $\mathcal{N}(G)$. The next theorem explains the origin of the term precompact, and relates it to the completion.

Theorem 1.1 ([42, 3.5]). *Let G be a topological group, and $X \subseteq G$ a subset. Then X is precompact if and only if $\mathrm{cl}_{\widetilde{G}} X$ is compact.*

For a space X, we denote by βX and υX its Stone-Čech compactification and Hewitt realcompactification, respectively (cf. [22, 6.5, 8.4] and [21, 3.6.1, 3.11.16]). A G_δ*-subset* of a space $(X, \mathcal{T})$ is a set of the form $\bigcap\limits_{n<\omega} U_n$ with each $U_n \in \mathcal{T}$. The G_δ*-topology* on X is the topology generated by the G_δ-subsets of $(X, \mathcal{T})$. A subset of X is G_δ*-open* (respectively, G_δ*-closed*, G_δ*-dense*) if it is open (respectively, closed, dense) in the G_δ-topology on G.

Definition 1.2. A space X is *pseudocompact* if it satisfies the following equivalent conditions:

(i) every continuous real-valued map on X has bounded range;
(ii) every locally finite family of non-empty open subsets of X is finite;
(iii) X is G_δ-dense in βX.

Definition 1.2(i) was introduced by Hewitt, who established the equivalence of (i) and (iii) (cf. [30] and [22, 1.4]). The equivalence of conditions (i) and (ii) was shown by Glicksberg (cf. [23, Theorem 2] and [21, 3.10.22]).

Definition 1.3. A topological group G is said to be *locally pseudocompact* if there is $U \in \mathcal{N}(G)$ such that $\mathrm{cl}_G U$ is pseudocompact.

Since every pseudocompact subset of a topological group is precompact (cf. [9, 1.1] and [10, 1.11]), every locally pseudocompact group is locally precompact. Numerous equivalent definitions of local pseudocompactness are provided in Theorem 1.4 below, which summarizes the main results of [10]. (The paper [10] generalizes to the local context the results of [9].)

Theorem 1.4 ([10]). *Let G be a topological group. The following statements are equivalent:*

(i) *G is locally pseudocompact;*
(ii) *for every $V \in \mathcal{N}(G)$, there is $U \in \mathcal{N}(G)$ such that $\mathrm{cl}_G U$ is pseudocompact and $\mathrm{cl}_G U \subseteq V$;*
(iii) *there is $U \in \mathcal{N}(G)$ such that $\beta(\mathrm{cl}_G U) = \mathrm{cl}_{\widetilde{G}} U$;*
(iv) *G is locally precompact, and $\beta(\mathrm{cl}_G U) = \mathrm{cl}_{\widetilde{G}} U$ for every precompact $U \in \mathcal{N}(G)$;*
(v) *G is locally precompact, and $\beta G = \beta \widetilde{G}$;*
(vi) *G is locally precompact, and $\upsilon G = \upsilon \widetilde{G}$;*
(vii) *G is locally precompact, and G_δ-dense in $\widetilde{G}$.*

Next, for the sake of completeness, we recall a well-known technical lemma concerning open subgroups of dense subgroups, which will be used several times in this paper. We denote by $\mathcal{H}(G)$ the set of open subgroups of a topological group G, and we set $o(G) := \bigcap \mathcal{H}(G)$. We note for emphasis that in Lemma 1.5, no normality conditions are imposed on any subgroups.

Lemma 1.5. *Let G be a topological group, and D a dense subgroup. Then:*

(a) *the maps*

$$\begin{aligned} \Phi\colon \mathcal{H}(G) &\longrightarrow \mathcal{H}(D) \\ M &\longmapsto M \cap D \end{aligned} \qquad\qquad \begin{aligned} \Psi\colon \mathcal{H}(D) &\longrightarrow \mathcal{H}(G) \\ H &\longmapsto \mathrm{cl}_G H \end{aligned}$$

satisfy $\Psi \circ \Phi = \mathrm{id}_{\mathcal{H}(G)}$ and $\Phi \circ \Psi = \mathrm{id}_{\mathcal{H}(D)}$, and thus they are order-preserving bijections;
(b) *for every $M \in \mathcal{H}(G)$, one has $|G/M| = |D/(M \cap D)|$;*
(c) *for every $H \in \mathcal{H}(D)$, one has $|G/\mathrm{cl}_G H| = |D/H|$;*
(d) *$o(D) = o(G) \cap D$.* ■

2. Local and global realcompactness and Dieudonné completeness

Definition 2.1.

(a) A space is *realcompact* if it is homeomorphic to a closed subspace of $\mathbb{R}^\lambda$ for some cardinal λ.

(b) A space is *Dieudonné complete* if it is homeomorphic to a closed subspace of a space of the form $\prod\limits_{\alpha\in I} M_\alpha$ with each M_α metrizable.

Remark 2.2.

(a) It is clear from Definition 2.1 that every realcompact space is Dieudonné complete. For limitations concerning the converse statement, see Discussion 2.6 and Corollary 2.9 below.

(b) Responding to a question posed in the fundamental memoire of Weil [63], Dieudonné proved that a space has a compatible complete uniformity if and only if it is (in our terminology) Dieudonné complete (cf. [11, p. 286]). For this reason, many authors prefer to call such spaces *topologically complete* (cf. [22] and [8]). The class has been studied broadly, for example, by Kelley (cf. [39, Chapter 15]) and Isbell (cf. [37, I.10-22]).

(c) It is obvious from the definitions that a product of realcompact (respectively, Dieudonné complete) spaces is realcompact (respectively, Dieudonné complete), and that a closed subspace of a realcompact (respectively, Dieudonné complete) space is realcompact (respectively, Dieudonné complete). Since the intersection $\bigcap\limits_{\alpha\in I} A_\alpha$ of subspaces of any (fixed) space is homeomorphic to a closed subspace of $\prod\limits_{\alpha\in I} A_\alpha$, it is further immediate from the definitions that in any space Y, each subspace of the form $\bigcap\limits_{\alpha\in I} A_\alpha$ with each A_α a realcompact (respectively, Dieudonné complete) subspace of Y is itself realcompact (respectively, Dieudonné complete).

The statements given in the next theorems, which are all basic in the study of realcompact spaces and of Dieudonné complete spaces, are less obvious; we will rely on these properties in what follows.

Theorem 2.3.

(a) ([5, 2.3]) *Every G_δ-closed subspace of a realcompact (respectively Dieudonné complete) space is realcompact (respectively, Dieudonné complete).*

(b) ([22, 8.2], [21, 3.11.12]) *Every Lindelöf space is realcompact.*

(c) ([5, 3.6]) *Every locally compact topological group is Dieudonné complete.*

Notation 2.4. With each space X are associated spaces υX and γX defined as follows:

$$\upsilon X := \{p\in\beta X \mid \text{each continuous map from } X \text{ to } \mathbb{R} \text{ extends continuously to } p\};$$
$$\gamma X := \{p\in\beta X \mid \text{each continuous map from } X \text{ to a metric space extends continuously to } p\}.$$

Theorem 2.5 ([22, Chapter 8], [21, 3.11.16, 8.5.13], and [8, pp. 1-20]). *Let X be a space. Then:*

(a) *υX is realcompact and $\upsilon X = \bigcap\{X' \mid X \subseteq X' \subseteq \beta X,\ X'$ is realcompact$\}$;*

(b) *γX is Dieudonné complete and*

$$\gamma X = \bigcap\{X' \mid X \subseteq X' \subseteq \beta X,\ X' \textit{ is Dieudonné complete}\}.$$

Realcompact spaces (under the name of *Q-spaces*) as well as the space υX were introduced by Hewitt (cf. [30, Definition 12, Theorems 56-60]). Accordingly, the space υX is called the *Hewitt realcompactification* of X. Similarly, honoring Dieudonné, the space γX is called the *Dieudonné completion* of X (cf. [11]).

The definitions of realcompactness and Dieudonné completeness are similar, yet different. The distinction is best described by using the set-theoretic notion of Ulam-measurable cardinals. A cardinal number λ is said to be *Ulam-measurable* if there is a non-atomic countably additive measure $\mu\colon \mathcal{P}(\lambda) \rightarrow \{0,1\}$ such that $\mu(\lambda) = 1$. Ulam-measurable cardinals are called *measurable* in the text [22], but we follow standard procedure in reserving that term for cardinals λ with a measure $\mu\colon \mathcal{P}(\lambda) \twoheadrightarrow \{0,1\}$ that is $<\lambda$-additive in the sense that every $A \subseteq \lambda$ with $|A| < \lambda$ satisfies $\mu(A) = 0$.

Discussion 2.6. The existence of Ulam-measurable cardinals cannot be proven in ZFC—that is, their non-existence is consistent with the axioms of ZFC (cf. [40, IV.6.9, VI.4.13]). Most set theorists (appear to) believe that the existence of an Ulam-measurable cardinal is consistent with the axioms of ZFC, but that has not been established (cf. [40]). It is known that an Ulam-measurable cardinal exists if and only if an uncountable measurable cardinal exists. Indeed, the first Ulam-measurable cardinal $\mathfrak{m}$ (if it exists) is measurable (cf. [59], [56], and [22, 12.5(ii)]). Henceforth, we write $\lambda < \mathfrak{m}$ instead of "λ is not Ulam-measurable." Such statements are to be read with some good will: If no Ulam-measurable cardinal exists, then the expression $\lambda < \mathfrak{m}$ is vacuously true for every cardinal λ.

The relevance of Ulam-measurable cardinals to our work is given by the following consequence of a theorem of Mackey (cf. [44]).

Theorem 2.7 ([22, 12.2]). *A discrete space D is realcompact if and only if $|D| < \mathfrak{m}$.*

Recall that a *cellular family* in a space X is a collection of non-empty, pairwise disjoint open subsets of X. The *cellularity* of X is defined by the relation

$$c(X) := \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a cellular family in } X\}.$$

The following consequence of a theorem of Shirota (cf. [53]) provides a sufficient condition for Dieudonné complete spaces to be realcompact.

Theorem 2.8 ([22, 15.20]). *If X is Dieudonné complete and $c(X) < \mathfrak{m}$, then X is realcompact.*

It is easily seen that a metrizable space of Ulam-measurable cardinality contains a closed, discrete subspace of Ulam-measurable cardinality (cf. [8, 6.2]). Thus, combining Theorems 2.7 and 2.8 yields the following useful result.

Corollary 2.9 ([22], [8]). *The following statements are equivalent:*

(i) *there is no Ulam-measurable cardinal;*

(ii) *the class of realcompact spaces coincides with the class of Dieudonné complete spaces.*

It is well known that a space is compact if and only if it is pseudocompact and realcompact (cf. [30, Theorem 54] and [21, 3.11.1]). Thus, the notion of realcompactness is a natural complement to that of pseudocompactness. While Theorem 1.4 provides a complete "internal" characterization of locally pseudocompact groups, we are aware of no parallel intrinsic characterization of (locally) realcompact groups. In this section, we remedy this deficiency for locally precompact groups. Since every Lindelöf space is realcompact (cf. [22, 8.2] and [21, 3.11.12]), a complete description of realcompact groups is beyond the scope of this paper. Our approach is based on an argument that was used in [5, Section 4], which we formulate here explicitly.

For a topological space X, a *zero-set in* X is a set of the form $f^{-1}(0)$, where f is a real-valued continuous function on X. A subset $Y \subseteq X$ is z*-embedded* in X if for every zero-set Z in Y, there is a zero-set W in X such that $Z = W \cap Y$. (To our best knowledge, this concept was first introduced into the literature by Isbell [36], and explicitly by Henriksen and Johnson [28]; see also Hager [27] for additional citations and applications.) One says that X is an Oz*-space* if every open subset of X is z-embedded (cf. [3]). Recall that a subset $F \subseteq X$ is *regular-closed* if $F = \mathrm{cl}_X(\mathrm{int}_X F)$, or equivalently, if $F = \mathrm{cl}_X U$ for an open subset $U \subseteq X$. Blair has characterized Oz-spaces in several ways.

Theorem 2.10 ([3, 5.1]). *For every space X, the following statements are equivalent:*

(i) *X is an Oz-space;*

(ii) *every dense subset of X is z-embedded in X;*

(iii) *every regular-closed subset of X is a zero-set in X.*

Theorem 2.11.

(a) ([3, 5.3]) *If X is an Oz-space and $S \subseteq X$ is dense or open or regular-closed, then S is an Oz-space.*

(b) ([4, 1.1(b)]) *If Y is z-embedded in X, then υY is the G_δ-closure of Y in υX; hence, $\upsilon Y \subseteq \upsilon X$.*

A key component of our treatment of locally compact groups (and their subgroups) is the following consequence of a result of Ross and Stromberg:

Theorem 2.12 ([50, 1.3, 1.6], [10, 1.10]). *Every locally compact group is an Oz-space.*

Since every locally precompact group is a dense subgroup of a locally compact group, Theorems 2.12 and 2.11(a) yield:

Corollary 2.13 ([10, 1.10]). *Every locally precompact group is an Oz-space.*

Lemma 2.14. *Let G be a locally precompact group, and $U \subseteq G$ an open subset. Put $F := \mathrm{cl}_G U$ and $K := \mathrm{cl}_{\widetilde{G}} U$. Then F is z-embedded in K, and υF is the G_δ-closure of F in υK.*

Proof. Since U is open in G, there is an open subset $V \subseteq \widetilde{G}$ such that $U = V \cap G$. Thus, one has $K = \mathrm{cl}_{\widetilde{G}} U = \mathrm{cl}_{\widetilde{G}}(V \cap G) = \mathrm{cl}_{\widetilde{G}} V$, because G is dense in $\widetilde{G}$. Therefore, K is a regular-closed subset of the Oz-space $\widetilde{G}$ (cf. Theorem 2.12), and by Theorem 2.11(a), K is an Oz-space. So, by Theorem 2.10, every dense subset of K is z-embedded in K. In particular, F is z-embedded in K. Hence, by Theorem 2.11(b), υF is the G_δ-closure of F in υK. ■

Remark 2.15. In developing our proof of Lemma 2.14, we have followed the authors of [10, 2.3] in relying on the results cited from [50], [4], and [3]. We note that alternative sources for equivalent statements are available in the literature: The fact that every locally compact group is (in our terminology) an Oz-space follows immediately from Ščepin's results (cf. [51] and [52]); Tkachenko has shown that every G_δ-dense subspace of an Oz-space is C-embedded (cf. [57, Theorem 2]).

A topological space X is said to be *locally realcompact* (respectively, *locally Dieudonné complete*) if for every $x \in X$, there is a neighborhood U of x such that $\mathrm{cl}_X U$ is realcompact (respectively, Dieudonné complete). Since our spaces are Tychonoff, and the properties in question are inherited by closed subspaces, it is clear that a space X is locally realcompact (respectively, locally Dieudonné complete) if and only if for each $x \in X$ and neighborhood U of x there is a neighborhood V of x such that $\mathrm{cl}_X V$ is realcompact (respectively, Dieudonné complete) and $\mathrm{cl}_X V \subseteq U$. Echoing the relationship between a locally compact space and its Stone-Čech compactification, a space X is locally realcompact (respectively, locally Dieudonné complete) if and only if X is open in its Hewitt realcompactification υX (respectively, in its Dieudonné completion γX) (cf. [43, 2.11]).

In order to characterize global and local realcompactness and Dieudonné completeness in the class of locally precompact groups, one introduces a cardinal invariant.

Definition 2.16. Let τ be an infinite cardinal, and G a topological group.

(a) A subset X of G is said to be *τ-precompact* if for every $U \in \mathcal{N}(G)$, there is $S \subseteq X$ that satisfies $|S| \leq \tau$ and $X \subseteq (SU) \cap (US)$ (cf. "τ-bounded" in [26]).

(b) The *precompactness index $ip(X)$* of a subset X of G is the least infinite cardinal τ such that X is τ-precompact (cf. "index of boundedness" in [58]).

Remark 2.17. We note that the precompactness index is not a topological invariant of a space X, but rather of the way a space X is placed in G. Indeed, homeomorphic subspaces of a given group G may have different precompactness indices, as the following example shows: Let $\lambda > \omega$ be a cardinal, E a discrete group of cardinality λ, put $G := (\mathbb{Z}/2\mathbb{Z})^\lambda \times E$, and let D be a discrete subset of cardinality λ of $(\mathbb{Z}/2\mathbb{Z})^\lambda$. (For instance, one can take D to be the set of elements with precisely one non-zero coordinate.) Then $D \times \{e\}$ and $\{e\} \times E$ are homeomorphic, but D is precompact, and so $ip(D \times \{e\}) = \omega$, while $ip(\{e\} \times E) = \lambda$. Nevertheless, if H is a subgroup of G that contains X, then X has the same precompactness index in H and in G (cf. [42, 2.24(d)]).

In what follows, we need the following elementary properties of the precompactness index. (Theorem 2.18 below has an obvious analogue for cardinals $\lambda \geq \omega$, but we require only the case $\lambda = \omega$.)

Theorem 2.18 ([42, 1.29]). *For every locally compact group L, the following statements are equivalent:*

(i) *L is ω-precompact;*
(ii) *L is σ-compact;*
(iii) *L is Lindelöf.*

Theorem 2.19. *Let G be a topological group, and X a subset of G.*

(a) ([42, 2.24(a)]) *If $Y \subseteq X$, then $ip(Y) \leq ip(X)$.*
(b) ([20, 3.2], [42, 2.24(c)]) *$ip(\mathrm{cl}_G X) = ip(X)$.*
(c) ([42, 2.30]) *$ip(\langle X \rangle) = ip(X)$.*

In [5], the authors used the *compact covering number* $\kappa(X)$ (i.e., the smallest number of compact subsets of X that cover X) to characterize realcompactness in the context of locally compact groups. It is easily seen that $ip(L) = \omega \cdot \kappa(L)$ for every infinite locally compact group L. Theorem 2.19(b) indicates that for locally precompact groups, the precompactness index is the correct cardinal invariant to consider.

Theorem 2.20. *Let G be a locally precompact group, and $U \subseteq G$ an open subset. Then:*

(a) *$c(U) \leq ip(U)$;*
(b) *if $\mathrm{cl}_G U$ is Dieudonné complete and $ip(U) < \mathfrak{m}$, then $\mathrm{cl}_G U$ is realcompact.*

Proof. (a) Since $ip(U)$ is independent of the ambient group G, by replacing the group G with the subgroup $\langle U \rangle$ generated by U if necessary, we may assume that $G = \langle U \rangle$. Thus, by Theorem 2.19, $ip(\widetilde{G}) = ip(G) = ip(U)$. Since G is locally precompact, its completion $\widetilde{G}$ is locally compact, and so $ip(\widetilde{G}) = \omega \cdot \kappa(\widetilde{G})$. Therefore, by a theorem of Tkachenko, $c(\widetilde{G}) \leq \omega \cdot \kappa(\widetilde{G}) = ip(\widetilde{G})$ (cf. [58, 4.8]). Hence,

$$c(U) \leq c(G) = c(\widetilde{G}) \leq ip(\widetilde{G}) = ip(U).$$

(b) By (a), $c(\mathrm{cl}_G U) = c(U) \leq ip(U) < \mathfrak{m}$. Thus, the statement follows by Theorem 2.8. ∎

Remark 2.21. The hypothesis in Theorem 2.20(a) that G is locally precompact cannot be omitted: Indeed, put $G := \bigoplus\limits_{\omega_1} \mathbb{Z}/2\mathbb{Z}$, and equip G with the group topology whose base at zero consists of subgroups $H_\alpha := \{x \in G \mid x_\beta = 0 \text{ for all } \beta < \alpha\}$, where $\alpha < \omega_1$. Since the quotient G/H_α is countable for every $\alpha < \omega_1$, it follows that $ip(G) = \omega$, and thus, by Theorem 2.19(a), $ip(U) = \omega$ for every open subset U of G. On the other hand, if $e^{(\gamma)} \in G$ is such that $e^{(\gamma)}_\beta = 1$ if and only if $\gamma = \beta$, then $\{H_\gamma + e^{(\gamma)}\}_{\alpha \leq \gamma < \omega_1}$ is a pairwise disjoint family of open subsets of H_α. Therefore, $c(H_\alpha) = \omega_1$ for every $\alpha < \omega_1$, and hence $ip(U) < c(U) = \omega_1$ for every non-empty open subset of G. (The group G was defined and considered in [9, 3.2] for a different, but related, purpose.)

We now turn to identifying locally realcompact groups within the class of locally precompact groups. Unexpectedly, these prove to be exactly the (locally) Dieudonné complete groups in the class. Therefore, Theorem 2.22 below provides a positive answer to a special case of a problem of Arhangel′skiĭ and Tkachenko (cf. [2, 3.2.2]), who asked whether every locally Dieudonné complete topological group is Dieudonné complete.

Theorem 2.22. *Let G be a locally precompact group. The following statements are equivalent:*

(i) *G is Dieudonné complete;*

(ii) *G is locally Dieudonné complete;*

(iii) *G is locally realcompact;*

(iv) *G is G_δ-closed in $\widetilde{G}$;*

(v) *every open subgroup of G is G_δ-closed in $\widetilde{G}$;*

(vi) *G contains an open subgroup that is G_δ-closed in $\widetilde{G}$;*

(vii) *every ω-precompact open subgroup of G is realcompact;*

(viii) *G contains a realcompact open subgroup;*

(ix) *G contains a Dieudonné complete open subgroup.*

Proof. The logical scheme of the proof is as follows:

$$\begin{tikzcd}
\text{(vi)} \arrow[r, Rightarrow] & \text{(iv)} \arrow[r, Rightarrow] \arrow[dl, Rightarrow] & \text{(i)} \arrow[r, Rightarrow] & \text{(vii)} \arrow[dr, Rightarrow] & \\
\text{(v)} \arrow[u, Rightarrow] & \text{(iii)} \arrow[u, Rightarrow] & \text{(ii)} \arrow[l, Rightarrow] & \text{(ix)} \arrow[l, Rightarrow] & \text{(viii)} \arrow[l, Rightarrow]
\end{tikzcd}$$

The implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi), and (viii) $\Rightarrow$ (ix) are obvious.

(vi) $\Rightarrow$ (iv): Let H be an open subgroup of G that is G_δ-closed in $\widetilde{G}$. By Lemma 1.5(a), $M := \mathrm{cl}_{\widetilde{G}} H$ is an open subgroup of $\widetilde{G}$. Thus, G and $\widetilde{G}$ are homeomorphic (as topological spaces) to $H \times (G/H)$ and $M \times (\widetilde{G}/M)$, respectively, where both G/H and $\widetilde{G}/M$ are discrete (cf. [31, 5.26]). By Lemma 1.5(c), one has $|G/H| = |\widetilde{G}/M|$. Therefore, we obtain the following commutative diagram with the horizontal arrows representing homeomorphisms (as topological spaces):

$$\begin{tikzcd} G \arrow[r, "\sim"] \arrow[d, hook] & H \times (G/H) \arrow[d, hook] \\ \widetilde{G} \arrow[r, "\sim"] & M \times (\widetilde{G}/M) \end{tikzcd}$$

Hence, the statement follows from the fact that H is G_δ-closed in M.

(iv) $\Rightarrow$ (i): Since G is locally precompact, $\widetilde{G}$ is locally compact, and by Theorem 2.3(c), $\widetilde{G}$ is Dieudonné complete. Thus, by Theorem 2.3(a), G is Dieudonné complete, being G_δ-closed in $\widetilde{G}$.

(i) $\Rightarrow$ (vii): Let H be an ω-precompact open subgroup of G. Then H is closed in G, and so by Remark 2.2(c), H is Dieudonné complete. As $ip(H)\leq\omega$, by Theorem 2.20(b), H is realcompact.

(vii) $\Rightarrow$ (viii): Since G is locally precompact, there exists $U\in\mathcal{N}(G)$ such that U is precompact. Then $ip(U)\leq\omega$, and so by Theorem 2.19(c), one has $ip(\langle U\rangle)=ip(U)\leq\omega$. Therefore, $H:=\langle U\rangle$ is an ω-precompact open subgroup of G. Hence, by (vii), H is realcompact.

(ix) $\Rightarrow$ (ii): Let H be a Dieudonné complete open subgroup of G. Then H is closed, and thus G is locally Dieudonné complete.

(ii) $\Rightarrow$ (iii): Let $U\in\mathcal{N}(G)$ be such that $\mathrm{cl}_G U$ is Dieudonné complete. Since G is locally precompact, there is $V\in\mathcal{N}(G)$ such that V is precompact. Put $W:=U\cap V$. By Theorem 2.19(a), one has $ip(W)\leq ip(V)\leq\omega<\mathfrak{m}$. By Remark 2.2(c), $\mathrm{cl}_G W$ is Dieudonné complete, being a closed subspace of $\mathrm{cl}_G U$. Therefore, by Theorem 2.20, $\mathrm{cl}_G W$ is realcompact. Hence, G is locally realcompact.

(iii) $\Rightarrow$ (iv): Let $U\in\mathcal{N}(G)$ be such that $F:=\mathrm{cl}_G U$ is realcompact. By replacing U with $U\cap U^{-1}$ if necessary, we may assume that U is *symmetric* (i.e., $U=U^{-1}$). Put $K:=\mathrm{cl}_{\widetilde{G}}U$ and $V:=\mathrm{int}_{\widetilde{G}}K$. Since U is symmetric, so are K and V. By Lemma 2.14, $F=\upsilon F$ is the G_δ-closure of F in υK. In particular, F is G_δ-closed in K. Let $x\in\widetilde{G}\backslash G$. We may pick $g\in(Vx)\cap G$, because G is dense in $\widetilde{G}$; one has $x\in Vg$, as V is symmetric. Since $Fg\subseteq G$ and $x\notin G$, clearly $x\notin Fg$. Thus, there is a G_δ-set A' in $\widetilde{G}$ such that $x\in A'$ and $A'\cap Fg=\emptyset$, because Kg is closed in $\widetilde{G}$ and Fg is G_δ-closed in Kg. Therefore, $A:=A'\cap(Vg)$ is a G_δ-set in $\widetilde{G}$ that contains x, and it satisfies

$$A\cap G = A'\cap(Vg)\cap G = A'\cap((V\cap G)g)\subseteq A'\cap((K\cap G)g) = A'\cap(Fg)=\emptyset.$$

Hence, G is G_δ-closed in $\widetilde{G}$, as desired. ■

The next theorem was inspired by [5, 3.8].

Theorem 2.23. *Let G be a locally precompact group. The following statements are equivalent:*

(i) *G is locally realcompact, and $ip(G)<\mathfrak{m}$;*
(ii) *G is Dieudonné complete, and $ip(G)<\mathfrak{m}$;*
(iii) *G is locally Dieudonné complete, and $ip(G)<\mathfrak{m}$;*

(iv) *G is G_δ-closed in $\widetilde{G}$, and $ip(G) < \mathfrak{m}$;*

(v) *$\widetilde{G}$ is realcompact, and G is G_δ-closed in $\widetilde{G}$;*

(vi) *G is realcompact.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follow by Theorem 2.22. We note that the implication (ii) $\Rightarrow$ (vi) can also be obtained as a consequence of Theorem 2.20(b).

(iv) $\Rightarrow$ (v): Let H be an ω-precompact open subgroup of G, and put $M := \mathrm{cl}_{\widetilde{G}} H$. (The existence of such a subgroup H follows from the local precompactness of G; see the proof of Theorem 2.22.) By Theorem 2.19(b), M is ω-precompact, and so by Theorem 2.18, M is Lindelöf. Therefore, by Theorem 2.3(b), M is realcompact. Furthermore, by Theorem 2.19(b), $|\widetilde{G}/M| \leq ip(\widetilde{G}) = ip(G)$, as $M \in \mathcal{H}(\widetilde{G})$. Thus, $|\widetilde{G}/M| < \mathfrak{m}$, and so by Theorem 2.7,the discrete space $\widetilde{G}/M$ is realcompact. On the other hand, by Lemma 1.5(a), $M \in \mathcal{H}(\widetilde{G})$, and so $\widetilde{G}$ is homeomorphic (as a topological space) to $M \times (\widetilde{G}/M)$. Hence, by Remark 2.2(c), $\widetilde{G}$ is realcompact, being homeomorphic to a product of realcompact spaces.

(v) $\Rightarrow$ (vi): By Lemma 2.14, υG is the G_δ-closure of G in $\upsilon\widetilde{G} = \widetilde{G}$. Thus, $\upsilon G = G$.

(vi) $\Rightarrow$ (i): Since G is realcompact, in particular, it is locally realcompact. In order to show that $ip(G) < \mathfrak{m}$, let $V \in \mathcal{N}(G)$. Pick an ω-precompact open subgroup H of G. (The existence of such a subgroup H follows from the local precompactness of G, as in the proof of Theorem 2.22.) Then H can be covered by countably many translates of V, and so G can be covered by at most $\omega \cdot |G/H|$-many translates of V. Thus, one has $ip(G) \leq \omega |G/H|$, and it suffices to show that $|G/H| < \mathfrak{m}$. Let X be a set of representatives for G/H, that is, $|X \cap (Hg)| = 1$ for every $g \in G$. Then X is discrete and closed in G (because each Hg is open), and consequently, X is a discrete realcompact space. Hence, by Theorem 2.7, $|X| = |G/H| < \mathfrak{m}$, as desired. ■

Remark 2.24.

(a) Suppose that Ulam-measurable cardinals exist, and put $G := (\mathbb{Z}/2\mathbb{Z})^{\mathfrak{m}}$, where G is equipped with the product topology. Since G is compact, it is realcompact and ω-precompact, and thus G satisfies all conditions of Theorem 2.23, but $|G| = 2^{\mathfrak{m}} > \mathfrak{m}$. This example shows that

(i′) G is locally realcompact, and $|G| < \mathfrak{m}$

cannot be added to the equivalent conditions listed in Theorem 2.23.

(b) We note in passing the availability of an alternative proof for the implication (vi) $\Rightarrow$ (i) in Theorem 2.23: If $ip(G) \geq \mathfrak{m}$, then there are $U \in \mathcal{N}(G)$ and a (recursively defined) $\mathfrak{m}$-sequence $X = \{x_\eta \mid \eta < \mathfrak{m}\}$ in G such that $x_0 = e$ and $x_\eta \notin \bigcup\limits_{\xi<\eta} x_\xi U$. Then for $V \in \mathcal{N}(G)$ chosen such that $V = V^{-1}$ and $V^2 \subseteq U$, one has $|gV \cap X| \leq 1$ for every $g \in G$. Therefore, X is discrete and closed in G, and of non-Ulam-measurable cardinality, contrary to (vi).

3. Hereditary realcompactness

A topological space X is *hereditarily realcompact* if every subspace of X is realcompact. In this section, we characterize hereditary realcompactness in the class of locally precompact groups with a "well-behaved" conjugation structure. We rely in this section on the following properties of hereditary realcompactness.

Theorem 3.1 ([22, 8.18]). *If the space X admits a coarser hereditarily realcompact topology, then X is hereditarily realcompact.*

Recall that a topological space X has *countable pseudocharacter* if every singleton in X is a G_δ-set (cf. [42, 2.1]).

Theorem 3.2 ([22, 8.15]). *If a space X is realcompact and has countable pseudocharacter, then X is hereditarily realcompact.*

Clearly, if X admits a coarser first-countable topology, then every singleton in X is the intersection of countably many open subsets, and thus X has countable pseudocharacter. For locally precompact groups, the converse is also true.

Theorem 3.3. *Let G be a locally precompact group. Then G has countable pseudocharacter if and only if G admits a coarser homogeneous metrizable topology. Moreover, in this case, the metric can be taken to be left invariant.*

In order to prove Theorem 3.3, we use the following classic result (see also the paragraph following the proof of the theorem).

Theorem 3.4 ([31, 8.14(d)]). *Let L be a topological group, and M a compact subgroup of L. Then the coset space L/M is metrizable if and only if it is first-countable. Moreover, in this case, the metric can be taken to be left invariant.*

Proof of Theorem 3.3. Since necessity is clear, we focus on sufficiency of the condition. Put $L := \widetilde{G}$, and suppose that G has countable pseudocharacter, that is, G is discrete in the G_δ-topology. Then there is a G_δ-set A in L such that $A \cap G = \{e\}$; there exist $U_n \in \mathcal{N}(L)$ such that $A = \bigcap\limits_{n=1}^{\infty} U_n$. Since G is locally precompact, its completion L is locally compact. Let $V_0 \in \mathcal{N}(L)$ be such that $\mathrm{cl}_L V_0$ is compact. For each $n \geq 1$, we pick recursively $V_n \in \mathcal{N}(L)$ that satisfies $V_n V_n \subseteq V_{n-1} \cap U_n$ and $V_n = V_n^{-1}$. Set $M = \bigcap\limits_{n=1}^{\infty} V_n$. It is easily seen that M is a closed subgroup; it is compact, because $M \subseteq \mathrm{cl}_L V_0$. We turn our attention to the coset space L/M. It follows from the construction that $M = \bigcap\limits_{n=1}^{\infty}(V_n M)$, and so L/M has countable pseudocharacter. Since L is locally compact and the canonical projection $\pi\colon L \to L/M$ is open, L/M is locally compact too (cf. [31, 5.22]). Therefore, L/M is first-countable, because every locally compact *space* of countable pseudocharacter is first-countable (cf. [21, 3.3.4]). By Theorem 3.4, this implies that L/M is metrizable, and its metric can be taken to be left invariant (under the action of L). Finally, it follows from property (ii) that $M \subseteq A$, and so $M \cap G = \{e\}$. Hence, the restriction $\pi_{|G}$ is injective; its image is metrizable and homogeneous, because G acts on it continuously (and transitively) from the left. This completes the proof, because the topology of $\pi(G)$ is the desired coarser homogeneous topology generated by a left invariant metric. ∎

We do not know whether every locally precompact group G with countable pseudocharacter admits a coarser metrizable group topology. The answer is clearly affirmative if the subgroup M constructed in the proof of Theorem 3.3 is normal in L, because then the quotient L/M is itself a metrizable topological group. This is obviously the case when G is abelian. The same conclusion can also be achieved by using a Kakutani-Kodaira style argument when G is a (not necessarily abelian) ω-precompact group. Indeed, in the latter case, by Theorems 2.19(b) and 2.18, the completion L of G is locally compact and σ-compact. (For the Kakutani-Kodaira theorem, we refer the reader to the second edition of [31, 8.7].)

Theorem 3.5. *Let G be a locally precompact group. The following statements are equivalent:*

(i) *G is hereditarily realcompact;*
(ii) *G has countable pseudocharacter, and $|G| < \mathfrak{m}$;*
(iii) *G has countable pseudocharacter, and $ip(G) < \mathfrak{m}$;*
(iv) *G admits a coarser homogeneous metrizable topology, and $|G| < \mathfrak{m}$;*
(v) *G admits a coarser homogeneous metrizable topology, and $ip(G) < \mathfrak{m}$.*

Proof. The equivalences (ii) $\Leftrightarrow$ (iv) and (iii) $\Leftrightarrow$ (v) follow by Theorem 3.3, while (ii) $\Rightarrow$ (iii) is clear, because $ip(G) \leq |G|$.

(i) $\Rightarrow$ (ii): If G is discrete, then clearly it has countable pseudocharacter, and so we may assume without loss of generality that G is not discrete. Let $g \in G$, and put $X := G \backslash \{g\}$. Since G is not discrete, X is dense in G. By Corollary 2.13, G is an Oz-space, and thus by Theorem 2.10, X is z-embedded in G. Therefore, by Theorem 2.11(b), υX is the G_δ-closure of X in υG. By (i), both X and G are realcompact, and so X is G_δ-closed in G. Hence, $G \backslash X = \{g\}$ is G_δ-open. Since every G_δ-open singleton is a G_δ-set, the group G has countable pseudocharacter.

The G_δ-topology on G is finer than the topology of G, and so by Theorem 3.1, the G_δ-topology on G is hereditarily realcompact. On the other hand, since G has countable pseudocharacter, the G_δ-topology is discrete on G. Therefore, by Theorem 2.7, $|G| < \mathfrak{m}$, as desired.

(iii) $\Rightarrow$ (i): The group G equipped with the G_δ-topology is discrete, because it has countable pseudocharacter. Thus, G is G_δ-closed in $\widetilde{G}$, since (in every topological group) every discrete *subgroup* is closed (cf. [42, 1.51]). Therefore, by Theorem 2.23, G is realcompact. Hence, by Theorem 3.2, G is hereditarily realcompact. ∎

If D is a discrete space such that $\omega < |D| < \mathfrak{m}$, then the Alexandroff one-point compactification of D is compact, hereditarily realcompact (by Theorem 2.7), but not metrizable. It follows from Corollary 3.6(b) below that no such example exists among topological groups. Since every locally compact *space* of countable pseudocharacter is first-countable (cf. [21, 3.3.4]), every locally compact *group* of countable pseudocharacter is metrizable (cf. [42, 1.23]). Thus, Theorem 3.5 has the following consequence:

Corollary 3.6. *Let L be a locally compact group. Then:*
(a) *L is hereditarily realcompact if and only if it is metrizable and $|L|<\mathfrak{m}$;*
(b) *if L is Lindelöf, then L is hereditarily realcompact if and only if it is metrizable.* ■

Theorem 3.5 guarantees only the existence of a coarser homogeneous metrizable topology, but falls short of providing a coarser metrizable group topology. So far as we are aware, such a group topology is available only under some additional assumptions on the algebraic and topological structure of the group.

A topological group G is said to be *ω-balanced* if for every $U\in\mathcal{N}(G)$, there is $\mathcal{V}_U\subseteq\mathcal{N}(G)$ such that for every $x\in G$, there is $V\in\mathcal{V}_U$ that satisfies $x^{-1}Vx\subseteq U$, and $|\mathcal{V}_U|\leq\omega$ (cf. [42, 2.7]). The class of ω-balanced groups was introduced by Kac (under the name of *groups with a quasi-invariant basis*), who also proved that a group is ω-balanced if and only if it embeds as a topological group into a product of metrizable groups (cf. [38] and [42, 2.18]). (For the sake of correct historical presentation, we note that questions related to embedding of topological groups into the product of groups of a certain class were first studied by Graev [24]; Kac's results were generalized later by Arhangel'skiĭ [1] and Guran [26].) Thanks to the following theorem due to Kac, ω-balanced groups lend themselves to a more elegant characterization of hereditary realcompactness.

Theorem 3.7 ([38], [42, 2.19])**.** *Let G be an ω-balanced topological group. Then G has countable pseudocharacter if and only if G admits a coarser metrizable group topology.*

Discussion 3.8. The class of ω-balanced groups contains all abelian groups, metrizable groups, ω-precompact groups (cf. [42, 2.27]), and also the so-called *balanced groups* (i.e., groups whose left and right uniform structures coincide; cf. [42, 1.25]). By Theorem 3.7, if G is an ω-balanced locally precompact group, then the conditions

(iv′) G admits a coarser metrizable group topology, and $|G|<\mathfrak{m}$, and
(v′) G admits a coarser metrizable group topology, and $ip(G)<\mathfrak{m}$,

may be added to the equivalent conditions listed in Theorem 3.5.

Since every metrizable precompact group has cardinality at most $\mathfrak{c}$, Theorem 3.5 can be stated in a simple form for precompact groups, and it implies portions of [5, 4.6] and [29, 3.3].

Corollary 3.9. *For every precompact group G, the following statements are equivalent:*
(i) *G is hereditarily realcompact;*

(ii) *G has countable pseudocharacter;*
(iii) *G admits a coarser metrizable group topology, and $|G| \leq \mathfrak{c}$.* ■

4. Connectedness properties

With Theorems 4.2 and 4.3 below in mind as motivation, we investigate in this section the relationship between connectedness properties of locally precompact (or locally pseudocompact) groups and their completions.

Notation 4.1. With each topological group G are associated functorial subgroups related to connectedness properties of G, defined as follows (cf. [15, 1.1.1]):
(a) G_0 denotes the connected component of the identity;
(b) $q(G)$ denotes the *quasi-component* of the identity, that is, the intersection of all clopen sets containing the identity;
(c) $o(G) := \bigcap \mathcal{H}(G)$, the intersection of all open subgroups of G.

It is well known and easily seen that all three of these subgroups are closed and normal (cf. [31, 7.1], [16, 2.2], and [42, 1.32(b)]). Clearly, $G_0 \subseteq q(G) \subseteq o(G)$, and for locally compact groups, all three are equal:

Theorem 4.2 ([31, 7.8]). *Let L be a locally compact group. Then*

$$L_0 = q(L) = o(L).$$

Following many authors, we say that a space is *zero-dimensional* if it has a base consisting of *clopen* (open-and-closed) sets. It is clear that a zero-dimensional (Hausdorff) group G satisfies $q(G) = \{e\}$.

Theorem 4.3 ([31, 3.5, 7.13]). *Let L be a locally compact group, and N a closed normal subgroup. Then the following statements are equivalent:*
(i) *L/N is zero-dimensional;*
(ii) *$(L/N)_0 = \{N\}$;*
(iii) *$L_0 \subseteq N$.*

One may wonder whether the conclusions of Theorems 4.2 and 4.3 hold for locally precompact groups. Examples 4.4(a)-(e) provide a negative answer to this question. Moreover, as Example 4.4(d) and Theorem 5.6 indicate, the relation $G_0 = q(G)$ fails for some pseudocompact abelian groups. When G and H are topological groups, we use the symbol $G \cong H$ to indicate that G and H are topologically isomorphic, that is, there is a bijection from G onto H that is simultaneously an algebraic isomorphism and a topological homeomorphism.

Examples 4.4.
(a) Comfort and van Mill showed that there exists a pseudocompact abelian group G such that $G_0 = q(G) = \{0\}$, but G is not zero-dimensional (cf. [7, 7.7]). Thus, Theorem 4.3 fails not only for locally precompact groups (or locally pseudocompact ones), but even for pseudocompact groups. Nevertheless, it is

possible to characterize zero-dimensional quotients of locally pseudocompact groups (Theorem 4.9).

(b) Ursul showed that there is a subgroup G of the group $\mathbb{R}^2$ in its usual topology such that $G_0=\{0\}$ and $q(G)\cong\mathbb{Z}$ (cf. [60]). Thus, the equality $G_0=q(G)$ in Theorem 4.2 fails for the (locally precompact) group G.

(c) Put $G=\mathbb{Q}/\mathbb{Z}$. It is obvious (and also follows from Theorem 4.6 below) that G has no proper open subgroups, and so $o(G)=G$. On the other hand, like every Tychonoff space of cardinality less than continuum, G is zero-dimensional (the argument given in [21, 6.2.8] suffices to show this); in particular, $q(G)=\{0\}$. Therefore, the equality $q(G)=o(G)$ in Theorem 4.2 fails for the (precompact) group G.

(d) By Theorem 5.6(b) below, there is a pseudocompact abelian group G such that $q(G)\cong\mathbb{Z}/2\mathbb{Z}$. Since $G_0\subseteq q(G)$ and $q(G)$ is discrete, it follows that $G_0=\{0\}$. Thus, the equality $G_0=q(G)$ in Theorem 4.2 fails even for some pseudocompact abelian groups.

(e) The iterated quasi-components $q(G), q(q(G)), \ldots, q_\alpha(G)$ of a topological group G define a descending chain of normal subgroups of G indexed by ordinals. Dikranjan showed that for every ordinal α, there is a pseudocompact abelian group H such that $H_0=q_\alpha(H)$, but $H_0\subsetneq q_\beta(H)$ for every $\beta<\alpha$ (cf. [13, Theorem 11] and [15, 1.4.10]). Dikranjan's construction is the most striking illustration known to the authors of how big the gap between G_0 and $q(G)$ can be.

We show now that the equality $q(G)=o(G)$ in Theorem 4.2 does hold for locally pseudocompact groups. The following theorem generalizes [12, 1.4], which treats the same property in the case of pseudocompact groups.

Theorem 4.5. *Let G be a locally pseudocompact group. Then:*

(a) $q(G)=q(\widetilde{G})\cap G$;

(b) $q(G)=(\widetilde{G})_0\cap G$;

(c) $q(G)=o(G)$.

Proof. (a) For every Tychonoff space X, the quasi-component of $x\in X$ is equal to the trace on X of the quasi-component of x in βX (cf. [17, 2.1]). By the implication (i) $\Rightarrow$ (v) of Theorem 1.4, one has $\beta G=\beta\widetilde{G}$, and the statement follows.

(b) As $\widetilde{G}$ is locally compact, by Theorem 4.2, $q(\widetilde{G})=(\widetilde{G})_0$. Thus, the statement follows by (a).

(c) Since $\widetilde{G}$ is locally compact, one has $q(\widetilde{G})=o(\widetilde{G})$ by Theorem 4.2. By (a) and Lemma 1.5(d),

$$q(G) = q(\widetilde{G}) \cap G = o(\widetilde{G}) \cap G = o(G),$$

as desired. ∎

Theorem 4.6. *Let G be a locally precompact group, and consider the following statements:*

(i) *G is connected;*

(ii) *G has no proper open subgroups;*

(iii) *$\widetilde{G}$ is connected.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). *Furthermore, if G is locally pseudocompact, then all three conditions are equivalent.*

In order to prove Theorem 4.6, we rely on a well-known relationship between the connectedness of a space and its Stone-Čech compactification:

Theorem 4.7 ([22, 6L.1]). *A Tychonoff space X is connected if and only if βX is connected.*

Proof of Theorem 4.6. The implication (i) $\Rightarrow$ (ii) is obvious, because every open subgroup is closed.

(ii) $\Rightarrow$ (iii): If G has no proper open subgroups, then $o(G)=G$. Consequently, $G\subseteq o(\widetilde{G})$ by Lemma 1.5(d), and so $G\subseteq(\widetilde{G})_0$ by Theorem 4.2. Since G is dense in $\widetilde{G}$ and $(\widetilde{G})_0$ is a closed subgroup, this implies that $(\widetilde{G})_0=\widetilde{G}$.

(iii) $\Rightarrow$ (ii): If $\widetilde{G}$ is connected, then by Theorem 4.2, $o(\widetilde{G})=\widetilde{G}$, and so by Lemma 1.5(d), $o(G)=G$.

(i) $\iff$ (iii): If G is locally pseudocompact, then by the implication (i) $\Rightarrow$ (v) of Theorem 1.4, one has $\beta G=\beta\widetilde{G}$, and thus the statement follows by Theorem 4.7. ∎

Connectedness is not the only property that holds for a locally pseudocompact group if and only if it holds for its completion. The same is true for the other extreme, namely, zero-dimensionality.

Theorem 4.8. *Let G be a locally pseudocompact group. Then G is zero-dimensional if and only if $\widetilde{G}$ is zero-dimensional.*

Proof. Suppose that G is zero-dimensional. Then its topology has a clopen base at e, and thus βG has a clopen base at e (cf. [22, 6L.2]). Since G is locally pseudocompact, by the implication (i) $\Rightarrow$ (v) of Theorem 1.4, one has $\beta G=\beta\widetilde{G}$. Therefore, $\widetilde{G}$ admits a clopen base at e. This shows that $\widetilde{G}$ is zero-dimensional. Since G is a subspace of $\widetilde{G}$, the converse is obvious. ∎

In connection with the proof of Theorem 4.8, it is well to recognize that although the zero-dimensional property is inherited by all subspaces, there are zero-dimensional spaces X for which βX is not zero-dimensional (cf. [21, 6.2.20] and [22, 16P.3]). In particular, a zero-dimensional space in our terminology need not have *Lebesgue covering dimension* zero.

We already noted in Example 4.4(a) that the conclusion of Theorem 4.3 fails for certain (locally) pseudocompact groups. Nevertheless, it is possible to obtain a meaningful characterization of zero-dimensional quotients of such groups.

Theorem 4.9. *Let G be a locally pseudocompact group, and M a closed normal subgroup. Then G/M is zero-dimensional if and only if $(\widetilde{G})_0 \subseteq \mathrm{cl}_{\widetilde{G}} M$.*

In the proof of Theorem 4.9, we rely on the following lemma, which is a variant on a theorem of Sulley that was extended to the non-abelian case by Grant (cf. [55] and [25, 1.3]).

Lemma 4.10 ([42, 1.19])**.** *Let G be a topological group, D a dense subgroup, and M a closed normal subgroup of D. Then $N = \mathrm{cl}_G M$ is a normal subgroup of G, and the canonical homomorphism $\bar{\pi}_{|D} \colon D/M \to DN/N$ is a topological isomorphism.*

Proof of Theorem 4.9. Put $N = \mathrm{cl}_{\widetilde{G}} M$. By Lemma 4.10, with $\widetilde{G}$ and G replacing G and D, respectively, there is a topological isomorphism φ from G/M onto a dense subgroup of the locally compact group $\widetilde{G}/N$. Thus, $\widetilde{G}/N$ is the completion of $\varphi(G/M)$. By the implication (i) $\Rightarrow$ (vii) of Theorem 1.4, G is G_δ-dense in $\widetilde{G}$. Consequently, the image $\varphi(G/M)$ is G_δ-dense in $\widetilde{G}/N$, and so by Theorem 1.4, $\varphi(G/M)$ is also locally pseudocompact. Therefore, by Theorem 4.8, $\varphi(G/M)$ is zero-dimensional if and only if $\widetilde{G}/N$ is zero-dimensional. By Theorem 4.3, the latter holds if and only if $(\widetilde{G})_0 \subseteq N$. ■

Corollary 4.11. *Let G be a locally pseudocompact group. Then:*

(a) *$G/q(G)$ is zero-dimensional if and only if $q(G)$ is dense in $(\widetilde{G})_0$;*

(b) *G/G_0 is zero-dimensional if and only if G_0 is dense in $(\widetilde{G})_0$, in which case $G_0 = q(G)$.* ■

Discussion 4.12. Theorems 4.8 and 4.9 were inspired by the work of Dikranjan (cf. [14]), and Corollary 4.11 generalizes [14, 1.7]. Dikranjan showed that if every closed subgroup of G is pseudocompact, then G/G_0 is zero-dimensional and G_0 is dense in $(\widetilde{G})_0$. (cf. [14, 1.2]). It is natural to ask whether a similar statement is true if one replaces "pseudocompact" with "locally pseudocompact."

Problem 4.13. Let G be a topological group such that every closed subgroup of G is locally pseudocompact. Is G/G_0 zero-dimensional? Equivalently, is G_0 dense in $(\widetilde{G})_0$?

After the present manuscript was submitted, Dikranjan and Lukács provided a positive answer to Problem 4.13 (cf. [18, Theorem A]).

We turn now to connectedness in the local context. Recall that a space X is *locally connected* if each connected component of every open subspace of X is open. The proof of the following easy lemma is omitted.

Lemma 4.14. *Let G be a topological group, and D a dense subgroup. If D is locally connected, then so is G.* ■

Theorem 4.15. *Let G be a locally pseudocompact group. Then G is locally connected if and only if $\widetilde{G}$ is locally connected.*

Proof. Lemma 4.14 proves the implication $\Rightarrow$. For $\Leftarrow$, let $V\in\mathcal{N}(G)$. There is $W\in\mathcal{N}(G)$ such that $\mathrm{cl}_G W\subseteq V$ and W is precompact. Then there is $W'\in\mathcal{N}(\widetilde{G})$ such that $W=W'\cap G$, and thus

$$W' \subseteq \mathrm{cl}_{\widetilde{G}} W' = \mathrm{cl}_{\widetilde{G}}(W' \cap G) = \mathrm{cl}_{\widetilde{G}} W.$$

Let C denote the connected component of the identity in W'. Since $\widetilde{G}$ is locally connected, one has $C\in\mathcal{N}(\widetilde{G})$. Consequently,

$$U=C\cap G\in\mathcal{N}(G), \text{ and } U\subseteq W'\cap G=W.$$

In particular, U is precompact. Since G is locally pseudocompact, using Theorem 1.4, we obtain that

$$\beta(\mathrm{cl}_G U) = \mathrm{cl}_{\widetilde{G}} U = \mathrm{cl}_{\widetilde{G}}(C \cap G) = \mathrm{cl}_{\widetilde{G}} C,$$

which is connected, being a closure of the connected set C. Therefore, by Theorem 4.7, $\mathrm{cl}_G U$ is connected. Finally, observe that $e\in U\subseteq \mathrm{cl}_G U\subseteq \mathrm{cl}_G W\subseteq V$, as desired. ■

Example 4.16. Let $G=\mathbb{Q}/\mathbb{Z}$. Clearly, $\widetilde{G}=\mathbb{R}/\mathbb{Z}$ is compact, connected, and locally connected; in particular, G is precompact. It is obvious (and also follows from Theorem 4.6) that the group G has no proper open subgroups. However, G is zero-dimensional (cf. [21, 6.2.8]). This shows that the assumption of local pseudocompactness in Theorems 4.6 (the implication (ii) $\Rightarrow$ (i)) and 4.15 cannot be omitted even for precompact groups.

5. Which locally precompact abelian groups occur as a quasi-component?

In order to answer the question in the title of this section, some further terminology is required. Recall that a topological group is *compactly generated* if it is generated algebraically by some compact subset. Since every connected group is generated by every neighborhood of its identity (cf. [42, 1.30]), every connected locally compact group is compactly generated.

We use additive notation for abelian topological groups. We put $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, which is the circle group written additively. Recall that if p is a prime number, then the group $\mathbb{Z}_p$ of *p-adic integers* is the (projective) limit of the quotients $\mathbb{Z}/p^n\mathbb{Z}$. The group $\mathbb{Z}_p$ is compact, zero-dimensional, and $\{p^n\mathbb{Z}_p\}_{n\in\mathbb{N}}$ is a base of open subgroups for the topology at zero (cf. [31, §10] and [19, §3.5]).

Definition 5.1. A topological group G is *precompactly generated* if there is a precompact set $X\subseteq G$ such that $G=\langle X\rangle$.

Lemma 5.2. *Let G be a locally precompact group. Then G is precompactly generated if and only if $\widetilde{G}$ is compactly generated.*

Proof. Let $X \subseteq G$ be a precompact set such that $G = \langle X \rangle$. Then, by Theorem 1.1, $Y := \mathrm{cl}_{\widetilde{G}} X$ is compact. Since G is locally precompact, $\widetilde{G}$ is locally compact, and so there is $V \in \mathcal{N}(\widetilde{G})$ such that $\mathrm{cl}_{\widetilde{G}} V$ is compact. It is easy to see that the compact set $K := (\mathrm{cl}_{\widetilde{G}} V)Y$ generates $\widetilde{G}$.

Conversely, suppose that $\widetilde{G}$ is compactly generated, that is, $\widetilde{G} = \langle K \rangle$, where K is compact. Since G is locally precompact, its completion $\widetilde{G}$ is locally compact, and so there is $V \in \mathcal{N}(\widetilde{G})$ such that $\mathrm{cl}_{\widetilde{G}} V$ is compact. Put $U := (VK) \cap G$. As VK is open in $\widetilde{G}$, the set U is open in G. The set U is precompact, because it is contained in the compact set $\mathrm{cl}_{\widetilde{G}}(VK) = (\mathrm{cl}_{\widetilde{G}} V)K$. It is easily seen that U generates G. ■

For a topological space X, we denote by $w(X)$ the *weight* of X, that is, the smallest possible cardinality of a base for the topology of X.

Theorem 5.3. *Let G be a locally precompact abelian group. The following statements are equivalent:*

(i) *G is precompactly generated;*
(ii) *G is topologically isomorphic to a subgroup of a connected locally compact abelian group C.*

Furthermore,

(a) *if G is infinite, then the group C in* (ii) *may be chosen such that $w(C) = w(G)$;*
(b) *if G is precompact, then the group C in* (ii) *may be chosen to be compact.*

Theorem 5.3 is a generalization to locally precompact groups of the statement that every compactly generated locally compact abelian group is a topological subgroup of a connected locally compact abelian group (cf. [31, 9.8] and [54, 23.11]).

In order to prove Theorem 5.3, we rely on the following known, albeit perhaps not sufficiently well-known, result of Morris. We are grateful to Kenneth A. Ross for directing us to the cited references.

Theorem 5.4 ([45, Corollary 2], [46], [47, p. 93, Exercise 1], and [54, 23.13])**.** *Every closed subgroup of a compactly generated locally compact abelian group is compactly generated.*

Remark 5.5. A recent result of K. H. Hofmann and K.-H. Neeb, which generalizes Morris's theorem, states that closed (almost) soluble subgroups of (almost) connected locally compact groups are compactly generated (cf. [35]). However, without such extra assumptions, statements parallel to Theorems 5.3 and 5.4 may fail for non-abelian groups:

(a) The semidirect product $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts by shifts, is locally compact, but is not pro-Lie (cf. [33]). Thus, it cannot be a (closed) topological subgroup of a connected locally compact group, as every connected locally compact group is pro-Lie (cf. [64, Theorem 5']).
(b) The commutator subgroup of the (discrete) free group on $n > 1$ generators is a free group of countable rank (cf. [41, Vol. II, p. 36, Theorem I]).

Proof of Theorem 5.3. (i) $\Rightarrow$ (ii): Lemma 5.2, $\widetilde{G}$ is compactly generated. Thus, by replacing G with $\widetilde{G}$, we may assume that G itself is locally compact and compactly generated. (Since $w(G)=w(\widetilde{G})$, doing so does not affect the statement concerning equality of weights.) Consequently, $G\cong M\times\mathbb{R}^a\times\mathbb{Z}^c$, where M is the maximal compact subgroup of G, and $a,c\in\mathbb{N}$ (cf. [31, 9.8] and [54, 23.11]). Since M is a subgroup of G, one has $w(M)\leq w(G)$, and so M is topologically isomorphic to a subgroup of $\mathbb{T}^{w(G)}$. The group $\mathbb{Z}^c$ is a subgroup of $\mathbb{R}^c$. Therefore, $M\times\mathbb{R}^a\times\mathbb{Z}^c$ is topologically isomorphic to a subgroup of the connected group $C:=\mathbb{T}^{w(G)}\times\mathbb{R}^{a+c}$.

(a) If G is infinite, then $w(G)$ is infinite, and hence $w(C)=w(\mathbb{T}^{w(G)})=w(G)$.

(b) If G is precompact, then $\widetilde{G}$ is compact, and so $a=c=0$. Thus, the group $C:=\mathbb{T}^{w(G)}$ is compact, and G is topologically isomorphic to a subgroup of C.

(ii) $\Rightarrow$ (i): Suppose that $G\cong S$, where S is a subgroup of C. Since C is connected, it is compactly generated. Thus, by Theorem 5.4, $\mathrm{cl}_C S$ is compactly generated too, and so $\widetilde{G}$ is compactly generated, because it is topologically isomorphic to $\mathrm{cl}_C S$. Hence, by Lemma 5.2, G is precompactly generated. ∎

We are now ready to answer the question in the title of the section.

Theorem 5.6. *Let A be a locally precompact abelian group. The following statements are equivalent:*

(a) *A is precompactly generated;*

(b) *there is a locally pseudocompact abelian group G such that*

$$A\cong q(G)=(\widetilde{G})_0\cap G.$$

Furthermore,

(i) *if $w(A)\geq\omega_1$ and* (i) *holds, then the group G in* (ii) *may be chosen so that $w(G)=w(A)$;*

(ii) *if A is precompact, then the group G in* (ii) *may be chosen to be pseudocompact; and*

(iii) *if A is connected, then $A\cong G_0=q(G)$.*

Theorem 5.6 follows the pattern of a number of known "embedding" results, which state that certain (locally) precompact groups embed into (locally) pseudocompact groups as a particular (e.g., functorial) closed subgroup (cf. [6, 2.1], [61], [7, 7.6], [62], and [12, 3.6]). We are grateful to Dikran Dikranjan for suggesting that Theorem 5.6 might also hold for locally precompact abelian groups (rather than simply for precompact abelian groups, as it appeared in an early version of this manuscript), and for drawing our attention to the possibility of choosing G so that $w(G)=w(A)$. In the proof of Theorem 5.6, we rely on a well-known theorem and a technical lemma that are presented below.

Theorem 5.7 ([31, 9.14 and 24.25], [54, 23.27]). *Every connected locally compact abelian group is divisible.*

In order to distinguish continuous homomorphisms from those that are not subject to topological assumptions, we refer to the latter as *group homomorphisms*.

Lemma 5.8. *Let E be a divisible abelian group, and λ an infinite cardinal such that $|E|\leq 2^\lambda$. Then for every (fixed) prime p, there is a group homomorphism $\varphi\colon \mathbb{Z}_p^{\omega_1\times\lambda}\to E$ such that:*

(a) *$\varphi^{-1}(x)$ is G_δ-dense in $\mathbb{Z}_p^{\omega_1\times\lambda}$ for every $x\in E$;*

(b) *for every abelian topological group C and group homomorphism $\psi\colon C\to E$, the pullback*

$$\mathbb{Z}_p^{\omega_1\times\lambda}\times_E C := \{(x,c)\in \mathbb{Z}_p^{\omega_1\times\lambda}\times C \mid \varphi(x)=\psi(c)\}$$

is G_δ-dense in $\mathbb{Z}_p^{\omega_1\times\lambda}\times C$.

Proof. Since the free rank of $\mathbb{Z}_p$ is 2^ω, the free rank of $\mathbb{Z}_p^\lambda$ is 2^λ. Thus, $\mathbb{Z}_p^\lambda$ contains a free abelian subgroup F of rank 2^λ. By our assumption, $|E|\leq 2^\lambda$, and so there exists a surjective group homomorphism $\varphi_0\colon F\to E$. One can extend φ_0 to a surjective group homomorphism $\varphi_1\colon \mathbb{Z}_p^\lambda\to E$, because E is divisible. Let $\varphi_2\colon \bigoplus\limits_{\omega_1}\mathbb{Z}_p^\lambda\to E$ denote the group homomorphism $\bigoplus\limits_{\omega_1}\varphi_1$. Since $\bigoplus\limits_{\omega_1}\mathbb{Z}_p^\lambda$ is naturally isomorphic to a subgroup of $(\mathbb{Z}_p^\lambda)^{\omega_1}=\mathbb{Z}_p^{\omega_1\times\lambda}$ and E is divisible, φ_2 extends to a group homomorphism $\varphi\colon \mathbb{Z}_p^{\omega_1\times\lambda}\to E$. We show that φ satisfies the stated properties.

(a) Since translation in $\mathbb{Z}_p^{\omega_1\times\lambda}$ is a homeomorphism, it suffices to show that each non-empty G_δ-subset of $\mathbb{Z}_p^{\omega_1\times\lambda}$ meets $\varphi^{-1}(0_E)$. Let B be a non-empty G_δ-subset, and let $z\in B$. There exists $K\subseteq\omega_1\times\lambda$ such that

$$N(z,K) := \{y\in \mathbb{Z}_p^{\omega_1\times\lambda}\mid \forall(\gamma,\delta)\in K, y_{\gamma,\delta}=z_{\gamma,\delta}\}\subseteq B,$$

and $|K|\leq\omega$. If $K\cap(\{\alpha\}\times\lambda)\neq\emptyset$ for every $\alpha\in\omega_1$, then $|K|\geq\omega_1>\omega$, contrary to the assumption that K is countable. Thus, there is $\alpha_0\in\omega_1$ such that

$$K\cap(\{\alpha_0\}\times\lambda)=\emptyset.$$

Since φ_1 is surjective, there is $w=(w_\beta)_{\beta\in\lambda}\in\mathbb{Z}_p^\lambda$ such that $\varphi_1(w)=\varphi(z)$. Let $r\in\mathbb{Z}_p^{\omega_1\times\lambda}$ denote the element defined by

$$r_{\alpha,\beta}=\begin{cases} w_\beta & \text{if } \alpha=\alpha_0\\ 0 & \text{otherwise.}\end{cases}$$

Since $r\in\bigoplus\limits_{\omega_1}\mathbb{Z}_p^\lambda$, one has $\varphi(r)=\varphi_2(r)=\varphi_1(w)=\varphi(z)$, and therefore $\varphi(z-r)=0$. To conclude, observe that $z-r\in N(z,K)$, because z and $z-r$ differ only at coordinates of the form (α_0,β), and α_0 was chosen such that $K\cap(\{\alpha_0\}\times\lambda)=\emptyset$. Hence, $z-r\in N(z,K)\cap\varphi^{-1}(0_E)\neq\emptyset$, as desired.

(b) By (a), the set $\varphi^{-1}(\psi(c))\times\{c\}$ is G_δ-dense in $\mathbb{Z}_p^{\omega_1\times\lambda}\times\{c\}$ for every $c\in C$. Consequently,

$$\mathbb{Z}_p^{\omega_1\times\lambda}\times_E C=\bigcup_{c\in C}(\varphi^{-1}(\psi(c))\times\{c\})$$

is G_δ-dense in $\mathbb{Z}_p^{\omega_1\times\lambda}\times C$. ∎

Proof of Theorem 5.6. (i) $\Rightarrow$ (ii): By Theorem 5.3, there is a connected locally compact abelian group C such that A is topologically isomorphic to a subgroup of C. Without loss of generality, we may assume that A is actually a subgroup of C. Put $E:=C/A$ and $\lambda=w(C)$, and let $\psi\colon C\to E$ denote the canonical projection. (Unless A is locally compact, this quotient is not Hausdorff, but we are interested in E only as an abstract group, and ignore its topological properties.) By Theorem 5.7, C is divisible, and thus E is divisible as well. Since $|E|\leq|C|\leq 2^{w(C)}=2^{\lambda}$, the subgroup $G:=\mathbb{Z}_p^{\omega_1\times\lambda}\times_E C$ of the group $L:=\mathbb{Z}_p^{\omega_1\times\lambda}\times C$ provided by Lemma 5.8(b) is G_δ-dense in L. Being a product of a compact and a locally compact group, L is locally compact, and consequently $L=\widetilde{G}$. Therefore, G is locally precompact, and by (the implication (vii) $\Rightarrow$ (i) of) Theorem 1.4, G is locally pseudocompact. One has $L_0=\{0\}\times C$, because $\mathbb{Z}_p$ is zero-dimensional. Hence,

$$L_0\cap G=\{(x,c)\in\mathbb{Z}_p^{\omega_1\times\lambda}\times C\mid\varphi(x)=\psi(c),x=0\}\tag{1}$$
$$=\{(0,c)\in\mathbb{Z}_p^{\omega_1\times\lambda}\times C\mid\psi(c)=0\}=\{0\}\times\ker\psi=\{0\}\times A,\tag{2}$$

where φ is the homomorphism constructed in Lemma 5.8.

(a) If $w(A)\geq\omega_1$, then A is infinite, and therefore by Lemma 5.3(a), C may be chosen such that $w(C)=w(A)$. Hence,

$$w(G)=w(L)=\omega_1\cdot\lambda\cdot w(C)=w(C)=w(A),\tag{3}$$

as required.

(b) If A is precompact, then by Lemma 5.3(b), C may be chosen to be compact. Thus, the group L is compact, being a product of two compact groups. Therefore, the G_δ-dense subgroup G of L is pseudocompact (cf. [9]).

(c) If A is connected and G is the group provided by (ii), then $q(G)$ is connected, and therefore $G_0=q(G)$. Hence, the statement follows by (ii).

(ii) $\Rightarrow$ (i): Since G is locally pseudocompact, its completion $\widetilde{G}$ is locally compact, and thus $(\widetilde{G})_0$ is a connected locally compact abelian group. By our assumption, A is topologically isomorphic to a subgroup of $(\widetilde{G})_0$, specifically, to $(\widetilde{G})_0\cap G$. Hence, by Theorem 5.3, A is precompactly generated. ■

Remark 5.9. We note that when A is metrizable, the indicated equivalence of Theorem 5.6 holds, but the choice of G with $w(G)=w(A)$ may be impossible. Indeed, if A is metrizable and precompactly generated, then $w(A)=\omega$. Consequently, if G is a locally pseudocompact group such that $w(G)=w(A)$, then G is locally compact. Thus, $q(G)$ is locally compact (being a closed subgroup of G), and by Theorem 4.2, $q(G)$ is connected. Hence, A can be topologically isomorphic to $q(G)$ only if A itself is connected and locally compact; in that case, one can take $G=A$. In particular, for $A:=\mathbb{Q}$, no locally pseudocompact G can satisfy both (ii) of Theorem 5.6 and $w(G)=w(A)$.

Acknowledgements

The authors wish to thank Dikran Dikranjan, Kenneth A. Ross, and Javier Trigos-Arrieta for valuable electronic correspondence, and James D. Reid and Grant Woods for valuable discussions. The authors are grateful to Karen Kipper for her kind help in proofreading this paper for grammar and punctuation. The authors appreciate the wealth of constructive comments of the anonymous referees that led to an improved presentation of this paper.

References

[1] Arhangel′skiĭ, A. V., *Cardinal invariants of topological groups. Embeddings and condensations.*, Dokl. Akad. Nauk SSSR **247** (1979), 779–782.

[2] Arhangel′skiĭ, A. V., and M. Tkachenko, "Topological groups and related structures," Atlantis Studies in Mathematics, World Scientific, 2008.

[3] Blair, R. L., *Spaces in which special sets are z-embedded*, Canad. J. Math. **28** (1976), 673–690.

[4] Blair, R. L., and A. W. Hager, *Notes on the Hewitt realcompactification of a product*, General Topology and Appl. **5** (1975), 1–8.

[5] Comfort, W. W., S. Hernández, and F. J. Trigos-Arrieta, *Relating a locally compact abelian group to its Bohr compactification*, Adv. Math. **120** (1996), 322–344.

[6] Comfort, W. W., and J. van Mill, *On the existence of free topological groups*, Topology Appl. **29** (1988), 245–265.

[7] Comfort, W. W. and J. van Mill, *Concerning connected, pseudocompact abelian groups*, Topology Appl. **33** (1989), 21–45.

[8] Comfort, W. W., and S. Negrepontis, "Continuous pseudometrics," Lecture Notes in Pure and Applied Mathematics, Vol. 14, Marcel Dekker Inc., New York, 1975.

[9] Comfort, W. W., and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific J. Math. **16** (1966), 483–496.

[10] Comfort, W. W., and F. J. Trigos-Arrieta, *Locally pseudocompact topological groups*, Topology Appl. **62** (1995), 263–280.

[11] Dieudonné, J., *Sur les espaces uniformes complets*, Ann. École Norm. **56** (1939), 277–291.

[12] Dikranjan, D., *Connectedness and disconnectedness in pseudocompact groups*, Rend. Accad. Naz. Sci. XL Mem. Mat. (5), **16** (1992), 211–221.

[13] —, *Dimension and connectedness in pseudo-compact groups*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), 309–314.

[14] —, *Zero-dimensionality of some pseudocompact groups*, Proc. Amer. Math. Soc. **120** (1994), 1299–1308.

[15] —, *Compactness and connectedness in topological groups*, Topology Appl. **84** (1998), 227–252.

[16] —, *Closure operators in topological groups related to von Neumann's kernel*, Topology Appl. **153** (2006), 1930–1955.

[17] —, *CLP-compactness for topological spaces and groups*, Topology Appl. **154** (2007), 1321–1340.

[18] Dikranjan, D., and G. Lukács, *On zero-dimensionality and the connected component of locally pseudocompact groups*, submitted, ArXiv: 0909.1390.

[19] Dikranjan, D., I. R. Prodanov, and L. N. Stoyanov, "Topological groups: Characters, dualities and minimal group topologies," volume **130** of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc., New York, 1990.

[20] Dikranjan, D., and M. Tkachenko, *Weakly complete free topological groups*, Topology Appl. **112** (2001), 259–287.

[21] Engelking, R., "General Topology," volume **6** of Sigma Series in Pure Mathematics, Heldermann Verlag, Berlin, second edition, 1989.

[22] Gillman, L., and M. Jerison, "Rings of continuous functions," The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960.

[23] Glicksberg, I., *The representation of functionals by integrals*, Duke Math. J. **19** (1952), 253–261.

[24] Graev, M. I., *Theory of topological groups I. Norms and metrics on groups. Complete groups. Free topological groups.*, Uspehi Matem. Nauk (N.S.), **5** (1950), 3–56.

[25] Grant, D. L., *Topological groups which satisfy an open mapping theorem*, Pacific J. Math. **68** (1977), 411–423.

[26] Guran, I. I., *On topological groups close to being Lindelöf*, Soviet Math. Dokl. **23** (1981), 173–175.

[27] Hager, A. W., *On inverse-closed subalgebras of $C(X)$*, Proc. London Math. Soc. (3) **19** (1969), 233–257.

[28] Henriksen, M., and D. G. Johnson, *On the structure of a class of archimedean lattice-ordered algebras*, Fund. Math. **50** (1961/1962), 73–94.

[29] Hernández, S., and S. Macario, *Dual properties in totally bounded abelian groups*, Arch. Math. (Basel) **80** (2003), 271–283.

[30] Hewitt, E., *Rings of real-valued continuous functions. I.*, Trans. Amer. Math. Soc. **64** (1948), 45–99.

[31] Hewitt, E., and K. A. Ross, "Abstract harmonic analysis," Academic Press Inc., Publishers, New York, 1963.

[32] Hofmann, K. H., and S. A. Morris, "The structure of compact groups: A primer for the student—a handbook for the expert," volume **25** of de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, 1998.

[33] —, *Projective limits of finite-dimensional Lie groups*, Proc. London Math. Soc. (3) **87** (2003), 647–676.

[34] —, "The Lie theory of connected pro-Lie groups: A structure theory for pro-Lie algebras, pro-Lie groups, and connected locally compact groups," volume **2** of EMS Tracts in Mathematics, European Mathematical Society (EMS), Zürich, 2007.

[35] Hofmann, K. H., and K.-H. Neeb, *The compact generation of closed subgroups of locally compact groups*, J. Group Theory **12** (2009), 555–559.

[36] Isbell, J. R., *Algebras of uniformly continuous functions*, Ann. of Math. (2) **68** (1958), 96–125.

[37] —, "Uniform spaces," American Mathematical Society, Providence, R.I., 1964.

[38] Kac, G. I., *Isomorphic mapping of topological groups into a direct product of groups satisfying the first countability axiom*, Uspehi Matem. Nauk (N.S.) **8** (1953), 107–113.

[39] Kelley, J. L., "General Topology," D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[40] Kunen, K., "Set theory: An introduction to independence proofs," volume **102** of Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1983.

[41] Kurosh, A. G., "The theory of groups," Chelsea Publishing Co., New York, 1960.

[42] Lukács, G., "Compact-like Topological Groups," volume **31** of Research and Exposition in Mathematics, Heldermann Verlag, Berlin, 2009.

[43] Mack, J., M. Rayburn, and G. Woods, *Local topological properties and one point extensions*, Canad. J. Math. **24** (1972), 338–348.

[44] Mackey, G. W., *Equivalence of a problem in measure theory to a problem in the theory of vector lattices*, Bull. Amer. Math. Soc. **50** (1944), 719–722.

[45] Morris, S. A., *Locally compact abelian groups and the variety of topological groups generated by the reals*, Proc. Amer. Math. Soc. **34** (1972), 290–292.

[46] —, *Erratum to: "Locally compact abelian groups and the variety of topological groups generated by the reals" (Proc. Amer. Math. Soc. 34 (1972), 290–292)*, Proc. Amer. Math. Soc. **51** (1975), 503.

[47] —, "Pontryagin duality and the structure of locally compact abelian groups," London Mathematical Society Lecture Note Series, No. **29**, Cambridge University Press, Cambridge, 1977.

[48] Raĭkov, D., *On the completion of topological groups*, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10** (1946), 513–528.

[49] Roelcke, W., and S. Dierolf, "Uniform structures on topological groups and their quotients," McGraw-Hill International Book Co., New York, 1981.

[50] Ross, K. A., and K. Stromberg, *Baire sets and Baire measures*, Ark. Mat. **6** (1965), 151–160.

[51] Ščepin, E. V., *Topological products, groups, and a new class of spaces that are more general than metric spaces*, Dokl. Akad. Nauk SSSR **226** (1976), 527–529.

[52] —, *κ-metrizable spaces*, Akad. Nauk SSSR Ser. Mat. **43** (1979), 442–478; English translation: Math. USSR-Izv. **14** (1980), 407–440.

[53] Shirota, T., *A class of topological spaces*, Osaka Math. J. **4** (1952), 23–40.

[54] Stroppel, M., "Locally compact groups," EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2006.

[55] Sulley, L. J., *A note on B- and B_r-complete topological Abelian groups*, Proc. Cambridge Philos. Soc. **66** (1969), 275–279.

[56] Tarski, A., *Drei Überdeckungssätze der allgemeinen Mengenlehre*, Fund. Math. **30** (1938), 132–155.

[57] Tkachenko, M., *The notion of* o*-tightness and C-embedded subspaces of products*, Topology Appl. **15** (1983), 93–98.

[58] —, *Introduction to topological groups*, Topology Appl. **86** (1998), 179–231.

[59] Ulam, S., *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. **16** (1930), 140–150.

[60] Ursul, M. I., *An example of a plane group whose quasicomponent does not coincide with its component*, Mat. Zametki **38** (1985), 517–522, 634.

[61] —, *Embedding of locally precompact groups into locally pseudocompact groups*, Izv. Akad. Nauk Moldav. SSR Ser. Fiz.-Tekhn. Mat. Nauk, (1989), 54–56, 77.

[62] —, *Embedding of precompact groups into pseudocompact groups*, Izv. Akad. Nauk Moldav. SSR Mat., (1991), 88–89, 93.

[63] Weil, A., “Sur les Espaces à Structure Uniforme et sur la Topologie Générale,” Publ. Math. Univ. Strasbourg. Hermann et Cie., Paris, 1937.

[64] Yamabe, H., *A generalization of a theorem of Gleason*, Ann. of Math. (2) **58** (1953), 351–365.

W. W. Comfort
Department of Mathematics
Wesleyan University
Middletown, CT 06459, USA
wcomfort@wesleyan.edu

Gábor Lukács
Department of Mathematics
University of Manitoba
Winnipeg, Manitoba, R3T 2N2, Canada
lukacs@cc.umanitoba.ca

Received November 23, 2009
and in final form April 2, 2010